Tik III lugh 6

(1) $D(n,n+1) = \frac{4^n}{4^n-1} D(n,n) - \frac{1}{4^n-1} D(n-1,n)$

$$= \frac{4^n}{4^n-1}\left(L + \sum_{k=n}^{\infty} A(k,n)\left(\frac{h}{2^n}\right)^{2k}\right) - \frac{1}{4^n-1}\left(L + \sum_{k=n}^{\infty} A(k,n)\left(\frac{h}{2^{n-1}}\right)^{2k}\right)$$

$$= \frac{4^n-1}{4^n-1} L + \frac{4^n}{4^n-1}\sum_{k=n}^{\infty} A(k,n)\left(\frac{h}{2^n}\right)^{2k} - \frac{1}{4^n-1}\sum_{k=n}^{\infty} A(k,n)\left(\frac{h}{2^{n-1}}\right)^{2k}$$

$$= L + \frac{4^n}{4^n-1}\sum_{k=n}^{\infty} A(k,n)\left(\frac{h}{2^n}\right)^{2k} - \frac{1}{4^n-1}\sum_{k=n}^{\infty} A(k,n)\, 2^{2k}\left(\frac{h}{2^n}\right)^{2k}$$

$$= L + \frac{4^n}{4^n-1}\sum_{k=n}^{\infty} A(k,n)\left(\frac{h}{2^n}\right)^{2k} - \frac{1}{4^n-1}\sum_{k=n}^{\infty} A(k,n)\, 4^k\left(\frac{h}{2^n}\right)^{2k}$$

$$= L + \sum_{k=n}^{\infty}\left(\frac{4^n}{4^n-1} A(k,n)\left(\frac{h}{2^n}\right)^{2k} - \frac{4^k}{4^n-1} A(k,n)\left(\frac{h}{2^n}\right)^{2k}\right)$$

$$= L + \sum_{k=n}^{\infty}\left(\left(A(k,n)\left(\frac{h}{2^n}\right)^{2k}\right)\left(\underbrace{\frac{4^n-4^k}{4^n-1}}_{=0 \text{ when } k=n}\right)\right)$$

As we see, lowest-order error term (which is also biggest when $h<1$) vanishes. Now we are left with something that looks like $D(n,n)$ with some coefficient $\frac{4^n-4^k}{4^n-1}$. Now leading error term is proportional to $h^{2(k+1)}$ instead of h^{2k} which means error is $O(h^2)$ smaller. $\square$

2. From re_der I get -2.4783497330, which agrees with manually calculated derivative for all but last shown decimal. Relative errors of derivatives with various n are shown below. They differ somewhat from re_der and WolframAlpha.

n	central diff	re_der
4	-4.5247938828e-09	4.1015831904e-11
6	-6.3249950314e-11	-5.2888637544e-09
8	-3.0310423649e-09	1.3720468244e-07
10	-2.1805600221e-07	2.9124720360e-05
12	6.7649196950e-05	2.5709034249e-03
14	-3.2697173674e-03	2.8927047776e-02

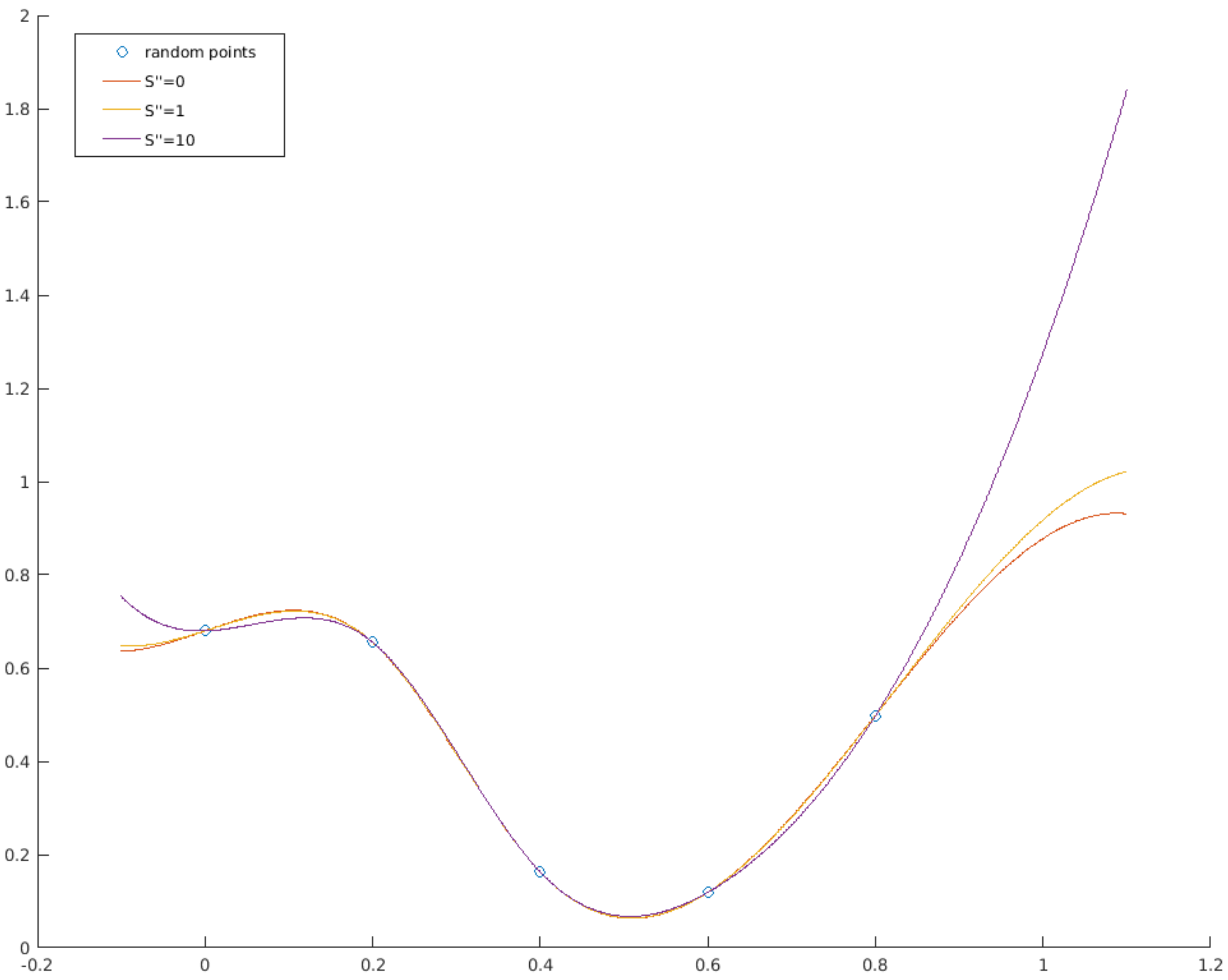
Error of re_der seems to grow when h gets smaller from 10^-4 whereas central difference method gives best results with h=10^-6.

3. I generated 5 or 20 random y-values in range [0,1] wit equally many evenly spaced values in range [0, 1[(spacing with which range [0,1] would have contained 11 points ie. 0.2 or 0.05). For getting the spline I used matlab function csape giving ‘second’ and wanted second derivatives like for S’’=0

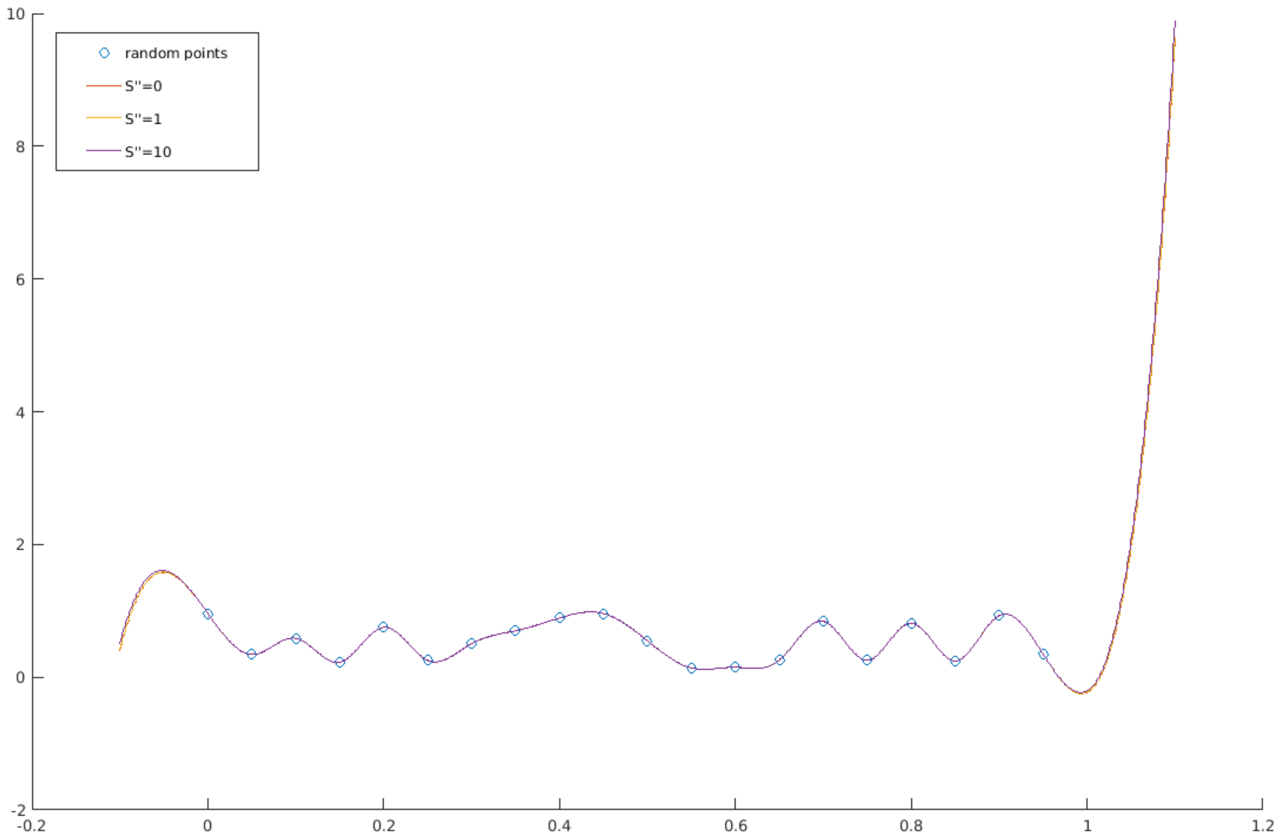
```
p1 = csape(x, y, 'second', [0, 0])
```

Csape returns a polynomial, values of which I evaluated and then plotted together with the original points.

Plot with 5 points:



plot with 20 points:



(4) $N_i^1 = \left(\frac{t-t_i}{t_{i+1}-t_i}\right) N_i^0(t) + \left(\frac{t_{i+2}-t}{t_{i+2}-t_{i+1}}\right) N_{i+1}^0(t)$

$= \left(\frac{t-i}{i+1-i}\right) N_i^0(t) + \left(\frac{i+2-t}{i+2-i-1}\right) N_{i+1}^0(t)$

$= (t-i)\, N_i^0(t) + (2+i-t)\, N_{i+1}^0(t)$

Smallest reasonable number of nodes for first order spline is 2 so

$N_0^1(t) = t N_0^0 + (2-t)\, N_1^0(t)$

$= \begin{cases} t, & t \in [0,1[\\ 2-t, & t \in [1,2[\end{cases}$ 1^{st} order spline

$N_1^1(t) = (t-1)\, N_1^0(t) + (2+1-t)\, N_2^0(t)$

$= \begin{cases} t-1, & t \in [1,2[\\ 3-t, & t \in [2,3[\end{cases}$

$N_2^1(t) = (t-2)\, N_2^0(t) + (2+2-t)\, N_3^0(t)$

$= \begin{cases} t-2, & t \in [2,3[\\ 4-t, & t \in [3,4[\end{cases}$

(4) $N_i^2(t) = \left(\frac{t-t_i}{t_{i+2}-t_i}\right) N_i^1(t) + \left(\frac{t_{i+3}-t}{t_{i+2}-t_{i+1}}\right) N_{i+1}^1(t)$

$$= \left(\frac{t-t_i}{t_{i+2}-t_i}\right)\left[\left(\frac{t-t_i}{t_{i+1}-t_i}\right) N_i^0 + \left(\frac{t_{i+2}-t}{t_{i+2}-t_{i+1}}\right) N_{i+1}^0\right] +$$

$$+ \left(\frac{t_{i+3}-t}{t_{i+2}-t_{i+1}}\right)\left[\left(\frac{t-t_{i+1}}{t_{i+2}-t_{i+1}}\right) N_{i+1}^0 + \left(\frac{t_{i+3}-t}{t_{i+3}-t_{i+2}}\right) N_{i+2}^0\right]$$

$$= \frac{(t-t_i)^2}{(t_{i+2}-t_i)(t_{i+1}-t_i)} N_i^0 + \frac{(t-t_i)(t_{i+2}-t)}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} N_{i+1}^0 +$$

$$+ \frac{(t_{i+3}-t)(t-t_{i+1})}{(t_{i+2}-t_{i+1})^2} N_{i+1}^0 + \frac{(t_{i+3}-t)^2}{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+2})} N_{i+2}^0$$

$$N_0^2(t) = \frac{t^2}{(2-0)(1-0)} N_0^0 + \frac{t(2-t)}{(2-0)(2-1)} N_1^0 + \frac{(3-t)(t-1)}{(2-1)^2} N_1^0 +$$

$$+ \frac{(3-t)^2}{(2-1)(3-2)} N_2^0$$

$$= \frac{t^2}{2} N_0^0 + \left(t - \frac{t^2}{2} + 3t - \frac{t^2}{2} - 3 - t\right) N_1^0 + \left(\frac{9}{2} - 3t + \frac{t^2}{2}\right) N_2^0$$

$$= \frac{t^2}{2} N_0^0 + \left(-t^2 + 3t - 3\right) N_1^0 + \left(\frac{t^2}{2} - 3t + \frac{9}{2}\right) N_2^0$$

$$= \begin{cases} \frac{t^2}{2} & t \in [0, 1[\\ -t^2 + 3t - 3 & t \in [1, 2[\\ \frac{t^2}{2} - 3t + \frac{9}{2} & t \in [2, 3[\end{cases}$$

2nd order spline

(4)

$$N_0^3 = \frac{t}{3} N_0^2 + \frac{4-t}{4-1} N_1^2$$

$$= \frac{t}{3} \cdot \left(\frac{t}{2} N_0^1(t) + \frac{3-t}{3-1} N_1^1 \right) + \frac{4-t}{3} \left(\frac{t-1}{3-1} N_1^1 + \frac{4-t}{3} N_2^1 \right)$$

$$= \frac{t^2}{6} \left(\frac{t}{1} N_0^0(t) + \frac{2-t}{2-1} N_1^0(t) \right) + \frac{3t-t^2}{6} \left(\frac{t-1}{2-1} N_1^0 + \frac{3-t}{3-2} N_2^0 \right) +$$

$$+ \frac{4t-4-t^2+t}{6} \left(\frac{t-1}{2-1} N_0^0(t) + \frac{3-t}{3-2} N_2^0(t) \right) +$$

$$+ \frac{16-8t+t^2}{9} \left(\frac{t-2}{3-2} N_2^0 + \frac{4-t}{4-3} N_3^0(t) \right)$$

$$= \frac{t^3}{6} N_0^0 + \frac{2t^2-t^3}{6} N_1^0 + \frac{3t^2-3t-t^3+t^2}{6} N_1^0 + \frac{9t-3t^2-3t^2+t^3}{6} N_2^0 +$$

$$+ \frac{4t^2-4t-t^3+t^2-4t+4+t^2-t}{6} N_0^0 + \frac{12t-4t^2-12+4t-3t^2+t^3+3t-t^2}{6} N_2^0 +$$

$$+ \frac{16t-8t^2+t^3-32+16t-2t^2}{9} N_2^0 + \frac{64-16t-32t+8t^2+4t^2-t^3}{9} N_3^0$$

$$= \left(\frac{t^3}{6} - \frac{t^3}{6} + t^2 - \frac{3}{2}t + \frac{4}{6} \right) N_0^0 + \left(-\frac{t^3}{6} + \frac{2}{3}t^2 - \frac{1}{2}t \right) N_1^0 +$$

$$+ \left(\frac{t^3}{6} - t^2 + \frac{3}{2}t + \frac{t^3}{6} - \frac{7}{6}t^2 + \frac{19}{6}t - 2 + \frac{t^3}{9} - \frac{10}{9}t^2 + \frac{32}{9}t - \frac{32}{9} \right) N_2^0 +$$

$$+ \left(-\frac{t^3}{9} + \frac{4}{3}t^2 - \frac{16}{3}t + \frac{64}{9} \right) N_3^0$$

$$= \begin{cases} t^2 - \frac{3}{2}t + \frac{4}{6} & t \in [0,1[\\ -\frac{t^3}{6} + \frac{2}{3}t^2 - \frac{1}{2}t & t \in [1,2[\\ \frac{4}{9}t^3 - \frac{59}{18}t^2 + \frac{74}{9}t - 2 & t \in [3,4[\\ -\frac{t^3}{9} + \frac{4}{3}t^2 - \frac{16}{7}t + \frac{64}{9} & t \in [4,5[\end{cases}$$