

①

Multiplication:

$$(x \pm e_x)(y \pm e_y) = xy \pm xe_y \pm e_x y \pm \underbrace{e_x e_y}_{\approx 0} = xy \pm xy r_y \pm xy r_x$$

$$\Rightarrow r_{xy} = \frac{xy - (x \pm e_x)(y \pm e_y)}{xy} = \frac{xy - xy \pm xy r_y \pm xy r_x}{xy} = \pm r_x \pm r_y \quad (\text{sum of errors})$$

Division:

$$r_{xy} = \frac{\frac{x}{y} - \frac{x \pm e_x}{y \pm e_y}}{\frac{x}{y}} = \frac{\frac{xy \pm xe_y - xy \pm ye_x}{y(y \pm e_y)}}{\frac{x}{y}} = \frac{\pm ye_y \mp ye_x}{xy \pm xe_y} = \pm \frac{e_y}{y} \mp \frac{e_x}{x} = \pm r_y \mp r_x$$

(small  $e_y$  compared to  $y$ ) (difference of errors)

③

Let's consider an original binary number  $B$  and its complement  $C$ . What we want to prove is that  $C + 1 + B = 0$ . Because  $C$  has been obtained by complementing  $B$ , it is clear\* that in all positions where  $B$  has 0  $C$  has 1 and vice versa. Therefore sum  $B + C$  contains only ones. Because summation is commutative  $C + 1 + B = (B + C) + 1$  and therefore we can just add 1 to the previously obtained sum  $B + C = \underbrace{111 \dots 111}_{n \text{ ones}}$  thus getting  $B + C + 1 = \underbrace{1000 \dots 000}_{n \text{ zeros}}$ .

Due to the nature of computers, leading 1 overflows and thus it won't affect the result and we are left with  $\underbrace{000 \dots 000}_{n \text{ zeros}}$ . Thus  $C + 1 + B = 0$  i.e. complementing all bits of a number, adding one and adding the original number produces zero.  $\square$

\*Think about column addition: in each column there is exactly one "1" so the result contains only ones



$$(4) (B) e^x \approx 1 + \sum_{i=1}^{n-1} \frac{x^i}{i!} \cdot 1 + R(n)$$

$$\approx 1 + \sum_{i=1}^{n-1} \frac{x^i}{i!} + \frac{x^n}{n!} e^\xi$$

Near value  $x=0$  there's a spike (top row plot)

(C) Using the approximation above we can write  $f_1$  and  $f_2$  easier to compute near zero:

$$f_1 = \frac{e^x - 1}{x}$$

$$\approx \frac{1 + \sum_{i=1}^{n-1} \frac{x^i}{i!} - 1}{x}$$

$$\approx \sum_{i=1}^{n-1} \frac{x^{i-1}}{i!}$$

$$f_2 = \frac{e^x - e^{-x}}{2x}$$

$$\approx \frac{1 + \sum_{i=1}^{n-1} \frac{x^i}{i!} - 1 - \sum_{i=1}^{n-1} \frac{(-x)^i}{i!}}{2x}$$

Even terms cancel each other out and we are left with

$$f_2 \approx \frac{2 \sum_{i=0}^{n-1} \frac{x^{2i+1}}{(2i+1)!}}{2x}$$

$$\approx \sum_{i=0}^{n-1} \frac{x^{2i}}{(2i+1)!}$$

Using these approximations we can see that spikes present in first plots have disappeared.



(2) B) Calculating value of harmonic series normally gives a value of approximately 15,40 whereas using harmonic-bunch gives following values:

N	value
50	18,6
100	19,3
500	20,9

This makes sense because even if a single term  $\frac{1}{n}$  is too small compared to the already calculated part of the sum to affect the value of the float, it is very much possible that summing N "too small" floats together results in a float that is big enough to affect the sum.