Tilo III lesh 6

(1) $$D(n,n+1) = \frac{4^n}{4^n-1} D(n,n) - \frac{1}{4^n-1} D(n-1,n)$$

$$= \frac{4^n}{4^n-1}\left(L + \sum_{k=n}^{\infty} A(k,n)\left(\frac{h}{2^n}\right)^{2k}\right) - \frac{1}{4^n-1}\left(L + \sum_{k=n}^{\infty} A(k,n)\left(\frac{h}{2^{n-1}}\right)^{2k}\right)$$

$$= \frac{4^n-1}{4^n-1} L + \frac{4^n}{4^n-1}\sum_{k=n}^{\infty} A(k,n)\left(\frac{h}{2^n}\right)^{2k} - \frac{1}{4^n-1}\sum_{k=n}^{\infty} A(k,n)\left(\frac{h}{2^{n-1}}\right)^{2k}$$

$$= L + \frac{4^n}{4^n-1}\sum_{k=n}^{\infty} A(k,n)\left(\frac{h}{2^n}\right)^{2k} - \frac{1}{4^n-1}\sum_{k=n}^{\infty} A(k,n)\, 2^{2k}\left(\frac{h}{2^n}\right)^{2k}$$

$$= L + \frac{4^n}{4^n-1}\sum_{k=n}^{\infty} A(k,n)\left(\frac{h}{2^n}\right)^{2k} - \frac{1}{4^n-1}\sum_{k=n}^{\infty} A(k,n)\, 4^k\left(\frac{h}{2^n}\right)^{2k}$$

$$= L + \sum_{k=n}^{\infty}\left(\frac{4^n}{4^n-1} A(k,n)\left(\frac{h}{2^n}\right)^{2k} - \frac{4^k}{4^n-1} A(k,n)\left(\frac{h}{2^n}\right)^{2k}\right)$$

$$= L + \sum_{k=n}^{\infty}\left(\left(A(k,n)\left(\frac{h}{2^n}\right)^{2k}\right)\underbrace{\left(\frac{4^n-4^k}{4^n-1}\right)}_{=0 \text{ when } k=n}\right)$$

As we see, lowest-order error term (which is also biggest when $h<1$) vanishes. Now we are left with something that looks like $D(n,n)$ with some coefficient $\frac{4^n-4^k}{4^n-1}$. Now leading error term is proportional to $h^{2(k+1)}$ instead of h^{2k} which means error is $O(h^2)$ smaller. $\square$