

(1) For orthogonal matrix  $A$   $AA^T = I$  where  $I$  is unit matrix. Matrix  $Q_{pq}$  has the form

$$Q_{pq} = \begin{bmatrix} \underbrace{1 \dots 1}_{p-1 \text{ rows before } c} & & & & \\ & \underbrace{1 \dots 1}_{q-1 \text{ cols before } c} & & & \\ & & s & & \\ & & & c & \\ & & & & \ddots \\ & & & & & 1 \end{bmatrix} \begin{matrix} \left. \vphantom{\begin{matrix} 1 \\ \vdots \\ 1 \end{matrix}} \right\} p-1 \text{ rows before } c \\ \left. \vphantom{\begin{matrix} 1 \\ \vdots \\ 1 \end{matrix}} \right\} q-1 \text{ cols before } c \\ \text{zeros everywhere else} \end{matrix}$$

Therefore row  $i$  of  $N \times N$  matrix  $Q_{pq}$  is

$$[ \underbrace{0, 0, \dots, 0}_{i-1}, 1, \underbrace{0, 0, \dots, 0}_{N-i} ]$$

if  $i \neq p$  and  $i \neq q$

And or

$$[ \underbrace{0, 0, \dots, 0}_{p-1}, s, \underbrace{0, 0, \dots, 0}_{q-p-1}, c, \underbrace{0, 0, \dots, 0}_{N-q} ]$$

if  $i = p$

or

$$[ \underbrace{0, 0, \dots, 0}_{p-1}, s, \underbrace{0, 0, \dots, 0}_{q-p-1}, -s, \underbrace{0, 0, \dots, 0}_{N-q} ]$$

if  $i = q$



① And for columns  $j$  in matrix  $Q_{pq}^T$

$i \neq p$  and  $j \neq q$ :

$$i-1 \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right.$$

$$N-j \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right.$$

$j = q$ :

$$p-1 \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right.$$

$$q-r-1 \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right.$$

$$N-q \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right.$$

$j = p$

$$\left[ \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{array} \right] \left\{ \begin{array}{l} i-1 \\ q-r-1 \\ N-q \end{array} \right.$$

So multiplying row  $i$  of  $Q$  and column  $j$  of  $Q^T$   
when  $i = j$  (diagonal of  $QQ^T$ ) we set

$$(i-1) \cdot 0 + 1 \cdot 1 + (N-i) \cdot 0 = 1$$

and when  $i \neq j$  and  $i \neq p$  and  $i \neq q$  every index  
with something nonzero in column  $j$  is zeroed  
out with zero in row  $i$  and vice versa. Now  
for for example  $6 \times 6$  matrix  $Q_{2,5}$  we have  
filled all but cells marked with  $\square$

$$QQ^T = \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \square & \square & 1 & \square & \square & \square \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \square & \square & \square & \square & 1 & \square \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Only rows  $i = p$  and  
 $i = q$  remain unknown.



① Next we'll check the case  $i=p$  and  $j \neq i$ .  
If  $j=q$  multiplication yields

$$\begin{aligned} & 0+0+\dots+0+cs+0+0+\dots+0-cs+0+\dots+0 \\ &= 0+cs-cs \\ &= 0 \end{aligned}$$

And for cells  $j \neq p$  we get

$$\begin{aligned} & 0+0+\dots+0+cs+0+\dots+0-cs+0+\dots+0 \\ &= cs-cs \\ &= 0 \end{aligned}$$

Therefore all for all  $i \neq j$ ,  $(QQ^T)_{ij} = 0$   
and for all  $i=j$ ,  $(QQ^T)_{ij} = 1$   
which means that  $QQ^T = I$  and  
therefore  $Q$  is orthogonal  $\square$