Introduction to Machine Learning
Homework 6: Support Vector Machines

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1. Consider the data set for four points with features $\mathbf{x}_i = (x_{i1}, x_{i2})$ and binary class labels $y_i = \pm 1$.

x_{i1}	0	1	1	2
x_{i2}	0	0.3	0.7	1
y_i	-1	-1	1	1

   (a) Find a linear classifier that separates the two classes. Your classifier should be of the form

$$\hat{y} = \begin{cases} 1 & \text{if } b + w_1 x_1 + w_2 x_2 > 0 \\ -1 & \text{if } b + w_1 x_1 + w_2 x_2 < 0 \end{cases}$$

   State the intercept b and weights w_1 and w_2 for your classifier. Note there is no unique answer as there are multiple linear classifiers that could separate the classes.

   (b) Find the maximum γ such that

$$y_i(b + w_1 x_{i1} + w_{i2} x_{i2}) \geq \gamma, \text{ for all } i,$$

   for the classifier in part (a)?

   (c) Compute the margin of the classifier

$$m = \frac{\gamma}{\|\mathbf{w}\|}, \quad \|\mathbf{w}\| = \sqrt{w_1^2 + w_2^2}.$$

   (d) Which samples i are on the margin for your classifier?

2. Consider the data set with scalar features x_i and binary class labels $y_i = \pm 1$.

x_i	0	1.3	2.1	2.8	4.2	5.7
y_i	-1	-1	-1	1	-1	1

   Consider a linear classifier for this data of the form,

$$\hat{y} = \begin{cases} 1 & z > 0 \\ -1 & z < 0, \end{cases} \quad z = x - t,$$

   where t is a threshold. For each threshold t, let $J(t)$ denote the sum hinge loss,

$$J(t) = \sum_i \epsilon_i, \quad \epsilon_i = \max(0, 1 - y_i z_i).$$

(a) Write a short python program to plot $J(t)$ vs. t for 100 values of t in the interval $t \in [0, 5]$.

(b) Based on the plot, what is one value of t that minimizes $J(t)$.

(c) For the value of t in part (b), find the corresponding slack variables ϵ_i.

(d) Which samples i violate the margin ($\epsilon_i > 0$) and which samples i are misclassified ($\epsilon_i > 1$).

3. Consider an image recognition problem, where an image $\mathbf{X}$ and filter $\mathbf{W}$ are 4×4 matrices:

$$\mathbf{X} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Recall that in linear classification, the 4×4 image matrices $\mathbf{X}$ and $\mathbf{W}$ can be represented as 16-dimensional vectors, $\mathbf{x} = \mathrm{vec}(\mathbf{X})$ and $\mathbf{w} = \mathrm{vec}(\mathbf{W})$ by stacking the columns of the matrices vertically. What are $\mathbf{x}$ and $\mathbf{w}$ for the matrices above.

(b) What is the inner product $z = \mathbf{w}^\mathsf{T}\mathbf{x}$.

(c) What is the inner product $z = \mathbf{w}^\mathsf{T}\mathbf{x}_{\text{right}}$ where $\mathbf{x}_{\text{right}}$ is the vector corresponding to the matrix $\mathbf{X}$ right shifted by one pixel with the left column filled with zeros.

(d) What is the inner product $z = \mathbf{w}^\mathsf{T}\mathbf{x}_{\text{left}}$ where $\mathbf{x}_{\text{left}}$ is the vector corresponding to the matrix $\mathbf{X}$ left shifted by one pixel with the right column filled with zeros.

(e) Write the python command that can covert a 4×4 image matrix, `Xmat` to the 16-dimensional vector, `x`. What is the python command to go from `x` to `Xmat`.

4. Consider the data set with scalar features x_i and binary class labels $y_i = \pm 1$.

x_i	0	1	2	3
y_i	1	-1	1	-1

A support vector classifier is of the form

$$\hat{y} = \begin{cases} 1 & z > 0 \\ -1 & z < 0, \end{cases} \qquad z = \sum_i \alpha_i y_i K(x_i, x),$$

where $K(x, x')$ is the radial basis function, $K(x, x') = e^{-\gamma(x-x')^2}$, and $\gamma > 0$ and $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_4]$ are parameters of the classifier.

(a) Use python to plot z vs. x and $\hat{y}$ vs. x when $\gamma = 3$ and $\boldsymbol{\alpha} = [0, 0, 1, 1]$.

(b) Repeat (a) with $\gamma = 0.3$ and $\boldsymbol{\alpha} = [1, 1, 1, 1]$.

(c) Which classifier makes more errors on the training data.