let three fns be f, g and h

$g * (f * h)(x) = g * (f * h)(x)$

$= g(x) * \int f(x-u)\, h(u)\, du$

$= \int g(x-u') \cdot \left[\int f(u'-u)\, h(u)\, du \right] du'$

$= \int\int g(x-u')\, f(u'-u)\, h(u)\, du\, du'$

let $u'-u = v$

$=$

$= \int\int g(x-u-v)\, f(v)\, h(u)\, du\, dv$

$= \int\int g(x-u-v)\, f(v)\, dv\quad h(u)\, du$

$= \cancel{\int g(x-u}$

$= \int (g * f)(x-u)\quad h(u)\, du$

$= (g * f) * h(x)$

ASSIGNMENT - 6

Let, f be a function that represents an intensity image and g be a gaussian kernel with standard deviation σ. $g(x) = N(x; \mu, \sigma)$

Now, convolving f two times with g gives:

$$(f \circ g) \circ g = f \circ (g \circ g) \quad [\text{distributive property}]$$

if we can prove that $g \circ g = g'$ (say) and g' is a gaussian kernel with standard deviation $\sigma\sqrt{2}$, then that will imply the statement given. ~~g = $N(x; \mu, \sigma\sqrt{2})$~~

Now, $g(x) = N(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$$\mathcal{F}(g) = e^{-i\omega\mu} e^{-\sigma^2\frac{\omega^2}{2}} \quad [\text{Fourier transform of } g(x)]$$

$$g'(z) = (g \circ g)(z)$$

$$= \mathcal{F}^{-1}\{\mathcal{F}(g) \cdot \mathcal{F}(g)\}$$

$$= \mathcal{F}^{-1}\left(e^{-i\omega\mu} e^{-\sigma^2\frac{\omega^2}{2}}\right)^2$$

$$= \mathcal{F}^{-1}\left[e^{-i\omega(2\mu)} \cdot e^{-(2\sigma^2)\frac{\omega^2}{2}}\right]$$

Let, $\mu' = 2\mu$ and $\sigma'^2 = 2\sigma^2$

$$g'(z) = \mathcal{F}^{-1}\left[e^{-i\omega\mu'} e^{-\sigma'^2\frac{\omega^2}{2}}\right]$$

~~By definition~~

$$= \frac{1}{\sqrt{2\pi}\,\sigma'} e^{-\frac{1}{2}\left(\frac{z-\mu'}{\sigma'}\right)^2}$$

$$= N(z; \mu', \sigma')$$

g' is a gaussian kernel with standard deviation $\sigma' = \sqrt{2\sigma^2} = \sigma\sqrt{2}$. [Proved]