Outline

- 1D and 2D Poisson Equations
- Solution using Finite Differences
- Solution using Spectral Methods
- Introduction to OpenMP

Numerical Solution of Poisson Equation

Poisson eqn is one elliptic PDE

Elliptic PDE : solution in the entire domain is tightly coupled

$\Rightarrow$ only smooth solutions

easy

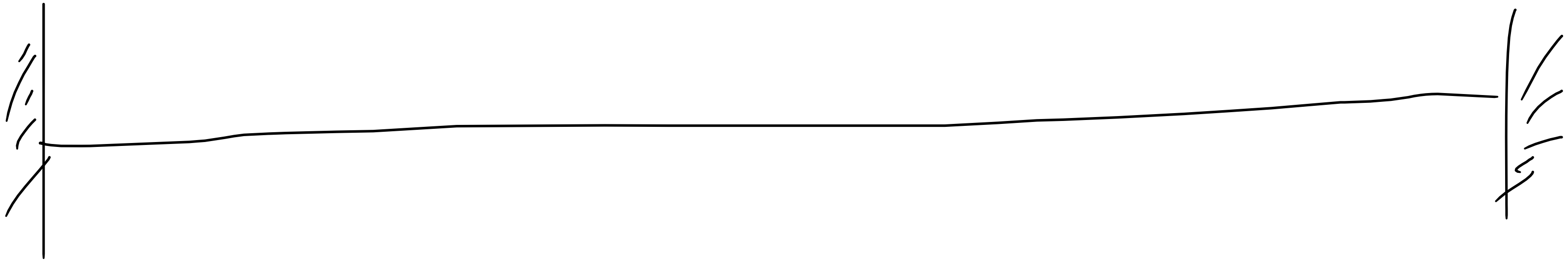
expensive

Numerical Solution of Poisson Equation

1D Poisson Eqn:

$$\frac{\partial^2 u}{\partial x^2} + \dot{q} = 0$$

$\dot{q}(x)$ is source ; given



$x = 0$ $\qquad$ L

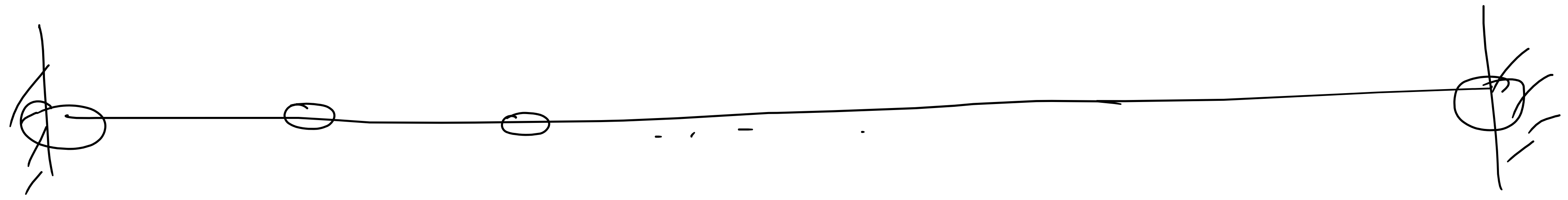
General B.C

$$p\frac{\partial u}{\partial x} + qu = r$$

(at 0, L)

Numerical Solution of Poisson Equation

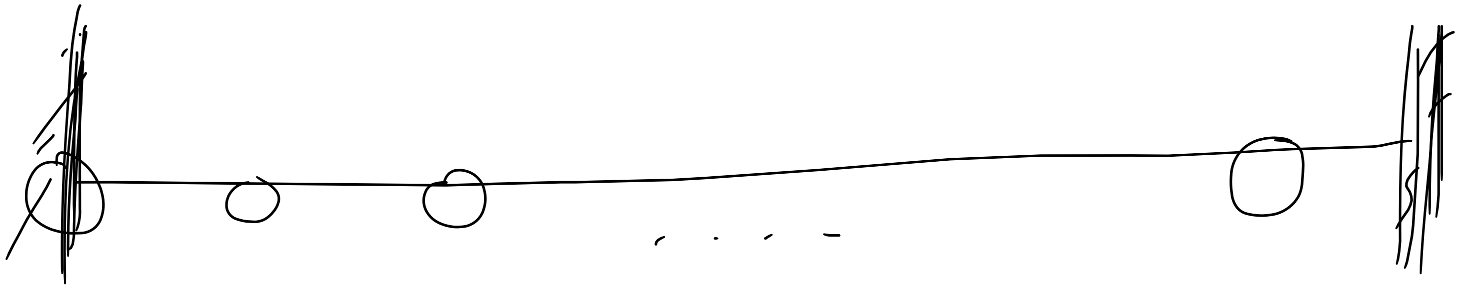
Numerical solution requires discretiz^{n}



$x_0 = 0$ $\qquad\qquad$ $x_{nx-1} = L$

$$h = \frac{L}{nx-1} \qquad x_i = i\,h \qquad i = 0, 1, \ldots nx-1$$

Numerical Solution of Poisson Equation



← Arrangement needed for periodic BCs

Finite Differences : 2^{nd}-order Central Diff.

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_i \approx \frac{u_{i+1} + u_{i-1} - 2u_i}{h^2} \qquad O(h^2)$$

Numerical Solution of Poisson Equation

$$\frac{u_{i+1} + u_{i-1} - 2u_i}{h^2} + q_i = 0$$

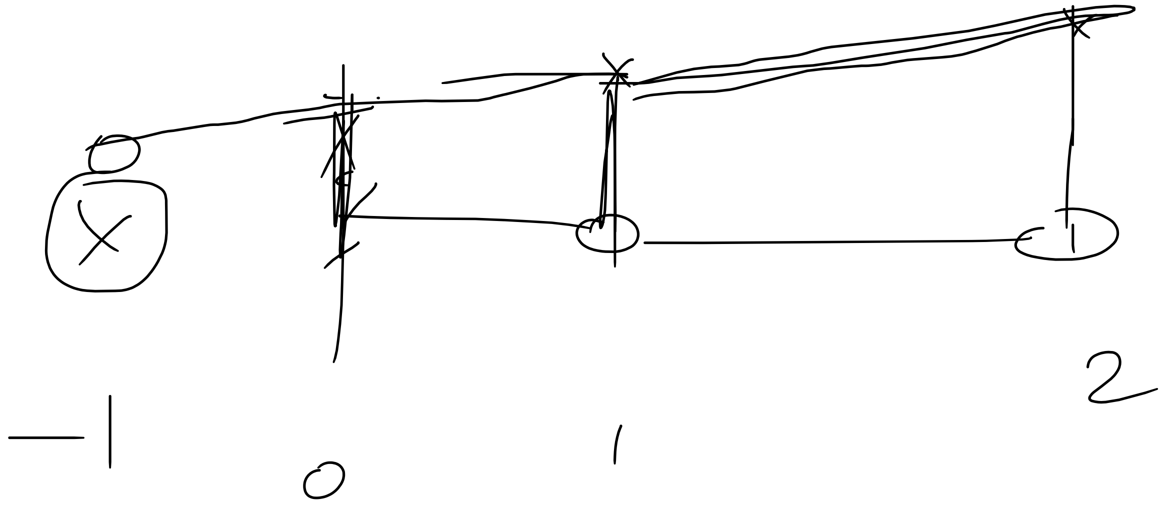
$$i = 1, 2, 3, \ldots nx-2$$

At $i=0$:

$$p_0 \left(\frac{u_1 - u_{-1}}{2h} \right) + q_0 u_0 = r_0$$

2^{nd}-ord CD for $\frac{\partial \varphi}{\partial x}$ $\longrightarrow$

Numerical Solution of Poisson Equation



$$\boxed{u_{-1} = 2u_0 - u_1}$$

Ghost Point

$$u_{-1} = \{\text{extrapolation from the interior}\}$$

$$= \{u_1, u_0\}$$

Numerical Solution of Poisson Equation

$$p_0 \left(\frac{u_1 - 2u_0 + u_1}{2h} \right) + q_0 u_0 = r_0$$

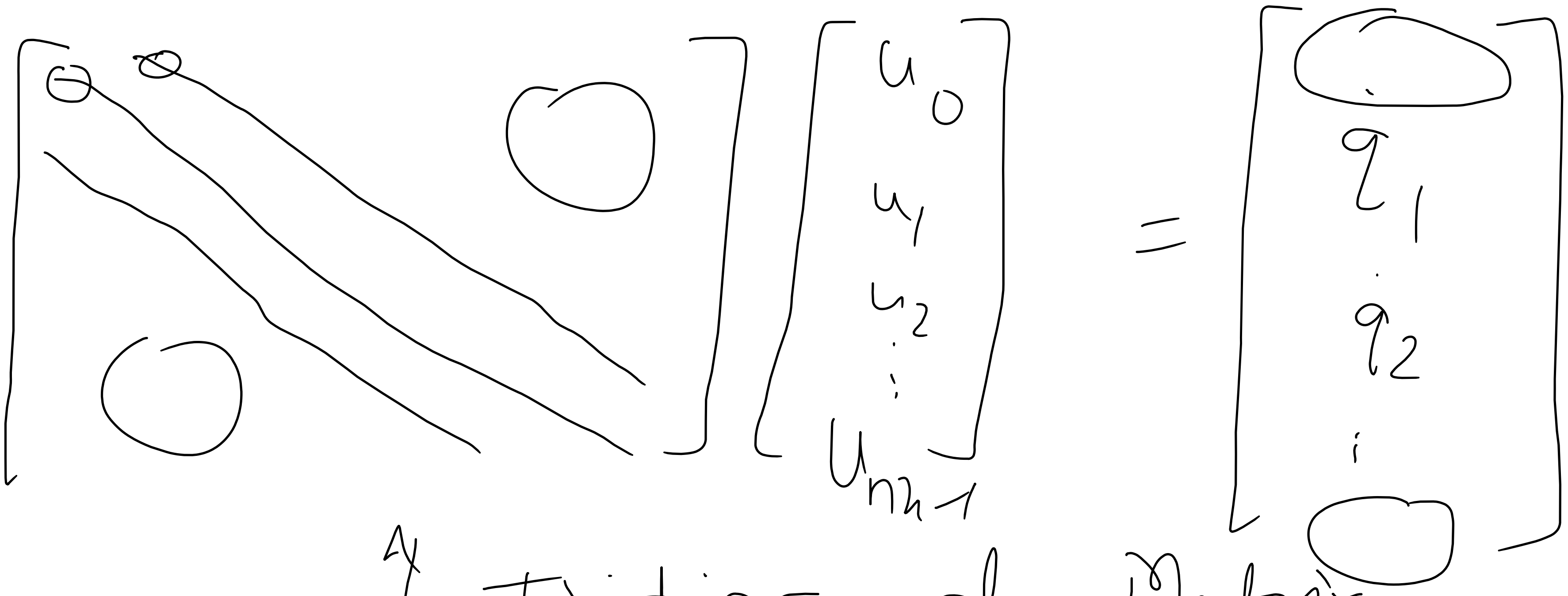
$$p_0 \frac{(u_1 - u_0)}{h} + q_0 u_0 = r_0 \quad \text{———} \quad \underline{\text{At } i = 0}$$

Same treatment at $i = nx - 1$

Numerical Solution of Poisson Equation

At $i = 0$ ______ At $i = 1, \ldots nx-2$ ______

$i = nx-1$



Tridiagonal Matrix

Numerical Solution of Poisson Equation

Extend this to 4th-order accuracy:

$$\frac{\partial^2 u}{\partial x^2} \approx \bigcirc\, u_{i+2} + \bigcirc\, u_{i+1} + \bigcirc\, u_i + \bigcirc\, u_{i-1}$$

$$+ \bigcirc\, u_{i-2}$$

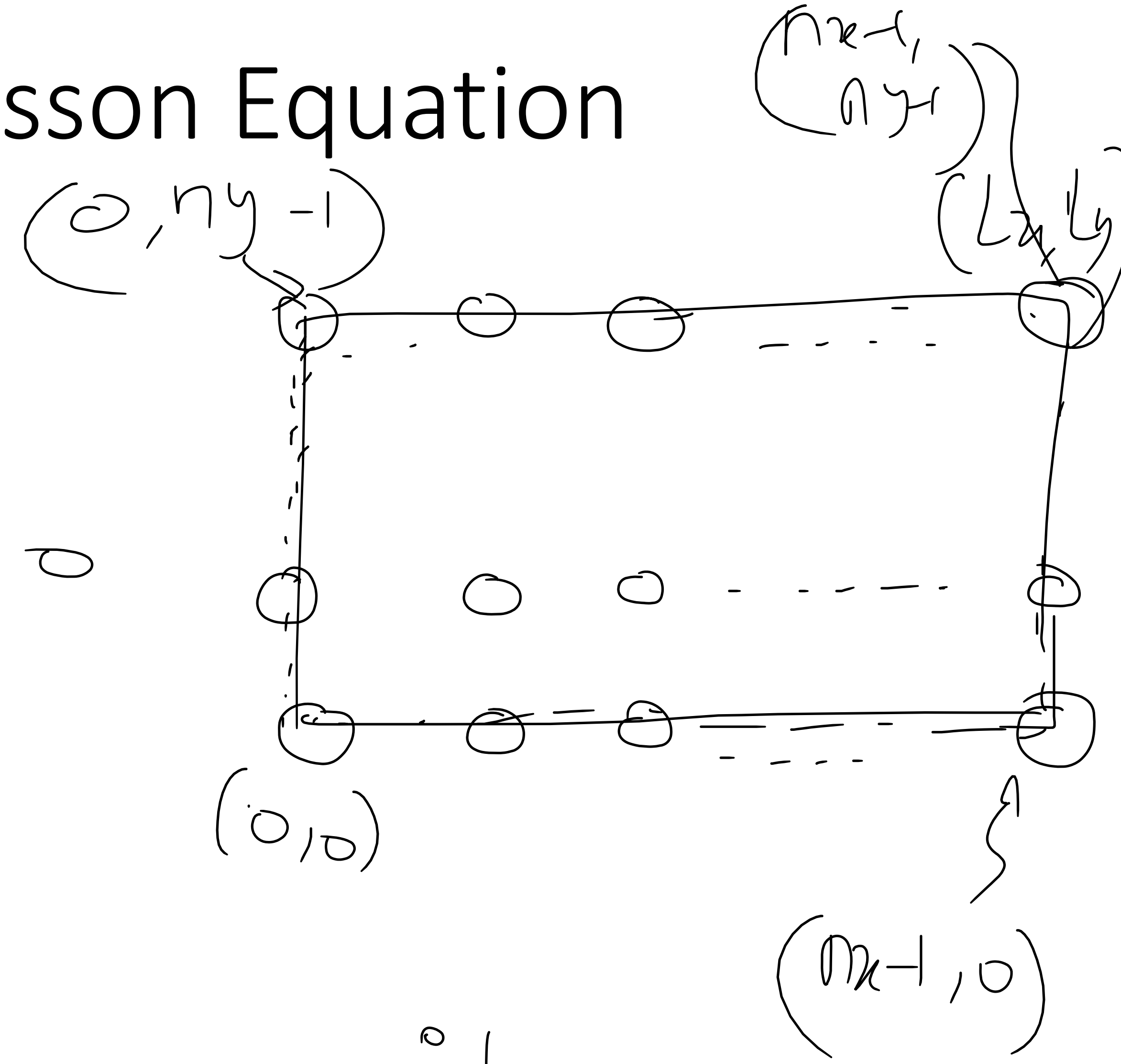
$$\frac{\partial u}{\partial x} \approx$$

Numerical Solution of Poisson Equation

$$\begin{bmatrix} \ddots & \ddots & \ddots & & 0 \\ \ddots & \ddots & \ddots & \ddots & \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{n_x-1} \end{bmatrix} = \begin{bmatrix} 0 \\ q_1 \\ q_2 \\ \vdots \\ 0 \end{bmatrix}$$

Pentadiagonal system can be solved directly. (no iterations)

Numerical Solution of Poisson Equation



2D Poisson Eqn

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + q = 0$$

BCs: $p\, u + q \frac{\partial u}{\partial x} = r$

$$h_x = \frac{L_x}{n_x - 1} \qquad h_y = \frac{L_y}{n_y - 1} \qquad x_i = i\, h_x$$

$i = 0, \ldots n_x - 1 \qquad j = 0 \ldots n_y - 1 \qquad y_j = j\, h_y$

Numerical Solution of Poisson Equation

Discrete eqn :

$$\frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{h_x^2} + \frac{u_{i,j+1} + u_{i,j-1} - 2u_{i,j}}{h_y^2}$$

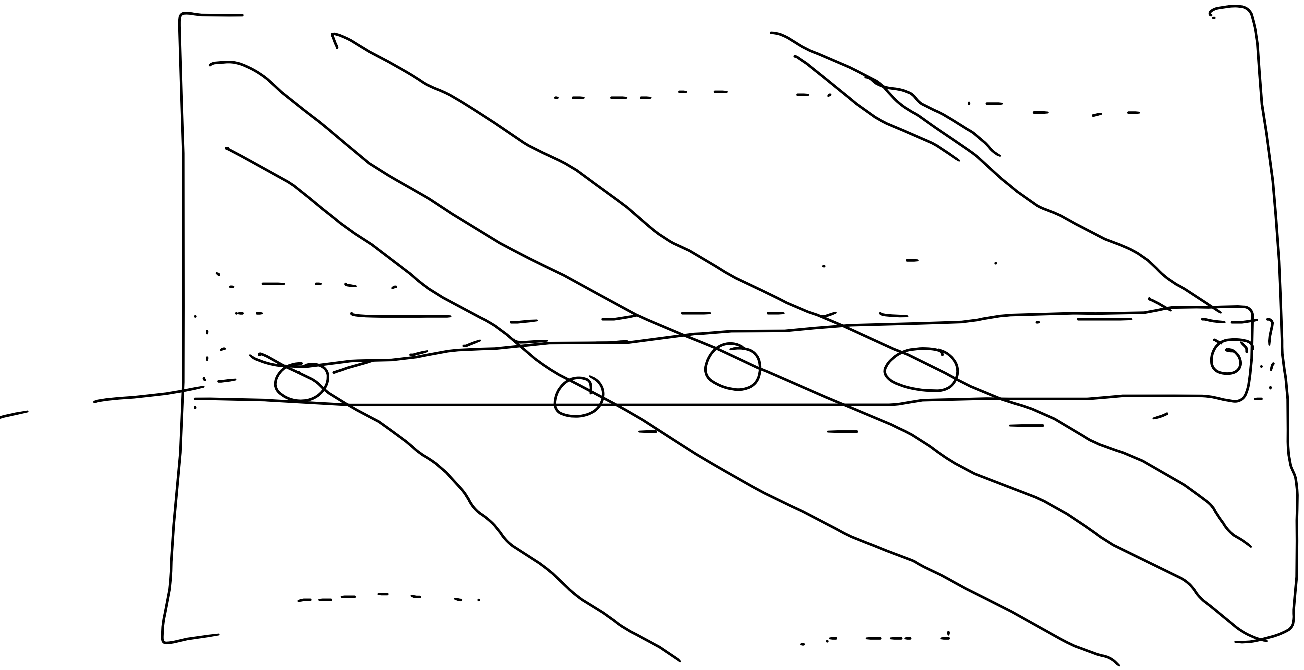
$$+ q_{(i,j)} = 0$$

$$\begin{bmatrix} i = 1, 2, \ldots nx-2 \\ j = 1, 2 \ldots ny-2 \end{bmatrix}$$

Numerical Solution of Poisson Equation

Discretize eqns on the boundaries

System of eqns $nx \cdot ny$



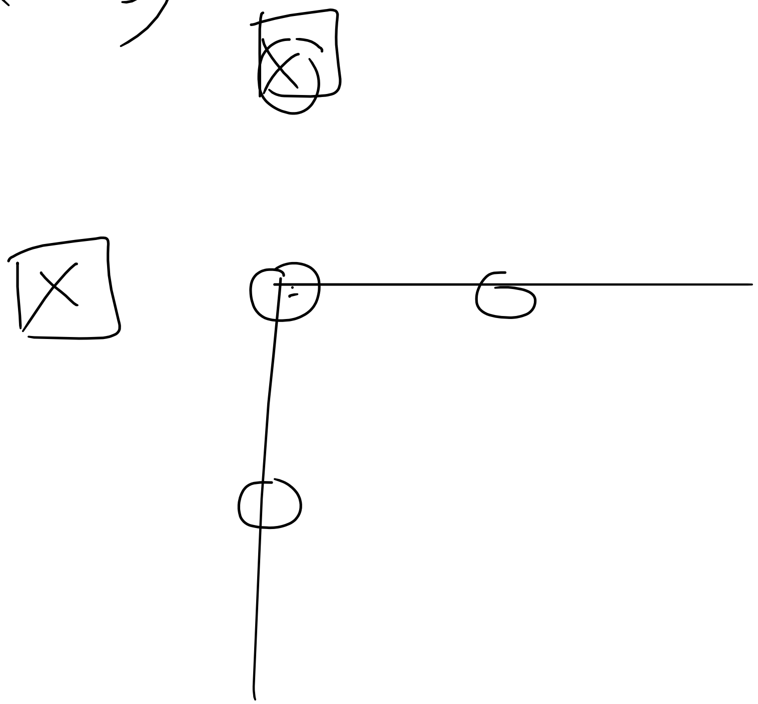
$$\begin{bmatrix} u_{0,0} \\ u_{1,0} \\ u_{2,0} \\ \vdots \\ u_{nx-1,0} \\ u_{0,1} \\ u_{1,1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \\ \\ \\ q_{i,j,r} \\ \\ \\ \end{bmatrix}$$

Numerical Solution of Poisson Equation

$(i, j+1)$

$(i-1, j)$ (i, j) $(i+1, j)$

$(i, j-1)$



Numerical Solution of Poisson Equation

Banded ; diagonals that are non-zero are not adjacent.

Linear Solvers for $\underline{A}\,\overline{u} = \overline{f}$

Direct vs Iterative

↳ Splitting → Jacobi, G-S

↳ Krylov Subspace

Numerical Solution of Poisson Equation

Root finding : $f(x) = 0$

Fixed-Point Iteration

$f(x) \Longleftrightarrow x = g(x)$

Come up with a $g(x)$ s.t. $x = g(x) \Longleftrightarrow f(x) = 0$

Numerical Solution of Poisson Equation

$$f(x) = x^3 + 3x^2 + x - 1$$

$$f(x) = 0 \Rightarrow x^3 + 3x^2 + x - 1 = 0$$

$$x = -x^3 - 3x^2 + 1 \quad : \quad x = g_1(x)$$

$$x = \pm\sqrt{\left(-x^3 - x + 1\right)/3} \qquad x = g_2(x)$$

$$\vdots$$

Numerical Solution of Poisson Equation

Fixed-point Itern : Start with p_0

$p_1 = g(p_0)$

$p_2 = g(p_1)$

$p_3 = g(p_2)$

$\vdots$

p_k

If g is chosen correctly, p_k for $k \gg 1$ is the root for $x = g(x)$; $f(x) = 0$

Numerical Solution of Poisson Equation

$$x = g(x) \quad \longleftrightarrow \quad \underline{A}\,\bar{u} \doteq \bar{f}$$

Splitting Methods: $\quad \underline{A} = \underline{M} + \underline{N}$

$$(\underline{M} + \underline{N})\,\bar{u} \doteq \bar{f}$$

$$\boxed{\underline{M}\,\bar{u} = \bar{f} - \underline{N}\,\bar{u}} \quad \text{—— } (*)$$

Numerical Solution of Poisson Equation

Fixed Point Iteration on $\textcircled{+}$

Given $\bar{u}^{(k)}$ get $\bar{u}^{(k+1)}$

$$\underline{M}\,\bar{u}^{(k+1)} = \bar{f} - \underline{H}\,\bar{u}^{(k)}$$

$k = 0, 1, 2 \ldots$

until conv.

Numerical Solution of Poisson Equation

1) $\underline{M}$ = Diagonal matrix $\Rightarrow$ Jacobi

$$\underbrace{\begin{bmatrix} \times & & & 0 \\ & \times & & \\ & & \times & \\ 0 & & & \ddots \end{bmatrix}}_{\underline{M}} + \underbrace{\begin{bmatrix} 0 & \times & \times & \cdots \\ \times & 0 & & \vdots \\ \vdots & & 0 & \\ \cdots & & \times & 0 \end{bmatrix}}_{\underline{N}}$$

2) $\underline{M}$: lower-triangle $\Rightarrow$ Gauss-Seidel

$$\underline{M} \sim \begin{bmatrix} \times & & & 0 \\ \times & \times & & \\ \times & \times & \times & \\ \times & \times & \times & \times \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & \times & \times & \times \\ & 0 & \times & \times \\ & & 0 & \times \\ 0 & & & 0 \end{bmatrix}}_{\leftarrow\ \underline{N}}$$

Numerical Solution of Poisson Equation

1D, 2D Poisson

Discrete Eqns ; Linear Solve using Jacobi, G-S

$$\underline{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 10 \end{bmatrix}$$

$$N = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$N = [$

Numerical Solution of Poisson Equation

$\underline{A}$ kn $\quad \bar{u}$ unkn $\quad \bar{f}$ kn

Jacobi:

$$u_i^{(k+1)} = \left[f_i - \sum_{j=1}^{i-1} a_{ij} u_j^{(k)} - \sum_{j=i+1}^{nsize} a_{ij} u_j^{(k)} \right] \frac{1}{a_{ii}}$$

$$u_i^{(k+1)} = \left[f_i - \sum_{j=1}^{i-1} a_{ij} u_j^{(k+1)} - \sum_{j=i+1}^{nsize} a_{ij} u_j^{(k)} \right] \frac{1}{a_{ii}}$$

as

Numerical Solution of Poisson Equation

$$\underline{A}\,\bar{u} = \bar{f}$$

$$\begin{array}{c c}
 & \\
i & \begin{bmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{insize} \\ & & & \end{bmatrix}
\end{array}
\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{nsize} \end{bmatrix}
=
\begin{bmatrix} f_1 \\ f_2 \\ f_i \\ \vdots \\ f_{nsize} \end{bmatrix}$$

Numerical Solution of Poisson Equation

Numerical Solution of Poisson Equation