Bayesian Statistical Methods: Homework 1

Due on Jan 29, 2013

Andrew Kurzawski

Problem 1

I have decided to use the PyMC package to create a model and sample from the posterior distribution. The code is as follows:

```
# Model
from pymc import *
from numpy import array, empty
from numpy.random import randint

theta = pymc.Beta('theta',alpha=10,beta=10)
d = pymc.Binomial('d', n=np.array([20]), p=theta, value=np.array([3]),observed=True)



# Sampler
import hw1_model
from pymc import MCMC
from pylab import hist, show
from pymc.Matplot import plot

M = MCMC(hw1_model)
M.sample(iter=10000, burn=1000, thin=10)
plot(M)
```

The resulting posterior distributions are as follows:

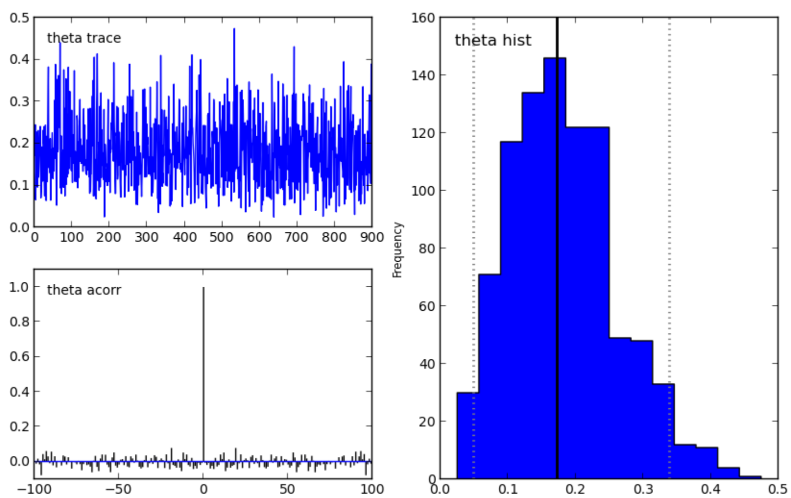

Figure 1: $\alpha = 1, \beta = 1$

It is notable that the mode of the posterior given an uninformed prior (Fig. 1), is closest to 0.15 which corresponds to $Y = 3$ after 20 trials. The influence of adding information to the prior in the form of the parameters α and β, can be seen in Fig. 2 and 3. In these instances the priors did not have a uniform shape.

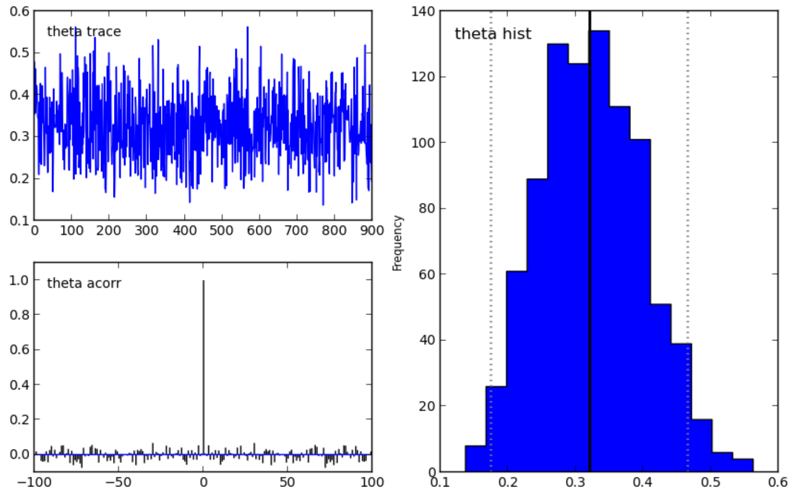

Figure 2: $\alpha = 10, \beta = 10$

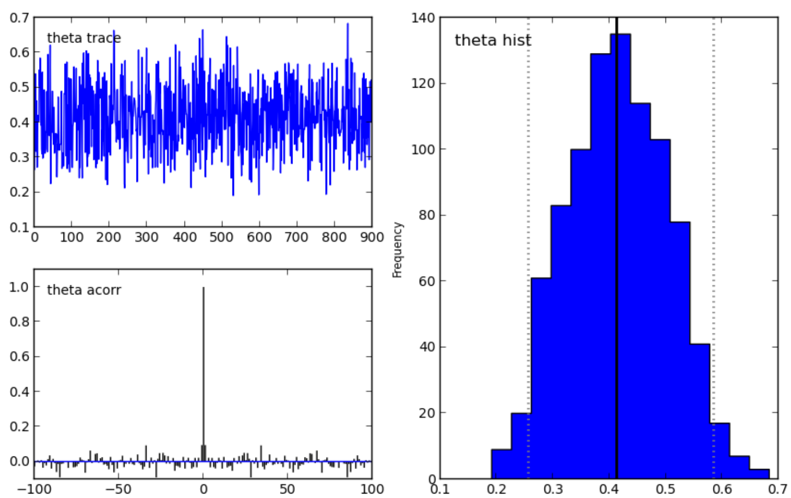

Figure 3: $\alpha = 10, \beta = 1$

Problem 2

For the case where a 20% response rate has been observed, α and β such that the mode of the prior distribution is 0.2.

$$0.2 = \frac{\alpha - 1}{\alpha + \beta - 2} \tag{1}$$

Then, we could scale α and β based on how much weight we wish to place on this previous observation.

Problem 3

Derive Jeffreys' prior for the parameter in the Binomial model.

$$Y \sim Bin(n, \theta) \tag{2}$$

Where Jeffreys' prior is given as

$$J(\theta) = -E[\frac{\partial^2 logP(y|\theta)}{\partial \theta^2}] \tag{3}$$

and

$$P(y|\theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y} \tag{4}$$

Then, we derive Jeffreys' prior taking the following steps.

$$logP(y|\theta) = log\binom{n}{y} + ylog\theta + (n-y)log(1-\theta) \tag{5}$$

$$\frac{\partial^2 logP(y|\theta)}{\partial \theta^2} = -\frac{y}{\theta^2} - \frac{n-y}{(1-\theta)^2} \tag{6}$$

$$J(\theta) = -E[-\frac{y}{\theta^2} - \frac{n-y}{(1-\theta)^2}] \tag{7}$$

$$J(\theta) = \frac{n\theta}{\theta^2} + \frac{n-n\theta}{(1-\theta)^2} \tag{8}$$

Simplifying, we get:

$$\boxed{J(\theta) = \frac{n}{\theta(1-\theta)} \tag{9}}$$

Problem 4

$$P(\theta|y) \propto P(\theta)L(\theta|y) \tag{10}$$

For a Gamma distribution the probability density function is

$$P(\theta) = \frac{1}{\Gamma(a)b^a}\theta^{a-1}exp(\frac{-\theta}{b}) \tag{11}$$

and the likelihood function n independent Poisson responses is

$$L(\theta|y) = \frac{exp(-n\theta)\theta^{\Sigma y_i}}{\prod y_i!} \tag{12}$$

Combining Eqn. 10, 11 and 12 while leaving out terms that do not depend on θ.

$$P(\theta|y) \propto \theta^{a-1}exp\left(\frac{-\theta}{b}\right)exp\left(-n\theta\right)\theta^{\Sigma y_i} \tag{13}$$

Thus, the posterior distribution is a Gamma

$$\boxed{P(\theta|y) \propto \theta^{a-1+\Sigma y_i}exp\left(-\theta(\frac{1}{b}+n)\right) \tag{14}}$$

with parameters $a_o = a + \Sigma y_i$ and $b_o = \frac{1}{n+(1/b)}$.

Problem 5

$$P\left(\frac{1}{\sigma^2}|y\right) \propto P\left(\frac{1}{\sigma^2}\right)L\left(\frac{1}{\sigma^2}|y\right) \tag{15}$$

The likelihood of a Normal with a known mean is

$$L\left(\frac{1}{\sigma^2}|y\right) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} exp\left[-\frac{1}{2\sigma^2}\sum(y_i-\mu)^2\right] \tag{16}$$

Using Eqn. 11 where θ is replaced by $\frac{1}{\sigma^2}$ and Eqn. 16, we can substitute into Eqn. **??**. Leaving out terms that do not depend on $\frac{1}{\sigma^2}$, we get that

$$P\left(\frac{1}{\sigma^2}|y\right) \propto \left(\frac{1}{\sigma^2}\right)^{a-1} exp\left(\frac{-1}{\sigma^2 b}\right)\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} exp\left[-\frac{1}{2\sigma^2}\sum(y_i-\mu)^2\right] \tag{17}$$

Simplifying, the posterior becomes

$$\boxed{P\left(\frac{1}{\sigma^2}|y\right) \propto \left(\frac{1}{\sigma^2}\right)^{a-1-\frac{n}{2}} exp\left[\frac{-1}{\sigma^2 b} - \frac{1}{2\sigma^2}\sum(y_i-\mu)^2\right] \tag{18}}$$

To bring in prior information on $\frac{1}{\sigma^2}$ to the Gamma distribution, we could take the mean, variance, and number of samples from a previous set of observations of $\frac{1}{\sigma^2}$ and calculate new values for a and b based on the form of Eqn. 18.