

# Synthetic Isotropic Turbulence based on a Specified Energy Spectrum

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## Abstract

Given a turbulent energy spectrum, the task is to generate an isotropic turbulent velocity field that reproduces the input spectrum. I will use Lars Davidson's [1] formulation for generating inlet turbulent data. His method is easily extendable to three dimensions as well as different resolutions in space.

## 1 Formulation

We start with a generalized Fourier series for a real valued scalar function

$$u = a_0 + \sum_{m=1}^M a_m \cos\left(\frac{2\pi m x}{L}\right) + b_m \sin\left(\frac{2\pi m x}{L}\right) \quad (1)$$

For simplicity, we set  $k_m \equiv \frac{2\pi m}{L}$  as the  $m^{\text{th}}$  wave number. Also, if the mean of  $f$  is known, we have

$$\int_0^L u \, dx = a_0 L \quad (2)$$

Hence, for a turbulent velocity field with zero mean (in space), we can set  $a_0 = 0$ . At the outset, we have

$$u = \sum_{m=1}^M a_m \cos(k_m x) + b_m \sin(k_m x) \quad (3)$$

We now introduce the following changes

$$a_m = \hat{u}_m \cos(\psi_m); \quad b_m = \hat{u}_m \sin(\psi_m); \quad \hat{u}_m^2 = a_m^2 + b_m^2, \quad \psi_m = \arctan\left(\frac{b_m}{a_m}\right) \quad (4)$$

then

$$\begin{aligned} a_m \cos(k_m x) + b_m \sin(k_m x) &= \hat{u}_m \cos(\psi_m) \cos(k_m x) + \hat{u}_m \sin(\psi_m) \sin(k_m x) \\ &= \hat{u}_m \cos(k_m x - \psi_m) \end{aligned} \quad (5)$$

so that

$$u = \sum_{m=1}^M \hat{u}_m \cos(k_m x - \psi_m) \quad (6)$$

The extension to 3D follows

$$u = \sum_{m=1}^M \hat{u}_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \quad (7)$$

$$v = \sum_{m=1}^M \hat{v}_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \quad (8)$$

$$w = \sum_{m=1}^M \hat{w}_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \quad (9)$$

where  $\mathbf{k}_m \equiv (k_{x,m}, k_{y,m}, k_{z,m})$  is the position vector in wave space and  $\mathbf{x} \equiv (x, y, z)$  is the position vector in physical space. Therefore,  $\mathbf{k}_m \cdot \mathbf{x}_m = k_{x,m}x + k_{y,m}y + k_{z,m}z$ . A condensed form is

$$\mathbf{u} = \sum_{m=1}^M \hat{\mathbf{u}}_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \quad (10)$$

where  $\hat{\mathbf{u}}_m \equiv (\hat{u}_m, \hat{v}_m, \hat{w}_m)$ . Continuity dictates that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (11)$$

This gives

$$-\sum_{m=1}^M (k_{x,m}\hat{u}_m + k_{y,m}\hat{v}_m + k_{z,m}\hat{w}_m) \sin(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) = 0 \quad (12)$$

or

$$\sum_m \mathbf{k}_m \cdot \hat{\mathbf{u}}_m \sin(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) = 0 \quad (13)$$

This equation can be enforced by setting

$$\mathbf{k}_m \cdot \hat{\mathbf{u}}_m = 0, \forall m \in \{0, 1, \dots, M\}$$

This means that the Fourier coefficients have different directions in wave space. We therefore write the Fourier coefficients as

$$\hat{\mathbf{u}}_m \equiv q_m \boldsymbol{\sigma}_m \mid \mathbf{k}_m \cdot \boldsymbol{\sigma}_m = 0 \quad (14)$$

where  $\boldsymbol{\sigma}_m$  is a unit vector computed such that  $\mathbf{k}_m \cdot \boldsymbol{\sigma}_m = 0$  at any point  $\mathbf{x}$ . Note that this is the original formulation presented in [1]. While (13) is true in the continuous sense, it becomes invalid when discretized leading to a diverging velocity field. I will show you how to fix this in the next paragraph.

The velocity vector at point  $\mathbf{x}$  is now at hand

$$\mathbf{u}(\mathbf{x}) = \sum_{m=1}^M q_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \boldsymbol{\sigma}_m \quad (15)$$

The last step is to link  $q_m$  to the energy spectrum. This can be computed from

$$q_m = 2 \sqrt{E(k_m) \Delta k} \quad (16)$$

## 2 Enforcing Continuity

Given an analytic vector field  $\mathbf{u}$  such that  $\nabla \cdot \mathbf{u} = 0$ , we show here that this does not hold for the discrete continuity equation. Since different codes use different discretization schemes for the dilatation term (staggered vs collocated), one must first write the divergence formula in the desired discrete form and then infer the condition that enforces discrete divergence. A classic example is the Taylor-Green vortex initialization. This velocity field is given by

$$u = \sin x \cos y \quad (17)$$

$$v = -\cos x \sin y \quad (18)$$

It is true that, for this velocity field,  $\nabla \cdot \mathbf{u} = 0$  because

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \cos x \cos y - \cos x \cos y = 0 \quad (19)$$

However, when used to initialize a discrete grid, the resulting discrete continuity equation does not always hold true. Take for instance the Taylor-Green vortex and initialize a staggered grid. Continuity, to second order in space on a staggered grid implies

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \approx \frac{u(x + \frac{\Delta x}{2}, y) - u(x - \frac{\Delta x}{2}, y)}{\Delta x} + \frac{v(x, y + \frac{\Delta y}{2}) - v(x, y - \frac{\Delta y}{2})}{\Delta y} \quad (20)$$

Then, using the formula for  $u$  and  $v$ , e.g.  $u(x + \frac{\Delta x}{2}, y) = \sin(x + \frac{\Delta x}{2}) \cos y$ , etc..., one recovers

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \approx 2 \cos x \cos y \left[ \frac{\sin(\frac{\Delta x}{2})}{\Delta x} - \frac{\sin(\frac{\Delta y}{2})}{\Delta y} \right] \quad (21)$$

which is guaranteed to be zero when  $\Delta x = \Delta y$ . A nonuniform grid spacing will always result in a diverging initial condition. The overall less that I'd like to convey here is that it is generally preferable to operate with the discrete form of equations since those usually bring up hidden issues that can be easily missed in the continuous sense.

Back to our isotropic velocity field, recall that

$$\mathbf{u}(\mathbf{x}) = \sum_{m=1}^M q_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \boldsymbol{\sigma}_m \quad (22)$$

Now, write the continuity equation in discrete form, assuming a staggered grid, we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \approx \frac{u(x + \frac{\Delta x}{2}, y, z) - u(x - \frac{\Delta x}{2}, y, z)}{\Delta x} + \frac{v(x, y + \frac{\Delta y}{2}, z) - v(x, y - \frac{\Delta y}{2}, z)}{\Delta y} + \frac{w(x, y, z + \frac{\Delta z}{2}) - w(x, y, z - \frac{\Delta z}{2})}{\Delta z} \quad (23)$$

Here, for example,

$$u(x + \frac{\Delta x}{2}, y, z) = \sum_{m=1}^M q_m \cos(k_{m,x}(x + \frac{\Delta x}{2}) + k_{m,y}y + k_{m,z}z - \psi_m) \sigma_{m,x} \quad (24)$$

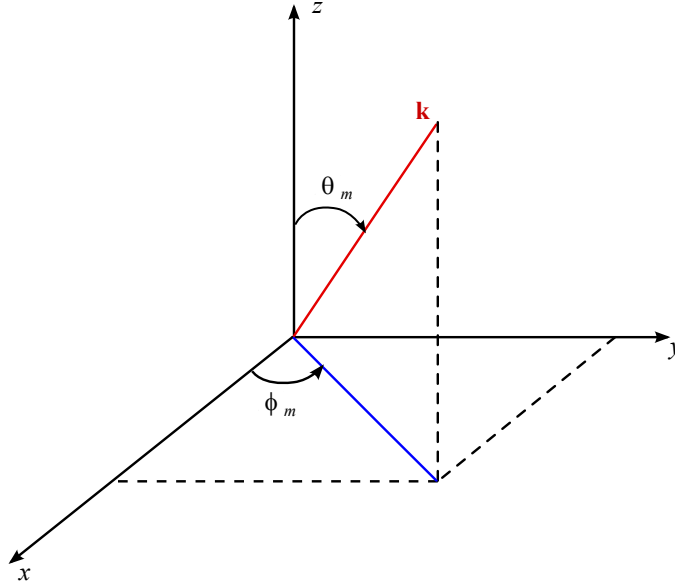


Figure 1: Angles associated with wave number  $\mathbf{k}_m$ .

Upon careful substitution and tedious trigonometric operations (which are rendered benign when using Mathematica, bless Stephen Wolfram), we recover the following

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \approx - \sum_{m=1}^M 2 \left[ \frac{\sigma_{m,x}}{\Delta x} \sin(\frac{1}{2} k_{m,x} \Delta x) + \frac{\sigma_{m,y}}{\Delta y} \sin(\frac{1}{2} k_{m,y} \Delta y) + \frac{\sigma_{m,z}}{\Delta z} \sin(\frac{1}{2} k_{m,z} \Delta z) \right] \sin(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \quad (25)$$

or, written in a more convenient form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \approx - \sum_{m=1}^M \sigma_m \cdot \tilde{\mathbf{k}}_m \sin(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \quad (26)$$

where

$$\tilde{\mathbf{k}}_m \equiv \frac{2}{\Delta x} \sin(\frac{1}{2} k_{m,x} \Delta x) \mathbf{i} + \frac{2}{\Delta y} \sin(\frac{1}{2} k_{m,y} \Delta y) \mathbf{j} + \frac{2}{\Delta z} \sin(\frac{1}{2} k_{m,z} \Delta z) \mathbf{k} \quad (27)$$

A sufficient condition for the discrete continuity equation given in (26) to be zero is to make

$$\sigma_m \cdot \tilde{\mathbf{k}}_m = 0, \quad \forall m \quad (28)$$

This means that instead of selecting  $\sigma_m$  such that it is perpendicular to  $\mathbf{k}_m$  ( $\sigma_m \cdot \mathbf{k}_m = 0$ ), we instead choose  $\sigma_m$  to be perpendicular to  $\tilde{\mathbf{k}}_m$ . Interestingly, in the limit as the grid spacing approaches zero,  $\tilde{\mathbf{k}}_m$  will approach  $\mathbf{k}_m$ . This is so cool!

### 3 In Practice

- Specify the number of modes  $M$ . This will determine the Fourier representation of the velocity field at every point in the spatial domain

- Compute or set a minimum wave number  $k_0$
- Compute a maximum wave number  $k_{\max} = \frac{\pi}{\Delta x}$ . For multiple dimensions, use  $k_{\max} = \max(\frac{\pi}{\Delta x}, \frac{\pi}{\Delta y}, \frac{\pi}{\Delta z})$
- Generate a list of  $M$  modes:  $k_m \equiv k(m) = k_0 + \frac{k_{\max} - k_0}{M}(m - 1)$ . Those will correspond to the magnitude of the vector  $\mathbf{k}_m$ . In other words,  $k_m$  is the radius of a sphere.
- Generate four arrays of random numbers, each of which is of size  $M$  (those will be needed next). Those will correspond to the angles:  $\theta_m$ ,  $\varphi_m$ ,  $\psi_m$ , and  $\alpha_m$ .
- Compute the wave vectors. To generate as much randomness as possible, we write the wave vector as a function of two angles in 3D space. This means

$$k_{x,m} = \sin(\theta_m) \cos(\varphi_m) k_m \quad (29)$$

$$k_{y,m} = \sin(\theta_m) \sin(\varphi_m) k_m \quad (30)$$

$$k_{z,m} = \cos(\theta_m) k_m \quad (31)$$

- Compute the unit vector  $\sigma_m$ . Note that  $\sigma_m$  lies in a plane perpendicular to the vector  $\mathbf{k}_m$ . We choose the following

$$\sigma_{x,m} = \cos(\theta_m) \cos(\varphi_m) \cos(\alpha_m) - \sin(\varphi_m) \sin(\alpha_m) \quad (32)$$

$$\sigma_{y,m} = \cos(\theta_m) \sin(\varphi_m) \cos(\alpha_m) + \cos(\varphi_m) \sin(\alpha_m) \quad (33)$$

$$\sigma_{z,m} = -\sin(\theta_m) \cos(\alpha_m) \quad (34)$$

- To enforce continuity, compute vector  $\tilde{\mathbf{k}}_m$  and make  $\sigma_m$  perpendicular to  $\tilde{\mathbf{k}}_m$ .
- Once those quantities are computed, loop over the mesh. For every point on the mesh, loop over all  $M$  modes. For every mode, compute  $q_m = 2 \sqrt{E(k_m) \Delta k}$  and  $\beta_m = \mathbf{k}_m \cdot \mathbf{x} - \psi_m$ . Finally, construct the following summations (at every point  $(x, y, z)$  you will have a summation of  $M$ -modes)

$$u(x, y, z) = \sum_{n=1}^M q_m \cos(\beta_m) \sigma_{x,m} \quad (35)$$

$$v(x, y, z) = \sum_{n=1}^M q_m \cos(\beta_m) \sigma_{y,m} \quad (36)$$

$$w(x, y, z) = \sum_{n=1}^M q_m \cos(\beta_m) \sigma_{z,m} \quad (37)$$

## References

- [1] Lars Davidson. Hybrid les-rans: Inlet boundary conditions for flows with recirculation. In *Advances in Hybrid RANS-LES Modelling*, pages 55–66. Springer, 2008.