

# Synthetic Isotropic Turbulence based on a Specified Energy Spectrum

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## Abstract

Given a turbulent energy spectrum, the task is to generate an isotropic turbulent velocity field that reproduces the input spectrum. I will use Lars Davidson's [1] formulation for generating inlet turbulent data. His method is easily extendable to three dimensions as well as different resolutions in space.

## 1 Formulation

We start with a generalized Fourier series for a real valued scalar function

$$u = a_0 + \sum_{m=1}^M a_m \cos\left(\frac{2\pi mx}{L}\right) + b_m \sin\left(\frac{2\pi mx}{L}\right) \quad (1)$$

For simplicity, we set  $k_m \equiv \frac{2\pi m}{L}$  as the  $m^{\text{th}}$  wave number. Also, if the mean of  $f$  is known, we have

$$\int_0^L u \, dx = a_0 L \quad (2)$$

Hence, for a turbulent velocity field with zero mean (in space), we can set  $a_0 = 0$ . At the outset, we have

$$u = \sum_{m=1}^M a_m \cos(k_m x) + b_m \sin(k_m x) \quad (3)$$

We now introduce the following changes

$$a_m = \hat{u}_m \cos(\psi_m); \quad b_m = \hat{u}_m \sin(\psi_m); \quad \hat{u}_m^2 = a_m^2 + b_m^2, \quad \psi_m = \arctan\left(\frac{b_m}{a_m}\right) \quad (4)$$

then

$$\begin{aligned} a_m \cos(k_m x) + b_m \sin(k_m x) &= \hat{u}_m \cos \alpha_m \cos(k_m x) + \hat{u}_m \sin \alpha_m \sin(k_m x) \\ &= \hat{u}_m \cos(k_m x - \psi_m) \end{aligned} \quad (5)$$

so that

$$u = \sum_{n=1}^M \hat{u}_m \cos(k_m x - \psi_m) \quad (6)$$

The extension to 3D follows

$$u = \sum_{n=1}^M \hat{u}_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \quad (7)$$

$$v = \sum_{n=1}^M \hat{v}_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \quad (8)$$

$$w = \sum_{n=1}^M \hat{w}_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \quad (9)$$

where  $\mathbf{k}_m \equiv (k_{x,n}, k_{y,n}, k_{z,n})$  is the position vector in wave space and  $\mathbf{x} \equiv (x, y, z)$  is the position vector in physical space. Therefore,  $\mathbf{k}_m \cdot \mathbf{x}_m = k_{x,n}x + k_{y,n}y + k_{z,n}z$ . A condensed form is

$$\mathbf{u} = \sum_{n=1}^N \hat{\mathbf{u}}_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \quad (10)$$

where  $\hat{\mathbf{u}}_m \equiv (\hat{u}_m, \hat{v}_m, \hat{w}_m)$ . Continuity dictates that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (11)$$

or

$$-\sum_{n=1}^N (k_{x,n}\hat{u}_m + k_{y,n}\hat{v}_m + k_{z,n}\hat{w}_m) \sin(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) = 0 \quad (12)$$

Finally, this means that the wave vector is perpendicular to the Fourier coefficients

$$\mathbf{k}_m \cdot \hat{\mathbf{u}}_m = 0$$

This means that the Fourier coefficients have different directions in the wave space. We therefore write the Fourier coefficients as

$$\hat{\mathbf{u}}_m \equiv q_m \boldsymbol{\sigma}_m; \quad \mathbf{k}_m \cdot \boldsymbol{\sigma}_m = 0 \quad (13)$$

so that

$$\mathbf{u} = \sum_{n=1}^N q_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \boldsymbol{\sigma}_m \quad (14)$$

The last step is to link  $q_m$  to the energy spectrum

$$q_m = 2\sqrt{E(k_m)\Delta k} \quad (15)$$

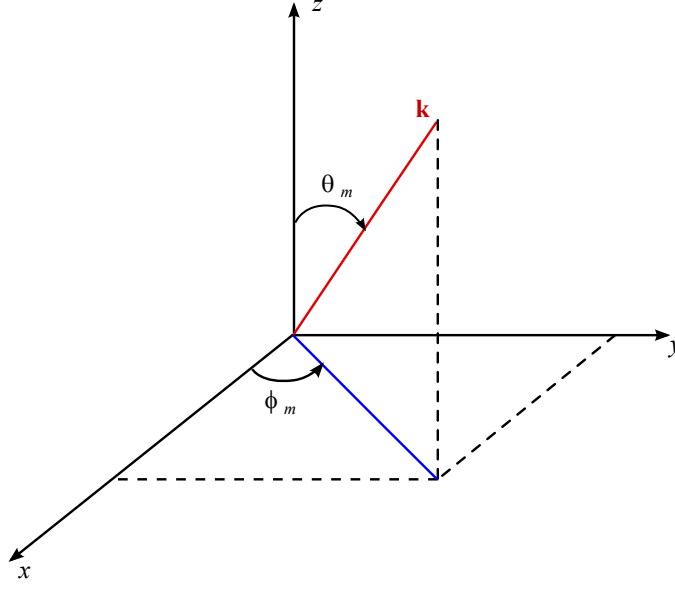


Figure 1: Angles associated with wave number  $\mathbf{k}_m$ .

## 2 In Practice

- Specify the number of modes  $M$ . This will determine the Fourier representation of the velocity field at every point in the spatial domain
- Compute or set a minimum wave number  $k_0$
- Compute a maximum wave number  $k_{\max} = \frac{\pi}{\Delta x}$ . For multiple dimensions, use  $k_{\max} = \max(\frac{\pi}{\Delta x}, \frac{\pi}{\Delta y}, \frac{\pi}{\Delta z})$
- Generate a list of  $M$  modes:  $k_m \equiv k(m) = k_0 + \frac{k_{\max} - k_0}{M}(m - 1)$ . Those will correspond to the magnitude of the vector  $\mathbf{k}_m$ . In other words,  $k_m$  is the radius of a sphere.
- Generate a four arrays of random numbers, each of which is of size  $M$  (those will be needed next). Those will correspond to the angles:  $\theta_m$ ,  $\varphi_m$ ,  $\psi_m$ , and  $\alpha_m$ .
- Compute the wave vectors. To generate as much randomness as possible, we write the wave vector as a function of two angles in 3D space. This means

$$k_{x,m} = \sin(\theta_m) \cos(\varphi_m) k_m \quad (16)$$

$$k_{y,m} = \sin(\theta_m) \sin(\varphi_m) k_m \quad (17)$$

$$k_{z,m} = \cos(\theta_m) k_m \quad (18)$$

- Compute the unit vector  $\boldsymbol{\sigma}_m$ . Note that  $\boldsymbol{\sigma}_m$  lies in a plane perpendicular to the vector  $\mathbf{k}_m$ . We choose the following

$$\sigma_{x,m} = \cos(\theta_m) \cos(\varphi_m) \cos(\alpha_m) - \sin(\varphi_m) \sin(\alpha_m) \quad (19)$$

$$\sigma_{y,m} = \cos(\theta_m) \sin(\varphi_m) \cos(\alpha_m) + \cos(\varphi_m) \sin(\alpha_m) \quad (20)$$

$$\sigma_{z,m} = -\sin(\theta_m) \cos(\alpha_m) \quad (21)$$

- Once those quantities are computed, loop over the mesh. For every point on the mesh, loop over all M modes. For every mode, compute  $q_m = 2\sqrt{E(k_m)\Delta k}$  and  $\beta_m = \mathbf{k}_m \cdot \mathbf{x} - \psi_m$ . Finally, construct the following summations (at every point you will have a summation of M-modes)

$$u = \sum_{n=1}^M q_m \cos(\beta_m) \sigma_{x,m} \quad (22)$$

$$v = \sum_{n=1}^M q_m \cos(\beta_m) \sigma_{y,m} \quad (23)$$

$$w = \sum_{n=1}^M q_m \cos(\beta_m) \sigma_{z,m} \quad (24)$$

## References

- [1] Lars Davidson. Hybrid les-rans: Inlet boundary conditions for flows with recirculation. In *Advances in Hybrid RANS-LES Modelling*, pages 55–66. Springer, 2008.