

Synthetic Isotropic Turbulence based on a Specified Energy Spectrum

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Abstract

Given a turbulent energy spectrum, the task is to generate an isotropic turbulent velocity field that reproduces the input spectrum. I will use Lars Davidson's [1] formulation for generating inlet turbulent data. His method is easily extendable to three dimensions as well as different resolutions in space.

1 Formulation

We start with a generalized Fourier series for a real valued scalar function

$$u = a_0 + \sum_{m=1}^M a_m \cos\left(\frac{2\pi m x}{L}\right) + b_m \sin\left(\frac{2\pi m x}{L}\right) \quad (1)$$

For simplicity, we set $k_m \equiv \frac{2\pi m}{L}$ as the m -th wave number. Also, if the mean of f is known, we have

$$\int_0^L u \, dx = a_0 L \quad (2)$$

Hence, for a turbulent velocity field with zero mean (in space), we can set $a_0 = 0$. At the outset, we have

$$u = \sum_{m=1}^M a_m \cos(k_m x) + b_m \sin(k_m x) \quad (3)$$

We now introduce the following changes

$$a_m = \hat{u}_m \cos(\psi_m); \quad b_m = \hat{u}_m \sin(\psi_m); \quad \hat{u}_m^2 = a_m^2 + b_m^2, \quad \psi_m = \arctan\left(\frac{b_m}{a_m}\right) \quad (4)$$

then

$$\begin{aligned} a_m \cos(k_m x) + b_m \sin(k_m x) &= \hat{u}_m \cos \alpha_m \cos(k_m x) + \hat{u}_m \sin \alpha_m \sin(k_m x) \\ &= \hat{u}_m \cos(k_m x - \psi_m) \end{aligned} \quad (5)$$

so that

$$u = \sum_{m=1}^M \hat{u}_m \cos(k_m x - \psi_m) \quad (6)$$

The extension to 3D follows

$$u = \sum_{n=1}^M \hat{u}_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \quad (7)$$

$$v = \sum_{n=1}^M \hat{v}_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \quad (8)$$

$$w = \sum_{n=1}^M \hat{w}_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \quad (9)$$

where $\mathbf{k}_m \equiv (k_{x,n}, k_{y,n}, k_{z,n})$ is the position vector in wave space and $\mathbf{x} \equiv (x, y, z)$ is the position vector in physical space. Therefore, $\mathbf{k}_m \cdot \mathbf{x} = k_{x,n}x + k_{y,n}y + k_{z,n}z$. A condensed form is

$$\mathbf{u} = \sum_{n=1}^N \hat{\mathbf{u}}_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \quad (10)$$

where $\hat{\mathbf{u}}_m \equiv (\hat{u}_m, \hat{v}_m, \hat{w}_m)$. Continuity dictates that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (11)$$

or

$$-\sum_{n=1}^N (k_{x,n}\hat{u}_m + k_{y,n}\hat{v}_m + k_{z,n}\hat{w}_m) \sin(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) = 0 \quad (12)$$

Finally, this means that the wave vector is perpendicular to the fourier coefficients

$$\mathbf{k}_m \cdot \hat{\mathbf{u}}_m = 0$$

This means that the fourier coefficients have different directions in the wave space. We therefore write the fourier coefficients as

$$\hat{\mathbf{u}}_m \equiv q_m \boldsymbol{\sigma}_m; \quad \mathbf{k}_m \cdot \boldsymbol{\sigma}_m = 0 \quad (13)$$

so that

$$\mathbf{u} = \sum_{n=1}^N q_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \boldsymbol{\sigma}_m \quad (14)$$

The last step is to link q_m to the energy spectrum

$$q_m = 2 \sqrt{E(k_m) \Delta k} \quad (15)$$

2 In Practice

- Specify the number of modes M . This will determine the Fourier representation of the velocity field at every point in the spatial domain
- Compute or set a minimum wave number k_0
- Compute a maximum wave number $k_{\max} = \frac{2\pi}{\Delta x}$. For multidimensions, use $k_{\max} = \max(\frac{2\pi}{\Delta x}, \frac{2\pi}{\Delta y}, \frac{2\pi}{\Delta z})$
- Generate a list of M modes: $k_m \equiv k(m) = k_0 + \frac{k_{\max} - k_0}{M}(m-1)$. Those will correspond to the magnitude of the vector \mathbf{k}_m . In other words, k_m is the radius of a sphere.
- Generate a four arrays of random numbers, each of which is of size M (those will be needed next). Those will correspond to the angles: θ_m , φ_m , ψ_m , and α_m .
- Compute the wave vectors. To generate as much randomness as possible, we write the wave vector as a function of two angles in 3D space. This means

$$k_{x,m} = \sin(\theta_m) \cos(\varphi_m) k_m \quad (16)$$

$$k_{y,m} = \sin(\theta_m) \sin(\varphi_m) k_m \quad (17)$$

$$k_{z,m} = \cos(\theta_m) k_m \quad (18)$$

- Compute the unit vector σ_m . Note that σ_m lies in a plane perpendicular to the vector \mathbf{k}_m . We choose the following

$$\sigma_{x,m} = \cos(\theta_m) \cos(\varphi_m) \cos(\alpha_m) - \sin(\varphi_m) \sin(\alpha_m) \quad (19)$$

$$\sigma_{y,m} = \cos(\theta_m) \sin(\varphi_m) \cos(\alpha_m) + \cos(\varphi_m) \sin(\alpha_m) \quad (20)$$

$$\sigma_{z,m} = -\sin(\theta_m) \cos(\alpha_m) \quad (21)$$

- Once those quantities are computed, loop over the mesh. For every point on the mesh, loop over all M modes. For every mode, compute $q_m = 2 \sqrt{E(k_m) \Delta k}$ and $\beta_m = \mathbf{k}_m \cdot \mathbf{x} - \psi_m$. Finally, construct the following summations (at every point you will have a summation of M -modes)

$$u = \sum_{n=1}^M q_m \cos(\beta_m) \sigma_{x,m} \quad (22)$$

$$v = \sum_{n=1}^M q_m \cos(\beta_m) \sigma_{y,m} \quad (23)$$

$$w = \sum_{n=1}^M q_m \cos(\beta_m) \sigma_{z,m} \quad (24)$$

References

- [1] Lars Davidson. Hybrid les-rans: Inlet boundary conditions for flows with recirculation. In *Advances in Hybrid RANS-LES Modelling*, pages 55–66. Springer, 2008.