Synthetic Isotropic Turbulence based on a Specified Energy Spectrum

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Abstract

Given a turbulent energy spectrum, the task is to generate an isotropic turbulent velocity field that reproduces the input spectrum. I will use Lars Davidson's [1] formulation for generating inlet turbulent data. His method is easily extendable to three dimensions as well as different resolutions in space.

1 Formulation

We start with a generalized Fourier series for a real valued scalar function

$$u = a_0 + \sum_{m=1}^{M} a_m \cos(\frac{2\pi m x}{L}) + b_m \sin(\frac{2\pi m x}{L}) \tag{1}$$

For simplicity, we set $k_m \equiv \frac{2\pi m}{L}$ as the m^{th} wave number. Also, if the mean of f is known, we have

$$\int_0^L u \, \mathrm{d}x = a_0 \tag{2}$$

Hence, for a turbulent velocity field with zero mean (in space), we can set $a_0 = 0$. At the outset, we have

$$u = \sum_{m=1}^{M} a_m \cos(k_m x) + b_m \sin(k_m x) \tag{3}$$

We now introduce the following changes

$$a_m = \hat{u}_m \cos(\psi_m); \quad b_m = \hat{u}_m \sin(\psi_m); \quad \hat{u}_m^2 = a_m^2 + b_m^2, \quad \psi_m = \arctan(\frac{b_m}{a_m}) \tag{4}$$

then

$$\begin{aligned} a_m \cos(k_m x) + b_m \sin(k_m x) &= \hat{u}_m \cos\alpha_m \cos(k_m x) + \hat{u}_m \sin\alpha_m \sin(k_m x) \\ &= \hat{u}_m \cos(k_m x - \psi_m) \end{aligned} \tag{5}$$

so that

$$u = \sum_{m=1}^{M} \hat{u}_m \cos(k_m x - \psi_m) \tag{6}$$

The extension to 3D follows

$$u = \sum_{m=1}^{M} \hat{u}_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \tag{7}$$

$$v = \sum_{m=1}^{M} \hat{v}_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \tag{8}$$

$$w = \sum_{m=1}^{M} \hat{w}_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \tag{9}$$

where $\mathbf{k}_m \equiv (k_{x,m}, k_{y,m}, k_{z,m})$ is the position vector in wave space and $\mathbf{x} \equiv (x, y, z)$ is the position vector in physical space. Therefore, $\mathbf{k}_m \cdot \mathbf{x}_m = k_{x,m}x + k_{y,m}y + k_{z,m}z$. A condensed form is

$$\mathbf{u} = \sum_{m=1}^{M} \hat{\mathbf{u}}_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \tag{10}$$

where $\hat{\mathbf{u}}_m \equiv (\hat{u}_m, \hat{v}_m, \hat{w}_m)$. Continuity dictates that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{11}$$

This gives

$$-\sum_{m=1}^{m} \left(k_{x,m}\hat{u}_m + k_{y,m}\hat{v}_m + k_{z,m}\hat{w}_m \right) \sin(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) = 0 \tag{12}$$

or

$$\sum_{m} \mathbf{k}_m \cdot \hat{\mathbf{u}}_m \sin(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) = 0 \tag{13}$$

This equation can be enforced by setting

$$\mathbf{k}_m \cdot \hat{\mathbf{u}}_m = 0, \ \forall\, m \in \{0, 1, \cdots, M\}$$

This means that the Fourier coefficients have different directions in wave space. We therefore write the Fourier coefficients as

$$\hat{\mathbf{u}}_m \equiv q_m \boldsymbol{\sigma}_m \mid \mathbf{k}_m \cdot \boldsymbol{\sigma}_m = 0 \tag{14}$$

where $\boldsymbol{\sigma}_m$ is a unit vector computed such that $\mathbf{k}_m \cdot \boldsymbol{\sigma}_m = 0$ at any point $\mathbf{x}$. The velocity vector at point $\mathbf{x}$ is now at hand

$$\mathbf{u}(\mathbf{x}) = \sum_{m=1}^{M} q_m \cos(\mathbf{k}_m \cdot \mathbf{x} - \psi_m) \boldsymbol{\sigma}_m \tag{15}$$

The last step is to link q_m to the energy spectrum. This can be computed from

$$q_m = 2\sqrt{E(k_m)\Delta k} \tag{16}$$

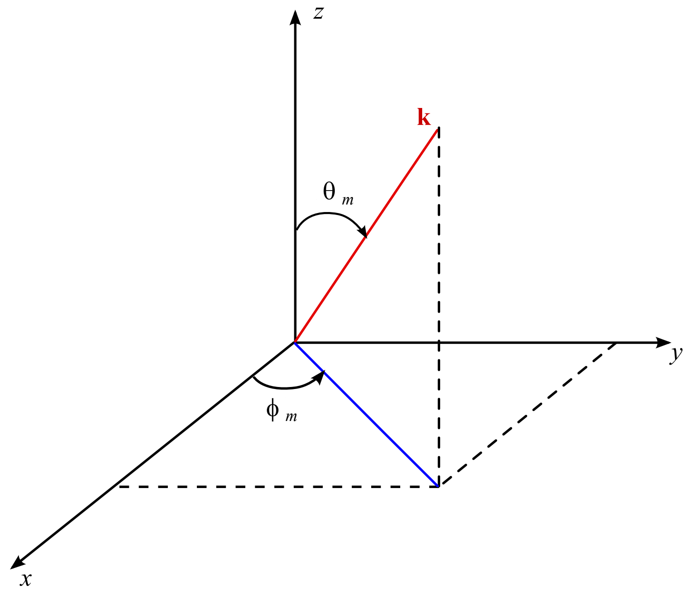

Figure 1: Angles associated with wave number $\mathbf{k}_m$.

2 In Practice

- Specify the number of modes M. This will determine the Fourier representation of the velocity field at every point in the spatial domain
- Compute or set a minimum wave number k_0
- Compute a maximum wave number $k_{\max} = \frac{\pi}{\Delta x}$. For multiple dimensions, use $k_{\max} = \max(\frac{\pi}{\Delta x}, \frac{\pi}{\Delta y}, \frac{\pi}{\Delta z})$
- Generate a list of M modes: $k_m \equiv k(m) = k_0 + \frac{k_{\max} - k_0}{M}(m-1)$. Those will correspond to the magnitude of the vector $\mathbf{k}_m$. In other words, k_m is the radius of a sphere.
- Generate four arrays of random numbers, each of which is of size M (those will be needed next). Those will correspond to the angles: θ_m, φ_m, ψ_m, and α_m.
- Compute the wave vectors. To generate as much randomness as possible, we write the wave vector as a function of two angles in 3D space. This means

$$k_{x,m} = \sin(\theta_m)\cos(\varphi_m)k_m \tag{17}$$

$$k_{y,m} = \sin(\theta_m)\sin(\varphi_m)k_m \tag{18}$$

$$k_{x,m} = \cos(\theta_m)k_m \tag{19}$$

- Compute the unit vector $\boldsymbol{\sigma}_m$. Note that $\boldsymbol{\sigma}_m$ lies in a plane perpendicular to the vector $\mathbf{k}_m$. We choose the following

$$\sigma_{x,m} = \cos(\theta_m)\cos(\varphi_m)\cos(\alpha_m) - \sin(\varphi_m)\sin(\alpha_m) \tag{20}$$

$$\sigma_{y,m} = \cos(\theta_m)\sin(\varphi_m)\cos(\alpha_m) + \cos(\varphi_m)\sin(\alpha_m) \tag{21}$$

$$\sigma_{z,m} = -\sin(\theta_m)\cos(\alpha_m) \tag{22}$$

- Once those quantities are computed, loop over the mesh. For every point on the mesh, loop over all M modes. For every mode, compute $q_m = 2\sqrt{E(k_m)\Delta k}$ and $\beta_m = \mathbf{k}_m \cdot \mathbf{x} - \psi_m$. Finally, construct the following summations (at every point (x, y, z) you will have a summation of M-modes)

$$u(x,y,z) = \sum_{n=1}^{M} q_m \cos(\beta_m)\sigma_{x,m} \tag{23}$$

$$v(x,y,z) = \sum_{n=1}^{M} q_m \cos(\beta_m)\sigma_{y,m} \tag{24}$$

$$w(x,y,z) = \sum_{n=1}^{M} q_m \cos(\beta_m)\sigma_{z,m} \tag{25}$$

References

[1] Lars Davidson. Hybrid les-rans: Inlet boundary conditions for flows with recirculation. In *Advances in Hybrid RANS-LES Modelling*, pages 55–66. Springer, 2008.