Quiz 1 Practice Problems: Cyclic Groups

Math 332, Spring 2010

These are not to be handed in. The quiz will be on Tuesday.

1. Find all generators of Z_6, Z_8, and Z_{20}.

2. List all elements of the subgroup $\langle 30 \rangle$ in Z_{80}.

3. If $|a| = 60$, what is the order of a^{24}?

4. How many subgroups does Z_{20} have? List the possible generators for each subgroup.

5. Suppose that $|a| = 24$. Find a generator for $\langle a^{21} \rangle \cap \langle a^{10} \rangle$. In general, what is a generator for the subgroup $\langle a^m \rangle \cap \langle a^n \rangle$?

6. In Z_{60}, list all generators for the subgroup of order 12.

7. Let G be a group and let a be an element of G.

a. If $a^{12} = e$, what can we say about the order of a?

b. If $a^m = e$, what can we say about the order of a?

8. List all elements of order 8 in $Z_{8000000}$. How do you know your list is complete?

9. Determine the subgroup lattice for $Z_{p^2 q}$, where p and q are distinct primes.

10. Determine the subgroup lattice for Z_{p^n}, where p is a prime and n is some positive integer.

11. If $|x| = 40$, list all elements of $\langle x \rangle$ that have order 10.

12. Determine the orders of the elements of D_{33} and how many there are of each.

13. If $|a^5| = 12$, what are the possibilities for $|a|$? What if $|a^5| = 15$?

Answers

1. For Z_6, generators are 1 and 5; for Z_8, generators are 1, 3, 5, and 7; for Z_{20}, generators are 1, 3, 7, 9, 11, 13, 17, and 19.

2. $0, 10, 20, 30, 40, 50, 60, 70$

3. Since $\langle a^{24} \rangle = \langle a^{12} \rangle$ is a subgroup of order 5, the element a^{24} must have order 5 as well.

4. Six subgroups: Z_{20} (generated by 1, 3, 7, 9, 11, 13, 17, or 19), the subgroup of even numbers (generated by 2, 6, 14, or 18), the subgroup of multiples of 4 (generated by 4, 8, 12, or 16), the subgroup of multiples of 5 (generated by 5 or 15), the subgroup of multiples of 10 (generated by 10), and the trivial subgroup (generated by 0).

5. $\langle a^{21} \rangle \cap \langle a^{10} \rangle = \langle a^6 \rangle$. In the general case $\langle a^m \rangle \cap \langle a^n \rangle = \langle a^k \rangle$, where $k = \operatorname{lcm}(m, n) \bmod 24$.

6. 5, 25, 35, and 55. **7.** a. $|a|$ divides 12. b. $|a|$ divides m.

8. 1000000, 3000000, 5000000, 7000000. By Theorem 4.3, $\langle 1000000 \rangle$ is the unique subgroup of order 8, and only those on the list are generators

9.

$$
\begin{tikzcd}[arrows=dash]
 & \langle 1 \rangle \arrow[dl] \arrow[dr] & \\
\langle p \rangle \arrow[d] \arrow[drr] & & \langle q \rangle \arrow[d] \\
\langle p^2 \rangle \arrow[dr] & & \langle pq \rangle \arrow[dl] \\
 & \langle 0 \rangle &
\end{tikzcd}
\qquad \text{or} \qquad
\begin{tikzcd}[arrows=dash]
 & Z_{p^2q} \arrow[dl] \arrow[dr] & \\
Z_{pq} \arrow[d] \arrow[drr] & & Z_{p^2} \arrow[d] \\
Z_q \arrow[dr] & & Z_p \arrow[dl] \\
 & \text{trivial} &
\end{tikzcd}
$$

10.

$$
\begin{tikzcd}[arrows=dash]
\langle 1 \rangle \arrow[d] \\
\langle p \rangle \arrow[d] \\
\langle p^2 \rangle \arrow[d] \\
\vdots \arrow[d] \\
\langle p^{n-1} \rangle \arrow[d] \\
\langle 0 \rangle
\end{tikzcd}
\qquad \text{or} \qquad
\begin{tikzcd}[arrows=dash]
Z_{p^n} \arrow[d] \\
Z_{p^{n-1}} \arrow[d] \\
Z_{p^{n-2}} \arrow[d] \\
\vdots \arrow[d] \\
Z_p \arrow[d] \\
\text{trivial}
\end{tikzcd}
$$

11. x^4, x^{12}, x^{28}, x^{36}.

12. 33 of order 2, 20 of order 33, 10 of order 11, 2 of order 3, one of order 1.

13. $|a| = 12$ or $|a| = 60$; $|a| = 75$.