

Complex Functions

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1 Lecturer Information

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2 Recommended Reading

1. James Ward Brown & Ruel V. Churchill, “Complex Variables and Applications”, McGraw-Hill, Inc. 1996.
2. D. Zill, P. Shanahan, “Complex Variables with Applications”, Jones and Bartlett Publishers.

3 Additional Reading

1. Saff, Edward B., and Arthur David Snider. Fundamentals of Complex Analysis with Applications to Engineering, Science, and Mathematics. 3rd ed. Upper Saddle River, NJ: Prentice Hall, 2002. ISBN: 0139078746.
2. Sarason, Donald. Complex Function Theory. American Mathematical Society. ISBN: 0821886223
3. Alfhors, Lars. Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable. McGraw-Hill Education, 1979. ISBN: 0070006571.

Part I

Complex Numbers

Definition 1. A number of the form

$$z = x + iy$$

where

$$i = \sqrt{-1}$$

$$x \in \mathbb{R}$$

$$y \in \mathbb{R}$$

is called a complex number.

Definition 2 (Real part of a complex number). If

$$z = x + iy$$

then x is called the real part of z , and is denoted as

$$x = \Re(z)$$

Definition 3 (Imaginary part of a complex number). If

$$z = x + iy$$

then y is called the imaginary part of z , and is denoted as

$$x = \Im(z)$$

Definition 4 (Complex conjugate). If

$$z = x + iy$$

then

$$\bar{z} = x - iy$$

is called the complex conjugate of z .

Theorem 1.

$$z\bar{z} = |z|^2$$

Proof.

$$\begin{aligned} z &= x + iy \\ \therefore \bar{z} &= x - iy \end{aligned}$$

Therefore,

$$\begin{aligned} z\bar{z} &= (x + iy)(x - iy) \\ &= x^2 - ixy + ixy + y^2 \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

□

Definition 5 (Polar representation). If

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

then (r, θ) is called the polar representation of (x, y) .

Theorem 2 (Euler's Formula).

$$r \cos \theta + ir \sin \theta = re^{i\theta}$$

Definition 6 (Absolute value or Norm).

$$\begin{aligned} |z| &= |x + iy| \\ &= \sqrt{x^2 + y^2} \end{aligned}$$

is called the absolute value, or the norm of z .

Theorem 3.

$$|z| \leq |\Re(z)| + |\Im(z)| \leq \sqrt{2}|z|$$

Proof.

$$\begin{aligned} \sqrt{x^2 + y^2} &\leq |x| + |y| \leq \sqrt{2x^2 + 2y^2} \\ \iff x^2 + y^2 &\leq x^2 + y^2 + 2|x||y| \leq 2x^2 + 2y^2 \\ \iff x^2 + y^2 - 2|x||y| &\geq 0 \\ \iff (|x| - |y|)^2 &\geq 0 \end{aligned}$$

□

Definition 7 (Argument). Let z be a complex number. Then, θ , such that $\theta \in (-\pi, \pi]$, and

$$z = (r, \theta)$$

is called the argument of z . It is denoted as

$$\theta = \text{Arg}(z)$$

If $\theta \notin (-\pi, \pi]$, but

$$z = (r, \theta)$$

then

$$\theta = \arg(z)$$

Theorem 4.

$$z^n = |z|^n e^{in \text{Arg}(z)}$$

Proof.

$$\begin{aligned} z &= |z| e^{i \text{Arg}(z)} \\ \therefore z^n &= \left(|z| e^{i \text{Arg}(z)} \right)^n \\ &= (|z|)^n \left(e^{i \text{Arg}(z)} \right)^n \\ &= |z|^n e^{in \text{Arg}(z)} \end{aligned}$$

□

Theorem 5. *Let*

$$\begin{aligned} z &= r e^{i\theta} \\ w &= \rho e^{i\varphi} \end{aligned}$$

The solutions to

$$w = \sqrt[n]{z}$$

are

$$\varphi_k = \frac{\theta}{n} + \frac{2\pi k}{n}$$

where $k \in \{0, \dots, n-1\}$.

Proof.

$$\begin{aligned}w &= \sqrt[n]{z} \\ \therefore w^n &= z\end{aligned}$$

Therefore,

$$\rho^n e^{in\varphi} = r e^{i\theta}$$

Therefore, for $k \in \{0, \dots, n-1\}$,

$$\begin{aligned}\rho &= \sqrt[n]{r} \\ n\varphi &= \theta + 2\pi k \\ \therefore \varphi &= \frac{\theta}{n} + \frac{2\pi k}{n}\end{aligned}$$

□

Part II

Complex Sequences and Series

Definition 8 (Convergence of complex sequences). Let

$$z_n = x_n + iy_n$$

The sequence $\{z_n\}$ is said to converge to the limit $z = x + iy$, if $\forall \varepsilon > 0, \exists N$, such that $\forall n > N, |z_n - z| < \varepsilon$, i.e. there is a circular region of radius ε , centred at z , in which z_n lies.

Theorem 6. $\{z_n\} \rightarrow z$, i.e. $\{z_n\}$ converges to z if and only if all subsequences of $\{z_n\}$ converge to z .

Exercise 1.

Find the limit $\lim_{n \rightarrow \infty} \frac{n+i}{2n-i}$.

Solution 1.

$$\begin{aligned} z_n &= \frac{n+i}{2n-i} \\ &= \frac{(n+i)(2n+i)}{4n^2+1} \\ &= \frac{2n^2+1}{4n^2+1} + i \frac{3n}{4n^2+1} \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} z_n &= \lim_{n \rightarrow \infty} \frac{2n^2+1}{4n^2+1} + i \frac{3n}{4n^2+1} \\ &= \frac{1}{2} \end{aligned}$$

Exercise 2.

Show that for

$$z_n = -2 + \frac{(-1)^n}{n}i$$

$\lim_{n \rightarrow \infty} \text{Arg}(z_n)$ does not exist, but $\lim_{n \rightarrow \infty} |z_n|$ exists.

Solution 2.

The magnitude of z_n is

$$\begin{aligned}|z_n| &= \left| -2 + \frac{(-1)^n}{n}i \right| \\&= \sqrt{4 + \frac{(-1)^{2n}}{n^2}} \\&= \sqrt{4 + \frac{1}{n^2}}\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{n \rightarrow \infty} |z_n| &= \lim_{n \rightarrow \infty} \sqrt{4 + \frac{1}{n^2}} \\&= 2\end{aligned}$$

The argument of z_{2n} is

$$\begin{aligned}\text{Arg}(z_{2n}) &= \text{Arg}\left(-2 + \frac{(-1)^{2n}}{2n}i\right) \\ \therefore \lim_{n \rightarrow \infty} \text{Arg}(z_{2n}) &= \lim_{n \rightarrow \infty} \text{Arg}\left(-2 + \frac{i}{2n}\right) \\&= \pi\end{aligned}$$

The argument of z_{2n+1} is

$$\begin{aligned}\text{Arg}(z_{2n+1}) &= \text{Arg}\left(-2 + \frac{(-1)^{2n+1}}{2n+1}i\right) \\ \therefore \lim_{n \rightarrow \infty} \text{Arg}(z_{2n}) &= \lim_{n \rightarrow \infty} \text{Arg}\left(-2 - \frac{i}{2n}\right) \\&= -\pi\end{aligned}$$

Therefore, as the limit of two subsequences are not equal, the limit does not exist.

Part III

Topology on the Complex Plane

Definition 9 (Neighbourhood of a complex number). A circular region of radius ε centred at z , is called the ε neighbourhood of z .

$$B(z, \varepsilon) = D(z, \varepsilon) = \{w \in \mathbb{C} : |w - z| < \varepsilon\}$$

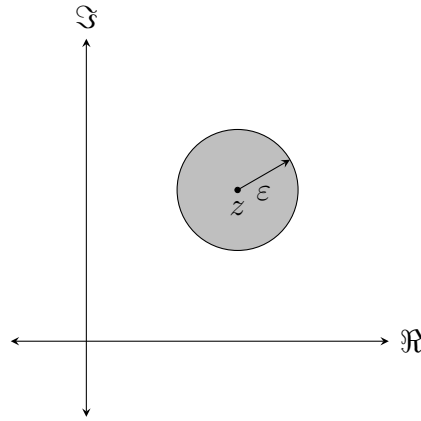


Figure 1: Neighbourhood of a complex number

Definition 10 (Interior point). Let $A \subseteq \mathbb{C}$.

$z \in \mathbb{C}$ is called an inner or interior point of A if there exists at least one $\varepsilon_z > 0$, such that $B(z, \varepsilon_z) \subset A$.

The set of all interior points of A is denoted by $\text{Int}(A)$ or A° .

Definition 11 (Exterior point). Let $A \subseteq \mathbb{C}$.

$z \in \mathbb{C}$ is called an outer or exterior point of A if there exists at least one $\varepsilon_z > 0$, such that $B(z, \varepsilon_z) \subset (\mathbb{C} \setminus A)$. The set of all exterior points of A is denoted by $\text{Ext}(A)$.

Definition 12 (Edge point). Let $A \subseteq \mathbb{C}$.

$z \in \mathbb{C}$ is called an edge or boundary point of A if it is neither an inner point of A , nor an outer point of A . The set of all boundary points of A is denoted by $\partial(A)$.

Definition 13 (Open set). A set $A \subseteq \mathbb{C}$ is called an open set if $A = A^\circ$, i.e. for any point $z \in A$, $\exists \varepsilon > 0$, such that $D(z, \varepsilon) \subset A$.

Definition 14 (Closur of a set). The closer of A is defined to be

$$\overline{A} = A^\circ \cup \partial A$$

Definition 15 (Closed set). A set A is called a closed set if $\partial A \subset A$, i.e. $A = \overline{A}$.

Definition 16 (Connected set). A set A is called a connected set if for any $z_1, z_2 \in A$, there exists a polygonal path, i.e. a finite set of connected straight lines, which connects z_1 and z_2 , and belongs to A .

Definition 17 (Domain). An open connected set is called a domain.

Definition 18 (Bound set). A set A is said to be a bound set if it is bound inside a disk.

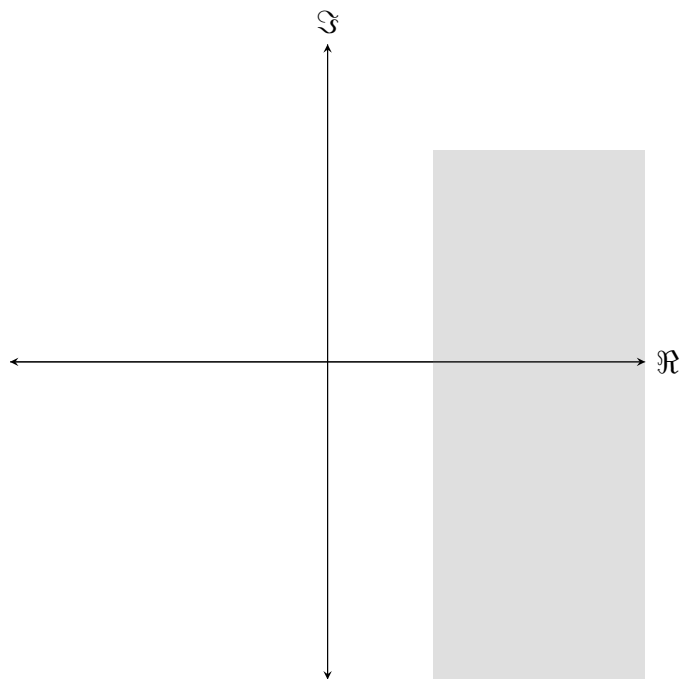
Exercise 3.

Describe geometrically and list the properties of the following sets.

1. $A = \{z \in \mathbb{C} : \Re(z) \geq 2, \Im(z) \leq 4\}$
2. $B = \{z \in \mathbb{C} : |z - 1 + 3i| > 3\}$

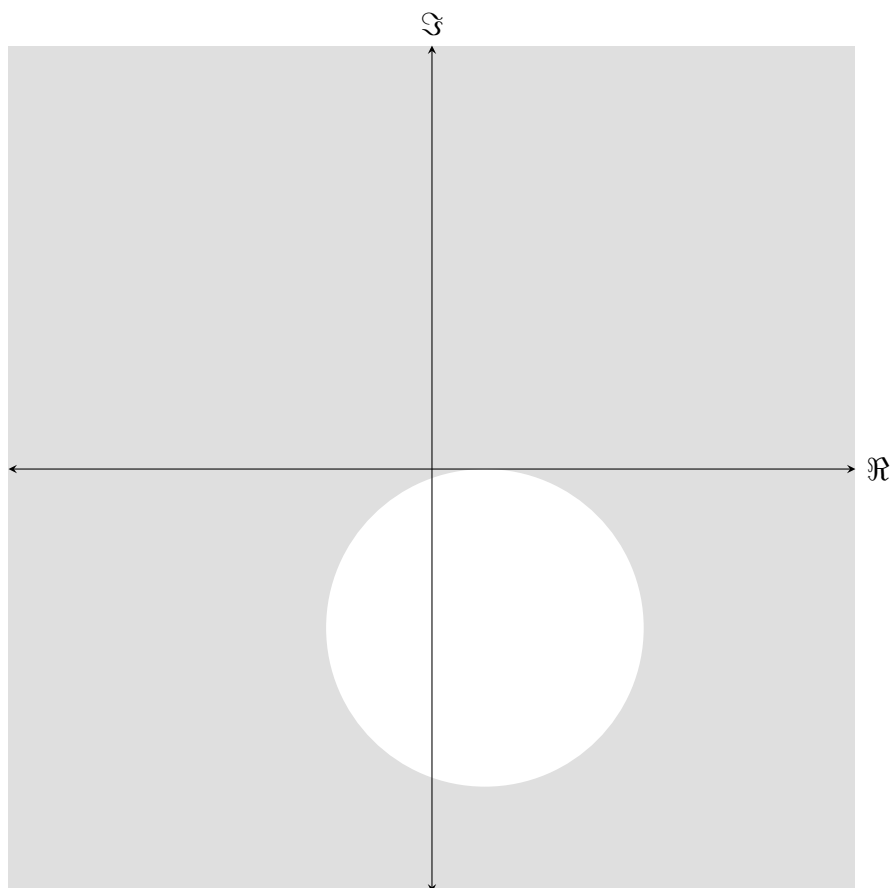
Solution 3.

1. A is the union of the bottom half plane with respect to the line $y = 4$, and the right half plane with respect to the line $x = 2$.



Therefore, as $A = A^\circ + \partial A$, it is a closer, unbounded set.

2. A is the complement of a disk, centred at $1 - 3i$, with radius 3.



Therefore, it is an open, unbounded set.

Exercise 4.

Prove that the upper half plane $U = \{z : \Im(z) > 0\}$ is open.

Solution 4.

Let

$$z = x + iy$$

Therefore, as $z \in U$, $y > 0$.

Therefore, consider the disk $D\left(z, \frac{y}{2}\right)$.

Let $w \in D\left(z, \frac{y}{2}\right)$. Therefore,

$$\begin{aligned} |w - z| &< \frac{y}{2} \\ \therefore |\Im(w - z)| &\leq |w - z| \\ &\leq \frac{y}{2} \end{aligned}$$

Therefore,

$$\begin{aligned} -\frac{y}{2} &\leq \Im(w) - \Im(z) \leq \frac{y}{2} \\ \therefore -\frac{y}{2} &\leq \Im(w) - y \leq \frac{y}{2} \\ \therefore \Im(w) &\geq \frac{y}{2} > 0 \end{aligned}$$

Therefore, as $\Im(w) > 0$, $w \in U$. Therefore, U is open. □

Part IV

Complex Functions

1 Complex Functions

Definition 19 (Complex function). Let $A \subseteq \mathbb{C}$. $f : A \rightarrow \mathbb{C}$ is called a complex function, which matches $z \in A$ to $f(z) \in \mathbb{C}$.

Theorem 7. Any complex function f can be written as

$$\begin{aligned} f(x + iy) &= \Re f(x + iy) + i\Im f(x + iy) \\ &= u(x, y) + iv(x, y) \end{aligned}$$

2 Limits

Definition 20 (Limit of a function). Let f be a complex function defined on a neighbourhood of z_0 , but may or may not be defined at z_0 . Then, the limit of $f(z)$ at z_0 is defined as

$$w = \lim_{z \rightarrow z_0} f(z)$$

if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $\forall z \in \mathbb{X}$ such that $|z - z_0| < \delta$, $|f(z) - w| < \varepsilon$.

Exercise 5.

Show that

$$\lim_{z \rightarrow 1} \frac{iz}{2} = \frac{i}{2}$$

Solution 5.

Let $|z - 1| < \delta$. Therefore, for $\varepsilon > 0$,

$$\begin{aligned} \left| f(z) - \frac{i}{2} \right| &= \left| \frac{iz}{2} - \frac{i}{2} \right| \\ &= \left| \frac{i}{2} \right| |z - 1| \\ &= \frac{1}{2} |z - i| \end{aligned}$$

Therefore, for $\delta \leq 2\varepsilon$, $\left| f(z) - \frac{i}{2} \right| < \varepsilon$. □

Theorem 8. *If*

$$\begin{aligned} f(z) &= f(x + iy) \\ &= u(x, y) + iv(x, y) \end{aligned}$$

then

$$\lim_{z \rightarrow z_0} f(z) = u_0 + iv_0$$

if and only if

$$\begin{aligned} \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) &= u_0 \\ \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) &= v_0 \end{aligned}$$

Theorem 9 (Limit arithmetics). *If*

$$\begin{aligned} \lim_{z \rightarrow z_0} f(z) &= w_1 \\ \lim_{z \rightarrow z_0} g(z) &= w_2 \end{aligned}$$

then, as long as all quantities are defined,

$$\begin{aligned} \lim_{z \rightarrow z_0} f(z) \pm g(z) &= w_1 \pm w_2 \\ \lim_{z \rightarrow z_0} f(z)g(z) &= w_1w_2 \\ \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} &= \frac{w_1}{w_2} \end{aligned}$$

Exercise 6.

For the function $f(z) = \bar{z}^2$, prove

$$\begin{aligned} \lim_{z \rightarrow z_0} f(z) &= f(z_0) \\ &= \bar{z}_0^2 \end{aligned}$$

Solution 6.

$$\begin{aligned} \bar{z} &= \overline{(x + iy)}^2 \\ &= (x - iy)^2 \\ &= x^2 - y^2 - 2xyi \end{aligned}$$

Therefore, let

$$\begin{aligned}u(x, y) &= x^2 - y^2 \\v(x, y) &= -2xy\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) &= x_0^2 - y_0^2 \\ \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) &= -2x_0y_0\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{z \rightarrow z_0} f(z) &= u_0 + iv_0 \\ &= x_0^2 - y_0^2 - 2x_0y_0i \\ &= \overline{z_0}^2\end{aligned}$$

□

Definition 21 (Infinite limit). The limit of $f(z)$ is said to be infinite, i.e.

$$\lim_{z \rightarrow z_0} f(z) = \infty$$

if and only if

$$\lim_{z \rightarrow z_0} |f(z)| = \infty$$

if and only if

$$\lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0$$

Definition 22 (Limit at infinity). The limit of a function $f(z)$,

$$\lim_{z \rightarrow \infty} f(z) = w$$

if

$$\lim_{|z| \rightarrow \infty} f(z) = w$$

Alternatively, $\forall \varepsilon > 0, \exists R > 0$, such that for $|z| > R$, $|f(x) - w| < \varepsilon$.

Exercise 7.

Show that

$$\lim_{z \rightarrow \infty} \frac{1}{z^2} = 0$$

Solution 7.

Let $\varepsilon > 0$. Let $R > 0$, such that $\frac{1}{R^2} < \varepsilon$.

Therefore, if $|z| > R$,

$$\begin{aligned} |f(z) - 0| &= \left| \frac{1}{z^2} \right| \\ &= \frac{1}{|z^2|} \\ &= \frac{1}{|z|^2} \\ &< \frac{1}{R^2} \\ &< \varepsilon \end{aligned}$$

Therefore, $\lim_{z \rightarrow \infty} \frac{1}{z^2} = 0$.

3 Continuity

Definition 23 (Continuous function). $f(z)$ is said to be continuous at z_0 if $f(z)$ is defined at z_0 and

$$\lim_{z \rightarrow z_0} f(z) = f(z_0)$$

Theorem 10 (Continuity arithmetics). *If*

$$\lim_{z \rightarrow z_0} f(z) = f(z_0)$$

$$\lim_{z \rightarrow z_0} g(z) = g(z_0)$$

then, as long as all quantities are defined,

$$\lim_{z \rightarrow z_0} f(z) \pm g(z) = f(z_0) \pm g(z_0)$$

$$\lim_{z \rightarrow z_0} f(z)g(z) = f(z_0)g(z_0)$$

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g(z_0)}$$

4 Differentiability

Definition 24 (Differentiable function). Let $f(z)$ be defined in a neighbourhood of z_0 . f is said to be differentiable at z_0 if the limit $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

Theorem 11 (Differentiation arithmetics). If $f(z)$ and $g(z)$ are differentiable, then, as long as all quantities are defined,

$$\begin{aligned}(f(z) \pm g(z))' &= f'(z) \pm g'(z) \\ (f(z)g(z))' &= f'(z)g(z) + f(z)g'(z) \\ \left(\frac{f(z)}{g(z)}\right)' &= \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}\end{aligned}$$

5 Cauchy-Riemann Equations

Theorem 12 (Cauchy-Riemann Equations). $u(x, y)$ and $v(x, y)$ are said to be satisfying Cauchy-Riemann Equations at a point $(a, b) \in \mathbb{R}^2$, if

$$\begin{aligned}u_x(a, b) &= v_y(a, b) \\ u_y(a, b) &= -v_x(a, b)\end{aligned}$$

Theorem 13. Let

$$f(x + iy) = u(x, y) + iv(x, y)$$

Then, u and v satisfying the Cauchy-Riemann Equations is a necessary condition for f to be differentiable at (x_0, y_0) .

Theorem 14. If $f = u + iv$ is differentiable at $z_0 = a + ib$, then (u, v) satisfies the Cauchy-Riemann Equations at (a, b) .

Definition 25 (Analytic functions). If $f = u + iv$ is differentiable at any $z \in W$, where W is a neighbourhood of z_0 except maybe at z_0 , then f is said to be analytic at z_0 . If f is analytic at all $z \in W$, then it is said to be analytic in W .

Exercise 8.

Let $f : U \rightarrow \mathbb{C}$ be an analytic function in U , such that \bar{f} is also analytic in U . Show that $f' = 0$, i.e. $f = c$.

Solution 8.

As $f = u + iv$ is analytic, by Cauchy-Riemann Equations, for $(x, y) \in U$,

$$\begin{aligned} u_x(x, y) &= v_y(x, y) \\ u_y(x, y) &= -v_x(x, y) \end{aligned}$$

As $\bar{f} = u - iv$ is analytic, by Cauchy-Riemann Equations, for $(x, y) \in U$,

$$\begin{aligned} u_x(x, y) &= -v_y(x, y) \\ u_y(x, y) &= v_x(x, y) \end{aligned}$$

Therefore,

$$\begin{aligned} v_y &= -v_y \\ &= 0 \\ v_x &= -v_x \\ &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} u_x(x, y) &= 0 \\ u_y(x, y) &= 0 \end{aligned}$$

Therefore, u and v are constant functions.

6 Harmonic Functions

Definition 26 (Laplacian). Let u be an equation in x and y . The Laplacian is defined to be

$$\begin{aligned} \Delta u &= \nabla^2 u \\ &= u_{xx} + u_{yy} \end{aligned}$$

Definition 27 (Harmonic function). A real function in two variables, $u(x, y)$, which is twice differentiable, is called a harmonic function if it satisfies

$$\begin{aligned} \Delta u &= u_{xx} + u_{yy} \\ &= 0 \end{aligned}$$

Theorem 15. *If u and v are twice differentiable, and satisfy Cauchy-Riemann Equations, then (u, v) are harmonic.*

Theorem 16 (Sufficient condition for differentiability). *Let $f = u + iv$ be defined in a neighbourhood of $z_0 = a + ib$. Assume that u_x, u_y, v_x, v_y exist in this neighbourhood and are continuous at the point (a, b) . If (u, v) satisfying Cauchy-Riemann Equations at (a, b) then $f'(z_0)$ exists.*

Definition 28 (Harmonic conjugate). Let $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a harmonic function. Its harmonic conjugate is defined to be $v : \mathbb{R}^2 \rightarrow \mathbb{R}$, such that $f = u + iv$ is analytic.

7 Analytic Functions

Definition 29. $f : D \rightarrow \mathbb{C}$ is said to be differentiable on $D \subset \mathbb{C}$, if f is differentiable at any $z \in D$.

Definition 30 (Analytic functions). If $f = u + iv$ is differentiable at any $z \in W$, where W is a neighbourhood of z_0 except maybe at z_0 , then f is said to be analytic at z_0 . If f is analytic at all $z \in W$, then it is said to be analytic in W .

Theorem 17. *Let $D \subset \mathbb{C}$ be an open set. Then, f is differentiable on D if and only if f is analytic on D .*

Theorem 18. *Let $D \subseteq \mathbb{C}$ be a domain. Assume that f is analytic on D , and for any $z \in D$, $f'(z) = 0$. Then, f is constant.*

Theorem 19. *Let $u(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function such that $\nabla u = 0$ in a domain $D \subset \mathbb{R}^2$. Then, u is constant in D .*

Exercise 9.

1. Prove that

$$v(x, y) = \ln \left((x - 1)^2 + (y - 2)^2 \right)$$

is harmonic in any domain that does not include the point $(1, 2)$.

2. Find $u(x, y)$ such that $u + iv$ is analytic in some domain. Note: v is the conjugate harmonic of u .
3. Express $u + iv$ as a function of z .

Solution 9.

1.

$$v_x = \frac{2(x-1)}{(x-1)^2 + (y-2)^2}$$
$$v_y = \frac{2(y-2)}{(x-1)^2 + (y-2)^2}$$

Therefore,

$$v_{xx} = \frac{2 \left((x-1)^2 + (y-2)^2 \right) - (2(x-1))^2}{((x-1)^2 + (y-2)^2)^2}$$
$$v_{yy} = \frac{2 \left((x-1)^2 + (y-2)^2 \right) - (2(y-2))^2}{((x-1)^2 + (y-2)^2)^2}$$

2. For $u + iv$ to be analytic, by Cauchy-Riemann Equations,

$$u_x = v_y$$
$$u_y = -v_x$$

Therefore,

$$u_x = v_y$$
$$= \frac{2(y-2)}{(x-1)^2 + (y-2)^2}$$

Therefore,

$$u = \int \frac{2(y-2)}{(x-1)^2 + (y-2)^2} dx$$
$$= \frac{2(y-2)}{(y-2)^2} \int \frac{1}{1 + \left(\frac{x-1}{y-2} \right)^2} dx$$
$$= 2 \tan^{-1} \left(\frac{x-1}{y-2} \right) + g(y)$$

Therefore,

$$u_y = -v_x$$
$$\therefore -\frac{2(x-1)}{(x-1)^2 + (y-2)^2} = \frac{2}{1 + \frac{(x-1)^2}{(y-2)^2}} \left(-\frac{x-1}{y-2} \right) + g'(y)$$

Therefore,

$$\begin{aligned} g'(y) &= 0 \\ \therefore g(y) &= c \end{aligned}$$

Therefore,

$$u = 2 \tan^{-1} \left(\frac{x-1}{y-2} \right) + c$$

3.

$$\begin{aligned} u + iv &= \tan^{-1} \left(\frac{x-1}{y-2} \right) + i \ln \left((x-1)^2 + (y-2)^2 \right) \\ &= 2i \operatorname{Log} (-i(x-1) + (y-2)) \\ &= 2i \operatorname{Log} (-iz - 2 + i) \end{aligned}$$

Exercise 10.

Prove that there is no $f = u + iv$ analytic in the unit disk, such that

$$xu(x, y) = yv(x, y) + 2013$$

Hint: Use the function $zf(z)$.

Solution 10.

If possible, let there exist $f(z)$ such that

$$xu(x, y) = yv(x, y) + 2013$$

Therefore, as $zf(z)$ is analytic,

$$\begin{aligned} zf(z) &= (x + iy)(u + iv) \\ &= xu - yv + i(yu + xv) \\ &= 2013 + i(yu + xv) \end{aligned}$$

By the polar form of Cauchy-Riemann Equations, $yu + xv$ is constant.

Therefore, $zf(z)$ is constant.

Therefore, this contradicts the assumption.

Therefore, such a f does not exist.

8 Elementary Functions

8.1 Exponential Functions

Theorem 20.

$$|e^z| = e^{\Re(z)}$$

Proof.

$$\begin{aligned} |e^z| &= \left| e^{\Re(z)} \right| \left| e^{\Im(z)} \right| \\ &= \left| e^{\Re(z)} \right| \left| \cos(\Im(z)) + i \sin(\Im(z)) \right| \\ &= e^{\Re(z)} \end{aligned}$$

□

Theorem 21. *Let z and w be complex. Then*

$$e^{z+w} = e^z e^w$$

Theorem 22. $\forall n \in \mathbb{Z}$,

$$(e^z)^n = e^{nz}$$

Theorem 23. *The function e^z is onto with respect to $\mathbb{C} \setminus \{0\}$.*

8.2 Trigonometric Functions

Definition 31 (Trigonometric functions of complex numbers). Trigonometric functions of complex numbers are defined as

$$\begin{aligned} \cos(z) &= \frac{e^{iz} + e^{-iz}}{2} \\ \sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} \\ \cosh(z) &= \frac{e^z + e^{-z}}{2} \\ \sinh(z) &= \frac{e^z - e^{-z}}{2} \end{aligned}$$

8.3 Logarithmic Functions

Definition 32 (Power set). The set of all subsets of a set is called the power set of the set. The power set of a set A is denoted as $P(A)$.

Definition 33 (Multiple valued function). A set which maps a set A to its power set $P(A)$ is called a multiple valued set.

A multiple valued function gets over \mathbb{C} gets a complex number as input and returns a set of complex numbers as output.

Definition 34 (Natural logarithmic function). The natural logarithmic function over the complex plane is defined to be

$$\log w = \{z : e^z = w\}$$

Theorem 24.

$$\log w = \ln |w| + i \arg(w)$$

Proof. Let

$$\begin{aligned} e^z &= w \\ &= |w|e^{i\theta} \end{aligned}$$

where

$$\theta = \arg(w)$$

Therefore,

$$\begin{aligned} e^{\Re(z)+i\Im(z)} &= |w|e^{i\theta} \\ \therefore e^{\Re(z)}e^{i\Im(z)} &= |w|e^{i\theta} \end{aligned}$$

Therefore,

$$\begin{aligned} e^{\Re(z)} &= |w| \\ \Im(z) &= \theta + 2\pi k \end{aligned}$$

where $k \in \mathbb{Z}$.

Therefore,

$$\begin{aligned} \ln e^{\Re(z)} &= \ln |w| \\ \therefore \Re(z) &= \ln |w| \end{aligned}$$

Therefore,

$$\begin{aligned} \log w &= \{z : e^z = w\} \\ &= \{\ln |w| + iy : y = \arg(w)\} \end{aligned}$$

For any $w \in \log z$,

$$\begin{aligned} e^w &= e^{\ln|z| + i(\operatorname{Arg} z + 2\pi k)} \\ &= e^{\ln|z|} e^{i(\operatorname{Arg} z + 2\pi k)} \\ &= |z| e^{i \operatorname{Arg} z} \\ &= z \end{aligned}$$

□

Definition 35 (Branch of $\log z$). A branch of $\log z$ is a continuous function $L(z)$ defined on a U , a connected open subset of \mathbb{C} such that $L(z)$ is a logarithm of z for each $z \in U$.

Definition 36 ($\operatorname{Log} z$). $\operatorname{Log} z$ is defined to be

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$$

As $\operatorname{Arg} z$ is not continuous on the negative real axis, in order to make it continuous, the line $\operatorname{Arg} z = \pi$ is excluded. Hence, $\log z$ is continuous on $U = \mathbb{C} \setminus \{0\} \cup \mathbb{R}^-$, and is a branch of $\log z$.

Similarly, any other ray can be excluded in order to get a branch of $\log z$.

Definition 37. For any $\alpha \in \mathbb{R}$, $\operatorname{Log}_\alpha z$ is defined to be

$$\operatorname{Log}_\alpha z = \ln|z| + i \operatorname{Arg}_\alpha z$$

where $\operatorname{Arg}_\alpha z = \theta$, such that $\theta \in (\alpha, \alpha + 2\pi]$ and $\theta = \arg z$.

Any choice of $\operatorname{Arg}_\alpha z$ defines a branch of logarithm.

Definition 38 (Branch cut). The boundary of the domain of a branch is called a branch cut.

Definition 39 (Principal value). The value returned by $\operatorname{Log} z = \operatorname{Log}_{-\pi} z$ is called the principal value.

Theorem 25. $\operatorname{Log} z$ is analytic on $\mathbb{C} \setminus \{0\} \cup \mathbb{R}^-$.

Exercise 11.

Find the principal value of \sqrt{i} .

Solution 11.

$$\begin{aligned} \operatorname{pv} \left(i^{\frac{1}{2}} \right) &= e^{\frac{1}{2} \operatorname{Log} i} \\ &= e^{\frac{1}{2} (\ln|i| + i \operatorname{Arg} i)} \\ &= e^{\frac{1}{2} i \frac{\pi}{2}} \\ &= e^{i \frac{\pi}{4}} \end{aligned}$$

8.4 Power

Definition 40 (Power function). Let $z, c \in \mathbb{C}$, such that $z \neq 0$. The power multifunction as

$$z^c = e^{c \log z}$$

The branch of the power multifunction for $c \in \mathbb{C}$ is defined as

$$z^w = e^{w \log z}$$

Theorem 26.

$$\operatorname{Log}_\alpha z - \operatorname{Log}_\beta z = i \left(\operatorname{Arg}_\alpha z - \operatorname{Arg}_\beta z \right)$$

Part V

Complex Integrals

1 Complex Integrals

Definition 41 (Integral of complex functions). Let $f : [a, b] \rightarrow \mathbb{C}$. Let

$$f(t) = u(t) + iv(t)$$

Therefore, the integrals of $u(t)$ and $v(t)$ are defined as

$$\int_a^b u(t) \, dt = \lim_{\Delta t \rightarrow 0} \sum_{i=1}^n u(t_i) \Delta x_i$$

where T is a splitting of $[a, b]$, such that

$$a = t_1 < \cdots < t_n = b$$

and

$$\int_a^b v(t) \, dt = \lim_{\Delta t \rightarrow 0} \sum_{i=1}^n v(t_i) \Delta x_i$$

where T is a splitting of $[a, b]$, such that

$$a = t_1 < \cdots < t_n = b$$

These integrals are defined when the limit exists without depending on T .

When they exist, the integral of $f(t)$ is defined as

$$\int_a^b f(t) \, dt = \int_a^b u(t) \, dt + i \int_a^b v(t) \, dt$$

Theorem 27. *All properties of real integrals are also valid for complex integrals.*

Theorem 28.

$$\left| \int_a^b f(t) \, dt \right| \leq \int_a^b |f(t)| \, dt$$

2 Curves in \mathbb{C}

Definition 42. A continuous function $\gamma : [a, b] \rightarrow \mathbb{C}$ is called a curve.

Definition 43 (Parametric representation of a curve). The curve $\gamma(t)$ can be represented as

$$\gamma(t) = x(t) + iy(t)$$

where t is a parameter.

Definition 44 (Differentiability). γ is said to be differentiable if x and y are both differentiable.

Theorem 29 (Parametric representation of a straight line). *Let $z_1, z_2 \in \mathbb{C}$. The straight line passing through z_1 and z_2 can be represented parametrically as*

$$\gamma(t) = z_1 + t(z_2 - z_1)$$

The slope of this line is $z_2 - z_1$.

Theorem 30 (Parametric representation of a circle). *A circle with radius r , centred at the origin, can be represented parametrically as*

$$\gamma(t) = re^{it}$$

with $0 \leq t \leq 2\pi$.

Exercise 12.

Parametrize the curve $\left\{ z = x + iy : \frac{x^2}{4} + y^2 = 1 \right\}$ starting from 2, and going anti-clockwise twice.

Solution 12.

The curve is an ellipse centred at $(0, 0)$, with $a = 2$, and $b = 1$.

$$\gamma(t) = 2 \cos t + i \sin t$$

Therefore, as the curve goes anti-clockwise twice, $t \in [0, 4\pi]$.

Definition 45 (Simple curve). A curve γ is said to be simple if it is non self-intersecting, i.e. it is one-to-one with respect to the parameter t , except maybe at the extreme values of t .

Definition 46 (Closed curve). A curve $\gamma : [a, b] \rightarrow \mathbb{C}$ is said to be closed, if and only if

$$\gamma(a) = \gamma(b)$$

Definition 47 (Jordan curve). A closed simple curve is called a Jordan curve.

Theorem 31. *A Jordan curve enclosed a region inside it.*

Definition 48 (Piecewise differentiability). γ is said to be piecewise differentiable if there exists a splitting

$$a = t_1 < \cdots < t_n = b$$

such that γ is differentiable on each segment $[t_i, t_{i+1}]$.

3 Complex Line Integrals

Definition 49 (Complex line integral). Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a curve, and let $f : D \rightarrow \mathbb{C}$, where $D \subseteq \mathbb{C}$, and $\gamma([a, b]) \subset D$. Then, the integral

$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \dot{\gamma}(t) dt$$

If γ is piecewise differentiable, then

$$\int_{\gamma} f(z) dz = \sum_{i=1}^n \int_{x_i}^{x_{i+1}} f(\gamma(t)) \dot{\gamma}(t) dt$$

Definition 50 (Oriented contour). An oriented contour for $\alpha > 0$, $z_0 \in \mathbb{C}$, is defined to be

$$C_{\alpha, z_0} = \{w \in \mathbb{C} : |w - z_0| = \alpha\}$$

oriented anti-clockwise, starting at $z_0 + \alpha$.

Theorem 32. $\forall \alpha > 0, z_0 \in \mathbb{C}$,

$$\oint_{C_{\alpha, z_0}} \frac{dz}{z - z_0} = 2\pi i$$

Proof. Let

$$\gamma(t) = z_0 + \alpha e^{it}$$

with $0 \leq t \leq 2\pi$.

Therefore,

$$\dot{\gamma}(t) = \alpha i e^{it}$$

Therefore,

$$\begin{aligned} \oint_{C_{\alpha, z_0}} \frac{dz}{z - z_0} &= \int_0^{2\pi} \frac{1}{z_0 + \alpha e^{it} - z_0} \alpha i e^{it} dt \\ &= \int_0^{2\pi} i dt \\ &= 2\pi i \end{aligned}$$

□

Theorem 33. *Line integrals are linear for all $\alpha, \beta \in \mathbb{C}$, i.e.*

$$\alpha \int_{\gamma} f dz \pm \beta \int_{\gamma} g dz = \int_{\gamma} \alpha f \pm \beta g dz$$

Theorem 34. *Let γ_1 and γ_2 be two curves such that the start point of γ_2 is the end point of γ_1 . Then, the curves can be composited to a curve $\gamma_1 + \gamma_2$, and*

$$\int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz = \int_{\gamma_1 + \gamma_2} f(z) dz$$

Theorem 35. *Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a curve. Then, $\bar{\gamma} : [-b, -a] \rightarrow \mathbb{C}$ has orientation opposite to that of γ , and*

$$\bar{\gamma}(t) = \gamma(-t)$$

$$\bar{\gamma}'(t) = -\dot{\gamma}(t)$$

Then,

$$\int_{\bar{\gamma}} f(z) dz = - \int_{\gamma} f(z) dz$$

Theorem 36 (Length of a curve). *The length of the curve $\gamma : [a, b] \rightarrow \mathbb{C}$ is given by*

$$\text{length}(\gamma) = \int_a^b |\dot{\gamma}(t)| \, dt$$

Exercise 13.

Find the length of the astroid given by

$$\gamma(t) = \cos^3 t + i \sin^3 t$$

where $\gamma : [0, 2\pi] \rightarrow \mathbb{C}$.

Solution 13.

$$\begin{aligned} \gamma(t) &= \cos^3 t + i \sin^3 t \\ \therefore \dot{\gamma}(t) &= -3 \sin t \cos^2 t + 3i \cos t \sin^2 t \\ \therefore |\dot{\gamma}(t)| &= \sqrt{9 (\cos^4 t \sin^2 t + \sin^4 t \cos^2 t)} \\ &= 3 |\sin t \cos t| \sqrt{\cos^2 t + \sin^2 t} \\ &= 3 |\sin t \cos t| \end{aligned}$$

Therefore,

$$\begin{aligned} \text{length}(\gamma) &= \int_a^b |\dot{\gamma}(t)| \, dt \\ &= 3 \int_0^{2\pi} |\sin t \cos t| \, dt \\ &= 12 \int_0^{\frac{\pi}{2}} \sin t \cos t \, dt \\ &= 6 \int_0^{\frac{\pi}{2}} \sin 2t \, dt \\ &= 6 \end{aligned}$$

Theorem 37. Let $f(z)$ be a function defined in a domain D including a curve γ . Let $\exists M > 0$, such that all values of f have $|f(z)| \leq M$, then

$$\left| \int_{\gamma} f(z) dz \right| \leq M \text{length}(\gamma)$$

Definition 51 (Primitive function). Let $D \subset \mathbb{C}$. $F(z)$ is said to be the primitive function of $f(z)$ in D , if $\forall z \in D$,

$$F'(z) = f(z)$$

Theorem 38 (Fundamental Theorem of Calculus). Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be piecewise continuous, and let f be continuous on γ , i.e. $f \circ \gamma$ is continuous. Let there exist an analytic function F , defined on a domain including γ , such that $\forall z \in \gamma$,

$$F'(z) = f(z)$$

Then,

$$\int_{\gamma} f(z) dz = F(\gamma(b)) - F(\gamma(a))$$

Theorem 39 (Equivalent conditions for existence of a primitive function). Let D be a domain. Let f be continuous on D . Then, the following conditions are equivalent.

1. f has a primitive function F in D .
2. For any closed path γ such that $\gamma \subset D$,

$$\int_{\gamma} f(z) dz = 0$$

3. For any curve γ such that $\gamma \subset D$, the integral $\int_{\gamma} f(z) dz$ depends only on the edges of γ .

Exercise 14.

Find $\int_{\gamma} \cos z \, dz$ where γ goes from π to i .

Solution 14.

$\sin z$ is the primitive of $\cos z$ over \mathbb{C} .
Therefore,

$$\begin{aligned} \int_{\gamma} \cos z \, dz &= \sin i - \sin \pi \\ &= \frac{e^{i^2} - e^{-i^2}}{2i} - 0 \\ &= \frac{e^{-1} - e}{2i} \\ &= i \frac{-\frac{1}{e} + e}{2} \end{aligned}$$

Exercise 15.

Calculate the integral of

$$f(z) = (z - z_0)^n$$

$\forall n \in \mathbb{Z}$, where $\gamma = C_{R, z_0}$.

Solution 15.

For $0 \leq t \leq 2\pi$,

$$\begin{aligned} \gamma(t) &= z_0 + Re^{it} \\ \therefore \dot{\gamma}(t) &= Rie^{it} \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\gamma} (z - z_0)^n \, dz &= \int_0^{2\pi} (z_0 + Re^{it} - z_0)^n (Rie^{it}) \, dt \\ &= iR^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt \end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\gamma} (z - z_0)^n dz &= \begin{cases} 2\pi i & ; \quad n = -1 \\ \frac{R^{n+1}}{n+1} e^{i(n+1)t} \Big|_0^{2\pi} & ; \quad n \neq -1 \end{cases} \\ &= \begin{cases} 2\pi i & ; \quad n = -1 \\ 0 & ; \quad n \neq -1 \end{cases}\end{aligned}$$

Theorem 40.

$$\int_{\gamma} P dx + Q dy = \int_a^b \left(P(\gamma(t)) \dot{x}(t) + Q(\gamma(t)) \dot{y}(t) \right) dt$$

where $t \in [a, b]$.

Theorem 41. *If*

$$f = u + iv$$

then,

$$\int_{\gamma} f(z) dz = \int_{\gamma} u dx - v dy + i \int_{\gamma} v dx + u dy$$

Theorem 42 (Green's Theorem). *Let*

$$F = P dx + Q dy$$

such that P_x, P_y, Q_x, Q_y are continuous in the domain D ,

$$\int_{\partial D} P dx + Q dy = \iint_D (Q_x - P_y) dx dy$$

Theorem 43 (Cauchy-Goursat Theorem). *Let D be a domain, such that ∂D is obtained by a finite number of curves, ie. ∂D is piecewise differentiable. If $f : \overline{D} \rightarrow \mathbb{C}$ is analytic, then*

$$\int_{\partial D} f(z) dz = 0$$

4 Cauchy Integral Formula

Theorem 44 (Cauchy Integral Formula/Mean Value Theorem). *Let C be a simple closed curve in positive orientation with respect to a domain, D_C , closed by a curve C . If f is analytic in D_C , then*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

Theorem 45 (Cauchy Differentiation Formula). *Let C be a simple closed curve in positive orientation with respect to a domain, D_C , closed by a curve C . If f is analytic in D_C , then*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Theorem 46. *If f is analytic in D , then f is infinitely differentiable.*

Proof. Let $z_0 \in D$. Therefore, $\exists \varepsilon > 0$, such that $D(z_0, \varepsilon) \in D$. Therefore, by Cauchy Differentiation Formula,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_{z_0, \varepsilon}} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

and particularly, exists. □

Theorem 47 (Morera's Theorem). *Let D be a domain, and let $f : D \rightarrow \mathbb{C}$ be continuous. If $\int_{\gamma} f(z) dz = 0$, for any closed curve γ , such that $\gamma \in D$, then f is analytic in $\overset{\gamma}{D}$*

Proof. By Equivalent conditions for existence of a primitive function, as

$$\int_{\gamma} f(z) dz = 0$$

there exists a primitive function F for f , i.e.,

$$F'(z) = f(z)$$

for all $z \in D$.

Therefore, as F is differentiable in D , and as D is a domain, and hence is open, F is analytic.

Therefore, as F is analytic in D , F is infinitely differentiable, with analytic derivatives. □

Theorem 48 (Cauchy Derivative Estimate). *Let f be analytic in $D_{z_0,r}$. Let $\partial D_{z_0,r}$ be denoted as $C_{z_0,r}$. Let*

$$M_R = \max_{z \in C_{z_0,R}} |f(z)|$$

Then, $\forall n \in \mathbb{N}$,

$$|f^{(n)}(z_0)| \leq \frac{n!M_R}{R^n}$$

Exercise 16.

Find $\int_{-\pi}^{\pi} \frac{1}{2 - \cos t} dt$.

Solution 16.

Let

$$\begin{aligned} z &= e^{it} \\ \therefore dz &= iz dt \end{aligned}$$

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{1}{2 - \cos t} dt &= \int_{\partial D_{0,1}} \frac{1}{2 - \frac{z+z^{-1}}{2}} \frac{dz}{iz} \\ &= \int_{\partial D_{0,1}} \frac{2 dz}{(4 - z - z^{-1}) iz} \\ &= \int_{\partial D_{0,1}} \frac{2 dz}{-i(z^2 - 4z + 1)} \\ &= \int_{\partial D_{0,1}} \frac{2 dz}{i(z - 2 + \sqrt{3})(z - 2 - \sqrt{3})} \\ &= 2i \int_{\partial D_{0,1}} \frac{dz}{(z - 2 + \sqrt{3})(z - 2 - \sqrt{3})} \end{aligned}$$

Let

$$\begin{aligned} z_1 &= 2 + \sqrt{3} \\ z_2 &= 2 - \sqrt{3} \end{aligned}$$

Therefore, as $z_1 \in D_{0,1}$, by Cauchy Integral Formula/Mean Value Theorem,

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{1}{2 - \cos t} dt &= 2i \int_{\partial D_{0,1}} \frac{dz}{(z - 2 + \sqrt{3})(z - 2 - \sqrt{3})} \\
&= 2i \left(2\pi i \left(\frac{1}{z - 2 - \sqrt{3}} \right) \right) \Big|_{z=2-\sqrt{3}} \\
&= -4\pi \left(\frac{1}{2 - \sqrt{3} - 2 - \sqrt{3}} \right) \\
&= \frac{2\pi}{\sqrt{3}}
\end{aligned}$$

Therefore, the integral is real, which is expected, as the function is real.

Exercise 17.

Calculate $\int_{C_{1,3}} \frac{\cos z}{(z-i)^3} dz$.

Solution 17.

$$\begin{aligned}
\int_{C_{1,3}} \frac{\cos z}{(z-i)^{2+1}} dz &= \frac{2\pi i}{2} \cos z|_{z=i} \\
&= -i\pi \cos(i) \\
&= -i\pi \frac{e^{-1} + e^1}{2} \\
&= -i\pi \cosh(1)
\end{aligned}$$

5 Liouville's Theorem

Theorem 49 (Liouville's Theorem). *If f is entire and bounded, then f is constant.*

Exercise 18.

If f is entire, such that $\forall z \in \mathbb{C}, \Re(f(z)) < M$, show that it is constant.

Solution 18.

As $e^{\Re(f(x))} < M$,

$$\begin{aligned} |e^{f(z)}| &= e^{\Re(f(z))} \\ \therefore |e^{f(z)}| &< e^M \end{aligned}$$

Therefore, $e^{\Re(f(z))}$ is an entire and bounded function. Therefore, by Liouville's Theorem, $e^{f(z)}$ is constant.

Let

$$e^{f(z)} = c$$

Therefore,

$$f(z) = \ln |c| + 2\pi ki$$

Therefore, even though k may be dependent on z , as $f(z)$ is continuous, k must be continuous, to ensure that there is no discontinuity in $f(z)$. Therefore, $f(z)$ is constant.