Complex Functions

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Contents

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1 Lecturer Information

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2 Recommended Reading

1. James Ward Brown & Ruel V. Churchill, “Complex Variables and Applications”, McGraw-Hill, Inc. 1996.
2. D. Zill, P. Shanahan, “Complex Variables with Applications”, Jones and Bartlett Publishers.

3 Additional Reading

1. Saff, Edward B., and Arthur David Snider. Fundamentals of Complex Analysis with Applications to Engineering, Science, and Mathematics. 3rd ed. Upper Saddle River, NJ: Prentice Hall, 2002. ISBN: 0139078746.
2. Sarason, Donald. Complex Function Theory. American Mathematical Society. ISBN: 0821886223
3. Alfhors, Lars. Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable. McGraw-Hill Education, 1979. ISBN: 0070006571.

Part I
Complex Numbers

Definition 1. A number of the form

$$z = x + iy$$

where

$$\begin{aligned} i &= \sqrt{-1} \\ x &\in \mathbb{R} \\ y &\in \mathbb{R} \end{aligned}$$

is called a complex number.

Definition 2 (Real part of a complex number)**.** If

$$z = x + iy$$

then x is called the real part of z, and is denoted as

$$x = \Re(z)$$

Definition 3 (Imaginary part of a complex number)**.** If

$$z = x + iy$$

then y is called the imaginary part of z, and is denoted as

$$x = \Im(z)$$

Definition 4 (Complex conjugate)**.** If

$$z = x + iy$$

then

$$\overline{z} = x - iy$$

is called the complex conjugate of z.

Theorem 1.

$$z\overline{z} = |z|^2$$

Proof.

$$\begin{aligned} z &= x + iy \\ \therefore \overline{z} &= x - iy \end{aligned}$$

Therefore,

$$\begin{aligned} z\overline{z} &= (x + iy)(x - iy) \\ &= x^2 - ixy + ixy + y^2 \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

□

Definition 5 (Polar representation)**.** If

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

then (r, θ) is called the polar representation of (x, y).

Theorem 2 (Euler's Formula)**.**

$$r cos\theta + ir\sin\theta = re^{i\theta}$$

Definition 6 (Absolute value or Norm)**.**

$$\begin{aligned} |z| &= |x + iy| \\ &= \sqrt{x^2 + y^2} \end{aligned}$$

is called the absolute value, or the norm of z.

Theorem 3.

$$|z| \leq |\Re(z)| + |\Im(z)| \leq \sqrt{2}|z|$$

Proof.

$$\begin{aligned} &\sqrt{x^2 + y^2} \leq |x| + |y| \leq \sqrt{2x^2 + 2y^2} \\ &\iff x^2 + y^2 \leq x^2 + y^2 + 2|x||y| \leq 2x^2 + 2y^2 \\ &\iff x^2 + y^2 - 2|x||y| \geq 0 \\ &\iff (|x| - |y|)^2 \geq 0 \end{aligned}$$

□

Definition 7 (Argument)**.** Let z be a complex number.
Then, θ, such that $\theta \in (-\pi, \pi]$, and

$$z = (r, \theta)$$

is called the argument of z.
It is denoted as

$$\theta = \mathrm{Arg}(z)$$

If $\theta \notin (-\pi, \pi]$, but

$$z = (r, \theta)$$

then

$$\theta = \arg(z)$$

Theorem 4.

$$z^n = |z|^n e^{in \,\mathrm{Arg}(z)}$$

Proof.

$$\begin{aligned} z &= |z| e^{i \,\mathrm{Arg}(z)} \\ \therefore z^n &= \left(|z| e^{i \,\mathrm{Arg}(z)}\right)^n \\ &= (|z|)^n \left(e^{i \,\mathrm{Arg}(z)}\right)^n \\ &= |z|^n e^{in \,\mathrm{Arg}(x)} \end{aligned}$$

□

Theorem 5. *Let*

$$\begin{aligned} z &= re^{i\theta} \\ w &= \rho e^{i\varphi} \end{aligned}$$

The solutions to

$$w = \sqrt[n]{z}$$

are

$$\varphi_k = \frac{\theta}{n} + \frac{2\pi k}{n}$$

where $k \in \{0, \ldots, n-1\}$.

Proof.

$$
\begin{aligned}
w &= \sqrt[n]{z} \\
\therefore w^n &= z
\end{aligned}
$$

Therefore,

$$
\rho^n e^{in\varphi} = re^{i\theta}
$$

Therefore, for $k \in \{0, \dots, n-1\}$,

$$
\begin{aligned}
\rho &= \sqrt[n]{r} \\
n\varphi &= \theta + 2\pi k \\
\therefore \varphi &= \frac{\theta}{n} + \frac{2\pi k}{n}
\end{aligned}
$$

$\square$

Part II
Complex Sequences and Series

Definition 8 (Convergence of complex sequences)**.** Let

$$z_n = x_n + iy_n$$

The sequence $\{z_n\}$ is said to converge to the limit $z = x + iy$, if $\forall \varepsilon > 0$, $\exists N$, such that $\forall n > N$, $|z_n - z| < \varepsilon$, i.e. there is a circular region of radius ε, centred at z, in which z_n lies.

Theorem 6. *$\{z_n\} \to z$, i.e. $\{z_n\}$ converges to z if and only if all subsequences of $\{z_n\}$ converge to z.*

Exercise 1.
Find the limit $\lim\limits_{n\to\infty} \frac{n+i}{2n-i}$.

Solution 1.

$$\begin{aligned} z_n &= \frac{n+i}{2n-i} \\ &= \frac{(n+i)(2n+i)}{4n^2+1} \\ &= \frac{2n^2+1}{4n^2+1} + i\frac{3n}{4n^2+1} \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} z_n &= \lim_{n\to\infty} \frac{2n^2+1}{4n^2+1} + i\frac{3n}{4n^2+1} \\ &= \frac{1}{2} \end{aligned}$$

Exercise 2.
Show that for

$$z_n = -2 + \frac{(-1)^n}{n} i$$

$\lim\limits_{n\to\infty} \operatorname{Arg}(z_n)$ does not exist, but $\lim\limits_{n\to\infty} |z_n|$ exists.

Solution 2.

The magnitude of z_n is

$$\begin{aligned}|z_n| &= \left|-2 + \frac{(-1)^n}{n} i\right| \\ &= \sqrt{4 + \frac{(-1)^{2n}}{n^2}} \\ &= \sqrt{4 + \frac{1}{n^2}}\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} |z_n| &= \lim_{n\to\infty} \sqrt{4 + \frac{1}{n^2}} \\ &= 2\end{aligned}$$

The argument of z_{2n} is

$$\begin{aligned}\mathrm{Arg}(z_{2n}) &= \mathrm{Arg}\left(-2 + \frac{(-1)^{2n}}{2n} i\right) \\ \therefore \lim_{n\to\infty} \mathrm{Arg}(z_{2n}) &= \lim_{n\to\infty} \mathrm{Arg}\left(-2 + \frac{i}{2n}\right) \\ &= \pi\end{aligned}$$

The argument of z_{2n+1} is

$$\begin{aligned}\mathrm{Arg}(z_{2n+1}) &= \mathrm{Arg}\left(-2 + \frac{(-1)^{2n+1}}{2n+1} i\right) \\ \therefore \lim_{n\to\infty} \mathrm{Arg}(z_{2n}) &= \lim_{n\to\infty} \mathrm{Arg}\left(-2 - \frac{i}{2n}\right) \\ &= -\pi\end{aligned}$$

Therefore, as the limit of two subsequences are not equal, the limit does not exist.

Part III
Topology on the Complex Plane

Definition 9 (Neighbourhood of a complex number)**.** A circular region of radius ε centred at z, is called the ε neighbourhood of z.

$$B(z, \varepsilon) = D(z, \varepsilon) = \{w \in \mathbb{C} : |w - z| < \varepsilon\}$$

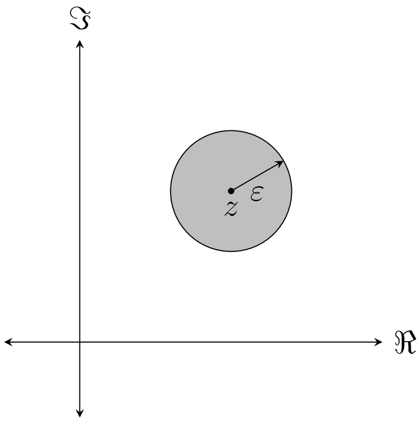


Figure 1: Neighbourhood of a complex number

Definition 10 (Interior point)**.** Let $A \subseteq \mathbb{C}$.
$z \in \mathbb{C}$ is called an inner or interior point of A if there exists at least one $\varepsilon_z > 0$, such that $B(z, \varepsilon_z) \subset A$.
The set of all interior points of A is denoted by $\mathrm{Int}(A)$ or A°.

Definition 11 (Exterior point)**.** Let $A \subseteq \mathbb{C}$.
$z \in \mathbb{C}$ is called an outer or exterior point of A if there exists at least one $\varepsilon_z > 0$, such that $B(z, \varepsilon_z) \subset (\mathbb{C} \setminus A)$. The set of all exterior points of A is denoted by $\mathrm{Ext}(A)$.

Definition 12 (Edge point)**.** Let $A \subseteq \mathbb{C}$.
$z \in \mathbb{C}$ is called an edge or boundary point of A if it is neither an inner point of A, nor an outer point of A. The set of all boundary points of A is denoted by $\partial(A)$.

Definition 13 (Open set)**.** A set $A \subseteq \mathbb{C}$ is called an open set if $A = A^\circ$, i.e. for any point $z \in A$, $\exists \varepsilon > 0$, such that $D(z, \varepsilon) \subset A$.

Definition 14 (Closer of a set)**.** The closer of A is defined to be

$$\overline{A} = A^{\circ} \cup \partial A$$

Definition 15 (Closed set)**.** A set A is called a closed set if $\partial A \subset A$, i.e. $A = \overline{A}$.

Definition 16 (Connected set)**.** A set A is called a connected set of for any $z_1, z_n \in A$, there exists a polygonal path, i.e. a finite set of connected straight lines, which connects z_1 and z_2, and belongs to A.

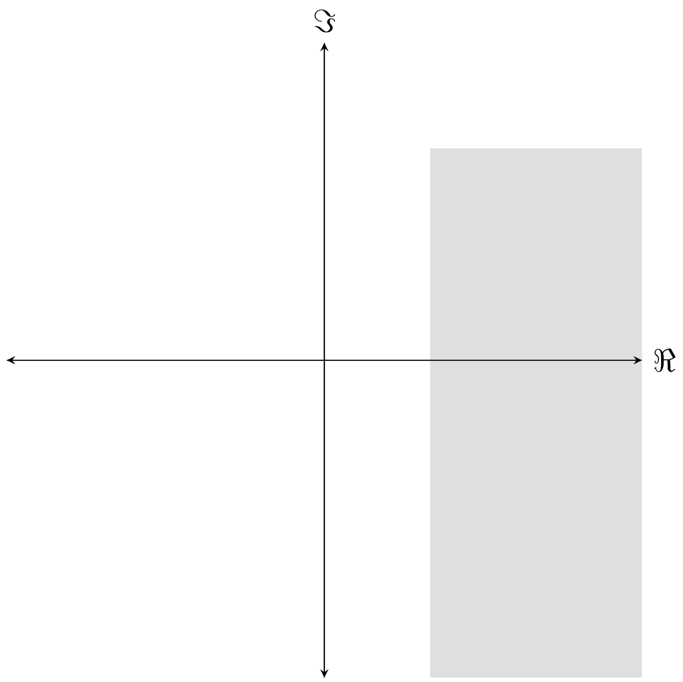
Definition 17 (Domain)**.** An open connected set is called a domain.

Definition 18 (Bound set)**.** A set A is said to be a bound set if it is bound inside a disk.

Exercise 3.
Describe geometrically and list the properties of the following sets.

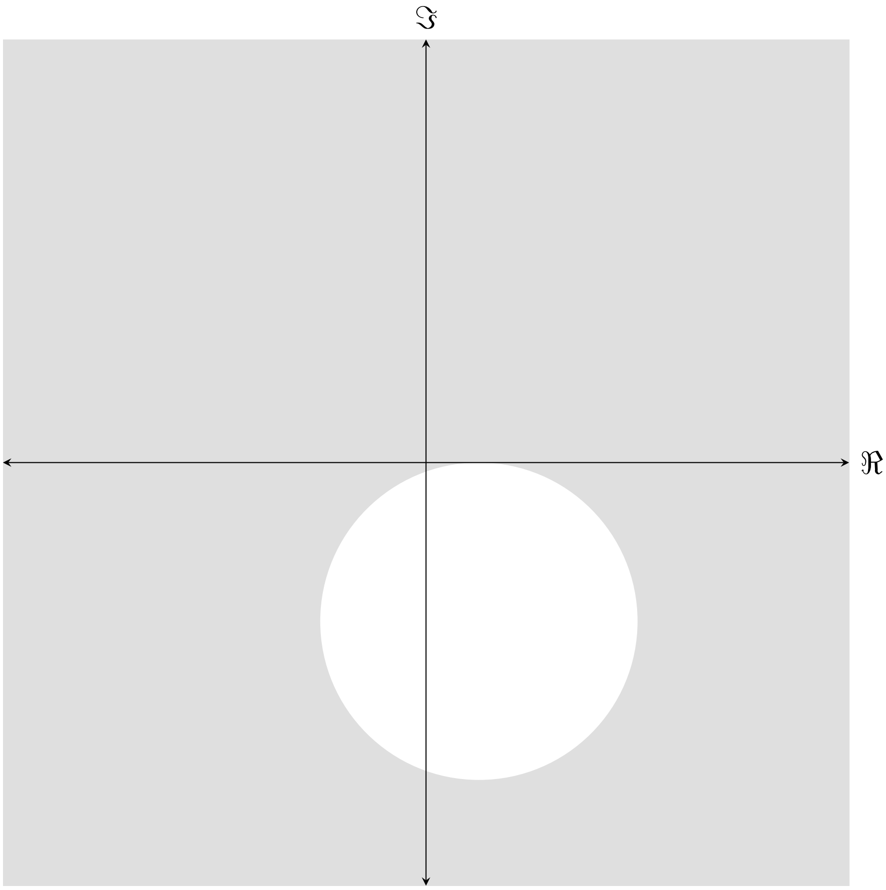
1. $A = \{z \in \mathbb{C} : \Re(z) \geq 2, \Im(z) \leq 4\}$
2. $B = \{z \in \mathbb{C} : |z - 1 + 3i| > 3\}$

Solution 3.
1. A is the union of the bottom half plane with respect to the line $y = 4$, and the right half plane with respect to the line $x = 2$.



Therefore, as $A = A^\circ + \partial A$, it is a closer, unbounded set.

2. A is the complement of a disk, centred at $1 - 3i$, with radius 3.



Therefore, it is an open, unbounded set.

Exercise 4.
Prove that the upper half plane $U = \{z : \Im(z) > 0\}$ is open.

Solution 4.
Let

$$z = x + iy$$

Therefore, as $z \in U$, $y > 0$.
Therefore, consider the disk $D\left(z, \frac{y}{2}\right)$.

Let $w \in D\left(z, \frac{y}{2}\right)$. Therefore,

$$\begin{aligned} |w - z| &< \frac{y}{2} \\ \therefore |\Im(w - z)| &\leq |w - z| \\ &\leq \frac{y}{2} \end{aligned}$$

Therefore,

$$\begin{aligned} &-\frac{y}{2} \leq \Im(w) - \Im(z) \leq \frac{y}{2} \\ &\therefore -\frac{y}{2} \leq \Im(w) - y \leq \frac{y}{2} \\ &\therefore \Im(w) \geq \frac{y}{2} > 0 \end{aligned}$$

Therefore, as $\Im(w) > 0$, $w \in U$. Therefore, U is open. $\square$

Part IV
Complex Functions

1 Complex Functions

Definition 19 (Complex function)**.** Let $A \subseteq \mathbb{C}$. $f : A \to \mathbb{C}$ is called a complex function, which matches $z \in A$ to $f(z) \in \mathbb{C}$.

Theorem 7. *Any complex function f can be written as*

$$\begin{aligned} f(x+iy) &= \Re f(x+iy) + i\Im f(x+iy) \\ &= u(x,y) + iv(x,y) \end{aligned}$$

2 Limits

Definition 20 (Limit of a function)**.** Let f be a complex function defined on a neighbourhood of z_0, but may or may not be defined at z_0. Then, the limit of $f(z)$ at z_0 is defined as

$$w = \lim_{z \to z_0} f(z)$$

if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $\forall z \in \mathbb{X}$such that $|z - z_0| < \delta$, $|f(z) - w| < \varepsilon$.

Exercise 5.
Show that

$$\lim_{z \to 1} \frac{iz}{2} = \frac{i}{2}$$

Solution 5.
Let $|z-1| < \delta$. Therefore, for $\varepsilon > 0$,

$$\begin{aligned} \left| f(z) - \frac{i}{2} \right| &= \left| \frac{iz}{2} - \frac{i}{2} \right| \\ &= \left| \frac{i}{2} \right| |z-1| \\ &= \frac{1}{2} |z-i| \end{aligned}$$

Therefore, for $\delta \leq 2\varepsilon$, $\left| f(z) - \frac{i}{2} \right| < \varepsilon$. □

Theorem 8. *If*

$$\begin{aligned} f(z) &= f(x+iy) \\ &= u(x,y) + iv(x,y) \end{aligned}$$

then

$$\lim_{z \to z_0} f(z) = u_0 + iv_0$$

if and only if

$$\begin{aligned} \lim_{(x,y)\to(x_0,y_0)} u(x,y) &= u_0 \\ \lim_{(x,y)\to(x_0,y_0)} v(x,y) &= v_0 \end{aligned}$$

Theorem 9 (Limit arithmetics)**.** *If*

$$\begin{aligned} \lim_{z \to z_0} f(z) &= w_1 \\ \lim_{z \to z_0} g(z) &= w_2 \end{aligned}$$

then, as long as all quantities are defined,

$$\begin{aligned} \lim_{z \to z_0} f(z) \pm g(z) &= w_1 \pm w_2 \\ \lim_{z \to z_0} f(z)g(z) &= w_1 w_2 \\ \lim_{z \to z_0} \frac{f(z)}{g(z)} &= \frac{w_1}{w_2} \end{aligned}$$

Exercise 6.
For the function $f(z) = \overline{z}^2$, prove

$$\begin{aligned} \lim_{z \to z_0} f(z) &= f(z_0) \\ &= \overline{z_0}^2 \end{aligned}$$

Solution 6.

$$\begin{aligned} \overline{z} &= \left(\overline{x+iy}\right)^2 \\ &= (x-iy)^2 \\ &= x^2 - y^2 - 2xyi \end{aligned}$$

Therefore, let

$$\begin{aligned}u(x,y) &= x^2 - y^2\\ v(x,y) &= -2xy\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{(x,y)\to(x_0,y_0)} u(x,y) &= {x_0}^2 - {y_0}^2\\ \lim_{(x,y)\to(x_0,y_0)} v(x,y) &= -2x_0y_0\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{z\to z_0} f(z) &= u_0 + iv_0\\ &= {x_0}^2 - {y_0}^2 - 2x_0y_0i\\ &= \overline{z_0}^2\end{aligned}$$

□

Definition 21 (Infinite limit)**.** The limit of $f(z)$ is said to be infinite, i.e.

$$\lim_{z\to z_0} f(z) = \infty$$

if and only if

$$\lim_{z\to z_0} |f(z)| = \infty$$

if and only if

$$\lim_{z\to z_0} \frac{1}{f(z)} = 0$$

Definition 22 (Limit at infinity)**.** The limit of a function $f(z)$,

$$\lim_{z\to\infty} f(z) = w$$

if

$$\lim_{|z|\to\infty} f(z) = w$$

Alternatively, $\forall \varepsilon > 0$, $\exists R > 0$, such that for $|z| > R$, $|f(x) - w| < \varepsilon$.

Exercise 7.
Show that

$$\lim_{z \to \infty} \frac{1}{z^2} = 0$$

Solution 7.
Let $\varepsilon > 0$. Let $R > 0$, such that $\frac{1}{R^2} < \varepsilon$.
Therefore, if $|z| > R$,

$$\begin{aligned} |f(z) - 0| &= \left|\frac{1}{z^2}\right| \\ &= \frac{1}{|z^2|} \\ &= \frac{1}{|z|^2} \\ &< \frac{1}{R^2} \\ &< \varepsilon \end{aligned}$$

Therefore, $\lim_{z \to \infty} \frac{1}{z^2} = 0$.

3 Continuity

Definition 23 (Continuous function)**.** $f(z)$ is said to be continuous at z_0 if $f(z)$ is defined at z_0 and

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Theorem 10 (Continuity arithmetics)**.** *If*

$$\lim_{z \to z_0} f(z) = f(z_0)$$
$$\lim_{z \to z_0} g(z) = g(z_0)$$

then, as long as all quantities are defined,

$$\begin{aligned} \lim_{z \to z_0} f(z) \pm g(z) &= f(z_0) \pm g(z_0) \\ \lim_{z \to z_0} f(z)g(z) &= f(z_0)g(z_0) \\ \lim_{z \to z_0} \frac{f(z)}{g(z)} &= \frac{f(z_0)}{g(z_0)} \end{aligned}$$

4 Differentiability

Definition 24 (Differentiable function)**.** Let $f(z)$ be defined in a neighbourhood of z_0. f is said to be differentiable at z_0 if the limit $\lim\limits_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists.

Theorem 11 (Differentiation arithmetics)**.** *If $f(z)$ and $g(z)$ are differentiable, then, as long as all quantities are defined,*

$$\begin{aligned}(f(z) \pm g(z))' &= f'(z) \pm g'(z) \\ (f(z)g(z))' &= f'(z)g(z) + f(z)g'(z) \\ \left(\frac{f(z)}{g(z)}\right)' &= \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}\end{aligned}$$

5 Cauchy-Riemann Equations

Theorem 12 (Cauchy-Riemann Equations)**.** *$u(x,y)$ and $v(x,y)$ are said to be satisfying Cauchy-Riemann Equations at a point $(a,b) \in \mathbb{R}^2$, if*

$$\begin{aligned}u_x(a,b) &= v_y(a,b) \\ u_y(a,b) &= -v_x(a,b)\end{aligned}$$

Theorem 13. *Let*

$$f(x+iy) = u(x,y) + iv(x,y)$$

Then, u and v satisfying the Cauchy-Riemann Equations is a necessary condition for f to be differentiable at $(x_0, y_0$.

Theorem 14. *If $f = u+iv$ is differentiable at $z_0 = a+ib$, then (u,v) satisfies the Cauchy-Riemann Equations at (a,b).*

Definition 25 (Analytic functions)**.** If $f = u + iv$ is differentiable at any $z \in W$, where W is a neighbourhood of z_0 except maybe at z_0, then f is said to be analytic at z_0. If f is analytic at all $z \in W$, then it is said to be analytic in W.

Exercise 8.
Let $f : U \to \mathbb{C}$ be an analytic function in U, such that $\overline{f}$ is also analytic in U. Show that $f' = 0$, i.e. $f = c$.

Solution 8.
As $f = u + iv$ is analytic, by Cauchy-Riemann Equations, for $(x, y) \in U$,

$$\begin{aligned} u_x(x, y) &= v_y(x, y) \\ u_y(x, y) &= -v_x(x, y) \end{aligned}$$

As $\overline{f} = u - iv$ is analytic, by Cauchy-Riemann Equations, for $(x, y) \in U$,

$$\begin{aligned} u_x(x, y) &= -v_y(x, y) \\ u_y(x, y) &= v_x(x, y) \end{aligned}$$

Therefore,

$$\begin{aligned} v_y &= -v_y \\ &= 0 \\ v_x &= -v_x \\ &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} u_x(x, y) &= 0 \\ u_y(x, y) &= 0 \end{aligned}$$

Therefore, u and v are constant functions.

6 Harmonic Functions

Definition 26 (Laplacian)**.** Let u be an equation in x and y.
The Laplacian is defined to be

$$\begin{aligned} \Delta u &= \nabla^2 u \\ &= u_{xx} + u_{yy} \end{aligned}$$

Definition 27 (Harmonic function)**.** A real function in two variables, $u(x, y)$, which is twice differentiable, is called a harmonic function if it satisfies

$$\begin{aligned} \Delta u &= u_{xx} + u_{yy} \\ &= 0 \end{aligned}$$

Theorem 15. *If u and v are twice differentiable, and satisfy Cauchy-Riemann Equations, then (u, v) are harmonic.*

Theorem 16. *Let $f = u + iv$ be defined in a neighbourhood of $z_0 = a + ib$. Assume that u_x, u_y, v_x, v_y exist in this neighbourhood and are continuous at the point (a, b). If (u, v) satisfying Cauchy-Riemann Equations at (a, b) then $f'(z_0)$ exists.*

Definition 28 (Harmonic conjugate)**.** Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a harmonic function. Its harmonic conjugate is defined to be $v : \mathbb{R}^2 \to \mathbb{R}$, such that $f = u + iv$ is analytic.

7 Analytic Functions

Definition 29. $f : D \to \mathbb{C}$ is said to be differentiable on $D \subset \mathbb{C}$, if f is differentiable at any $z \in D$.

Definition 30 (Analytic functions)**.** If $f = u + iv$ is differentiable at any $z \in W$, where W is a neighbourhood of z_0 except maybe at z_0, then f is said to be analytic at z_0. If f is analytic at all $z \in W$, then it is said to be analytic in W.

Theorem 17. *Let $D \subset \mathbb{C}$ be an open set. Then, f is differentiable on D if and only if f is analytic on D.*

Theorem 18. *Let $D \subseteq \mathbb{C}$ be a domain. Assume that f is analytic on D, and for any $z \in D$, $f'(z) = 0$. Then, f is constant.*

Theorem 19. *Let $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $\nabla u = 0$ in a domain $D \subset \mathbb{R}^2$. Then, u is constant in D.*

Exercise 9.

1. Prove that
$$v(x, y) = \ln\left((x-1)^2 + (y-2)^2\right)$$
is harmonic in any domain that does not include the point $(1, 2)$.

2. Find $u(x, y)$ such that $u + iv$ is analytic in some domain. Note: v is the conjugate harmonic of u.

3. Express $u + iv$ as a function of z.

Solution 9.

1.

$$\begin{aligned} v_x &= \frac{2(x-1)}{(x-1)^2+(y-2)^2} \\ v_y &= \frac{2(y-2)}{(x-1)^2+(y-2)^2} \end{aligned}$$

Therefore,

$$\begin{aligned} v_{xx} &= \frac{2\left((x-1)^2+(y-2)^2\right)-(2(x-1))^2}{((x-1)^2+(y-2)^2)^2} \\ v_{yy} &= \frac{2\left((x-1)^2+(y-2)^2\right)-(2(y-2))^2}{((x-1)^2+(y-2)^2)^2} \end{aligned}$$

2. For $u+iv$ to be analytic, by Cauchy-Riemann Equations,

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

Therefore,

$$\begin{aligned} u_x &= v_y \\ &= \frac{2(y-2)}{(x-1)^2+(y-2)^2} \end{aligned}$$

Therefore,

$$\begin{aligned} u &= \int \frac{2(y-2)}{(x-1)^2+(y-2)^2}\,\mathrm{d}x \\ &= \frac{2(y-2)}{(y-2)^2}\int \frac{1}{1+\left(\frac{x-1}{y-2}\right)^2}\,\mathrm{d}x \\ &= 2\tan^{-1}\left(\frac{x-1}{y-2}\right)+g(y) \end{aligned}$$

Therefore,

$$\begin{aligned} u_y &= -v_x \\ \therefore -\frac{2(x-1)}{(x-1)^2+(y-2)^2} &= \frac{2}{1+\frac{(x-1)^2}{(y-2)^2}}\left(-\frac{x-1}{y-2}\right)+g'(y) \end{aligned}$$

Therefore,

$$\begin{aligned} g'(y) &= 0 \\ \therefore g(y) &= c \end{aligned}$$

Therefore,

$$u = 2\tan^{-1}\left(\frac{x-1}{y-2}\right) + c$$

3.

$$\begin{aligned} u + iv &= \tan^{-1}\left(\frac{x-1}{y-2}\right) + i\ln\Big((x-1)^2 + (y-2)^2\Big) \\ &= 2i\,\mathrm{Log}\,(-i(x-1) + (y-2)) \\ &= 2i\,\mathrm{Log}\,(-iz - 2 + i) \end{aligned}$$

Exercise 10.
Prove that there is no $f = u + iv$ analytic in the unit disk, such that

$$xu(x,y) = yv(x,y) + 2013$$

Hint: Use the function $zf(z)$.

Solution 10.
If possible, let there exist $f(z)$ such that

$$xu(x,y) = yv(x,y) + 2013$$

Therefore, as $zf(z)$ is analytic,

$$\begin{aligned} zf(z) &= (x+iy)(u+iv) \\ &= xu - yv + i(yu + xv) \\ &= 2013 + i(yu + xv) \end{aligned}$$

By the polar form of Cauchy-Riemann Equations, $yu + xv$ is constant.
Therefore, $zf(z)$ is constant.
Therefore, this contradicts the assumption.
Therefore, such a f does not exist.

8 Elementary Functions

8.1 Exponential Functions

Theorem 20.

$$|e^z| = e^{\Re(z)}$$

Proof.

$$\begin{aligned}
|e^z| &= \left|e^{\Re(z)}\right| \left|e^{\Im(z)}\right| \\
&= \left|e^{\Re(z)}\right| \left|\cos\left(\Im(z)\right) + i\sin\left(\Im(z)\right)\right| \\
&= e^{\Re(z)}
\end{aligned}$$

□

Theorem 21. *Let z and w be complex. Then*

$$e^{z+w} = e^z e^w$$

Theorem 22. $\forall n \in \mathbb{Z}$,

$$(e^z)^n = e^{nz}$$

Theorem 23. *The function e^z is onto with respect to $\mathbb{C} \setminus \{0\}$.*

8.2 Trigonometric Functions

Definition 31 (Trigonometric functions of complex numbers)**.** Trigonometric functions of complex numbers are defined as

$$\begin{aligned}
\cos(z) &= \frac{e^{iz} + e^{-iz}}{2} \\
\sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} \\
\cosh(z) &= \frac{e^z + e^{-z}}{2} \\
\sinh(z) &= \frac{e^z - e^{-z}}{2}
\end{aligned}$$

8.3 Logarithmic Functions

Definition 32 (Power set)**.** The set of all subsets of a set is called the power set of the set. The power set of a set A is denoted as $\mathrm{P}(A)$.

Definition 33 (Multiple valued function)**.** A set which maps a set A to its power set $\mathrm{P}(A)$ is called a multiple valued set.

A multiple valued function gets over $\mathbb{C}$ gets a complex number as input and returns a set of complex numbers as output.

Definition 34 (Natural logarithmic function)**.** The natural logarithmic function over the complex plane is defined to be

$$\log w = \{z : e^z = w\}$$

Theorem 24.

$$\log w = \ln |w| + i \arg(w)$$

Proof. Let

$$\begin{aligned} e^z &= w \\ &= |w| e^{i\theta} \end{aligned}$$

where

$$\theta = \arg(w)$$

Therefore,

$$\begin{aligned} e^{\Re(z) + i\Im(z)} &= |w| e^{i\theta} \\ \therefore e^{\Re(z)} e^{i\Im(z)} &= |w| e^{i\theta} \end{aligned}$$

Therefore,

$$\begin{aligned} e^{\Re(z)} &= |w| \\ \Im(z) &= \theta + 2\pi k \end{aligned}$$

where $k \in \mathbb{Z}$.
Therefore,

$$\begin{aligned} \ln e^{\Re(z)} &= \ln |w| \\ \therefore \Re(z) &= \ln |w| \end{aligned}$$

Therefore,

$$\begin{aligned} \log w &= \{z : e^z = w\} \\ &= \{\ln |w| + iy : y = \arg(w)\} \end{aligned}$$

For any $w \in \log z$,

$$\begin{aligned} e^w &= e^{\ln|z| + i\left(\operatorname{Arg} z + 2\pi k\right)} \\ &= e^{\ln|z|} e^{i(\operatorname{Arg} z + 2\pi k)} \\ &= |z| e^{i \operatorname{Arg} z} \\ &= z \end{aligned}$$

$\square$

Definition 35 (Branch of $\log z$)**.** A branch of $\log z$ is a continuous function $\mathrm{L}(z)$ defined on a U, a connected open subset of $\mathbb{C}$ such that $\mathrm{L}(z)$ is a logarithm of z for each $z \in U$.

Definition 36 ($\operatorname{Log} z$)**.** $\operatorname{Log} z$ is defined to be

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$$

As $\operatorname{Arg} z$ is not continuous on the negative real axis, in order to make it continuous, the line $\operatorname{Arg} z = \pi$ is excluded. Hence, $\log z$ is continuous on $U = \mathbb{C} \setminus \{0\} \cup \mathbb{R}^-$, and is a branch of $\log z$.
Similarly, any other ray can be excluded in order to get a branch of $\log z$.

Definition 37. For any $\alpha \in \mathbb{R}$, $\operatorname{Log}_\alpha z$ is defined to be

$$\operatorname{Log}_\alpha z = \ln|z| + i \operatorname{Arg}_\alpha z$$

where $\operatorname{Arg}_\alpha z = \theta$, such that $\theta \in (\alpha, \alpha + 2\pi]$ and $\theta = \arg z$.
Any choice of $\operatorname{Arg}_\alpha z$ defines a branch of logarithm.

Definition 38 (Branch cut)**.** The boundary of the domain of a branch is called a branch cut.

Definition 39 (Principal value)**.** The value returned by $\operatorname{Log} z = \operatorname{Log}_{-\pi} z$ is called the principal value.

Theorem 25. $\operatorname{Log} z$ *is analytic on* $\mathbb{C} \setminus \{0\} \cup \mathbb{R}^-$.

Exercise 11.
Find the principal value of $\sqrt{i}$.

Solution 11.

$$\begin{aligned} \operatorname{pv}\left(i^{\frac{1}{2}}\right) &= e^{\frac{1}{2} \operatorname{Log} i} \\ &= e^{\frac{1}{2}\left(\ln|i| + i \operatorname{Arg} i\right)} \\ &= e^{\frac{1}{2} i \frac{\pi}{2}} \\ &= e^{i \frac{\pi}{4}} \end{aligned}$$

8.4 Power

Definition 40 (Power function)**.** Let $z, c \in \mathbb{C}$, such that $z \neq 0$. The power multifunction as

$$z^c = e^{c \log z}$$

The branch of the power multifunction for $c \in \mathbb{C}$ is defined as

$$z^w = e^{w \log z}$$

Theorem 26.

$$\operatorname{Log}_\alpha z - \operatorname{Log}_\beta z = i \left(\operatorname{Arg}_\alpha z - \operatorname{Arg}_\beta z\right)$$

Part V
Complex Integrals

1 Complex Integrals

Definition 41 (Integral of complex functions)**.** Let $f : [a, b] \to \mathbb{C}$. Let

$$f(t) = u(t) + iv(t)$$

Therefore, the integrals of $u(t)$ and $v(t)$ are defined as

$$\int_a^b u(t)\,\mathrm{d}t = \lim_{\Delta t \to 0} \sum_{i=1}^{n} u(t_i)\Delta x_i$$

where T is a splitting of $[a, b]$, such that

$$a = t_1 < \cdots < t_n = b$$

and

$$\int_a^b v(t)\,\mathrm{d}t = \lim_{\Delta t \to 0} \sum_{i=1}^{n} v(t_i)\Delta x_i$$

where T is a splitting of $[a, b]$, such that

$$a = t_1 < \cdots < t_n = b$$

These integrals are defined when the limit exists without depending on T. When they exist, the integral of $f(t)$ is defined as

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t$$

Theorem 27. *All properties of real integrals are also valid for complex integrals.*

Theorem 28.

$$\left|\int_a^b f(t)\,\mathrm{d}t\right| \le \int_a^b |f(t)|\,\mathrm{d}t$$

2 Curves in $\mathbb{C}$

Definition 42. A continuous function $\gamma : [a, b] \to \mathbb{C}$ is called a curve.

Definition 43 (Parametric representation of a curve)**.** The curve $\gamma(t)$ can be represented as

$$\gamma(t) = x(t) + iy(t)$$

where t is a parameter.

Definition 44 (Differentiability)**.** γ is said to be differentiable if x and y are both differentiable.

Theorem 29 (Parametric representation of a straight line)**.** *Let $z_1, z_2 \in \mathbb{C}$. The straight line passing through z_1 and z_2 can represented parametrically as*

$$\gamma(t) = z_1 + t(z_2 - z_1)$$

The slope of this line is $z_1 - z_2$.

Theorem 30 (Parametric representation of a circle)**.** *A circle with radius r, centred at the origin, can be represented parametrically as*

$$\gamma(t) = re^{it}$$

with $0 \leq t \leq 2\pi$.

Exercise 12.
Parametrize the curve $\left\{z = x + iy : \frac{x^2}{4} + y^2 = 1\right\}$ starting from 2, and going anti-clockwise twice.

Solution 12.
The curve is an ellipse centred at $(0, 0)$, with $a = 2$, and $b = 1$.

$$\gamma(t) = 2\cos t + i \sin t$$

Therefore, as the curve goes anti-clockwise twice, $t \in [0, 4\pi]$.

Definition 45 (Simple curve)**.** A curve γ is said to be simple if it is non self-intersecting, i.e. it is one-to-one with respect to the parameter t, except maybe at the extreme values of t.

Definition 46 (Closed curve)**.** A curve $\gamma : [a, b] \to \mathbb{C}$ is said to be closed, if and only if

$$\gamma(a) = \gamma(b)$$

Definition 47 (Jordan curve)**.** A closed simple curve is called a Jordan curve.

Theorem 31. *A Jordan curve enclosed a region inside it.*

Definition 48 (Piecewise differentiability)**.** γ is said to be piecewise differentiable if there exists a splitting

$$a = t_1 < \cdots < t_n = b$$

such that γ is differentiable on each segment $[t_i, t_{i+1}]$.

3 Complex Line Integrals

Definition 49 (Complex line integral)**.** Let $\gamma : [a, b] \to \mathbb{C}$ be a curve, and let $f : D \to \mathbb{C}$, where $D \subseteq \mathbb{C}$, and $\gamma([a, b]) \subset D$. Then, the integral

$$\int\limits_{\gamma} f(z)\,\mathrm{d}z = \int\limits_{a}^{b} f(\gamma(t))\,\dot{\gamma}(t)\,\mathrm{d}t$$

If γ is piecewise differentiable, then

$$\int\limits_{\gamma} f(z)\,\mathrm{d}z = \sum_{i=1}^{n} \int\limits_{x_i}^{x_{i+1}} f(\gamma(t))\,\dot{\gamma}(t)\,\mathrm{d}t$$