Complex Functions

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2015-16

Contents

1 Lecturer Information

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2 Recommended Reading

1. James Ward Brown & Ruel V. Churchill, “Complex Variables and Applications”, McGraw-Hill, Inc. 1996.
2. D. Zill, P. Shanahan, “Complex Variables with Applications”, Jones and Bartlett Publishers.

3 Additional Reading

1. Saff, Edward B., and Arthur David Snider. Fundamentals of Complex Analysis with Applications to Engineering, Science, and Mathematics. 3rd ed. Upper Saddle River, NJ: Prentice Hall, 2002. ISBN: 0139078746.
2. Sarason, Donald. Complex Function Theory. American Mathematical Society. ISBN: 0821886223
3. Alfhors, Lars. Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable. McGraw-Hill Education, 1979. ISBN: 0070006571.

Part I
Complex Numbers

Definition 1. A number of the form

$$z = x + iy$$

where

$$i = \sqrt{-1}$$
$$x \in \mathbb{R}$$
$$y \in \mathbb{R}$$

is called a complex number.

Definition 2 (Real part of a complex number)**.** If

$$z = x + iy$$

then x is called the real part of z, and is denoted as

$$x = \Re(z)$$

Definition 3 (Imaginary part of a complex number)**.** If

$$z = x + iy$$

then y is called the imaginary part of z, and is denoted as

$$x = \Im(z)$$

Definition 4 (Complex conjugate)**.** If

$$z = x + iy$$

then

$$\overline{z} = x - iy$$

is called the complex conjugate of z.

Theorem 1.

$$z\overline{z} = |z|^2$$

Proof.

$$\begin{aligned} z &= x + iy \\ \therefore \overline{z} &= x - iy \end{aligned}$$

Therefore,

$$\begin{aligned} z\overline{z} &= (x + iy)(x - iy) \\ &= x^2 - ixy + ixy + y^2 \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

□

Definition 5 (Polar representation)**.** If

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

then (r, θ) is called the polar representation of (x, y).

Theorem 2 (Euler's Formula)**.**

$$rcos\theta + ir\sin\theta = re^{i\theta}$$

Definition 6 (Absolute value or Norm)**.**

$$\begin{aligned} |z| &= |x + iy| \\ &= \sqrt{x^2 + y^2} \end{aligned}$$

is called the absolute value, or the norm of z.

Theorem 3.

$$|z| \leq |\Re(z)| + |\Im(z)| \leq \sqrt{2}|z|$$

Proof.

$$\begin{aligned} &\sqrt{x^2 + y^2} \leq |x| + |y| \leq \sqrt{2x^2 + 2y^2} \\ \iff\ & x^2 + y^2 \leq x^2 + y^2 + 2|x||y| \leq 2x^2 + 2y^2 \\ \iff\ & x^2 + y^2 - 2|x||y| \geq 0 \\ \iff\ & (|x| - |y|)^2 \geq 0 \end{aligned}$$

□

Definition 7 (Argument)**.** Let z be a complex number.
Then, θ, such that $\theta \in (-\pi, \pi]$, and

$$z = (r, \theta)$$

is called the argument of z.
It is denoted as

$$\theta = \operatorname{Arg}(z)$$

If $\theta \notin (-\pi, \pi]$, but

$$z = (r, \theta)$$

then

$$\theta = \arg(z)$$

Theorem 4.

$$z^n = |z|^n e^{in\operatorname{Arg}(z)}$$

Proof.

$$\begin{aligned} z &= |z|e^{i\operatorname{Arg}(z)} \\ \therefore z^n &= \left(|z|e^{i\operatorname{Arg}(z)}\right)^n \\ &= (|z|)^n \left(e^{i\operatorname{Arg}(z)}\right)^n \\ &= |z|^n e^{in\operatorname{Arg}(x)} \end{aligned}$$

□

Theorem 5. *Let*

$$\begin{aligned} z &= re^{i\theta} \\ w &= \rho e^{i\varphi} \end{aligned}$$

The solutions to

$$w = \sqrt[n]{z}$$

are

$$\varphi_k = \frac{\theta}{n} + \frac{2\pi k}{n}$$

where $k \in \{0, \ldots, n-1\}$.

Proof.

$$
\begin{aligned}
w &= \sqrt[n]{z} \\
\therefore w^n &= z
\end{aligned}
$$

Therefore,

$$
\rho^n e^{in\varphi} = re^{i\theta}
$$

Therefore, for $k \in \{0, \ldots, n-1\}$,

$$
\begin{aligned}
\rho &= \sqrt[n]{r} \\
n\varphi &= \theta + 2\pi k \\
\therefore \varphi &= \frac{\theta}{n} + \frac{2\pi k}{n}
\end{aligned}
$$

$\square$

Part II
Complex Sequences and Series

Definition 8 (Convergence of complex sequences)**.** Let

$$z_n = x_n + iy_n$$

The sequence $\{z_n\}$ is said to converge to the limit $z = x + iy$, if $\forall \varepsilon > 0$, $\exists N$, such that $\forall n > N$, $|z_n - z| < \varepsilon$, i.e. there is a circular region of radius ε, centred at z, in which z_n lies.

Theorem 6. *$\{z_n\} \to z$, i.e. $\{z_n\}$ converges to z if and only if all subsequences of $\{z_n\}$ converge to z.*

Part III
Topology on the Complex Plane

Definition 9 (Neighbourhood of a complex number)**.** A circular region of radius ε centred at z, is called the ε neighbourhood of z.

$$B(z, \varepsilon) = \{w \in \mathbb{C} : |w - z| < \varepsilon\}$$

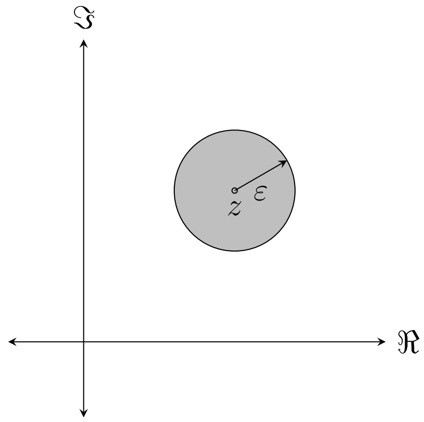

Figure 1: Neighbourhood of a complex number

Definition 10 (Inner point)**.** Let $A \subseteq \mathbb{C}$.
$z \in \mathbb{C}$ is called an inner point of A if there exists at least one $\varepsilon_z > 0$, such that $B(z, \varepsilon_z) \subset A$.

Definition 11 (Outer point)**.** Let $A \subseteq \mathbb{C}$.
$z \in \mathbb{C}$ is called an outer point of A if there exists at least one $\varepsilon_z > 0$, such that $B(z, \varepsilon_z) \subset (\mathbb{C} \setminus A)$.

Definition 12 (Edge point)**.** Let $A \subseteq \mathbb{C}$.
$z \in \mathbb{C}$ is called an edge point of A if it is neither an inner point of A, nor an outer point of A.