COMPLEX FUNCTIONS : ASSIGNMENT 5

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Exercise 1.
Prove

(1) $(\sin z)' = \cos z$
(2) $(\cos z)' = -\sin z$

Solution 1.
(1)

$$\sin z = \frac{e^{i \operatorname{Arg} z} - e^{-i \operatorname{Arg} z}}{2i}$$

Therefore,

$$\begin{aligned}
(\sin z)' &= \left(\frac{e^{i \operatorname{Arg} z} - e^{-i \operatorname{Arg} z}}{2i} \right)' \\
&= \frac{ie^{i \operatorname{Arg} z} + ie^{i \operatorname{Arg} z}}{2i} \\
&= \frac{e^{i \operatorname{Arg} z} + e^{i \operatorname{Arg} z}}{2} \\
&= \cos z
\end{aligned}$$

(2)

$$\cos z = \frac{e^{i \operatorname{Arg} z} + e^{-i \operatorname{Arg} z}}{2}$$

Therefore,

$$\begin{aligned}
(\cos z)' &= \left(\frac{e^{i \operatorname{Arg} z} + e^{-i \operatorname{Arg} z}}{2} \right)' \\
&= \frac{ie^{i \operatorname{Arg} z} - ie^{-i \operatorname{Arg} z}}{2} \\
&= -\frac{e^{i \operatorname{Arg} z} + e^{i \operatorname{Arg} z}}{2i} \\
&= -\sin z
\end{aligned}$$

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Exercise 3.
Calculate

(3) $\operatorname{pv}\left((1-i)^{4i}\right)$
(4) $(-1)^i$

Solution 3.
(3)

$$\begin{aligned}
\operatorname{pv}\left((1-i)^{4i}\right) &= \operatorname{Log}_{-\pi}\left((1-i)^{4i}\right) \\
&= \operatorname{Log}\left((1-i)^{4i}\right) \\
&= \operatorname{Log}\left(e^{4i\operatorname{Log}(1-i)}\right) \\
&= \operatorname{Log}\left(e^{4i\left(\ln|1-i|+i\operatorname{Arg}(1-i)\right)}\right) \\
&= \operatorname{Log}\left(e^{4i\left(\ln\sqrt{2}-i\frac{\pi}{2}\right)}\right) \\
&= 4i\left(\ln\sqrt{2}-i\frac{\pi}{2}\right) \\
&= 4i\left(\frac{\ln 2}{2}-i\frac{\pi}{2}\right) \\
&= 2i\ln 2+2\pi
\end{aligned}$$

(4)

$$\begin{aligned}
(-1)^i &= i^{2i} \\
&= \left(i^i\right)^2 \\
&= \left(\left(e^{i\frac{\pi}{2}}\right)^i\right)^2 \\
&= \left(e^{i^2\frac{\pi}{2}}\right)^2 \\
&= \left(e^{-\frac{\pi}{2}}\right)^2 \\
&= e^{-\pi} \\
&= -1
\end{aligned}$$

Exercise 4.
Show that

$$\operatorname{Log}(1+i)^2 = 2\operatorname{Log}(1+i)$$

Solution 4.

$$\begin{aligned}
\operatorname{Log}(1+i)^2 &= \ln\left|(1+i)^2\right| + i\operatorname{Arg}\left((1+i)^2\right)\\
&= \ln|1+2i-1| + i\operatorname{Arg}(1+2i-1)\\
&= \ln|2i| + i\operatorname{Arg}(2i)\\
&= \ln 2 + i\frac{\pi}{2}\\
&= 2\left(\ln\left|\sqrt{2}\right| + i\frac{\pi}{4}\right)\\
&= 2\big(\ln|1+i| + i\operatorname{Arg}(1+i)\big)\\
&= 2\operatorname{Log}(1+i)
\end{aligned}$$

□

Exercise 5.

In class we showed that

$$\frac{\mathrm{d}\operatorname{Log} z}{\mathrm{d}z} = \frac{1}{z}$$

using the polar Cauchy-Riemann equations. Prove it again using the chain rule and the fact that $z = e^{\operatorname{Log} z}$.

Solution 5.

$$\begin{aligned}
\frac{\mathrm{d}e^{\operatorname{Log} z}}{\mathrm{d}z} &= \frac{\mathrm{d}e^{\operatorname{Log} z}}{\mathrm{d}\operatorname{Log} z}\,\frac{\mathrm{d}\operatorname{Log} z}{\mathrm{d}z}\\
\therefore \frac{\mathrm{d}z}{\mathrm{d}z} &= e^{\operatorname{Log} z}\,\frac{\mathrm{d}\operatorname{Log} z}{\mathrm{d}z}\\
\therefore 1 &= e^{\operatorname{Log} z}\,\frac{\mathrm{d}\operatorname{Log} z}{\mathrm{d}z}\\
&= z\,\frac{\mathrm{d}\operatorname{Log} z}{\mathrm{d}z}\\
\therefore \frac{\mathrm{d}\operatorname{Log} z}{\mathrm{d}z} &= \frac{1}{z}
\end{aligned}$$

□