COMPLEX FUNCTIONS : ASSIGNMENT 9

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Exercise 3.
In this question f is an entire function. You need to prove that under the following conditions, for each condition separately, it must be constant.

(1) For every $z \in \mathbb{C}$, $\mathrm{Re}(f) \geq 0$.
(2) For every $z \in \mathbb{C}$, $\left|f(z)\right| \neq 1$.
(3) For every $z \in \mathbb{C}$, $f(z) \notin [0, 1]$.

Solution 3.
(1)

$$\begin{aligned} \mathrm{Re}(f(z)) &\geq 0 \\ \therefore \mathrm{Re}(f(z) + 1) &\geq 1 \\ \therefore \left|f(z) + 1\right| &\geq 1 \\ \therefore 1 &\geq \frac{1}{\left|f(z) + 1\right|} \end{aligned}$$

Therefore, as $\frac{1}{f(z)+1}$ is entire, and is bounded by 1, by Liouville's theorem, it is constant. Hence, $f(z)$ is also constant.

(2)
(3) Let

$$g(z) = \frac{1}{f(z)}$$

Therefore, $g(z)$ is entire, and bounded by 1. Therefore, by Liouville's theorem, $g(z)$ is constant. Hence, $f(z)$ is constant.

Exercise 4.
Calculate $\displaystyle\int\limits_0^{2\pi} \frac{\mathrm{d}t}{a \cos t + b \sin t + c}$, where a, b, c satisfy $\sqrt{a^2 + b^2} = 1 < c$.

Solution 4.
Let

$$\begin{aligned} a &= \cos \alpha \\ b &= \sin \alpha \end{aligned}$$

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Therefore,

$$\begin{aligned} a\cos t + b\sin t &= \cos\alpha\cos t + \sin\alpha\sin t \\ &= \cos(\alpha - t) \end{aligned}$$

Therefore,

$$\int\limits_0^{2\pi} \frac{\mathrm{d}t}{\cos(\alpha - t) + c} = \int\limits_0^{2\pi} \frac{\mathrm{d}t}{\frac{e^{i(\alpha-t)}+e^{i(t-\alpha)}}{2} + c}$$

Let

$$\begin{aligned} z &= i(\alpha - t) \\ \therefore \mathrm{d}z &= -i\,\mathrm{d}t \\ \therefore \mathrm{d}t &= \frac{\mathrm{d}z}{-i} \\ &= i\,\mathrm{d}z \end{aligned}$$

Therefore,

$$\begin{aligned} \int\limits_0^{2\pi} \frac{\mathrm{d}t}{\cos(\alpha - t) + c} &= \int\limits_{i\alpha}^{i\alpha-2\pi i} \frac{i\,\mathrm{d}z}{\frac{e^{iz}+e^{-iz}}{2} + c} \\ &= \int\limits_{i\alpha}^{i\alpha-2\pi i} \frac{2i\,\mathrm{d}z}{e^{iz} + e^{-iz} + 2c} \\ &= \left. \frac{2\tan^{-1}\left(\frac{c+e^{iz}}{\sqrt{1-c^2}}\right)}{\sqrt{1-c^2}} \right|_{i\alpha}^{i\alpha-2\pi i} \\ &= \frac{2\tan^{-1}\left(\frac{c+e^{i\alpha}e^{-2\pi i}}{\sqrt{1-c^2}}\right)}{\sqrt{1-c^2}} - \frac{2\tan^{-1}\left(\frac{c+e^{i\alpha}}{\sqrt{1-c^2}}\right)}{\sqrt{1-c^2}} \\ &= \frac{2\tan^{-1}\left(\frac{c+e^{i\alpha}}{\sqrt{1-c^2}}\right)}{\sqrt{1-c^2}} - \frac{2\tan^{-1}\left(\frac{c+e^{i\alpha}}{\sqrt{1-c^2}}\right)}{\sqrt{1-c^2}} \\ &= 0 \end{aligned}$$

Exercise 5.

If $p(z)$ is a polynomial of degree $n \geq 1$, and there exists $\alpha > 0$, such that

$$\left|p(z)\right| \leq \alpha|z|$$

Then, there exists $c \in \mathbb{C}$, such that

$$p(z) = cz$$

Solution 5.

$$\big|p(z)\big| \leq \alpha |z|$$

Therefore,

$$\operatorname{Re}\big(p(z)\big) \leq \alpha \operatorname{Re}(z)$$

Similarly,

$$\operatorname{Im}\big(p(z)\big) \leq \alpha \operatorname{Im}(z)$$

Therefore,

$$p(z) \leq \operatorname{Re}(z) + i\operatorname{Im}(z)$$

Therefore, $\exists c \in \mathbb{C}$, such that

$$p(z) = cz$$