COMPLEX FUNCTIONS : ASSIGNMENT 4

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Exercise 2.
Prove that the following functions are harmonic and find their conjugates.

(2) $u(x, y) = e^{-x}(x \sin y - y \cos y)$

Solution 2.

(2)

$$u(x, y) = e^{-x}(x \sin y - y \cos y)$$

Therefore,

$$\begin{aligned}
u_x &= -e^{-x}(x \sin y - y \cos y) + e^{-x}(\sin y) \\
\therefore u_{xx} &= e^{-x}(x \sin y - y \cos y) - e^{-x}(\sin y) - e^{-x} \sin y \\
&= e^{-x} \left(x \sin y - y \cos y - 2 \sin y\right) \\
u_y &= e^{-x}(x \cos y - \cos y + y \sin y) \\
u_{yy} &= e^{-x} \left(-x \sin y + \sin y + \sin y + y \cos y\right) \\
&= e^{-x} \left(-x \sin y + 2 \sin y + y \cos y\right)
\end{aligned}$$

Therefore,

$$u_{xx} + u_{yy} = 0$$

Therefore, the function is harmonic.

Exercise 3.
Express $\Re(\sin z)$ and $\Im(\sin z)$ as real functions dependent on x and y where $z = x + iy$. Simplify your answer using the definitions of the hyperbolic trigonometric functions.

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Solution 3.

$$
\begin{aligned}
\Re(\sin z) &= \Re\left(\frac{e^{iz}-e^{-iz}}{2i}\right) \\
&= \Re\left(\frac{e^{i(x+iy)}-e^{-i(x+iy)}}{2i}\right) \\
&= \Re\left(\frac{e^{ix-y}-e^{-ix+y}}{2i}\right) \\
&= \Re\left(\frac{e^{ix}e^{-y}-e^{-ix}e^{y}}{2i}\right) \\
&= \Re\left(\frac{e^{-y}\cos x + ie^{-y}\sin x - e^{y}\cos(-x) - ie^{y}\sin(-x)}{2i}\right) \\
&= \Re\left(\frac{e^{-y}\cos x - e^{y}\cos(-x)}{2i} + i\frac{e^{-y}\sin x - e^{y}\sin(-x)}{2i}\right) \\
&= \Re\left(\frac{e^{-y}\sin x + e^{y}\sin x}{2} - i\frac{e^{-y}\cos x - e^{y}\cos x}{2}\right) \\
&= \frac{e^{-y}\sin x + e^{y}\sin x}{2} \\
&= \sin x\left(\frac{e^{-y}+e^{y}}{2}\right) \\
&= \sin x\cosh y
\end{aligned}
$$

$$
\begin{aligned}
\Im(\sin z) &= \Im\left(\frac{e^{iz} - e^{-iz}}{2i}\right) \\
&= \Im\left(\frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}\right) \\
&= \Im\left(\frac{e^{ix-y} - e^{-ix+y}}{2i}\right) \\
&= \Im\left(\frac{e^{ix}e^{-y} - e^{-ix}e^{y}}{2i}\right) \\
&= \Im\left(\frac{e^{-y}\cos x + ie^{-y}\sin x - e^{y}\cos(-x) - ie^{y}\sin(-x)}{2i}\right) \\
&= \Im\left(\frac{e^{-y}\cos x - e^{y}\cos(-x)}{2i} + i\frac{e^{-y}\sin x - e^{y}\sin(-x)}{2i}\right) \\
&= \Im\left(\frac{e^{-y}\sin x + e^{y}\sin x}{2} - i\frac{e^{-y}\cos x - e^{y}\cos x}{2}\right) \\
&= \frac{e^{y}\cos x - e^{-y}\cos x}{2} \\
&= \cos x\frac{e^{y} - e^{-y}}{2} \\
&= \cos x \sinh y
\end{aligned}
$$

Exercise 4.
Find the image of the strip $|y| < \pi$ under the map $f(z) = e^z$.

Solution 4.

$$
\begin{aligned}
f(z) &= e^z \\
&= e^{x+iy} \\
&= e^x e^{iy}
\end{aligned}
$$

Therefore, as $|y| < \pi$,

$$
\begin{aligned}
&-\pi < y < \pi \\
&\therefore e^{-i\pi} < e^{iy} < e^{i\pi}
\end{aligned}
$$

Therefore, the image is a disk with radius e^x, except for the negative real axis.
Therefore, as $x \in (-\infty, \infty)$, the image of the strip is $\mathbb{C} \setminus \mathbb{R}^-$.

Exercise 6.
Prove the following identities.

(1) $\cos^2 z + \sin^2 z = 1$
(2) $\sin z \cos w = \frac{1}{2}\big(\sin(z+w) + \sin(z-w)\big)$

Solution 6.

(1)

$$\begin{aligned}
\cos^2 z + \sin^2 z &= \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 \\
&= \frac{e^{2iz} + 2 + e^{-2iz}}{4} - \frac{e^{2iz} - 2 + e^{-2iz}}{4} \\
&= \frac{4}{4} \\
&= 1
\end{aligned}$$

□

$$\begin{aligned}
\sin z \cos w &= \left(\frac{e^{iz} - e^{-iz}}{2i}\right)\left(\frac{e^{iw} + e^{-iw}}{2}\right) \\
&= \frac{e^{iz+iw} + e^{iz-iw} - e^{-iz+iw} - e^{-iz-iw}}{4i} \\
&= \frac{1}{2}\left(\frac{e^{iz+iw} - e^{-iz-iw}}{2i} + \frac{e^{iz-iw} - e^{-iz+iw}}{2i}\right) \\
&= \frac{1}{2}\big(\sin(z+w) + \sin(z-w)\big)
\end{aligned}$$