COMPLEX FUNCTIONS : ASSIGNMENT 2

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Exercise 1.
Draw the following sets in the complex plane.

(f) $\left\{z : -1 < \Im(z) < 1, -1 < \Re(z) < 1\right\}$
(h) $\left\{z : |z| < 1, 0 \leq \mathrm{Arg}(z) \leq \frac{\pi}{4}\right\}$

Solution 1.
(f)

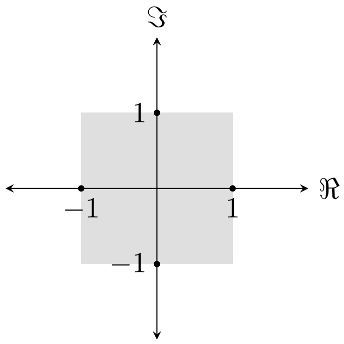

Figure 1. $\left\{z : -1 < \Im(z) < 1, -1 < \Re(z) < 1\right\}$

(h)

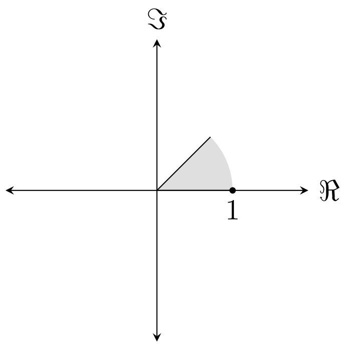

Figure 2. $\left\{z : |z| < 1, 0 \leq \mathrm{Arg}(z) \leq \frac{\pi}{4}\right\}$

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Exercise 3.
For every mapping, find where it maps the given set to.

(b) The set $\left\{z : 0 \leq \operatorname{Arg}(z) \leq \frac{\pi}{4}\right\}$ under the map $f(z) = z^2$

Solution 3.
(b) Let

$$z = re^{i\theta}$$

Therefore,

$$z^2 = r^2 e^{2i\theta}$$

Therefore, as $0 \leq \theta \leq \frac{\pi}{4}$, $0 \leq 2\theta \leq \frac{\pi}{2}$.
Therefore, the set $\left\{z : 0 \leq \operatorname{Arg}(z) \leq \frac{\pi}{4}\right\}$ maps to $\left\{z : 0 \leq \operatorname{Arg}(z) \leq \frac{\pi}{2}\right\}$ under the map $f(z) = z^2$.

Exercise 5.
Write the following functions in the form

$$f(z) = u(x, y) + iv(x, y)$$

Are u and v continuous? Where?

(c) $f(z) = ze^{2z}$

Solution 5.
(c) Let

$$z = x + iy$$

Therefore,

$$\begin{aligned} f(z) &= ze^{2z} \\ &= (x + iy)e^{2x}e^{2iy} \\ &= (x + iy)e^{2x} \big(\cos(2y) + i\sin(2y)\big) \\ &= e^{2x} \big(x\cos(2y) + ix\sin(2y) + iy\cos(2y) - y\sin(2y)\big) \\ &= e^{2x} \big(x\cos(2y) - y\sin(2y)\big) + ie^{2x} \big(x\sin(2y) + y\cos(2y)\big) \end{aligned}$$

Therefore,

$$\begin{aligned} u(x, y) &= e^{2x} \big(x\cos(2y) - y\sin(2y)\big) \\ v(x, y) &= e^{2x} \big(x\sin(2y) - y\cos(2y)\big) \end{aligned}$$

u and v are continuous over $\mathbb{R}$.

Exercise 7.
Prove, using the differentiability definition only, that $f_1(z) = \Re(z)$ and $f_2(z) = \Im(z)$ are not differentiable at any point.

Solution 7.

$$f_1(z) = \Re(z)$$

Therefore, for any $z \in \mathbb{C}$,

$$\begin{aligned}
{f_1}'(z) &= \lim_{h\to 0} \frac{f_1(z+h) - f(z)}{h} \\
&= \lim_{h\to 0} \frac{\Re(z+h) - \Re(z)}{h} \\
&= \lim_{h\to 0} \frac{\Re(z) + \Re(h) - \Re(z)}{h} \\
&= \lim_{h\to 0} \frac{\Re(h)}{h}
\end{aligned}$$

Therefore, as this limit depends on the argument of h, it does not exist.

$$f_2(z) = \Im(z)$$

Therefore, for any $z \in \mathbb{C}$,

$$\begin{aligned}
{f_1}'(z) &= \lim_{h\to 0} \frac{f_2(z+h) - f(z)}{h} \\
&= \lim_{h\to 0} \frac{\Im(z+h) - \Im(z)}{h} \\
&= \lim_{h\to 0} \frac{\Im(z) + \Im(h) - \Im(z)}{h} \\
&= \lim_{h\to 0} \frac{\Im(h)}{h}
\end{aligned}$$

Therefore, as this limit depends on the argument of h, it does not exist. □