

Complex Functions

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Contents

1	Lecturer Information	iii
2	Recommended Reading	iii
3	Additional Reading	iii
I	Complex Numbers	1
II	Complex Sequences and Series	5
III	Topology on the Complex Plane	7
IV	Complex Functions	11
1	Complex Functions	11
2	Limits	11
3	Continuity	14
4	Differentiability	15
5	Cauchy-Riemann Equations	15



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1 Lecturer Information

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2 Recommended Reading

1. James Ward Brown & Ruel V. Churchill, “Complex Variables and Applications”, McGraw-Hill, Inc. 1996.
2. D. Zill, P. Shanahan, “Complex Variables with Applications”, Jones and Bartlett Publishers.

3 Additional Reading

1. Saff, Edward B., and Arthur David Snider. Fundamentals of Complex Analysis with Applications to Engineering, Science, and Mathematics. 3rd ed. Upper Saddle River, NJ: Prentice Hall, 2002. ISBN: 0139078746.
2. Sarason, Donald. Complex Function Theory. American Mathematical Society. ISBN: 0821886223
3. Alfhors, Lars. Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable. McGraw-Hill Education, 1979. ISBN: 0070006571.

Part I

Complex Numbers

Definition 1. A number of the form

$$z = x + iy$$

where

$$i = \sqrt{-1}$$

$$x \in \mathbb{R}$$

$$y \in \mathbb{R}$$

is called a complex number.

Definition 2 (Real part of a complex number). If

$$z = x + iy$$

then x is called the real part of z , and is denoted as

$$x = \Re(z)$$

Definition 3 (Imaginary part of a complex number). If

$$z = x + iy$$

then y is called the imaginary part of z , and is denoted as

$$y = \Im(z)$$

Definition 4 (Complex conjugate). If

$$z = x + iy$$

then

$$\bar{z} = x - iy$$

is called the complex conjugate of z .

Theorem 1.

$$z\bar{z} = |z|^2$$

Proof.

$$\begin{aligned} z &= x + iy \\ \therefore \bar{z} &= x - iy \end{aligned}$$

Therefore,

$$\begin{aligned} z\bar{z} &= (x + iy)(x - iy) \\ &= x^2 - ixy + ixy + y^2 \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

□

Definition 5 (Polar representation). If

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

then (r, θ) is called the polar representation of (x, y) .

Theorem 2 (Euler's Formula).

$$r \cos \theta + ir \sin \theta = re^{i\theta}$$

Definition 6 (Absolute value or Norm).

$$\begin{aligned} |z| &= |x + iy| \\ &= \sqrt{x^2 + y^2} \end{aligned}$$

is called the absolute value, or the norm of z .

Theorem 3.

$$|z| \leq |\Re(z)| + |\Im(z)| \leq \sqrt{2}|z|$$

Proof.

$$\begin{aligned} \sqrt{x^2 + y^2} &\leq |x| + |y| \leq \sqrt{2x^2 + 2y^2} \\ \iff x^2 + y^2 &\leq x^2 + y^2 + 2|x||y| \leq 2x^2 + 2y^2 \\ \iff x^2 + y^2 - 2|x||y| &\geq 0 \\ \iff (|x| - |y|)^2 &\geq 0 \end{aligned}$$

□

Definition 7 (Argument). Let z be a complex number. Then, θ , such that $\theta \in (-\pi, \pi]$, and

$$z = (r, \theta)$$

is called the argument of z . It is denoted as

$$\theta = \text{Arg}(z)$$

If $\theta \notin (-\pi, \pi]$, but

$$z = (r, \theta)$$

then

$$\theta = \arg(z)$$

Theorem 4.

$$z^n = |z|^n e^{in\text{Arg}(z)}$$

Proof.

$$\begin{aligned} z &= |z|e^{i\text{Arg}(z)} \\ \therefore z^n &= \left(|z|e^{i\text{Arg}(z)}\right)^n \\ &= (|z|)^n \left(e^{i\text{Arg}(z)}\right)^n \\ &= |z|^n e^{in\text{Arg}(x)} \end{aligned}$$

□

Theorem 5. *Let*

$$\begin{aligned} z &= re^{i\theta} \\ w &= \rho e^{i\varphi} \end{aligned}$$

The solutions to

$$w = \sqrt[n]{z}$$

are

$$\varphi_k = \frac{\theta}{n} + \frac{2\pi k}{n}$$

where $k \in \{0, \dots, n-1\}$.

Proof.

$$\begin{aligned}w &= \sqrt[n]{z} \\ \therefore w^n &= z\end{aligned}$$

Therefore,

$$\rho^n e^{in\varphi} = re^{i\theta}$$

Therefore, for $k \in \{0, \dots, n-1\}$,

$$\begin{aligned}\rho &= \sqrt[n]{r} \\ n\varphi &= \theta + 2\pi k \\ \therefore \varphi &= \frac{\theta}{n} + \frac{2\pi k}{n}\end{aligned}$$

□

Part II

Complex Sequences and Series

Definition 8 (Convergence of complex sequences). Let

$$z_n = x_n + iy_n$$

The sequence $\{z_n\}$ is said to converge to the limit $z = x + iy$, if $\forall \varepsilon > 0$, $\exists N$, such that $\forall n > N$, $|z_n - z| < \varepsilon$, i.e. there is a circular region of radius ε , centred at z , in which z_n lies.

Theorem 6. $\{z_n\} \rightarrow z$, i.e. $\{z_n\}$ converges to z if and only if all subsequences of $\{z_n\}$ converge to z .

Exercise 1.

Find the limit $\lim_{n \rightarrow \infty} \frac{n+i}{2n-i}$.

Solution 1.

$$\begin{aligned} z_n &= \frac{n+i}{2n-i} \\ &= \frac{(n+i)(2n+i)}{4n^2+1} \\ &= \frac{2n^2+1}{4n^2+1} + i \frac{3n}{4n^2+1} \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} z_n &= \lim_{n \rightarrow \infty} \frac{2n^2+1}{4n^2+1} + i \frac{3n}{4n^2+1} \\ &= \frac{1}{2} \end{aligned}$$

Exercise 2.

Show that for

$$z_n = -2 + \frac{(-1)^n}{n}i$$

$\lim_{n \rightarrow \infty} \text{Arg}(z_n)$ does not exist, but $\lim_{n \rightarrow \infty} |z_n|$ exists.

Solution 2.

The magnitude of z_n is

$$\begin{aligned} |z_n| &= \left| -2 + \frac{(-1)^n}{n}i \right| \\ &= \sqrt{4 + \frac{(-1)^{2n}}{n^2}} \\ &= \sqrt{4 + \frac{1}{n^2}} \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} |z_n| &= \lim_{n \rightarrow \infty} \sqrt{4 + \frac{1}{n^2}} \\ &= 2 \end{aligned}$$

The argument of z_{2n} is

$$\begin{aligned} \text{Arg}(z_{2n}) &= \text{Arg} \left(-2 + \frac{(-1)^{2n}}{2n}i \right) \\ \therefore \lim_{n \rightarrow \infty} \text{Arg}(z_{2n}) &= \lim_{n \rightarrow \infty} \text{Arg} \left(-2 + \frac{i}{2n} \right) \\ &= \pi \end{aligned}$$

The argument of z_{2n+1} is

$$\begin{aligned} \text{Arg}(z_{2n+1}) &= \text{Arg} \left(-2 + \frac{(-1)^{2n+1}}{2n+1}i \right) \\ \therefore \lim_{n \rightarrow \infty} \text{Arg}(z_{2n}) &= \lim_{n \rightarrow \infty} \text{Arg} \left(-2 - \frac{i}{2n} \right) \\ &= -\pi \end{aligned}$$

Therefore, as the limit of two subsequences are not equal, the limit does not exist.

i++i

Part III

Topology on the Complex Plane

Definition 9 (Neighbourhood of a complex number). A circular region of radius ε centred at z , is called the ε neighbourhood of z .

$$B(z, \varepsilon) = D(z, \varepsilon) = \{w \in \mathbb{C} : |w - z| < \varepsilon\}$$

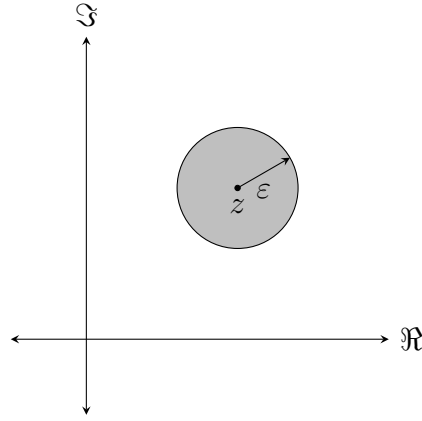


Figure 1: Neighbourhood of a complex number

Definition 10 (Interior point). Let $A \subseteq \mathbb{C}$.

$z \in \mathbb{C}$ is called an inner or interior point of A if there exists at least one $\varepsilon_z > 0$, such that $B(z, \varepsilon_z) \subset A$.

The set of all interior points of A is denoted by $\text{Int}(A)$ or A° .

Definition 11 (Exterior point). Let $A \subseteq \mathbb{C}$.

$z \in \mathbb{C}$ is called an outer or exterior point of A if there exists at least one $\varepsilon_z > 0$, such that $B(z, \varepsilon_z) \subset (\mathbb{C} \setminus A)$. The set of all exterior points of A is denoted by $\text{Ext}(A)$.

Definition 12 (Edge point). Let $A \subseteq \mathbb{C}$.

$z \in \mathbb{C}$ is called an edge or boundary point of A if it is neither an inner point of A , nor an outer point of A . The set of all boundary points of A is denoted by $\partial(A)$.

Definition 13 (Open set). A set $A \subseteq \mathbb{C}$ is called an open set if $A = A^\circ$, i.e. for any point $z \in A$, $\exists \varepsilon > 0$, such that $D(z, \varepsilon) \subset A$.

Definition 14 (Closur of a set). The closer of A is defined to be

$$\overline{A} = A^\circ \cup \partial A$$

Definition 15 (Closed set). A set A is called a closed set if $\partial A \subset A$, i.e. $A = \overline{A}$.

Definition 16 (Connected set). A set A is called a connected set if for any $z_1, z_2 \in A$, there exists a polygonal path, i.e. a finite set of connected straight lines, which connects z_1 and z_2 , and belongs to A .

Definition 17 (Domain). An open connected set is called a domain.

Definition 18 (Bound set). A set A is said to be a bound set if it is bound inside a disk.

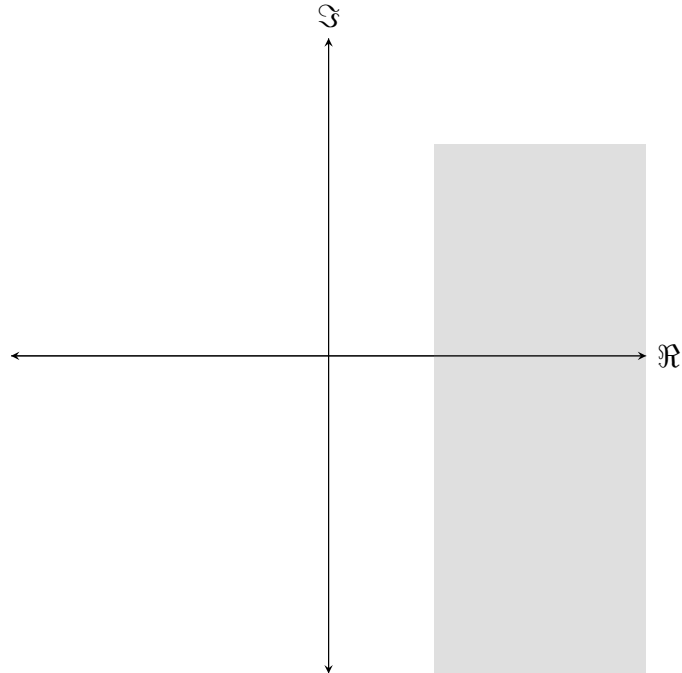
Exercise 3.

Describe geometrically and list the properties of the following sets.

1. $A = \{z \in \mathbb{C} : \Re(z) \geq 2, \Im(z) \leq 4\}$
2. $B = \{z \in \mathbb{C} : |z - 1 + 3i| > 3\}$

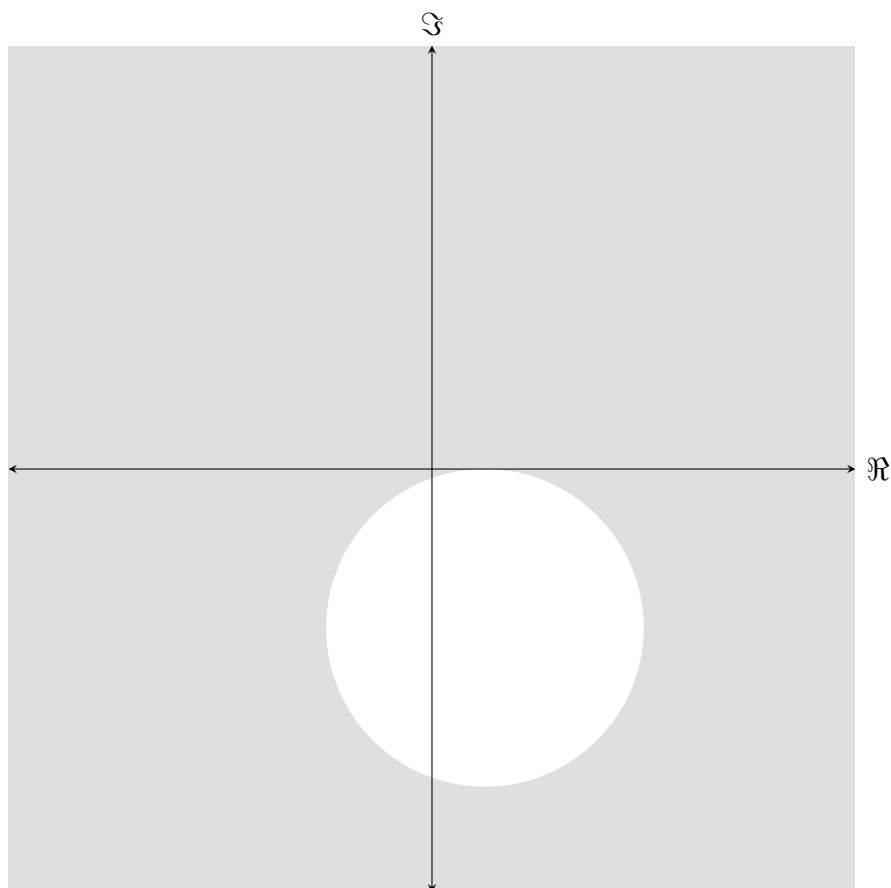
Solution 3.

1. A is the union of the bottom half plane with respect to the line $y = 4$, and the right half plane with respect to the line $x = 2$.



Therefore, as $A = A^\circ + \partial A$, it is a closer, unbounded set.

2. A is the complement of a disk, centred at $1 - 3i$, with radius 3.



Therefore, it is an open, unbounded set.

Exercise 4.

Prove that the upper half plane $U = \{z : \Im(z) > 0\}$ is open.

Solution 4.

Let

$$z = x + iy$$

Therefore, as $z \in U$, $y > 0$.

Therefore, consider the disk $D\left(z, \frac{y}{2}\right)$.

Let $w \in D\left(z, \frac{y}{2}\right)$. Therefore,

$$\begin{aligned} |w - z| &< \frac{y}{2} \\ \therefore |\Im(w - z)| &\leq |w - z| \\ &\leq \frac{y}{2} \end{aligned}$$

Therefore,

$$\begin{aligned} -\frac{y}{2} &\leq \Im(w) - \Im(z) \leq \frac{y}{2} \\ \therefore -\frac{y}{2} &\leq \Im(w) - y \leq \frac{y}{2} \\ \therefore \Im(w) &\geq \frac{y}{2} > 0 \end{aligned}$$

Therefore, as $\Im(w) > 0$, $w \in U$. Therefore, U is open. □

Part IV

Complex Functions

1 Complex Functions

Definition 19 (Complex function). Let $A \subseteq \mathbb{C}$. $f : A \rightarrow \mathbb{C}$ is called a complex function, which matches $z \in A$ to $f(z) \in \mathbb{C}$.

Theorem 7. Any complex function f can be written as

$$\begin{aligned} f(x + iy) &= \Re f(x + iy) + i\Im f(x + iy) \\ &= u(x, y) + iv(x, y) \end{aligned}$$

2 Limits

Definition 20 (Limit of a function). Let f be a complex function defined on a neighbourhood of z_0 , but may or may not be defined at z_0 . Then, the limit of $f(z)$ at z_0 is defined as

$$w = \lim_{z \rightarrow z_0} f(z)$$

if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $\forall z \in \mathbb{X}$ such that $|z - z_0| < \delta$, $|f(z) - w| < \varepsilon$.

Exercise 5.

Show that

$$\lim_{z \rightarrow 1} \frac{iz}{2} = \frac{i}{2}$$

Solution 5.

Let $|z - 1| < \delta$. Therefore, for $\varepsilon > 0$,

$$\begin{aligned} \left| f(z) - \frac{i}{2} \right| &= \left| \frac{iz}{2} - \frac{i}{2} \right| \\ &= \left| \frac{i}{2} \right| |z - 1| \\ &= \frac{1}{2} |z - i| \end{aligned}$$

Therefore, for $\delta \leq 2\varepsilon$, $\left| f(z) - \frac{i}{2} \right| < \varepsilon$. □

Theorem 8. *If*

$$\begin{aligned} f(z) &= f(x + iy) \\ &= u(x, y) + iv(x, y) \end{aligned}$$

then

$$\lim_{z \rightarrow z_0} f(z) = u_0 + iv_0$$

if and only if

$$\begin{aligned} \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) &= u_0 \\ \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) &= v_0 \end{aligned}$$

Theorem 9 (Limit arithmetics). *If*

$$\begin{aligned} \lim_{z \rightarrow z_0} f(z) &= w_1 \\ \lim_{z \rightarrow z_0} g(z) &= w_2 \end{aligned}$$

then, as long as all quantities are defined,

$$\begin{aligned} \lim_{z \rightarrow z_0} f(z) \pm g(z) &= w_1 \pm w_2 \\ \lim_{z \rightarrow z_0} f(z)g(z) &= w_1w_2 \\ \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} &= \frac{w_1}{w_2} \end{aligned}$$

Exercise 6.

For the function $f(z) = \bar{z}^2$, prove

$$\begin{aligned} \lim_{z \rightarrow z_0} f(z) &= f(z_0) \\ &= \bar{z}_0^2 \end{aligned}$$

Solution 6.

$$\begin{aligned} \bar{z} &= \overline{(x + iy)}^2 \\ &= (x - iy)^2 \\ &= x^2 - y^2 - 2xyi \end{aligned}$$

Therefore, let

$$\begin{aligned}u(x, y) &= x^2 - y^2 \\v(x, y) &= -2xy\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) &= x_0^2 - y_0^2 \\ \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) &= -2x_0y_0\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{z \rightarrow z_0} f(z) &= u_0 + iv_0 \\ &= x_0^2 - y_0^2 - 2x_0y_0i \\ &= \overline{z_0}^2\end{aligned}$$

□

Definition 21 (Infinite limit). The limit of $f(z)$ is said to be infinite, i.e.

$$\lim_{z \rightarrow z_0} f(z) = \infty$$

if and only if

$$\lim_{z \rightarrow z_0} |f(z)| = \infty$$

if and only if

$$\lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0$$

Definition 22 (Limit at infinity). The limit of a function $f(z)$,

$$\lim_{z \rightarrow \infty} f(z) = w$$

if

$$\lim_{|z| \rightarrow \infty} f(z) = w$$

Alternatively, $\forall \varepsilon > 0, \exists R > 0$, such that for $|z| > R$, $|f(x) - w| < \varepsilon$.

Exercise 7.

Show that

$$\lim_{z \rightarrow \infty} \frac{1}{z^2} = 0$$

Solution 7.

Let $\varepsilon > 0$. Let $R > 0$, such that $\frac{1}{R^2} < \varepsilon$.

Therefore, if $|z| > R$,

$$\begin{aligned} |f(z) - 0| &= \left| \frac{1}{z^2} \right| \\ &= \frac{1}{|z^2|} \\ &= \frac{1}{|z|^2} \\ &< \frac{1}{R^2} \\ &< \varepsilon \end{aligned}$$

Therefore, $\lim_{z \rightarrow \infty} \frac{1}{z^2} = 0$.

3 Continuity

Definition 23 (Continuous function). $f(z)$ is said to be continuous at z_0 if $f(z)$ is defined at z_0 and

$$\lim_{z \rightarrow z_0} f(z) = f(z_0)$$

Theorem 10 (Continuity arithmetics). *If*

$$\lim_{z \rightarrow z_0} f(z) = f(z_0)$$

$$\lim_{z \rightarrow z_0} g(z) = g(z_0)$$

then, as long as all quantities are defined,

$$\lim_{z \rightarrow z_0} f(z) \pm g(z) = f(z_0) \pm g(z_0)$$

$$\lim_{z \rightarrow z_0} f(z)g(z) = f(z_0)g(z_0)$$

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g(z_0)}$$

4 Differentiability

Definition 24 (Differentiable function). Let $f(z)$ be defined in a neighbourhood of z_0 . f is said to be differentiable at z_0 if the limit $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

Theorem 11 (Differentiation arithmetics). If $f(z)$ and $g(z)$ are differentiable, then, as long as all quantities are defined,

$$\begin{aligned}(f(z) \pm g(z))' &= f'(z) \pm g'(z) \\ (f(z)g(z))' &= f'(z)g(z) + f(z)g'(z) \\ \left(\frac{f(z)}{g(z)}\right)' &= \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}\end{aligned}$$

5 Cauchy-Riemann Equations

Theorem 12. $u(x, y)$ and $v(x, y)$ are said to be satisfying Cauchy-Riemann Equations at a point $(a, b) \in \mathbb{R}^2$, if

$$\begin{aligned}u_x(a, b) &= v_y(a, b) \\ u_y(a, b) &= -v_x(a, b)\end{aligned}$$

Theorem 13. Let

$$f(x + iy) = u(x, y) + iv(x, y)$$

Then, u and v satisfying the is a necessary condition for f to be differentiable at (x_0, y_0) .

Theorem 14. If $f = u + iv$ is differentiable at $z_0 = a + ib$, then (u, v) satisfies the at (a, b) .

Definition 25 (Analytic functions). If $f = u + iv$ is differentiable at any $z \in W$, where W is a neighbourhood of z_0 except maybe at z_0 , then f is said to be analytic at z_0 . If f is analytic at all $z \in W$, then it is said to be analytic in W .

Exercise 8.

Let $f : U \rightarrow \mathbb{C}$ be an analytic function in U , such that \bar{f} is also analytic in U . Show that $f' = 0$, i.e. $f = c$.

Solution 8.

As $f = u + iv$ is analytic, by , for $(x, y) \in U$,

$$\begin{aligned}u_x(x, y) &= v_y(x, y) \\u_y(x, y) &= -v_x(x, y)\end{aligned}$$

As $\bar{f} = u - iv$ is analytic, by , for $(x, y) \in U$,

$$\begin{aligned}u_x(x, y) &= -v_y(x, y) \\u_y(x, y) &= v_x(x, y)\end{aligned}$$

Therefore,

$$\begin{aligned}v_y &= -v_y \\&= 0 \\v_x &= -v_x \\&= 0\end{aligned}$$

Therefore,

$$\begin{aligned}u_x(x, y) &= 0 \\u_y(x, y) &= 0\end{aligned}$$

Therefore, u and v are constant functions.