Complex Functions : Review Session

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2015-16

Exercise 1. 10 P.

Let

$$D = \{z \in \mathbb{C} : \Re(z) \geq 0\}$$

Is $f(z) = \cos z$ bounded, where $f : D \to \mathbb{C}$?

Solution 1.

$$\begin{aligned} \cos z &= \frac{e^{iz} - e^{-iz}}{2} \\ &= \frac{e^{ix}e^{-y} - e^{-ix}e^{y}}{2} \end{aligned}$$

Therefore, if $x = 0$,

$$\cos z = \frac{e^{-y} + e^{y}}{2}$$

$$\therefore \lim_{y \to \pm\infty} \cos z = \infty$$

Therefore, $\cos z$ is not bounded.

Exercise 2. 15 P.

f, g, h are entire, such that $\forall n \in \mathbb{N}$,

$$\begin{aligned} f\left(1 + \frac{\pi}{n}\right) &= \cos\left(1 + \frac{\pi}{n}\right) \\ g(2\pi n) &= \cos(2\pi n) \\ h\left(\frac{\pi i}{n}\right) &= \cosh\left(\frac{\pi}{n}\right) \end{aligned}$$

Two of these functions are necessarily identical to each other, and the third is not necessarily identical to the other two. Determine which two are identical, and explain. Explain why the third is not necessarily identical to the other two.

Solution 2.

Let

$$z_n = 1 + \frac{\pi}{n}$$

Therefore, as $\{z_n\}$ converges to 1, and as

$$f(z_n) = \cos(z_n)$$

by the Second Identity Theorem,

$$f(z) \equiv \cos(z)$$

Let

$$z_n = \frac{\pi i}{n}$$

Therefore,

$$\begin{aligned} \cos\left(\frac{\pi i}{n}\right) &= \frac{e^{-\frac{\pi}{n}} + e^{\frac{\pi}{n}}}{2} \\ &= \cosh\left(\frac{\pi}{n}\right) \end{aligned}$$

Therefore, as $\{z_n\}$ converges to 0, and as

$$h(z_n) = \cos(z_n)$$

by the Second Identity Theorem,

$$\begin{aligned} h(z) &\equiv \cos(z) \\ \therefore h\left(\frac{\pi i}{n}\right) &\equiv \cos\left(\frac{\pi i}{n}\right) \\ &\equiv \cosh\left(\frac{\pi}{n}\right) \end{aligned}$$

Therefore,

$$f \equiv h$$

g is not necessarily identical to the other two, as the same argument cannot be used for g, as the sequence $\{2\pi n\}$ does not converge.

Exercise 3. 13 P.

Develop the Laurent series of

$$g(z) = \frac{1}{z^2(z-i)(z-1-i)}$$

around $z = 0$ in the ring $1 < |z| < 2$.

Solution 3.

$$\frac{1}{(z-i)(z-1-i)} = \frac{-1}{z-i} + \frac{1}{z-1-i}$$

Therefore,

$$g(z) = \frac{1}{z^2}\frac{1}{z-i} + \frac{1}{z^2}\frac{1}{z-1-i}$$

Let

$$g_1(z) = \frac{1}{z^2}\frac{1}{z-i}$$
$$g_2(z) = \frac{1}{z^2}\frac{1}{z-1-i}$$

Therefore,

$$g_1(z) = -\frac{1}{z^3}\frac{1}{1-\frac{i}{z}}$$

As $\left|\frac{i}{z}\right| < 1$, the infinite summation of $\frac{i^n}{z^n}$ is $\frac{1}{1-\frac{i}{z}}$.

$$= -\frac{1}{z^3}\sum_{n=0}^{\infty}\frac{i^n}{z^n}$$
$$= \sum_{n=0}^{\infty} -\frac{i^n}{z^{n+3}}$$
$$= \sum_{n=3}^{\infty} -\frac{i^{n-3}}{z^n}$$

Therefore,

$$\begin{aligned}
g_2(z) &= \frac{1}{z^2}\frac{1}{z-(1+i)} \\
&= -\frac{1}{(1+i)z^2}\frac{1}{1-\frac{z}{1+i}} \\
&= -\frac{1}{(1+i)z^2}\sum_{n=0}^{\infty}\frac{z^n}{(1+i)^n} \\
&= \sum_{n=0}^{\infty}\frac{-z^{n-2}}{(1+i)^{n+1}} \\
&= \sum_{n=-2}^{\infty}\frac{-z^n}{(1+i)^{n+3}}
\end{aligned}$$

Exercise 4. 10 P.

f is analytic in $\overline{D_{0,3}}$, satisfying

$$|f(z)| \le \left|\frac{1}{z^2}\right|$$

for all $z \in \overline{D_{0,3}}$. Prove that $\forall z \in \overline{D_{0,3}}$,

$$|f(z)| \le \frac{1}{9}$$

Solution 4.

As $\frac{1}{z^2}$ is not analytic, the maximum modulus principle cannot be applied, and hence it cannot be bounded.

Let

$$g(z) = f(z)z^2$$

Therefore, as both $f(z)$ and z^2 are analytic, $g(z)$ is also analytic. Therefore, $\forall z \in \overline{D_{0,3}}$,

$$\begin{aligned}
|g(z)| &= |f(z)|\left|z^2\right| \\
&\le 1
\end{aligned}$$

Therefore,

$$\begin{aligned}
\max_{z\in\overline{D_{0,3}}}|g(z)| &= \max_{z\in\partial D_{0,3}}|g(z)| \\
&= \max_{z\in\partial D_{0,3}}|f(z)|\left|z^2\right| \\
&= 9\max_{z\in\partial D_{0,3}}|f(z)|
\end{aligned}$$

Therefore,

$$\begin{aligned}
\max_{z\in\overline{D_{0,3}}} |f(z)| &= \max_{z\in\partial D_{0,3}} |f(z)| \\
&\leq \frac{1}{9}
\end{aligned}$$

Exercise 5. 14 P.

$$f(z) = \frac{\sin(i\pi z)}{(z-1)\,(z^4-1)}$$

Find the isolated singular points of f and determine their types.

Solution 5.
The points $z = \pm 1$ and $z = \pm i$ are isolated singular points of f.
Therefore, $\lim\limits_{z\to 1} f(z)(z-1)^2$ is finite and non-zero. Therefore, $z = 1$ is a pole of order 2.
Therefore, $\lim\limits_{z\to -1} f(z)(z+1)$ is finite and non-zero. Therefore, $z = -1$ is a pole of order 1.
Similarly, as the limits of the function at $\pm i$ are finite, $\pm i$ are removable isolated singular points.

Exercise 6. 11 P.

$$f(z) = \frac{\sin(i\pi z)}{(z-1)\,(z^4-1)}$$

Calculate $\displaystyle\int\limits_{C_{1,\sqrt{3}}} f(z)\,\mathrm{d}z$.

Solution 6.
The isolated singular points of $f(z)$ are ± 1 and $\pm i$. Therefore, as $\pm i$ and 1 are in $D_{1,\sqrt{3}}$,

$$\int\limits_{C_{1,\sqrt{3}}} f(z)\,\mathrm{d}z = 2\pi i \left(\mathrm{Res}_f(1) + \mathrm{Res}_f(i) + \mathrm{Res}_f(-i)\right)$$