COMPLEX FUNCTIONS : ASSIGNMENT 3

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Exercise 1.
Prove or disprove.

(1) $\lim\limits_{z \to 0} e^{\frac{1}{z}} = \infty$
(2) $\lim\limits_{z \to 0} e^{\frac{1}{|z|}} = \infty$

Solution 1.
(1)

$$\begin{aligned}\lim_{z \to 0} e^{\frac{1}{z}} &= \lim_{r \to 0} e^{\frac{1}{re^{i\theta}}} \\ &= \lim_{r \to 0} e^{\frac{1}{r}^{\frac{1}{e^{i\theta}}}} \\ &= \infty\end{aligned}$$

(2)

$$\begin{aligned}\lim_{z \to 0} e^{\frac{1}{|z|}} &= \lim_{r \to 0} e^{\frac{1}{r}} \\ &= \infty\end{aligned}$$

Exercise 2.
Find the domain of analyticity of the following functions.

(2) $f(z) = \frac{z^2+1}{(z+2)\left(z^2+2z+2\right)}$

Solution 2.
(2)

$$\begin{aligned}f(z) &= \frac{z^2+1}{(z+2)\left(z^2+2z+2\right)} \\ &= \frac{z^2+1}{(z+2)(z+1-i)(z+1+i)}\end{aligned}$$

Therefore, $f(z)$ is defined and is differentiable over $\mathbb{C} \setminus \{-2, -1+i, -1-i\}$.
Hence, $f(z)$ is analytic at $(-2,0)$, $(-1,1)$, $(-1,-1)$.

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Exercise 4.
Calculate $f'(z)$.

(2) $f(z) = \frac{8z^2-3}{z^2+1}$

Solution 4.

(2)

$$\begin{aligned} f(z) &= \frac{8z^2 - 3}{z^2 + 1} \\ &= \frac{8z^2 + 8 - 11}{z^2 + 1} \\ &= 8 - \frac{11}{z^2 + 1} \end{aligned}$$

Therefore, as $f(z)$ is defined on $\mathbb{C} \setminus \{(0,1),(0,-1)\}$, it is analytic at $(0,1)$ and $(0,-1)$. Therefore,

$$f'(z) = \frac{22}{\left(z^2 + 1\right)^2}$$

Exercise 6.
Let $f(z) = \overline{z}$. Show that f doesn't satisfy the polar Cauchy-Riemann equations and conclude that f isn't differentiable at any point in the plane.

Solution 6.

$$\begin{aligned} f(z) &= \overline{z} \\ &= x - iy \end{aligned}$$

Therefore,

$$\begin{aligned} u(r,\theta) &= x \\ &= r\cos\theta \\ v(r,\theta) &= -y \\ &= -r\sin\theta \end{aligned}$$

Therefore,

$$\begin{aligned} u_r &= \cos\theta \\ u_\theta &= -r\sin\theta \\ v_r &= -\sin\theta \\ v_\theta &= -r\cos\theta \end{aligned}$$

Therefore,

$$\begin{aligned} u_r &= v_\theta \\ \iff \cos\theta &= -r\cos\theta \\ \iff r &= -1 \\ u_\theta &= -v_r \\ \iff -r\sin\theta &= \sin\theta \\ \iff r &= -1 \end{aligned}$$

However, r cannot be negative. Therefore, the function does not satisfy the Cauchy-Riemann equations, and hence is not differentiable at any point in the plane.