

COMPLEX FUNCTIONS : ASSIGNMENT 1

AAKASH JOG
ID : 989323563

Exercise 1.

Calculate

$$(1) \frac{i}{1-i} + \frac{1-i}{i}$$

$$(2) \left| \frac{1+3i}{3+i} \right|$$

Solution 1.

(1)

$$\begin{aligned} \frac{i}{1-i} + \frac{1-i}{i} &= \frac{i^2}{i(1-i)} + \frac{(1-i)^2}{i(1-i)} \\ &= \frac{-1+1-2i-1}{i+1} \\ &= -\frac{2i+1}{i+1} \\ &= -\frac{(2i+1)(i-1)}{-1-1} \\ &= -\frac{-2-2i+i-1}{-2} \\ &= -\frac{-3-i}{-2} \\ &= \frac{-3-i}{2} \end{aligned}$$

(2)

$$\begin{aligned}
\left| \frac{1+3i}{3+i} \right| &= \left| \frac{(1+3i)(3-i)}{9+1} \right| \\
&= \left| \frac{3-i+9i+3}{10} \right| \\
&= \left| \frac{6+8i}{10} \right| \\
&= \left| \frac{3}{5} + \frac{4}{5}i \right| \\
&= \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} \\
&= \sqrt{\frac{9+16}{25}} \\
&= 1
\end{aligned}$$

Exercise 4.

Prove

- (1) $\bar{z} = \frac{1}{z} \iff |z| = 1$
(2) $z + \frac{1}{z} \in \mathbb{R} \iff |z| = 1 \text{ or } z \in \mathbb{R}$

Solution 4.

(1)

$$\begin{aligned}
\frac{1}{z} &= \frac{\bar{z}}{z\bar{z}} \\
&= \frac{\bar{z}}{|z|^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{z} &= \bar{z} \\
\iff |z|^2 &= 1 \\
\iff |z| &= 1
\end{aligned}$$

□

(2) Let

$$z = x + iy$$

Therefore,

$$\begin{aligned}
 z + \frac{1}{z} &= x + iy + \frac{1}{x + iy} \\
 &= x + iy + \frac{x - iy}{x^2 + y^2} \\
 &= \frac{x(x^2 + y^2) + iy(x^2 + y^2) + x - iy}{x^2 + y^2} \\
 &= \frac{x^3 + xy^2 + ix^2y + iy^3 + x - iy}{x^2 + y^2} \\
 &= \frac{x^3 + xy^2 + x}{x^2 + y^2} + i \frac{x^2y + y^3 - y}{x^2 + y^2}
 \end{aligned}$$

Therefore, $z + \frac{1}{z} \in \mathbb{R}$, if and only if

$$\begin{aligned}
 &x^2y + y^3 - y = 0 \\
 \iff &y(x^2 + y^2 - 1) = 0 \\
 \iff &y(|z|^2 - 1) = 0
 \end{aligned}$$

If and only if

$$\begin{array}{lll}
 y = 0 & \text{or} & |z|^2 - 1 = 0 \\
 \iff z \in \mathbb{R} & \text{or} & |z|^2 = 1 \\
 \iff z \in \mathbb{R} & \text{or} & |z| = 1
 \end{array}$$

□

Exercise 5.

Write the following in polar coordinates and find the argument set $\arg(z)$.

- (1) $-4i$
- (2) $-2 + 2i$
- (3) $1 - i$
- (4) $\frac{3-4i}{2-i}$

Solution 5.

(1)

$$z = -4i$$

Let

$$z = x + iy$$

Therefore,

$$x = 0$$

$$y = -4$$

Therefore,

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{0 + (-4)^2} \\ &= 4 \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1} \left(\frac{y}{x} \right) \\ &= \tan^{-1} \left(\frac{-4}{0} \right) \\ &= -\frac{\pi}{2} \end{aligned}$$

Therefore,

$$-4i = \left(4, -\frac{\pi}{2} \right)$$

The argument set is

$$\arg(z) = \left\{ -\frac{\pi}{2} + 2\pi k : k \in \mathbb{Z} \right\}$$

(2)

$$z = -2 + 2i$$

Let

$$z = x + iy$$

Therefore,

$$\begin{aligned} x &= -2 \\ y &= 2 \end{aligned}$$

Therefore,

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{4 + 4} \\ &= 2\sqrt{2} \\ \theta &= \tan^{-1} \left(\frac{y}{x} \right) \\ &= \tan^{-1} \left(\frac{2}{-2} \right) \\ &= \frac{3\pi}{2} \end{aligned}$$

Therefore,

$$-2 + 2i = \left(2\sqrt{2}, \frac{3\pi}{2} \right)$$

The argument set is

$$\arg(z) = \left\{ \frac{3\pi}{2} + 2\pi k : k \in \mathbb{Z} \right\}$$

(3)

$$z = 1 - i$$

Let

$$z = x + iy$$

Therefore,

$$x = 1$$

$$y = -1$$

Therefore,

$$r = \sqrt{x^2 + y^2}$$

$$= \sqrt{1 + 1}$$

$$= \sqrt{2}$$

$$\theta = \tan^{-1} \left(\frac{y}{x} \right)$$

$$= \tan^{-1} \left(\frac{-1}{1} \right)$$

$$= -\frac{\pi}{2}$$

Therefore,

$$1 - i = \left(\sqrt{2}, -\frac{\pi}{2} \right)$$

The argument set is

$$\arg(z) = \left\{ -\frac{\pi}{2} + 2\pi k : k \in \mathbb{Z} \right\}$$

(4) Let

$$z_1 = 3 - 4i$$

$$z_2 = 2 - i$$

Let

$$z_1 = x_1 + iy_1$$

$$z_2 = x_2 + iy_2$$

Therefore,

$$x_1 = 3$$

$$y_1 = -4$$

$$x_2 = 2$$

$$y_2 = -1$$

Therefore,

$$\begin{aligned} r_1 &= \sqrt{x_1^2 + y_1^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

$$\begin{aligned} \theta_1 &= \tan^{-1} \left(\frac{y_1}{x_1} \right) \\ &= \tan^{-1} \left(\frac{-4}{3} \right) \\ &= -\tan^{-1} \left(\frac{4}{3} \right) \end{aligned}$$

$$\begin{aligned} r_2 &= \sqrt{x_2^2 + y_2^2} \\ &= \sqrt{4 + 1} \\ &= \sqrt{5} \end{aligned}$$

$$\begin{aligned} \theta_2 &= \tan^{-1} \left(\frac{y_2}{x_2} \right) \\ &= \tan^{-1} \left(\frac{-1}{2} \right) \\ &= -\tan^{-1} \left(\frac{1}{2} \right) \end{aligned}$$

Therefore,

$$\begin{aligned} 3 - 4i &= \left(5, -\tan^{-1} \left(\frac{4}{3} \right) \right) \\ 2 - i &= \left(\sqrt{5}, -\tan^{-1} \left(\frac{1}{2} \right) \right) \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{z_1}{z_2} &= \left(\frac{r_1}{r_2}, \theta_1 - \theta_2 \right) \\ \therefore \frac{3 - 4i}{2 - i} &= \left(\frac{5}{\sqrt{5}}, -\tan^{-1} \left(\frac{4}{3} \right) + \tan^{-1} \left(\frac{1}{2} \right) \right) \\ &= \left(\sqrt{5}, -\tan^{-1} \left(\frac{4}{3} \right) + \tan^{-1} \left(\frac{1}{2} \right) \right) \end{aligned}$$

The argument set is

$$\arg(z) = \left\{ -\tan^{-1} \left(\frac{4}{3} \right) + \tan^{-1} \left(\frac{1}{2} \right) + 2\pi k : k \in \mathbb{Z} \right\}$$

Exercise 8.

Prove that for every $z_1, z_2 \in \mathbb{C}$

$$|z_1 + z_2| \geq \left| |z_1| - |z_2| \right|$$

Solution 8.

$$\begin{aligned} |z_1| &= |z_1 + z_2 - z_2| \\ &\leq |z_1 + z_2| + |-z_2| \\ \therefore |z_1| &\leq |z_1 + z_2| + |z_2| \\ \therefore |z_1| - |z_2| &\leq |z_1 + z_2| \\ \therefore |z_1 + z_2| &\geq \left| |z_1| - |z_2| \right| \end{aligned}$$