COMPLEX FUNCTIONS : ASSIGNMENT 1

AAKASH JOG
ID : 989323563

Exercise 1.
Calculate

(1) $\frac{i}{1-i} + \frac{1-i}{i}$
(2) $\left|\frac{1+3i}{3+i}\right|$

Solution 1.
(1)

$$\begin{aligned}
\frac{i}{1-i} + \frac{1-i}{i} &= \frac{i^2}{i(1-i)} + \frac{(1-i)^2}{i(1-i)} \\
&= \frac{-1+1-2i-1}{i+1} \\
&= -\frac{2i+1}{i+1} \\
&= -\frac{(2i+1)(i-1)}{-1-1} \\
&= -\frac{-2-2i+i-1}{-2} \\
&= -\frac{-3-i}{-2} \\
&= \frac{-3-i}{2}
\end{aligned}$$

Date: Wednesday 28$^{\text{th}}$ October, 2015.

(2)

$$\begin{aligned}
\left|\frac{1+3i}{3+i}\right| &= \left|\frac{(1+3i)(3-i)}{9+1}\right| \\
&= \left|\frac{3-i+9i+3}{10}\right| \\
&= \left|\frac{6+8i}{10}\right| \\
&= \left|\frac{3}{5}+\frac{4}{5}i\right| \\
&= \sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2} \\
&= \sqrt{\frac{9+16}{25}} \\
&= 1
\end{aligned}$$

Exercise 4.
Prove

(1) $\overline{z} = \frac{1}{z} \iff |z| = 1$
(2) $z + \frac{1}{z} \in \mathbb{R} \iff |z| = 1$ or $z \in \mathbb{R}$

Solution 4.
(1)

$$\begin{aligned}
\frac{1}{z} &= \frac{\overline{z}}{z\overline{z}} \\
&= \frac{\overline{z}}{|z^2|}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{z} = \overline{z} \\
\iff &|z^2| = 1 \\
\iff &|z| = 1
\end{aligned}$$

$\square$

(2) Let

$$z = x + iy$$

Therefore,

$$\begin{aligned}
z + \frac{1}{z} &= x + iy + \frac{1}{x+iy} \\
&= x + iy + \frac{x - iy}{x^2 + y^2} \\
&= \frac{x(x^2+y^2) + iy(x^2+y^2) + x - iy}{x^2+y^2} \\
&= \frac{x^3 + xy^2 + ix^2y + iy^3 + x - iy}{x^2+y^2} \\
&= \frac{x^3 + xy^2 + x}{x^2+y^2} + i\frac{x^2y + y^3 - y}{x^2+y^2}
\end{aligned}$$

Therefore, $z + \frac{1}{z} \in \mathbb{R}$, if and only if

$$\begin{aligned}
x^2y + y^3 - y &= 0 \\
\iff y(x^2 + y^2 - 1) &= 0 \\
\iff y(|z|^2 - 1) &= 0
\end{aligned}$$

If and only if

$$\begin{array}{rlcr}
& y = 0 & \text{or} & |z|^2 - 1 = 0 \\
\iff & z \in \mathbb{R} & \text{or} & |z|^2 = 1 \\
\iff & z \in \mathbb{R} & \text{or} & |z| = 1
\end{array}$$

□

Exercise 5.
Write the following in polar coordinates and find the argument set $\arg(z)$.

(1) $-4i$
(2) $-2 + 2i$
(3) $1 - i$
(4) $\frac{3-4i}{2-i}$

Solution 5.
(1)

$$z = -4i$$

Let

$$z = x + iy$$

Therefore,

$$\begin{aligned}
x &= 0 \\
y &= -4
\end{aligned}$$

Therefore,

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{0 + (-4)^2} \\ &= 4 \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{y}{x}\right) \\ &= \tan^{-1}\left(\frac{-4}{0}\right) \\ &= -\frac{\pi}{2} \end{aligned}$$

Therefore,

$$-4i = \left(4, -\frac{\pi}{2}\right)$$

The argument set is

$$\arg(z) = \left\{-\frac{\pi}{2} + 2\pi k : k \in \mathbb{Z}\right\}$$

(2)

$$z = -2 + 2i$$

Let

$$z = x + iy$$

Therefore,

$$\begin{aligned} x &= -2 \\ y &= 2 \end{aligned}$$

Therefore,

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{4 + 4} \\ &= 2\sqrt{2} \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{y}{x}\right) \\ &= \tan^{-1}\left(\frac{2}{-2}\right) \\ &= \frac{3\pi}{2} \end{aligned}$$

Therefore,

$$-2 + 2i = \left(2\sqrt{2}, \frac{3\pi}{2}\right)$$

The argument set is

$$\arg(z) = \left\{\frac{3\pi}{2} + 2\pi k : k \in \mathbb{Z}\right\}$$

(3)

$$z = 1 - i$$

Let

$$z = x + iy$$

Therefore,

$$\begin{aligned} x &= 1 \\ y &= -1 \end{aligned}$$

Therefore,

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1+1} \\ &= \sqrt{2} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \\ &= \tan^{-1}\left(\frac{-1}{1}\right) \\ &= -\frac{\pi}{2} \end{aligned}$$

Therefore,

$$1 - i = \left(\sqrt{2}, -\frac{\pi}{2}\right)$$

The argument set is

$$\arg(z) = \left\{-\frac{\pi}{2} + 2\pi k : k \in \mathbb{Z}\right\}$$

(4) Let

$$\begin{aligned} z_1 &= 3 - 4i \\ z_2 &= 2 - i \end{aligned}$$

Let

$$\begin{aligned} z_1 &= x_1 + iy_1 \\ z_2 &= x_2 + iy_2 \end{aligned}$$

Therefore,

$$\begin{aligned} x_1 &= 3 \\ y_1 &= -4 \\ x_2 &= 2 \\ y_2 &= -1 \end{aligned}$$

Therefore,

$$\begin{aligned}
r_1 &= \sqrt{{x_1}^2 + {y_1}^2} \\
&= \sqrt{9+16} \\
&= \sqrt{25} \\
&= 5 \\
\theta_1 &= \tan^{-1}\left(\frac{y_1}{x_1}\right) \\
&= \tan^{-1}\left(\frac{-4}{3}\right) \\
&= -\tan^{-1}\left(\frac{4}{3}\right) \\
r_2 &= \sqrt{{x_2}^2 + {y_2}^2} \\
&= \sqrt{4+1} \\
&= \sqrt{5} \\
\theta_2 &= \tan^{-1}\left(\frac{y_2}{x_2}\right) \\
&= \tan^{-1}\left(\frac{-1}{2}\right) \\
&= -\tan^{-1}\left(\frac{1}{2}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
3-4i &= \left(5, -\tan^{-1}\left(\frac{4}{3}\right)\right) \\
2-i &= \left(\sqrt{5}, -\tan^{-1}\left(\frac{1}{2}\right)\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{z_1}{z_2} &= \left(\frac{r_1}{r_2}, \theta_1 - \theta_2\right) \\
\therefore \frac{3-4i}{2-i} &= \left(\frac{5}{\sqrt{5}}, -\tan^{-1}\left(\frac{4}{3} - \tan^{-1}\left(\frac{1}{2}\right)\right)\right) \\
&= \left(\sqrt{5}, -\tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right)
\end{aligned}$$

The argument set is

$$\arg(z) = \left\{-\tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{2}\right) + 2\pi k : k \in \mathbb{Z}\right\}$$

Exercise 8.

Prove that for every $z_1, z_2 \in \mathbb{C}$

$$|z_1 + z_2| \geq \big||z_1| - |z_2|\big|$$

Solution 8.

$$\begin{aligned} |z_1| &= |z_1 + z_2 - z_2| \\ &\leq |z_1 + z_2| + |-z_2| \\ \therefore |z_1| &\leq |z_1 + z_2| + |z_2| \\ \therefore |z_1| - |z_2| &\leq |z_1 + z_2| \\ \therefore |z_1 + z_2| &\geq \big||z_1| - |z_2|\big| \end{aligned}$$