COMPLEX FUNCTIONS : ASSIGNMENT 7

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Exercise 1.
Calculate the length of the following curves using the correct parametrization.

(4) $\gamma(t) = (t - \sin t) + i(1 - \cos t)$, $t \in [0, 2\pi]$.

Solution 1.
(4)

$$\begin{aligned}\gamma(t) &= (t - \sin t) + i(1 - \cos t)\\ \therefore \dot{\gamma}(t) &= 1 - \cos t + i \sin t\end{aligned}$$

Therefore,

$$\begin{aligned}\left|\dot{\gamma}(t)\right| &= \sqrt{(1 - \cos t)^2 + \sin^2 t}\\ &= \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\\ &= \sqrt{2 - 2\cos t}\end{aligned}$$

Therefore,

$$\begin{aligned}\text{length}(\gamma) &= \int\limits_a^b \left|\dot{\gamma}(t)\right| \mathrm{d}t\\ &= \int\limits_0^{2\pi} \sqrt{2 - 2\cos t}\, \mathrm{d}t\\ &= \int\limits_0^{2\pi} \sqrt{4\sin^2 \frac{t}{2}}\, \mathrm{d}t\\ &= \int\limits_0^{2\pi} 2\sin \frac{t}{2}\, \mathrm{d}t\\ &= -4\cos \frac{t}{2}\Big|_0^{2\pi}\\ &= 8\end{aligned}$$

Date: Wednesday 16^{th} December, 2015.

Exercise 2.

Calculate $\frac{1}{2\pi}\int\limits_0^{2\pi} e^{im\theta}e^{-in\theta}\,\mathrm{d}\theta$, $m, n \in \mathbb{Z}$. Hint: Divide to the cases $m = n$, and $m \neq n$.

Solution 2.

If $m = n$,

$$\begin{aligned}\frac{1}{2\pi}\int\limits_0^{2\pi} e^{im\theta}e^{-in\theta}\,\mathrm{d}\theta &= \frac{1}{2\pi}\int\limits_0^{2\pi}\mathrm{d}\theta\\ &= 1\end{aligned}$$

If $m \neq n$,

$$\begin{aligned}\frac{1}{2\pi}\int\limits_0^{2\pi} e^{im\theta}e^{-in\theta}\,\mathrm{d}\theta &= \frac{1}{2\pi}\int\limits_0^{2\pi} e^{i(m-n)\theta}e^{in\theta}e^{-in\theta}\,\mathrm{d}\theta\\ &= \frac{1}{2\pi}\int\limits_0^{2\pi} e^{i(m-n)\theta}\,\mathrm{d}\theta\\ &= \frac{1}{2\pi}\left.\frac{e^{i(m-n)\theta}}{i(m-n)}\right|_0^{2\pi}\\ &= 0\end{aligned}$$

Exercise 3.

Prove

(1) The function $f(z) = \overline{z}$ has no analytic primitive which is analytic at every point.

Solution 3.

(1) Let γ_1 be the line joining 0 and 1.
Therefore,

$$\begin{aligned}\gamma_1(t) &= t\\ \therefore \dot{\gamma_1}(t) &= 1\end{aligned}$$

where $t \in (0, 1)$.
Let γ_2 be the line joining 1 and $1 + i$.
Therefore,

$$\begin{aligned}\gamma_2(t) &= 1 + it\\ \therefore \dot{\gamma_2}(t) &= i\end{aligned}$$

where $t \in (0, 1)$.
Let γ_3 be the line joining $1 + i$ and 0.

Therefore,

$$\gamma_3(t) = t + it$$
$$\therefore \dot{\gamma_3}(t) = 1 + i$$

where $t \in (0, 1)$.
Therefore,

$$\begin{aligned}
\int_\gamma \overline{z}\,\mathrm{d}z = & \int_0^1 f\left(\gamma_1(t)\right)\dot{\gamma_1}(t)\,\mathrm{d}t \\
& + \int_0^1 f\left(\gamma_2(t)\right)\dot{\gamma_2}(t)\,\mathrm{d}t \\
& + \int_1^0 f\left(\gamma_3(t)\right)\dot{\gamma_3}(t)\,\mathrm{d}t \\
& = \int_0^1 \overline{t}\,\mathrm{d}t + \int_0^1 \left(\overline{1+it}\right) i\,\mathrm{d}t + \int_1^0 \left(\overline{t+it}\right)(1+i)\,\mathrm{d}t \\
& = \int_0^1 t\,\mathrm{d}t + \int_0^1 (1-it)i\,\mathrm{d}t - \int_0^1 (t-it)(1+i)\,\mathrm{d}t \\
& = \int_0^1 t\,\mathrm{d}t + \int_0^1 i + t\,\mathrm{d}t - \int_0^1 2t\,\mathrm{d}t \\
& = \left.\frac{t^2}{2}\right|_0^1 + it + \left.\frac{t^2}{2}\right|_0^1 - \left.t^2\right|_0^1 \\
& = \frac{1}{2} + i + \frac{1}{2} - 1 \\
& = i
\end{aligned}$$

Therefore, the integral of $f(z) = \overline{z}$ is non zero over a closed path. Hence, it cannot have an analytic primitive over $\mathbb{C}$.

Exercise 6.

(2) γ is the triangle whose vertices are 0, 1, $1 + i$. Calculate $\int_\gamma \overline{z}\,\mathrm{d}z$.

Solution 6.

(1) Let γ_1 be the line joining 0 and 1.
Therefore,

$$\gamma_1(t) = t$$
$$\therefore \dot{\gamma_1}(t) = 1$$

where $t \in (0,1)$.
Let γ_2 be the line joining 1 and $1+i$.
Therefore,

$$\begin{aligned}\gamma_2(t) &= 1 + it \\ \therefore \dot{\gamma_2}(t) &= i\end{aligned}$$

where $t \in (0,1)$.
Let γ_3 be the line joining $1+i$ and 0.
Therefore,

$$\begin{aligned}\gamma_3(t) &= t + it \\ \therefore \dot{\gamma_3}(t) &= 1 + i\end{aligned}$$

where $t \in (0,1)$.
Therefore,

$$\begin{aligned}\int\limits_{\gamma} \overline{z}\,\mathrm{d}z = &\int\limits_0^1 f\left(\gamma_1(t)\right)\dot{\gamma_1}(t)\,\mathrm{d}t \\ &+ \int\limits_0^1 f\left(\gamma_2(t)\right)\dot{\gamma_2}(t)\,\mathrm{d}t \\ &+ \int\limits_1^0 f\left(\gamma_3(t)\right)\dot{\gamma_3}(t)\,\mathrm{d}t \\ &= \int\limits_0^1 \overline{t}\,\mathrm{d}t + \int\limits_0^1 \left(\overline{1+it}\right) i\,\mathrm{d}t + \int\limits_1^0 \left(\overline{t+it}\right)(1+i)\,\mathrm{d}t \\ &= \int\limits_0^1 t\,\mathrm{d}t + \int\limits_0^1 (1-it)i\,\mathrm{d}t - \int\limits_0^1 (t-it)(1+i)\,\mathrm{d}t \\ &= \int\limits_0^1 t\,\mathrm{d}t + \int\limits_0^1 i + t\,\mathrm{d}t - \int\limits_0^1 2t\,\mathrm{d}t \\ &= \left.\frac{t^2}{2}\right|_0^1 + it + \left.\frac{t^2}{2}\right|_0^1 - \left.t^2\right|_0^1 \\ &= \frac{1}{2} + i + \frac{1}{2} - 1 \\ &= i\end{aligned}$$

Exercise 7.

$$f(z) = \frac{1}{2\sqrt{z}}$$

Calculate the integral over the curve starting at 1 and ending at 9, going over the polygon whose vertices are 1, $1+8i$, $9+8i$, and then going through the right half part of a circle, with radius 4, from $9+8i$ to 9.

Solution 7.

$$f(z) = \frac{1}{2\sqrt{z}}$$

Therefore,

$$F(z) = \sqrt{z}$$

is a primitive of $f(z)$.
Let the given curve be γ.
Therefore,

$$\begin{aligned}\int\limits_{\gamma} f(z)\,\mathrm{d}z &= F(9) - F(1) \\ &= 3 - 1 \\ &= 2\end{aligned}$$