Complex Functions

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2015-16

Contents

1 Lecturer Information

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2 Recommended Reading

1. James Ward Brown & Ruel V. Churchill, “Complex Variables and Applications”, McGraw-Hill, Inc. 1996.
2. D. Zill, P. Shanahan, “Complex Variables with Applications”, Jones and Bartlett Publishers.

3 Additional Reading

1. Saff, Edward B., and Arthur David Snider. Fundamentals of Complex Analysis with Applications to Engineering, Science, and Mathematics. 3rd ed. Upper Saddle River, NJ: Prentice Hall, 2002. ISBN: 0139078746.
2. Sarason, Donald. Complex Function Theory. American Mathematical Society. ISBN: 0821886223
3. Alfhors, Lars. Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable. McGraw-Hill Education, 1979. ISBN: 0070006571.

Part I
Complex Numbers

Definition 1. A number of the form

$$z = x + iy$$

where

$$i = \sqrt{-1}$$
$$x \in \mathbb{R}$$
$$y \in \mathbb{R}$$

is called a complex number.

Definition 2 (Real part of a complex number)**.** If

$$z = x + iy$$

then x is called the real part of z, and is denoted as

$$x = \Re(z)$$

Definition 3 (Imaginary part of a complex number)**.** If

$$z = x + iy$$

then y is called the imaginary part of z, and is denoted as

$$x = \Im(z)$$

Definition 4 (Complex conjugate)**.** If

$$z = x + iy$$

then

$$\overline{z} = x - iy$$

is called the complex conjugate of z.

Theorem 1.

$$z\overline{z} = |z|^2$$

Proof.

$$
\begin{aligned}
z &= x + iy \\
\therefore \overline{z} &= x - iy
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
z\overline{z} &= (x + iy)(x - iy) \\
&= x^2 - ixy + ixy + y^2 \\
&= x^2 + y^2 \\
&= |z|^2
\end{aligned}
$$

□

Definition 5 (Polar representation)**.** If

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta
\end{aligned}
$$

then (r, θ) is called the polar representation of (x, y).

Theorem 2 (Euler's Formula)**.**

$$
rcos\theta + ir\sin\theta = re^{i\theta}
$$

Definition 6 (Absolute value or Norm)**.**

$$
\begin{aligned}
|z| &= |x + iy| \\
&= \sqrt{x^2 + y^2}
\end{aligned}
$$

is called the absolute value, or the norm of z.

Theorem 3.

$$
|z| \leq |\Re(z)| + |\Im(z)| \leq \sqrt{2}|z|
$$

Proof.

$$
\begin{aligned}
&\sqrt{x^2 + y^2} \leq |x| + |y| \leq \sqrt{2x^2 + 2y^2} \\
&\iff x^2 + y^2 \leq x^2 + y^2 + 2|x||y| \leq 2x^2 + 2y^2 \\
&\iff x^2 + y^2 - 2|x||y| \geq 0 \\
&\iff (|x| - |y|)^2 \geq 0
\end{aligned}
$$

□

Definition 7 (Argument)**.** Let z be a complex number.
Then, θ, such that $\theta \in (-\pi, \pi]$, and

$$z = (r, \theta)$$

is called the argument of z.
It is denoted as

$$\theta = \operatorname{Arg}(z)$$

If $\theta \notin (-\pi, \pi]$, but

$$z = (r, \theta)$$

then

$$\theta = \arg(z)$$

Theorem 4.

$$z^n = |z|^n e^{in \operatorname{Arg}(z)}$$

Proof.

$$\begin{aligned} z &= |z| e^{i \operatorname{Arg}(z)} \\ \therefore z^n &= \left(|z| e^{i \operatorname{Arg}(z)}\right)^n \\ &= (|z|)^n \left(e^{i \operatorname{Arg}(z)}\right)^n \\ &= |z|^n e^{in \operatorname{Arg}(x)} \end{aligned}$$

□

Theorem 5. *Let*

$$\begin{aligned} z &= re^{i\theta} \\ w &= \rho e^{i\varphi} \end{aligned}$$

The solutions to

$$w = \sqrt[n]{z}$$

are

$$\varphi_k = \frac{\theta}{n} + \frac{2\pi k}{n}$$

where $k \in \{0, \dots, n-1\}$.

Proof.

$$
\begin{aligned}
w &= \sqrt[n]{z} \\
\therefore w^n &= z
\end{aligned}
$$

Therefore,

$$\rho^n e^{in\varphi} = re^{i\theta}$$

Therefore, for $k \in \{0, \ldots, n-1\}$,

$$
\begin{aligned}
\rho &= \sqrt[n]{r} \\
n\varphi &= \theta + 2\pi k \\
\therefore \varphi &= \frac{\theta}{n} + \frac{2\pi k}{n}
\end{aligned}
$$

$\square$

Part II
Complex Sequences and Series

Definition 8 (Convergence of complex sequences)**.** Let

$$z_n = x_n + iy_n$$

The sequence $\{z_n\}$ is said to converge to the limit $z = x + iy$, if $\forall \varepsilon > 0$, $\exists N$, such that $\forall n > N$, $|z_n - z| < \varepsilon$, i.e. there is a circular region of radius ε, centred at z, in which z_n lies.

Theorem 6. *$\{z_n\} \to z$, i.e. $\{z_n\}$ converges to z if and only if all subsequences of $\{z_n\}$ converge to z.*

Exercise 1.
Find the limit $\lim\limits_{n\to\infty} \frac{n+i}{2n-i}$.

Solution 1.

$$\begin{aligned} z_n &= \frac{n+i}{2n-i} \\ &= \frac{(n+i)(2n+i)}{4n^2+1} \\ &= \frac{2n^2+1}{4n^2+1} + i\frac{3n}{4n^2+1} \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} z_n &= \lim_{n\to\infty} \frac{2n^2+1}{4n^2+1} + i\frac{3n}{4n^2+1} \\ &= \frac{1}{2} \end{aligned}$$

Exercise 2.
Show that for

$$z_n = -2 + \frac{(-1)^n}{n} i$$

$\lim\limits_{n\to\infty} \operatorname{Arg}(z_n)$ does not exist, but $\lim\limits_{n\to\infty} |z_n|$ exists.

Solution 2.

The magnitude of z_n is

$$\begin{aligned} |z_n| &= \left|-2 + \frac{(-1)^n}{n} i\right| \\ &= \sqrt{4 + \frac{(-1)^{2n}}{n^2}} \\ &= \sqrt{4 + \frac{1}{n^2}} \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} |z_n| &= \lim_{n\to\infty} \sqrt{4 + \frac{1}{n^2}} \\ &= 2 \end{aligned}$$

The argument of z_{2n} is

$$\begin{aligned} \operatorname{Arg}(z_{2n}) &= \operatorname{Arg}\left(-2 + \frac{(-1)^{2n}}{2n} i\right) \\ \therefore \lim_{n\to\infty} \operatorname{Arg}(z_{2n}) &= \lim_{n\to\infty} \operatorname{Arg}\left(-2 + \frac{i}{2n}\right) \\ &= \pi \end{aligned}$$

The argument of z_{2n+1} is

$$\begin{aligned} \operatorname{Arg}(z_{2n+1}) &= \operatorname{Arg}\left(-2 + \frac{(-1)^{2n+1}}{2n+1} i\right) \\ \therefore \lim_{n\to\infty} \operatorname{Arg}(z_{2n}) &= \lim_{n\to\infty} \operatorname{Arg}\left(-2 - \frac{i}{2n}\right) \\ &= -\pi \end{aligned}$$

Therefore, as the limit of two subsequences are not equal, the limit does not exist.

Part III
Topology on the Complex Plane

Definition 9 (Neighbourhood of a complex number)**.** A circular region of radius ε centred at z, is called the ε neighbourhood of z.

$$B(z, \varepsilon) = D(z, \varepsilon) = \{w \in \mathbb{C} : |w - z| < \varepsilon\}$$

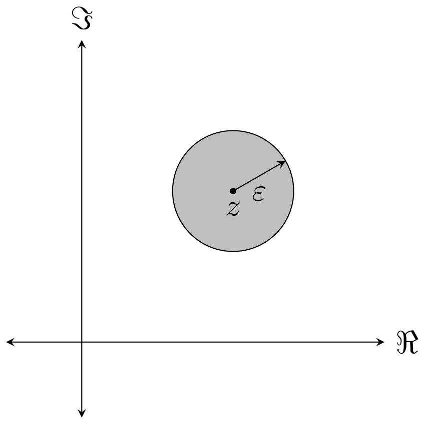


Figure 1: Neighbourhood of a complex number

Definition 10 (Interior point)**.** Let $A \subseteq \mathbb{C}$.
$z \in \mathbb{C}$ is called an inner or interior point of A if there exists at least one $\varepsilon_z > 0$, such that $B(z, \varepsilon_z) \subset A$.
The set of all interior points of A is denoted by $\mathrm{Int}(A)$ or A°.

Definition 11 (Exterior point)**.** Let $A \subseteq \mathbb{C}$.
$z \in \mathbb{C}$ is called an outer or exterior point of A if there exists at least one $\varepsilon_z > 0$, such that $B(z, \varepsilon_z) \subset (\mathbb{C} \setminus A)$. The set of all exterior points of A is denoted by $\mathrm{Ext}(A)$.

Definition 12 (Edge point)**.** Let $A \subseteq \mathbb{C}$.
$z \in \mathbb{C}$ is called an edge or boundary point of A if it is neither an inner point of A, nor an outer point of A. The set of all boundary points of A is denoted by $\partial(A)$.

Definition 13 (Open set)**.** A set $A \subseteq \mathbb{C}$ is called an open set if $A = A^\circ$, i.e. for any point $z \in A$, $\exists \varepsilon > 0$, such that $D(z, \varepsilon) \subset A$.

Definition 14 (Closer of a set)**.** The closer of A is defined to be

$$\overline{A} = A^{\circ} \cup \partial A$$

Definition 15 (Closed set)**.** A set A is called a closed set if $\partial A \subset A$, i.e. $A = \overline{A}$.

Definition 16 (Connected set)**.** A set A is called a connected set of for any $z_1, z_n \in A$, there exists a polygonal path, i.e. a finite set of connected straight lines, which connects z_1 and z_2, and belongs to A.

Definition 17 (Domain)**.** An open connected set is called a domain.

Definition 18 (Bound set)**.** A set A is said to be a bound set if it is bound inside a disk.

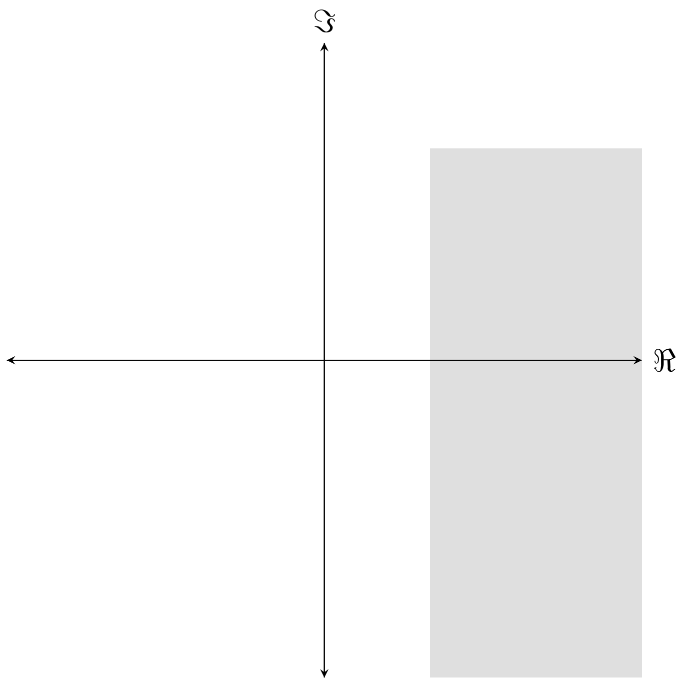
Exercise 3.

Describe geometrically and list the properties of the following sets.

1. $A = \{z \in \mathbb{C} : \Re(z) \geq 2, \Im(z) \leq 4\}$
2. $B = \{z \in \mathbb{C} : |z - 1 + 3i| > 3\}$

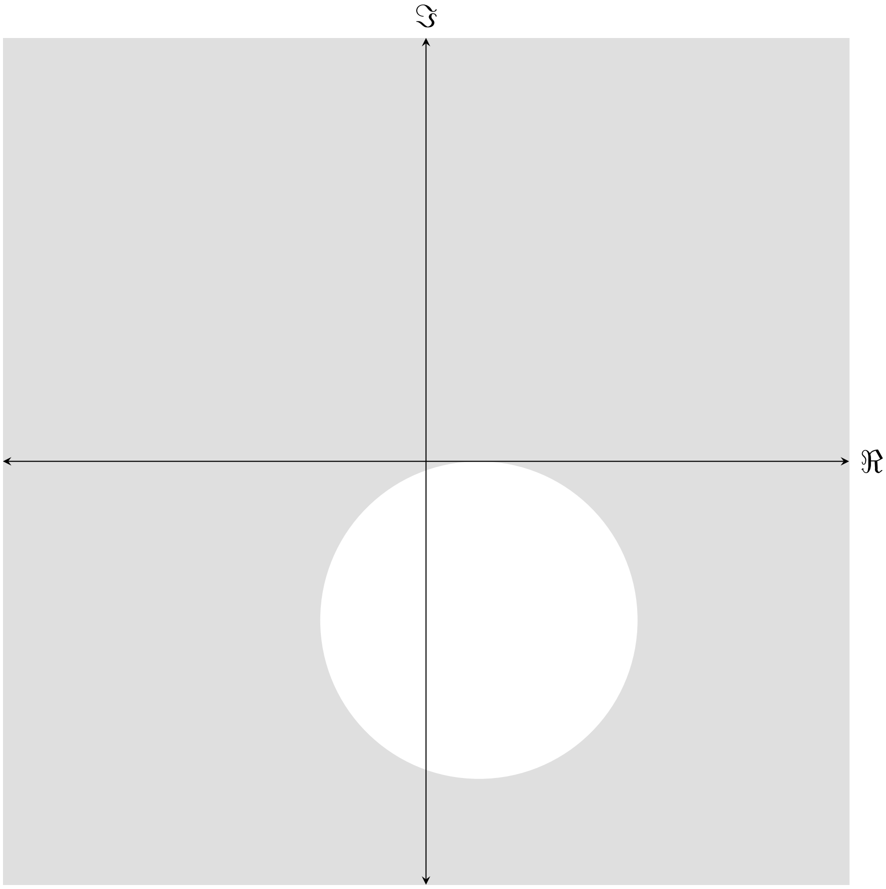
Solution 3.

1. A is the union of the bottom half plane with respect to the line $y = 4$, and the right half plane with respect to the line $x = 2$.



Therefore, as $A = A^\circ + \partial A$, it is a closer, unbounded set.

2. A is the complement of a disk, centred at $1 - 3i$, with radius 3.



Therefore, it is an open, unbounded set.

Exercise 4.
Prove that the upper half plane $U = \{z : \Im(z) > 0\}$ is open.

Solution 4.
Let

$$z = x + iy$$

Therefore, as $z \in U$, $y > 0$.
Therefore, consider the disk $D\left(z, \frac{y}{2}\right)$.

Let $w \in D\left(z, \frac{y}{2}\right)$. Therefore,

$$\begin{aligned}
|w - z| &< \frac{y}{2} \\
\therefore |\Im(w - z)| &\leq |w - z| \\
&\leq \frac{y}{2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&-\frac{y}{2} \leq \Im(w) - \Im(z) \leq \frac{y}{2} \\
&\therefore -\frac{y}{2} \leq \Im(w) - y \leq \frac{y}{2} \\
&\therefore \Im(w) \geq \frac{y}{2} > 0
\end{aligned}$$

Therefore, as $\Im(w) > 0$, $w \in U$. Therefore, U is open. $\square$

Part IV
Complex Functions

1 Complex Functions

Definition 19 (Complex function)**.** Let $A \subseteq \mathbb{C}$. $f : A \to \mathbb{C}$ is called a complex function, which matches $z \in A$ to $f(z) \in \mathbb{C}$.

Theorem 7. *Any complex function f can be written as*

$$\begin{aligned} f(x+iy) &= \Re f(x+iy) + i\Im f(x+iy) \\ &= u(x,y) + iv(x,y) \end{aligned}$$

2 Limits

Definition 20 (Limit of a function)**.** Let f be a complex function defined on a neighbourhood of z_0, but may or may not be defined at z_0. Then, the limit of $f(z)$ at z_0 is defined as

$$w = \lim_{z \to z_0} f(z)$$

if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $\forall z \in \mathbb{X}$such that $|z - z_0| < \delta$, $|f(z) - w| < \varepsilon$.

Exercise 5.
Show that

$$\lim_{z \to 1} \frac{iz}{2} = \frac{i}{2}$$

Solution 5.
Let $|z - 1| < \delta$. Therefore, for $\varepsilon > 0$,

$$\begin{aligned} \left| f(z) - \frac{i}{2} \right| &= \left| \frac{iz}{2} - \frac{i}{2} \right| \\ &= \left| \frac{i}{2} \right| |z - 1| \\ &= \frac{1}{2} |z - i| \end{aligned}$$

Therefore, for $\delta \leq 2\varepsilon$, $\left| f(z) - \frac{i}{2} \right| < \varepsilon$. □

Theorem 8. *If*

$$\begin{aligned} f(z) &= f(x+iy) \\ &= u(x,y) + iv(x,y) \end{aligned}$$

then

$$\lim_{z \to z_0} f(z) = u_0 + iv_0$$

if and only if

$$\begin{aligned} \lim_{(x,y)\to(x_0,y_0)} u(x,y) &= u_0 \\ \lim_{(x,y)\to(x_0,y_0)} v(x,y) &= v_0 \end{aligned}$$

Theorem 9 (Limit arithmetics)**.** *If*

$$\begin{aligned} \lim_{z\to z_0} f(z) &= w_1 \\ \lim_{z\to z_0} g(z) &= w_2 \end{aligned}$$

then, as long as all quantities are defined,

$$\begin{aligned} \lim_{z\to z_0} f(z) \pm g(z) &= w_1 \pm w_2 \\ \lim_{z\to z_0} f(z)g(z) &= w_1 w_2 \\ \lim_{z\to z_0} \frac{f(z)}{g(z)} &= \frac{w_1}{w_2} \end{aligned}$$

Exercise 6.
For the function $f(z) = \overline{z}^2$, prove

$$\begin{aligned} \lim_{z\to z_0} f(z) &= f(z_0) \\ &= \overline{z_0}^2 \end{aligned}$$

Solution 6.

$$\begin{aligned} \overline{z} &= \left(\overline{x+iy}\right)^2 \\ &= (x-iy)^2 \\ &= x^2 - y^2 - 2xyi \end{aligned}$$

Therefore, let

$$\begin{aligned} u(x, y) &= x^2 - y^2 \\ v(x, y) &= -2xy \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{(x,y)\to(x_0,y_0)} u(x, y) &= {x_0}^2 - {y_0}^2 \\ \lim_{(x,y)\to(x_0,y_0)} v(x, y) &= -2x_0y_0 \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{z\to z_0} f(z) &= u_0 + iv_0 \\ &= {x_0}^2 - {y_0}^2 - 2x_0y_0i \\ &= \overline{z_0}^2 \end{aligned}$$

□

Definition 21 (Infinite limit)**.** The limit of $f(z)$ is said to be infinite, i.e.

$$\lim_{z\to z_0} f(z) = \infty$$

if and only if

$$\lim_{z\to z_0} |f(z)| = \infty$$

if and only if

$$\lim_{z\to z_0} \frac{1}{f(z)} = 0$$

Definition 22 (Limit at infinity)**.** The limit of a function $f(z)$,

$$\lim_{z\to\infty} f(z) = w$$

if

$$\lim_{|z|\to\infty} f(z) = w$$

Alternatively, $\forall\varepsilon > 0$, $\exists R > 0$, such that for $|z| > R$, $|f(x) - w| < \varepsilon$.

Exercise 7.
Show that

$$\lim_{z\to\infty} \frac{1}{z^2} = 0$$

Solution 7.
Let $\varepsilon > 0$. Let $R > 0$, such that $\frac{1}{R^2} < \varepsilon$.
Therefore, if $|z| > R$,

$$\begin{aligned} |f(z) - 0| &= \left|\frac{1}{z^2}\right| \\ &= \frac{1}{|z^2|} \\ &= \frac{1}{|z|^2} \\ &< \frac{1}{R^2} \\ &< \varepsilon \end{aligned}$$

Therefore, $\lim_{z\to\infty} \frac{1}{z^2} = 0$.

3 Continuity

Definition 23 (Continuous function)**.** $f(z)$ is said to be continuous at z_0 if $f(z)$ is defined at z_0 and

$$\lim_{z\to z_0} f(z) = f(z_0)$$

Theorem 10 (Continuity arithmetics)**.** *If*

$$\lim_{z\to z_0} f(z) = f(z_0)$$
$$\lim_{z\to z_0} g(z) = g(z_0)$$

then, as long as all quantities are defined,

$$\lim_{z\to z_0} f(z) \pm g(z) = f(z_0) \pm g(z_0)$$
$$\lim_{z\to z_0} f(z)g(z) = f(z_0)g(z_0)$$
$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g(z_0)}$$

4 Differentiability

Definition 24 (Differentiable function)**.** Let $f(z)$ be defined in a neighbourhood of z_0. f is said to be differentiable at z_0 if the limit $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists.

Theorem 11 (Differentiation arithmetics)**.** *If $f(z)$ and $g(z)$ are differentiable, then, as long as all quantities are defined,*

$$\begin{aligned} (f(z) \pm g(z))' &= f'(z) \pm g'(z) \\ (f(z)g(z))' &= f'(z)g(z) + f(z)g'(z) \\ \left(\frac{f(z)}{g(z)}\right)' &= \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2} \end{aligned}$$

5 Cauchy-Riemann Equations

Theorem 12 (Cauchy-Riemann Equations)**.** *$u(x, y)$ and $v(x, y)$ are said to be satisfying Cauchy-Riemann Equations at a point $(a, b) \in \mathbb{R}^2$, if*

$$\begin{aligned} u_x(a, b) &= v_y(a, b) \\ u_y(a, b) &= -v_x(a, b) \end{aligned}$$

Theorem 13. *Let*

$$f(x + iy) = u(x, y) + iv(x, y)$$

Then, u and v satisfying the Cauchy-Riemann Equations is a necessary condition for f to be differentiable at (x_0, y_0).

Theorem 14. *If $f = u+iv$ is differentiable at $z_0 = a+ib$, then (u, v) satisfies the Cauchy-Riemann Equations at (a, b).*

Definition 25 (Analytic functions)**.** If $f = u + iv$ is differentiable at any $z \in W$, where W is a neighbourhood of z_0 except maybe at z_0, then f is said to be analytic at z_0. If f is analytic at all $z \in W$, then it is said to be analytic in W.

Exercise 8.
Let $f : U \to \mathbb{C}$ be an analytic function in U, such that $\overline{f}$ is also analytic in U. Show that $f' = 0$, i.e. $f = c$.

Solution 8.
As $f = u + iv$ is analytic, by Cauchy-Riemann Equations, for $(x, y) \in U$,

$$\begin{aligned} u_x(x, y) &= v_y(x, y) \\ u_y(x, y) &= -v_x(x, y) \end{aligned}$$

As $\overline{f} = u - iv$ is analytic, by Cauchy-Riemann Equations, for $(x, y) \in U$,

$$\begin{aligned} u_x(x, y) &= -v_y(x, y) \\ u_y(x, y) &= v_x(x, y) \end{aligned}$$

Therefore,

$$\begin{aligned} v_y &= -v_y \\ &= 0 \\ v_x &= -v_x \\ &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} u_x(x, y) &= 0 \\ u_y(x, y) &= 0 \end{aligned}$$

Therefore, u and v are constant functions.

6 Harmonic Functions

Definition 26 (Laplacian)**.** Let u be an equation in x and y.
The Laplacian is defined to be

$$\begin{aligned} \Delta u &= \nabla^2 u \\ &= u_{xx} + u_{yy} \end{aligned}$$

Definition 27 (Harmonic function)**.** A real function in two variables, $u(x, y)$, which is twice differentiable, is called a harmonic function if it satisfies

$$\begin{aligned} \Delta u &= u_{xx} + u_{yy} \\ &= 0 \end{aligned}$$

Theorem 15. *If u and v are twice differentiable, and satisfy Cauchy-Riemann Equations, then (u, v) are harmonic.*

Theorem 16 (Sufficient condition for differentiability)**.** *Let $f = u + iv$ be defined in a neighbourhood of $z_0 = a + ib$. Assume that u_x, u_y, v_x, v_y exist in this neighbourhood and are continuous at the point (a, b). If (u, v) satisfying Cauchy-Riemann Equations at (a, b) then $f'(z_0)$ exists.*

Definition 28 (Harmonic conjugate)**.** Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a harmonic function. Its harmonic conjugate is defined to be $v : \mathbb{R}^2 \to \mathbb{R}$, such that $f = u + iv$ is analytic.

7 Analytic Functions

Definition 29. $f : D \to \mathbb{C}$ is said to be differentiable on $D \subset \mathbb{C}$, if f is differentiable at any $z \in D$.

Definition 30 (Analytic functions)**.** If $f = u + iv$ is differentiable at any $z \in W$, where W is a neighbourhood of z_0 except maybe at z_0, then f is said to be analytic at z_0. If f is analytic at all $z \in W$, then it is said to be analytic in W.

Theorem 17. *Let $D \subset \mathbb{C}$ be an open set. Then, f is differentiable on D if and only if f is analytic on D.*

Theorem 18. *Let $D \subseteq \mathbb{C}$ be a domain. Assume that f is analytic on D, and for any $z \in D$, $f'(z) = 0$. Then, f is constant.*

Theorem 19. *Let $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $\nabla u = 0$ in a domain $D \subset \mathbb{R}^2$. Then, u is constant in D.*

Exercise 9.

1. Prove that
$$v(x, y) = \ln\left((x-1)^2 + (y-2)^2\right)$$
is harmonic in any domain that does not include the point $(1, 2)$.

2. Find $u(x, y)$ such that $u + iv$ is analytic in some domain. Note: v is the conjugate harmonic of u.

3. Express $u + iv$ as a function of z.

Solution 9.

1.

$$
\begin{aligned}
v_x &= \frac{2(x-1)}{(x-1)^2+(y-2)^2} \\
v_y &= \frac{2(y-2)}{(x-1)^2+(y-2)^2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
v_{xx} &= \frac{2\left((x-1)^2+(y-2)^2\right)-(2(x-1))^2}{\left((x-1)^2+(y-2)^2\right)^2} \\
v_{yy} &= \frac{2\left((x-1)^2+(y-2)^2\right)-(2(y-2))^2}{\left((x-1)^2+(y-2)^2\right)^2}
\end{aligned}
$$

2. For $u+iv$ to be analytic, by Cauchy-Riemann Equations,

$$
\begin{aligned}
u_x &= v_y \\
u_y &= -v_x
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
u_x &= v_y \\
&= \frac{2(y-2)}{(x-1)^2+(y-2)^2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
u &= \int \frac{2(y-2)}{(x-1)^2+(y-2)^2} \, \mathrm{d}x \\
&= \frac{2(y-2)}{(y-2)^2} \int \frac{1}{1+\left(\frac{x-1}{y-2}\right)^2} \, \mathrm{d}x \\
&= 2\tan^{-1}\left(\frac{x-1}{y-2}\right) + g(y)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
u_y &= -v_x \\
\therefore -\frac{2(x-1)}{(x-1)^2+(y-2)^2} &= \frac{2}{1+\frac{(x-1)^2}{(y-2)^2}}\left(-\frac{x-1}{y-2}\right) + g'(y)
\end{aligned}
$$

Therefore,

$$\begin{aligned} g'(y) &= 0 \\ \therefore g(y) &= c \end{aligned}$$

Therefore,

$$u = 2\tan^{-1}\left(\frac{x-1}{y-2}\right) + c$$

3.

$$\begin{aligned} u + iv &= \tan^{-1}\left(\frac{x-1}{y-2}\right) + i\ln\left((x-1)^2 + (y-2)^2\right) \\ &= 2i\,\mathrm{Log}\,(-i(x-1) + (y-2)) \\ &= 2i\,\mathrm{Log}\,(-iz - 2 + i) \end{aligned}$$

Exercise 10.
Prove that there is no $f = u + iv$ analytic in the unit disk, such that

$$xu(x, y) = yv(x, y) + 2013$$

Hint: Use the function $zf(z)$.

Solution 10.
If possible, let there exist $f(z)$ such that

$$xu(x, y) = yv(x, y) + 2013$$

Therefore, as $zf(z)$ is analytic,

$$\begin{aligned} zf(z) &= (x + iy)(u + iv) \\ &= xu - yv + i(yu + xv) \\ &= 2013 + i(yu + xv) \end{aligned}$$

By the polar form of Cauchy-Riemann Equations, $yu + xv$ is constant.
Therefore, $zf(z)$ is constant.
Therefore, this contradicts the assumption.
Therefore, such a f does not exist.

8 Elementary Functions

8.1 Exponential Functions

Theorem 20.

$$|e^z| = e^{\Re(z)}$$

Proof.

$$\begin{aligned}
|e^z| &= \left|e^{\Re(z)}\right| \left|e^{\Im(z)}\right| \\
&= \left|e^{\Re(z)}\right| \left|\cos\left(\Im(z)\right) + i \sin\left(\Im(z)\right)\right| \\
&= e^{\Re(z)}
\end{aligned}$$

□

Theorem 21. *Let z and w be complex. Then*

$$e^{z+w} = e^z e^w$$

Theorem 22. $\forall n \in \mathbb{Z}$,

$$(e^z)^n = e^{nz}$$

Theorem 23. *The function e^z is onto with respect to $\mathbb{C} \setminus \{0\}$.*

8.2 Trigonometric Functions

Definition 31 (Trigonometric functions of complex numbers)**.** Trigonometric functions of complex numbers are defined as

$$\begin{aligned}
\cos(z) &= \frac{e^{iz} + e^{-iz}}{2} \\
\sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} \\
\cosh(z) &= \frac{e^z + e^{-z}}{2} \\
\sinh(z) &= \frac{e^z - e^{-z}}{2}
\end{aligned}$$

8.3 Logarithmic Functions

Definition 32 (Power set)**.** The set of all subsets of a set is called the power set of the set. The power set of a set A is denoted as $\mathrm{P}(A)$.

Definition 33 (Multiple valued function)**.** A set which maps a set A to its power set $\mathrm{P}(A)$ is called a multiple valued set.

A multiple valued function gets over $\mathbb{C}$ gets a complex number as input and returns a set of complex numbers as output.

Definition 34 (Natural logarithmic function)**.** The natural logarithmic function over the complex plane is defined to be

$$\log w = \{z : e^z = w\}$$

Theorem 24.

$$\log w = \ln|w| + i\arg(w)$$

Proof. Let

$$\begin{aligned} e^z &= w \\ &= |w|e^{i\theta} \end{aligned}$$

where

$$\theta = \arg(w)$$

Therefore,

$$\begin{aligned} e^{\Re(z)+i\Im(z)} &= |w|e^{i\theta} \\ \therefore e^{\Re(z)}e^{i\Im(z)} &= |w|e^{i\theta} \end{aligned}$$

Therefore,

$$\begin{aligned} e^{\Re(z)} &= |w| \\ \Im(z) &= \theta + 2\pi k \end{aligned}$$

where $k \in \mathbb{Z}$.
Therefore,

$$\begin{aligned} \ln e^{\Re(z)} &= \ln|w| \\ \therefore \Re(z) &= \ln|w| \end{aligned}$$

Therefore,

$$\begin{aligned} \log w &= \{z : e^z = w\} \\ &= \{\ln|w| + iy : y = \arg(w)\} \end{aligned}$$

For any $w \in \log z$,

$$\begin{aligned} e^w &= e^{\ln|z|} + i\left(\operatorname{Arg} z + 2\pi k\right) \\ &= e^{\ln|z|} e^{i(\operatorname{Arg} z + 2\pi k)} \\ &= |z| e^{i \operatorname{Arg} z} \\ &= z \end{aligned}$$

$\square$

Definition 35 (Branch of $\log z$)**.** A branch of $\log z$ is a continuous function $\mathrm{L}(z)$ defined on a U, a connected open subset of $\mathbb{C}$ such that $\mathrm{L}(z)$ is a logarithm of z for each $z \in U$.

Definition 36 ($\operatorname{Log} z$)**.** $\operatorname{Log} z$ is defined to be

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$$

As $\operatorname{Arg} z$ is not continuous on the negative real axis, in order to make it continuous, the line $\operatorname{Arg} z = \pi$ is excluded. Hence, $\log z$ is continuous on $U = \mathbb{C} \setminus \{0\} \cup \mathbb{R}^-$, and is a branch of $\log z$.
Similarly, any other ray can be excluded in order to get a branch of $\log z$.

Definition 37. For any $\alpha \in \mathbb{R}$, $\operatorname{Log}_\alpha z$ is defined to be

$$\operatorname{Log}_\alpha z = \ln|z| + i \operatorname{Arg}_\alpha z$$

where $\operatorname{Arg}_\alpha z = \theta$, such that $\theta \in (\alpha, \alpha + 2\pi]$ and $\theta = \arg z$.
Any choice of $\operatorname{Arg}_\alpha z$ defines a branch of logarithm.

Definition 38 (Branch cut)**.** The boundary of the domain of a branch is called a branch cut.

Definition 39 (Principal value)**.** The value returned by $\operatorname{Log} z = \operatorname{Log}_{-\pi} z$ is called the principal value.

Theorem 25. $\operatorname{Log} z$ *is analytic on* $\mathbb{C} \setminus \{0\} \cup \mathbb{R}^-$.

Exercise 11.
Find the principal value of $\sqrt{i}$.

Solution 11.

$$\begin{aligned} \operatorname{pv}\left(i^{\frac{1}{2}}\right) &= e^{\frac{1}{2}\operatorname{Log} i} \\ &= e^{\frac{1}{2}\left(\ln|i| + i \operatorname{Arg} i\right)} \\ &= e^{\frac{1}{2} i \frac{\pi}{2}} \\ &= e^{i\frac{\pi}{4}} \end{aligned}$$

8.4 Power

Definition 40 (Power function)**.** Let $z, c \in \mathbb{C}$, such that $z \neq 0$. The power multifunction as

$$z^c = e^{c \log z}$$

The branch of the power multifunction for $c \in \mathbb{C}$ is defined as

$$z^w = e^{w \log z}$$

Theorem 26.

$$\operatorname{Log}_\alpha z - \operatorname{Log}_\beta z = i \left(\operatorname{Arg}_\alpha z - \operatorname{Arg}_\beta z \right)$$

Part V
Complex Integrals

1 Complex Integrals

Definition 41 (Integral of complex functions)**.** Let $f : [a, b] \to \mathbb{C}$. Let

$$f(t) = u(t) + iv(t)$$

Therefore, the integrals of $u(t)$ and $v(t)$ are defined as

$$\int\limits_a^b u(t)\,\mathrm{d}t = \lim_{\Delta t \to 0} \sum_{i=1}^{n} u(t_i)\Delta x_i$$

where T is a splitting of $[a, b]$, such that

$$a = t_1 < \cdots < t_n = b$$

and

$$\int\limits_a^b v(t)\,\mathrm{d}t = \lim_{\Delta t \to 0} \sum_{i=1}^{n} v(t_i)\Delta x_i$$

where T is a splitting of $[a, b]$, such that

$$a = t_1 < \cdots < t_n = b$$

These integrals are defined when the limit exists without depending on T. When they exist, the integral of $f(t)$ is defined as

$$\int\limits_a^b f(t)\,\mathrm{d}t = \int\limits_a^b u(t)\,\mathrm{d}t + i\int\limits_a^b v(t)\,\mathrm{d}t$$

Theorem 27. *All properties of real integrals are also valid for complex integrals.*

Theorem 28.

$$\left|\int\limits_a^b f(t)\,\mathrm{d}t\right| \le \int\limits_a^b |f(t)|\,\mathrm{d}t$$

2 Curves in $\mathbb{C}$

Definition 42. A continuous function $\gamma : [a, b] \to \mathbb{C}$ is called a curve.

Definition 43 (Parametric representation of a curve)**.** The curve $\gamma(t)$ can be represented as

$$\gamma(t) = x(t) + iy(t)$$

where t is a parameter.

Definition 44 (Differentiability)**.** γ is said to be differentiable if x and y are both differentiable.

Theorem 29 (Parametric representation of a straight line)**.** *Let $z_1, z_2 \in \mathbb{C}$. The straight line passing through z_1 and z_2 can represented parametrically as*

$$\gamma(t) = z_1 + t(z_2 - z_1)$$

The slope of this line is $z_1 - z_2$.

Theorem 30 (Parametric representation of a circle)**.** *A circle with radius r, centred at the origin, can be represented parametrically as*

$$\gamma(t) = re^{it}$$

with $0 \leq t \leq 2\pi$.

Exercise 12.
Parametrize the curve $\left\{z = x + iy : \frac{x^2}{4} + y^2 = 1\right\}$ starting from 2, and going anti-clockwise twice.

Solution 12.
The curve is an ellipse centred at $(0, 0)$, with $a = 2$, and $b = 1$.

$$\gamma(t) = 2\cos t + i \sin t$$

Therefore, as the curve goes anti-clockwise twice, $t \in [0, 4\pi]$.

Definition 45 (Simple curve)**.** A curve γ is said to be simple if it is non self-intersecting, i.e. it is one-to-one with respect to the parameter t, except maybe at the extreme values of t.

Definition 46 (Closed curve)**.** A curve $\gamma : [a, b] \to \mathbb{C}$ is said to be closed, if and only if

$$\gamma(a) = \gamma(b)$$

Definition 47 (Jordan curve)**.** A closed simple curve is called a Jordan curve.

Theorem 31. *A Jordan curve enclosed a region inside it.*

Definition 48 (Piecewise differentiability)**.** γ is said to be piecewise differentiable if there exists a splitting

$$a = t_1 < \cdots < t_n = b$$

such that γ is differentiable on each segment $[t_i, t_{i+1}]$.

3 Complex Line Integrals

Definition 49 (Complex line integral)**.** Let $\gamma : [a, b] \to \mathbb{C}$ be a curve, and let $f : D \to \mathbb{C}$, where $D \subseteq \mathbb{C}$, and $\gamma\left([a, b]\right) \subset D$. Then, the integral

$$\int_{\gamma} f(z)\,\mathrm{d}z = \int_{a}^{b} f\left(\gamma(t)\right)\dot{\gamma}(t)\,\mathrm{d}t$$

If γ is piecewise differentiable, then

$$\int_{\gamma} f(z)\,\mathrm{d}z = \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} f\left(\gamma(t)\right)\dot{\gamma}(t)\,\mathrm{d}t$$

Definition 50 (Oriented contour)**.** An oriented contour for $\alpha > 0$, $z_0 \in \mathbb{C}$, is defined to be

$$C_{\alpha,z_0} = \{w \in \mathbb{C} : |w - z_0| = \alpha\}$$

oriented anti-clockwise, starting at $z_0 + \alpha$.

Theorem 32. $\forall \alpha > 0,\ z_0 \in \mathbb{C}$,

$$\oint_{C_{\alpha,z_0}} \frac{\mathrm{d}z}{z - z_0} = 2\pi i$$

Proof. Let

$$\gamma(t) = z_0 + \alpha e^{it}$$

with $0 \leq t \leq 2\pi$.
Therefore,

$$\dot{\gamma}(t) = \alpha i e^{it}$$

Therefore,

$$\begin{aligned}
\oint\limits_{C_{\alpha,z_0}} \frac{\mathrm{d}z}{z - z_0} &= \int\limits_0^{2\pi} \frac{1}{z_0 + \alpha e^{it} - z_0} \alpha i e^{it} \, \mathrm{d}t \\
&= \int\limits_0^{2\pi} i \, \mathrm{d}t \\
&= 2\pi i
\end{aligned}$$

$\square$

Theorem 33. *Line integrals are linear for all $\alpha, \beta \in \mathbb{C}$, i.e.*

$$\alpha \int\limits_\gamma f \, \mathrm{d}z \pm \beta \int\limits_\gamma g \, \mathrm{d}z = \int\limits_\gamma \alpha f \pm \beta g \, \mathrm{d}z$$

Theorem 34. *Let γ_1 and γ_2 be two curves such that the start point of γ_2 is the end point of γ_1. Then, the curves can be composited to a curve $\gamma_1 + \gamma_2$, and*

$$\int\limits_{\gamma_1} f(z) \, \mathrm{d}z + \int\limits_{\gamma_2} f(z) \, \mathrm{d}z = \int\limits_{\gamma_1 + \gamma_2} f(z) \, \mathrm{d}z$$

Theorem 35. *Let $\gamma : [a, b] \to \mathbb{C}$ be a curve. Then, $\overline{\gamma} : [-b, -a] \to \mathbb{C}$ has orientation opposite to that of γ, and*

$$\begin{aligned}
\overline{\gamma}(t) &= \gamma(-t) \\
\dot{\overline{\gamma}}(t) &= -\dot{\gamma}(t)
\end{aligned}$$

Then,

$$\int\limits_{\overline{\gamma}} f(z) \, \mathrm{d}z = -\int\limits_\gamma f(z) \, \mathrm{d}z$$

Theorem 36 (Length of a curve)**.** *The length of the curve $\gamma : [a, b] \to \mathbb{C}$ is given by*

$$\text{length}(\gamma) = \int\limits_a^b |\dot{\gamma}(t)| \,\mathrm{d}t$$

Exercise 13.
Find the length of the astroid given by

$$\gamma(t) = \cos^3 t + i \sin^3 t$$

where $\gamma : [0, 2\pi] \to \mathbb{C}$.

Solution 13.

$$\begin{aligned}
\gamma(t) &= \cos^3 t + i \sin^3 t \\
\therefore \dot{\gamma}(t) &= -3 \sin t \cos^2 t + 3i \cos t \sin^2 t \\
\therefore |\dot{\gamma}(t)| &= \sqrt{9 \left(\cos^4 t \sin^2 t + \sin^4 t \cos^2 t\right)} \\
&= 3|\sin t \cos t| \sqrt{\cos^2 t + \sin^2 t} \\
&= 3|\sin t \cos t|
\end{aligned}$$

Therefore,

$$\begin{aligned}
\text{length}(\gamma) &= \int\limits_a^b |\dot{\gamma}(t)| \,\mathrm{d}t \\
&= 3 \int\limits_0^{2\pi} |\sin t \cos t| \,\mathrm{d}t \\
&= 12 \int\limits_0^{\frac{\pi}{2}} \sin t \cos t \,\mathrm{d}t \\
&= 6 \int\limits_0^{\frac{\pi}{2}} \sin 2t \,\mathrm{d}t \\
&= 6
\end{aligned}$$

Theorem 37. *Let $f(z)$ be a function defined in a domain D including a curve γ. Let $\exists M > 0$, such that all values of f have $|f(z)| \leq M$, then*

$$\left| \int\limits_{\gamma} f(z) \, \mathrm{d}z \right| \leq M \, \mathrm{length}(\gamma)$$

Definition 51 (Primitive function)**.** Let $D \subset \mathbb{C}$. $F(z)$ is said to be the primitive function of $f(z)$ in D, if $\forall z \in D$,

$$F'(z) = f(z)$$

Theorem 38 (Fundamental Theorem of Calculus)**.** *Let $\gamma : [a, b] \to \mathbb{C}$ be piecewise continuous, and let f be continuous on γ, i.e. $f \circ \gamma$ is continuous. Let there exist an analytic function F, defined on a domain including γ, such that $\forall z \in \gamma$,*

$$F'(z) = f(z)$$

Then,

$$\int\limits_{\gamma} f(z) \, \mathrm{d}z = F\left(\gamma(b)\right) - F\left(\gamma(a)\right)$$

Theorem 39 (Equivalent conditions for existence of a primitive function)**.** *Let D be a domain. Let f be continuous on D. Then, the following conditions are equivalent.*

1. *f has a primitive function F in D.*

2. *For any closed path γ such that $\gamma \subset D$,*

   $$\int\limits_{\gamma} f(z) \, \mathrm{d}z = 0$$

3. *For any curve γ such that $\gamma \subset D$, the integral $\int\limits_{\gamma} f(z) \, \mathrm{d}z$ depends only on the edges of γ.*

Exercise 14.
Find $\int\limits_{\gamma} \cos z \, \mathrm{d}z$ where γ goes from π to i.

Solution 14.
$\sin z$ is the primitive of $\cos z$ over $\mathbb{C}$.
Therefore,

$$
\begin{aligned}
\int\limits_{\gamma} \cos z \, \mathrm{d}z &= \sin i - \sin \pi \\
&= \frac{e^{i^2} - e^{-i^2}}{2i} - 0 \\
&= \frac{e^{-1} - e}{2i} \\
&= i \frac{-\frac{1}{e} + e}{2}
\end{aligned}
$$

Exercise 15.
Calculate the integral of

$$f(z) = (z - z_0)^n$$

$\forall n \in \mathbb{Z}$, where $\gamma = C_{R,z_0}$.

Solution 15.
For $0 \leq t \leq 2\pi$,

$$
\begin{aligned}
\gamma(t) &= z_0 + Re^{it} \\
\therefore \dot{\gamma}(t) &= Rie^{it}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int\limits_{\gamma} (z - z_0)^n \, \mathrm{d}z &= \int\limits_0^{2\pi} \left(z_0 + Re^{it} - z_0\right)^n \left(Rie^{it}\right) \mathrm{d}t \\
&= iR^{n+1} \int\limits_0^{2\pi} e^{i(n+1)t} \, \mathrm{d}t
\end{aligned}
$$

Therefore,

$$\int_\gamma (z-z_0)^n \, \mathrm{d}z = \begin{cases} 2\pi i & ; \quad n=-1 \\ \frac{R^{n+1}}{n+1} e^{i(n+1)t}\Big|_0^{2\pi} & ; \quad n \neq -1 \end{cases}$$
$$= \begin{cases} 2\pi i & ; \quad n=-1 \\ 0 & ; \quad n \neq -1 \end{cases}$$

Theorem 40.

$$\int_\gamma P \, \mathrm{d}x + Q \, \mathrm{d}y = \int_a^b \Big(P\left(\gamma(t)\right) \dot{x}(t) + Q\left(\gamma(t)\right) \dot{y}(t) \Big) \, \mathrm{d}t$$

where $t \in [a,b]$.

Theorem 41. *If*

$$f = u + iv$$

then,

$$\int_\gamma f(z) \, \mathrm{d}z = \int_\gamma u \, \mathrm{d}x - v \, \mathrm{d}y + i \int_\gamma v \, \mathrm{d}x + u \, \mathrm{d}y$$

Theorem 42 (Green's Theorem)**.** *Let*

$$F = P \, \mathrm{d}x + Q \, \mathrm{d}y$$

such that P_x*,* P_y*,* Q_x*,* Q_y *are continuous in the domain* D*,*

$$\int_{\partial D} P \, \mathrm{d}x + Q \, \mathrm{d}y = \iint_D \Big(Q_x - P_y \Big) \, \mathrm{d}x \, \mathrm{d}y$$

Theorem 43 (Cauchy-Goursat Theorem)**.** *Let* D *be a domain, such that* ∂D *is obtained by a finite number of curves, ie.* ∂D *is piecewise differentiable. If* $f : \overline{D} \to \mathbb{C}$ *is analytic, then*

$$\int_{\partial D} f(z) \, \mathrm{d}z = 0$$

4 Cauchy Integral Formula

Theorem 44 (Cauchy Integral Formula/Mean Value Theorem)**.** *Let C be a simple closed curve in positive orientation with respect to a domain, D_C, closed by a curve C. If f is analytic in D_C, then*

$$f(z_0) = \frac{1}{2\pi i} \int\limits_C \frac{f(z)}{z - z_0} \, \mathrm{d}z$$

Theorem 45 (Cauchy Differentiation Formula)**.** *Let C be a simple closed curve in positive orientation with respect to a domain, D_C, closed by a curve C. If f is analytic in D_C, then*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int\limits_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z$$

Theorem 46. *If f is analytic in D, then f is infinitely differentiable.*

Proof. Let $z_0 \in D$. Therefore, $\exists \varepsilon > 0$, such that $D(z_0, \varepsilon) \in D$. Therefore, by Cauchy Differentiation Formula,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int\limits_{C_{z_0,\varepsilon}} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}x$$

and particularly, exists. □

Theorem 47 (Morera's Theorem)**.** *Let D be a domain, and let $f : D \to \mathbb{C}$ be continuous. If $\int\limits_\gamma f(z) \, \mathrm{d}z = 0$, for any closed curve γ, such that $\gamma \in D$, then f is analytic in D*

Proof. By Equivalent conditions for existence of a primitive function, as

$$\int\limits_\gamma f(z) \, \mathrm{d}z = 0$$

there exists a primitive function F for f, i.e.,

$$F'(z) = f(z)$$

for all $z \in D$.
Therefore, as F is differentiable in D, and as D is a domain, and hence is open, F is analytic.
Therefore, as F is analytic in D, F is infinitely differentiable, with analytic derivatives. □

Theorem 48 (Cauchy Derivative Estimate)**.** *Let f be analytic in $D_{z_0,r}$. Let $\partial D_{z_0,r}$ be denoted as $C_{z_0,r}$.*
Let

$$M_R = \max_{z \in C_{z_0,R}} |f(z)|$$

Then, $\forall n \in \mathbb{N}$,

$$\left| f^{(n)}(z_0) \right| \leq \frac{n! M_R}{R^n}$$

Exercise 16.
Find $\int\limits_{-\pi}^{\pi} \frac{1}{2-\cos t} \, \mathrm{d}t$.

Solution 16.
Let

$$\begin{aligned} z &= e^{it} \\ \therefore \mathrm{d}z &= iz \, \mathrm{d}t \end{aligned}$$

$$\begin{aligned} \int\limits_{-\pi}^{\pi} \frac{1}{2-\cos t} \, \mathrm{d}t &= \int\limits_{\partial D_{0,1}} \frac{1}{2 - \frac{z+z^{-1}}{2}} \frac{\mathrm{d}z}{iz} \\ &= \int\limits_{\partial D_{0,1}} \frac{2 \, \mathrm{d}z}{(4 - z - z^{-1}) \, iz} \\ &= \int\limits_{\partial D_{0,1}} \frac{2 \, \mathrm{d}z}{-i \left(z^2 - 4z + 1\right)} \\ &= \int\limits_{\partial D_{0,1}} \frac{2 \, \mathrm{d}z}{i \left(z - 2 + \sqrt{3}\right) \left(z - 2 - \sqrt{3}\right)} \\ &= 2i \int\limits_{\partial D_{0,1}} \frac{\mathrm{d}z}{\left(z - 2 + \sqrt{3}\right) \left(z - 2 - \sqrt{3}\right)} \end{aligned}$$

Let

$$\begin{aligned} z_1 &= 2 + \sqrt{3} \\ z_2 &= 2 - \sqrt{3} \end{aligned}$$

Therefore, as $z_1 \in D_{0,1}$, by Cauchy Integral Formula/Mean Value Theorem,

$$\begin{aligned}
\int\limits_{-\pi}^{\pi} \frac{1}{2-\cos t}\,\mathrm{d}t &= 2i \int\limits_{\partial D_{0,1}} \frac{\mathrm{d}z}{\left(z-2+\sqrt{3}\right)\left(z-2-\sqrt{3}\right)} \\
&= 2i\left(2\pi i\left(\frac{1}{z-2-\sqrt{3}}\right)\right)\Bigg|_{z=2-\sqrt{3}} \\
&= -4\pi\left(\frac{1}{2-\sqrt{3}-2-\sqrt{3}}\right) \\
&= \frac{2\pi}{\sqrt{3}}
\end{aligned}$$

Therefore, the integral is real, which is expected, as the function is real.

Exercise 17.
Calculate $\int\limits_{C_{1,3}} \frac{\cos z}{(z-i)^3}\,\mathrm{d}z$.

Solution 17.

$$\begin{aligned}
\int\limits_{C_{1,3}} \frac{\cos z}{(z-i)^{2+1}}\,\mathrm{d}z &= \frac{2\pi i}{2} \cos z\big|_{z=1} \\
&= -i\pi\cos(i) \\
&= -i\pi\frac{e^{-1}+e^{1}}{2} \\
&= -i\pi\cosh(1)
\end{aligned}$$

5 Liouville's Theorem

Theorem 49 (Liouville's Theorem)**.** *If f is entire and bounded, then f is constant.*

Exercise 18.
If f is entire, such that $\forall z \in \mathbb{C}$, $\Re\left(f(z)\right) < M$, show that it is constant.

Solution 18.
As $e^{\Re\left(f(x)\right)} < M$,

$$\left|e^{f(z)}\right| = e^{\Re\left(f(z)\right)}$$
$$\therefore \left|e^{f(z)}\right| < e^{M}$$

Therefore, $e^{\Re\left(f(z)\right)}$ is an entire and bounded function. Therefore, by Liouville's Theorem, $e^{f(z)}$ is constant.
Let

$$e^{f(z)} = c$$

Therefore,

$$f(z) = \ln|c| + 2\pi ki$$

Therefore, even though k may be dependent on z, as $f(z)$ is continuous, k must be continuous, to ensure that there is no discontinuity in $f(z)$. Therefore, $f(z)$ is constant.

Exercise 19.
Let f be entire and periodic, with two periods, 1 and i, i.e. $\forall z \in \mathbb{C}$,

$$\begin{aligned} f(z) &= f(z+1) \\ &= f(z+i) \end{aligned}$$

Then, f is constant.

Solution 19.
Let

$$D = \{z : 0 \leq \Re(z) \leq 1, 0 \leq \Im(z) \leq 1\}$$

be a compact set.
f is continuous over D, and hence, $|f|$ is also continuous over D.
Therefore, by Weierstrass theorem, f is bounded in D.
As the function is periodic with periods 1 and i,

$$f(x+iy) = f\left(x - \lfloor x \rfloor + i\left(y - \lfloor y \rfloor\right)\right)$$
$$\therefore f(D) = f(\mathbb{C})$$

Therefore, f is bounded in $\mathbb{C}$, and by Liouville's Theorem, it is constant.

6 Fundamental Theorem of Algebra

Theorem 50. *$\exists R > 0$, such that, $\forall |z| > R$,*

$$\begin{aligned}|\rho(z)| &= \left|\sum_{k=0}^{n} a_k z^k\right| \\ &\geq \frac{|a_n||z|^n}{2}\end{aligned}$$

Theorem 51 (Fundamental Theorem of Algebra)**.** *Any polynomial $p(z)$, of degree $n \geq 1$, over $\mathbb{C}$ has at least one root in $\mathbb{C}$, i.e. $\exists z_0$, such that*

$$p(z_0) = 0$$

Proof. If possible, $\forall z \in \mathbb{C}$, let

$$p(z) \neq 0$$

As $p(z)$ is a polynomial, it is an entire function.
Therefore,

$$f(z) = \frac{1}{p(z)}$$

is also entire.
Therefore, $\exists R > 0$, such that $\forall |z| > R$,

$$\begin{aligned}|p(z)| &\geq \frac{|a_n||z|^n}{2} \\ \therefore |p(z)| &\geq \frac{|a_n|R^n}{2}\end{aligned}$$

Therefore, $\forall |z| > R$,

$$\begin{aligned}|f(z)| &= \frac{1}{|p(z)|} \\ \therefore |f(z)| &\leq \frac{1}{\frac{|a_n|R^n}{2}}\end{aligned}$$

Let

$$\begin{aligned}m_1 &= \frac{1}{\frac{|a_n|R^n}{2}} \\ &= \frac{2}{|a_n|R^n}\end{aligned}$$

Therefore, $\forall |z| > R$,

$$|f(z)| \leq m_1$$

Let the closed disk D be

$$D = \{z : |z| \leq R\}$$

Therefore, f is continuous in D. Hence, $|f|$ is also continuous in D.
By Weierstrass theorem, $|f|$ is bounded in D.
Therefore, let

$$|f(z)| \leq m_2$$

Therefore, $\forall z \in \mathbb{C}$,

$$|f(z)| \leq \max\{m_1, m_2\}$$

Therefore, as $f(z)$ is entire and bounded, by Liouville's Theorem, it is constant.
Therefore,

$$p(z) = \frac{1}{f(z)}$$

is constant. Hence, the degree of $p(z)$ is 0.
This contradicts the assumption the condition of $n \geq 1$. Hence, $p(z)$ has at least one root in $\mathbb{C}$. $\square$

Theorem 52. *Any polynomial of degree $n \geq 1$ has exactly n roots, not necessarily distinct. Particularly,*

$$p(z) = a_n \prod_{k=1}^{n} (z - z_k)$$

where each z_k is a root of $p(z)$.

7 Maximum Modulus Principle

Theorem 53. *Let f be an analytic function in a domain D, and $\forall z \in D_{z_0,\varepsilon} \subset D$, let*

$$|f(z)| \leq |f(z_0)|$$

Then, f is constant on $D_{z_0,\varepsilon}$, i.e., $\forall z \in D_{z_0,\varepsilon}$,

$$f(z) = f(z_0)$$

Proof. For $\rho < \varepsilon$, let

$$C_\rho = \{z : |z - z_0| = \rho\}$$

Therefore, f is analytic inside and on C_ρ.
Therefore, by Cauchy Integral Formula/Mean Value Theorem,

$|f(z)| \leq |f(z_0)|$

$$\begin{aligned}
|f(z_0)| &= \left| \frac{1}{2\pi i} \int\limits_{C_\rho} \frac{f(z)}{z - z_0} \, \mathrm{d}z \right| \\
&= \left| \frac{1}{2\pi i} \int\limits_0^{2\pi} \frac{f\left(z_0 + \rho e^{it}\right)}{z_0 + \rho e^{it} - z_0} i\rho e^{it} \, \mathrm{d}t \right| \\
&= \left| \frac{1}{2\pi} \int\limits_0^{2\pi} f\left(z_0 + \rho e^{it}\right) \mathrm{d}t \right| \\
&\leq \frac{1}{2\pi} \int\limits_0^{2\pi} \left| f\left(z_0 + \rho e^{it}\right) \right| \mathrm{d}t \\
&\leq \frac{1}{2\pi} \int\limits_0^{2\pi} |f(z_0)| \, \mathrm{d}t
\end{aligned}$$

Also,

$$|f(z_0)| \geq \left| f\left(z_0 + \rho e^{it}\right) \right|$$

$$\therefore |f(z_0)| \geq \frac{1}{2\pi} \int\limits_0^{2\pi} \left| f\left(z_0 + \rho e^{it}\right) \right| \mathrm{d}t$$

Therefore,

$$|f(z_0)| = \frac{1}{2\pi} \int\limits_0^{2\pi} \left| f\left(z_0 + \rho e^{it}\right) \right| \mathrm{d}t$$

$$\therefore \frac{1}{2\pi} \int\limits_0^{2\pi} |f(z_0)| \, \mathrm{d}t = \frac{1}{2\pi} \int\limits_0^{2\pi} \left| f\left(z_0 + \rho e^{it}\right) \right| \mathrm{d}t$$

$$\therefore 0 = \frac{1}{2\pi} \int\limits_0^{2\pi} \left(|f(z_0)| - \left| f\left(z_0 + \rho e^{it}\right) \right| \right) \mathrm{d}t$$

Therefore,

$$|f(z_0)| - \left|f\left(z_0 - \rho e^{it}\right)\right| \geq 0$$

Therefore, as the integral this non-negative expression is zero, the expression must be zero. Hence,

$$|f(z_0)| = \left|f\left(z_0 + \rho e^{it}\right)\right|$$

Similarly, by Cauchy-Riemann Equations, if $\forall z \in D_{z_0,\varepsilon}$,

$$|f(z_0)| = |f(z)|$$

then

$$f(z_0) = f(z)$$

$\square$

Theorem 54 (Maximum Modulus Principle)**.** *Let f be analytic in D and continuous on ∂D, and non-constant, then f has no local maximum in D, and the global maximum in $\overline{D}$, i.e. the closer of D, must be on ∂D.*

Exercise 20.
Find the maximum of

$$f(z) = e^z$$

in $\{z : |z| \leq 3\}$.

Solution 20.
$f(z)$ is entire and hence analytic in $D_{0,3}$. Also, it is non-constant. Hence, by Maximum Modulus Principle, the global maximum must be on $\{z : |z| < 3\}$.
Let

$$\gamma(t) = 3e^{it}$$

where $0 \leq t \leq 2\pi$.
Therefore, $\forall z \in \partial D$,

$$\begin{aligned}
|e^z| &= \left|e^{3e^{it}}\right| \\
&= \left|e^{3(\cos t + i\sin t)}\right| \\
&= \left|e^{3\cos t}\right|\left|e^{3i\sin t}\right| \\
&= e^{3\cos t} \\
&\leq e^3
\end{aligned}$$

Therefore, $z = 3$ is the global maximum.

Theorem 55 (Minimum Modulus Principle)**.** *If f is analytic in D, continuous on ∂D such that $\forall z \in D$, $f(z) \neq 0$, then show that f has a global minimum in ∂D.*

Proof. As $f(z) \neq 0$, let

$$g(z) = \frac{1}{f(z)}$$

Therefore, by Maximum Modulus Principle, $g(z)$ has a global maximum in ∂D, which corresponds to the global minimum of $f(z)$. □

Exercise 21.
Let D be a bounded domain and f be a non-constant, analytic function in $\overline{D}$, the closer of D, such that $\forall z \in \partial D$,

$$|f(z)| = 1$$

Prove that $\exists z_0 \in D$, such that

$$f(z_0) = 0$$

Solution 21.
By Maximum Modulus Principle, $\forall z \in D$,

$$|f(z)| \leq 1$$

If possible, $\forall z \in D$, let

$$f(z) \neq 0$$

Therefore, by Minimum Modulus Principle,

$$|f(z)| \geq 1$$

Therefore,

$$|f(z)| = 1$$

Therefore, by Cauchy-Riemann Equations, f is constant.
This contradicts that f is non-constant. Therefore, $\exists z_0 \in D$, such that

$$f(z_0) = 0$$

Exercise 22.
Let f be analytic on

$$D = \{z : |z| < 1\}$$

a and on ∂D.
Assuming $\forall z \in D$,

$$|f(z)| \leq \left|f\left(z^2\right)\right|$$

show that f is constant.

Solution 22.
Let $0 < r < 1$. Let

$$D_r = \{z : |z| \leq r\}$$

Therefore,

$$D_{r^2} = \left\{z : |z| \leq r^2\right\}$$

Therefore, as $0 < r < 1$,

$$D_{r^2} \subset D_r$$

As $|f(z)| \leq \left|f\left(z^2\right)\right|$, by Maximum Modulus Principle,

$$\max_{D_r} |f(z)| \leq \max_{D_{r^2}} |f(z)|$$

As $D_{r^2} \subset D_r$,

$$\max_{D_{r^2}} |f(z)| \leq \max_{D_r} |f(z)|$$

Therefore,

$$\max_{D_r} |f(z)| = \max_{D_{r^2}} |f(z)|$$

Therefore, the maximum $|f(z)|$ on D_r is at a point in the interior of D_r. Therefore, by Maximum Modulus Principle, f is constant on D_r. Therefore, as $0 < r < 1$, f is constant on D.

Part VI
Complex Sequences and Series

1 Complex Series

Definition 52 (Convergence of complex series)**.** The complex series $\sum z_n$ is said to converge to L, if and only if

$$\begin{aligned}\lim_{N\to\infty} S_N &= \lim_{N\to\infty}\sum_{n=0}^{N} z_n \\ &= L\end{aligned}$$

Theorem 56. *If*

$$z_n = x_n + iy_n$$

then,

$$\sum_{n=0}^{\infty} z_n = \sum_{n=0}^{\infty} x_n + i\sum_{n=0}^{\infty} y_n$$

Definition 53 (Absolute convergence of complex series)**.** The series $\sum\limits_{n=1}^{\infty} z_n$ is said to converge absolutely, if

$$\sum_{n=1}^{\infty} |z_n| < \infty$$

2 Series of Complex Functions

Theorem 57. *If a series converges converges absolutely, then it also converges.*

Definition 54 (Pointwise convergence of series of functions)**.** Let $f_n : \Omega \to \mathbb{C}$, where $\Omega \subseteq \mathbb{C}$. The series $\sum\limits_{n=0}^{\infty} f_n$ is said to converge pointwise to $f \in \Omega$, if $\forall z \in \Omega$,

$$\sum_{n=0}^{\infty} f_n(z) = f(z)$$

Definition 55 (Uniform convergence of series of functions)**.** Let $f_n : \Omega \to \mathbb{C}$, where $\Omega \subseteq \mathbb{C}$. The series $\sum\limits_{n=0}^{\infty} f_n$ is said to converge uniformly to $f \in \Omega$, if

$$\lim_{N\to\infty} \sup_{z\in\Omega} |S_N(z) - f(z)| = 0$$

where

$$S_N(z) = \sum_{n=0}^{N} z_n$$

2.1 Criteria for Uniform Convergence of Series of Functions

Theorem 58 (Weierstrass M-test)**.** *Let $f_n : \Omega \to \mathbb{C}$, where $\Omega \subseteq \mathbb{C}$. Let $M_n \geq 0$ be a sequence which converges, such that, $\forall z \in \Omega$,*

$$|f_n(z)| \leq M_n$$

Then $f_n(z)$ converges uniformly in Ω.

3 Power Series

Definition 56 (Power series)**.** A series of the form $\sum\limits_{n=0}^{\infty} a_n(z - z_0)^n$ is called a power series. All a_n are called the coefficients, and z_0 is called the centre.

Theorem 59. *A power series*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

converges in a disk $\{z : |z - z_0| < R\}$ and diverges in $\{z : |z - z_0| > R\}$, where

$$\frac{1}{R} = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}$$

Also, the series converges uniformly in the set $\{z : |z - z_0| < R'\}$, $\forall R'$, such that $0 < R' < R$.

3.1 Integration of Power Series

Theorem 60. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

be convergent in $D_{z_0,R}$.
Let Γ *be a curve in* $D_{z_0,R}$.
Let $g(z) : \Gamma \to \mathbb{C}$ *be continuous in* Γ.
Then,

$$\int_{\Gamma} g(z) f(z) \, \mathrm{d}z = \sum_{n=0}^{\infty} a_n \int_{\Gamma} g(z)(z - z_0)^n \, \mathrm{d}z$$

Theorem 61. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

be convergent in $D_{z_0,R}$.
Let Γ *be a curve in* $D_{z_0,R}$.
If

$$\begin{aligned} \int_{\Gamma} f(z) \, \mathrm{d}z &= \sum_{n=0}^{\infty} a_n \int_{\Gamma} (z - z_0)^n \, \mathrm{d}z \\ &= 0 \end{aligned}$$

then f *is analytic in* $D_{z_0,R}$.

3.2 Differentiation of Power Series

Theorem 62. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Then, in $D_{z_0,R}$,

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

where

$$\frac{1}{R} = \lim_{n \to \infty} \sup |a_n|^{\frac{1}{n}}$$

Theorem 63. *All functions of the form* $\frac{1}{n^z}$*, which converge uniformly, are analytic.*

Definition 57 (Riemann zeta function)**.** The Riemann zeta function is defined to be

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

Exercise 23.
Show that $\zeta(z)$, the Riemann zeta function is analytic in $\{z : \Re(z) > 1\}$.

Solution 23.

$$\begin{aligned}
\zeta(z) &= \left|\sum_{n=1}^{\infty} \frac{1}{n^z}\right| \\
&\leq \sum_{n=1}^{\infty} \left|\frac{1}{n^z}\right| \\
&\leq \sum_{n=1}^{\infty} \left|\frac{1}{n^{x+iy}}\right| \\
&\leq \sum_{n=1}^{\infty} \left|\frac{1}{n^x n^{iy}}\right| \\
&\leq \sum_{n=1}^{\infty} \frac{1}{n^x}
\end{aligned}$$

Let $\varepsilon > 0$.
Let

$$M_n = \frac{1}{n^{1+\varepsilon}}$$

Therefore, for $z \in \{z : \Re(z) > 1 + \varepsilon\}$, as $\left\{M_n = \frac{1}{n^{1+\varepsilon}}\right\}$ converges, and as

$$\frac{1}{n^z} \leq \frac{1}{n^{1+\varepsilon}}$$

by the Weierstrass M-test, $\zeta(z)$ converges in $\{z : \Re(z) \geq 1 + \varepsilon\}$. As this holds for all $\varepsilon > 0$, $\zeta(z)$ is also analytic in $\{z : \Re(z) > 1\}$.

4 Taylor Series for Complex Functions

Theorem 64 (Taylor Series for Complex Functions)**.** *Let f be analytic in $D_{z_0,R}$. Then,*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$\begin{aligned} a_n &= \frac{f^{(n)}(z_0)}{n!} \\ &= \frac{1}{2\pi i} \int\limits_{\partial D_{z_0,R'}} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z \end{aligned}$$

where $R' < R$.

Theorem 65 (First Uniqueness Theorem)**.** *Let f and g be analytic functions in a domain D, such that for $z_0 \in D$, $\forall n \in \mathbb{N}$,*

$$f^{(n)}(z_0) = g^{(n)}(z_0)$$

Then,

$$f(z) = g(z)$$

in D.

Theorem 66 (Second Uniqueness Theorem)**.** *Let f and g be analytic functions in a domain D. Let there exist $\{z_n\}_{n=1}^{\infty} \subset D$ which converges to $z_0 \in D$, such that $\forall n$,*

$$f(z_n) = g(z_n)$$

Then,

$$f(z) = g(z)$$

in D.

Proof. As f and g are analytic in D, they are continuous in D. Therefore,

$$\lim_{n \to \infty} f(z_n) = f(z_0)$$
$$\lim_{n \to \infty} g(z_n) = g(z_0)$$

As $\forall n$,

$$f(z_n) = g(z_n)$$

Let

$$f(z_0) = a_0$$
$$g(z_0) = a_0$$

Therefore, let

$$\lim_{n\to\infty} \frac{f(z_n) - f(z_0)}{z_n - z_0} = f'(z_0)$$
$$= a_1$$

Therefore,

$$\lim_{n\to\infty} \frac{g(z_n) - g(z_0)}{z_n - z_0} = f'(z_0)$$
$$= a_1$$

Similarly, let

$$f''(z_0) = \frac{f(z_n) - a_0 - a_1(z_n - z_0)}{(z_n - z_0)^2}$$
$$= a_2$$
$$g''(z_0) = \frac{g(z_n) - a_0 - a_1(z_n - z_0)}{(z_n - z_0)^2}$$
$$= a_2$$

Therefore,

$$\frac{f(z_n) - \sum_{k=0}^{N} a_k(z_n - z_0)^k}{(z_n - z_0)^{N+1}}$$
$$= \frac{f^{(N+1)}(z_0)}{(N+1)!}$$
$$= a_{N+1}$$
$$\frac{g(z_n) - \sum_{k=0}^{N} a_k(z_n - z_0)^k}{(z_n - z_0)^{N+1}}$$
$$= \frac{g^{(N+1)}(z_0)}{(N+1)!}$$
$$= a_{N+1}$$

Therefore the Taylor series coefficients of f and g are equal. Therefore,

$$f = g$$

in D. $\square$

Exercise 24.
Let $f(z)$ be analytic in $D_{0,1}$, such that $\forall n \in \mathbb{N} \geq 2$,

$$f\left(\frac{1}{n}\right) = \frac{1}{n}$$

Find $f(z)$.

Solution 24.
$\forall n \in \mathbb{N} \geq 2$,

$$\left|\frac{1}{n}\right| \leq 1$$

Therefore,

$$\left\{\frac{1}{n}\right\} \subset D_{0,1}$$

The limit of the sequence is

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

Therefore, the sequences converges to 0.
Let

$$\begin{aligned} g(z) &= z \\ \therefore g\left(\frac{1}{n}\right) &= \frac{1}{n} \end{aligned}$$

Therefore, by the Second Uniqueness Theorem,

$$\begin{aligned} f(z) &= g(z) \\ &= z \end{aligned}$$

5 Loren Series

Theorem 67 (Loren Theorem)**.** *Let f be analytic in an annulus $r < |z - z_0| < R$. Let C be a simple closed curve around z_0, with positive orientation, inside the annulus. Then, f has a unique Loren series around z_0, which converges to f in this ring, i.e.,*

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \\ &= \sum_{n=1}^{\infty} \frac{c_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} c_n (z - z_0)^n \end{aligned}$$

where

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z$$

and

$$\begin{aligned} r &= \lim_{n\to\infty} \sup \sqrt[n]{|c_n|} \\ \frac{1}{R} &= \lim_{n\to\infty} \sup \sqrt[n]{|c_n|} \end{aligned}$$

Exercise 25.

$$f(z) = -\frac{1}{(z-1)(z-2)}$$

Find the Loren series of $f(z)$ around $z = 0$.

Solution 25.

f is analytic everywhere except at $z = 1$ and $z = 2$.
For $|z| < 1$, converting to partial fractions,

$$\begin{aligned} -\frac{1}{(z-1)(z-2)} &= \frac{1}{z-1} + \frac{-1}{z-2} \\ &= -\frac{1}{1-z} + \frac{1}{2}\frac{1}{1-\frac{z}{2}} \\ &= -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^2 \\ &= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}} - 1\right) z^n \end{aligned}$$

For $1 < |z| < 2$, converting to partial fractions,

$$\begin{aligned} -\frac{1}{(z-1)(z-2)} &= \frac{1}{z}\frac{1}{1-\frac{1}{z}} + \frac{1}{2}\frac{1}{1-\frac{z}{2}} \\ &= \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \\ &= \sum_{n=1}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \end{aligned}$$

For $2 < |z|$, converting to partial fractions,

$$\begin{aligned} -\frac{1}{(z-1)(z-1)} &= \frac{1}{z-1} + \frac{-1}{z-2} \\ &= \frac{1}{z}\frac{1}{1-\frac{1}{z}} - \frac{1}{z}\frac{1}{1-\frac{2}{z}} \\ &= \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^2 \\ &= \sum_{n=1}^{\infty}\frac{1}{z^n} - \sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n} \\ &= \sum_{n=1}^{\infty}\left(1 - 2^{n-1}\right)\frac{1}{z^n} \end{aligned}$$

6 Isolated Singularity Points

Definition 58 (Isolated singular point)**.** A point z_0 is said to be an isolated singular point of $f(z)$ is f is analytic in a perforated neighbourhood of z_0, i.e. if $\exists \varepsilon > 0$ such that f is analytic in $D_{z_0,\varepsilon} \setminus \{z_0\}$.

Exercise 26.
Find all isolated singular points of

1. $f(z) = \frac{1}{z}$
2. $f(z) = \frac{\sin z}{z}$
3. $f(z) = \operatorname{Log} z$

Solution 26.
1.

$$f(z) = \frac{1}{z}$$

Therefore, $\forall \varepsilon > 0$ around $z = 0$, f is analytic. Therefore, $z = 0$ is an isolated singular point for $f(z)$.

2.

$$f(z) = \frac{\sin z}{z}$$

Therefore, $\forall \varepsilon > 0$ around $z = 0$, f is analytic. Therefore, $z = 0$ is an isolated singular point for $f(z)$.

3.

$$f(z) = \operatorname{Log} z$$

As $\operatorname{Log} z$ is not defined on a ray in $\mathbb{C}$, f is not analytic on $D_{0,\varepsilon}$. Therefore, $z = 0$ is not an isolated singular point.

6.1 Characterization of Isolated Singular Points

Definition 59 (Characterization of isolated singular points)**.** Let z_0 be an isolated singular point of f. Therefore, by Loren Theorem,f has a Loren series around z_0 with $r = 0$, which converges in the ring $0 < |z - z_0| < R$.

1. z_0 is said to be a removable isolated singular point, if $\forall n < 0$, $c_n = 0$.
2. z_0 is said to be a pole on order N, if $\forall n < -N$, $c_n = 0$, and $C_{-N} \neq 0$.
3. z_0 is said to be a principle removable isolated singular point, if $\forall n < 0$, $c_n \neq 0$.

Definition 60 (Residue)**.** Let f have an isolated singular point at $z = 0$. The residue of f at z_0 is defined to be the coefficient c_{-1}, of $\frac{1}{z-z_0}$.
It is denoted as

$$\begin{aligned} c_{-1} &= \operatorname{Res}_f(z_0) \\ &= \frac{1}{2\pi i} \int\limits_C f(z)\,\mathrm{d}z \end{aligned}$$

where c_{-1} is a Loren coefficient of f.

Definition 61. For any $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$, the order of the zero is defined to be $n \in \mathbb{N}$, such that

$$f^{(n)} \neq 0$$

and

$$f^{(k)}(z_0) = 0$$

where $k = 0, \ldots, n-1$.
A pole of order 1 is said to be a single pole.

Exercise 27.
Find the order of the zero at $z = 0$ for

1. $f(z) = z \sin z$
2. $f(z) = 1 - \cos z$

Solution 27.

1.

$$\begin{aligned} f(z) &= z \sin z \\ \therefore f(0) &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} f'(z) &= \sin z + z \cos z \\ \therefore f'(0) &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} f''(z) &= \cos z + \cos z - z \sin z \\ \therefore f''(0) &= 2 \\ &\neq 0 \end{aligned}$$

Therefore, the order of the zero at $z = 0$ is 2.

2.

$$\begin{aligned} f(z) &= 1 - \cos z \\ \therefore f(0) &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} f'(z) &= \sin z \\ \therefore f'(0) &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} f''(z) &= \cos z \\ \therefore f''(0) &= 1 \end{aligned}$$

Therefore, the order of the zero at $z = 0$ is 2.

Exercise 28.

Let $f(z)$ and $g(z)$ be functions analytic at z_0. Let z_0 be a zero of order m for $f(z)$, and n for $g(z)$. Then, prove that z_0 is a zero of order $m + n$ for the function $f(z)g(z)$.

Solution 28.
As z_0 is a zero of order m with respect to $f(z)$,

$$f(z) = (z - z_0)^m h_1(z)$$

where $h_1(z)$ is an analytic function, such that

$$h_1(z_0) \neq 0$$

As z_0 is a zero of order n with respect to $g(z)$,

$$g(z) = (z - z_0)^n h_2(z)$$

where $h_2(z)$ is an analytic function, such that

$$h_2(z_0) \neq 0$$

Therefore,

$$\begin{aligned} f(z)g(z) &= (z - z_0)^m h_1(z)(z - z_0)^n h_2(z) \\ &= (z - z_0)^{m+n} h_1(z) h_2(z) \end{aligned}$$

Therefore, z_0 is a zero of order $m + n$ for the function $f(z)g(z)$.