Lecture 14

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Contents

1 Applications of Definite Integrals

1.1 Area under a curve

An area S of the region which is situated between two functions $f(x)$ and $g(x)$ over $[a, b]$ is

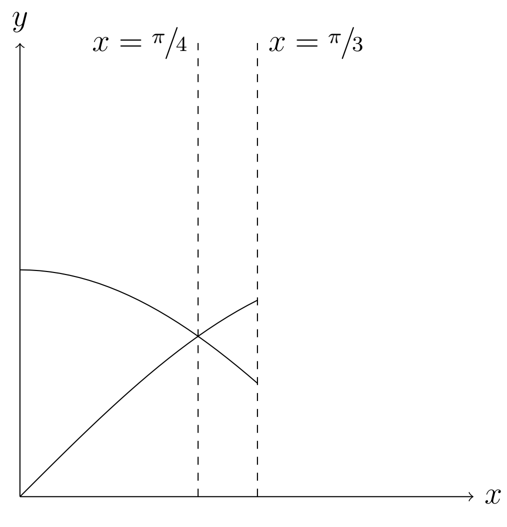
$$S = \int_a^b |f(x) - g(x)| \,\mathrm{d}x$$

Example 1. Calculate the area S of the region which is between $y = \cos x$ and $y = \sin x$ over $[0, \pi/3]$.

Solution.

$$S = \int_0^{\pi/3} |\cos x - \sin x| \,\mathrm{d}x$$

Therefore,



$$S = \int_0^{\pi/4} (\cos x - \sin x) \,\mathrm{d}x + \int_{\pi/4}^{\pi/3} (\sin x - \cos x) \,\mathrm{d}x$$

$$= 2\sqrt{2} - \frac{3 + \sqrt{3}}{2}$$

1.2 Length of a curve

Example 2. Calculate the length L of a curve which is given by a differentiable function $y = f(x)$ over $[a, b]$.

Solution.

$$\begin{aligned} L &= \lim_{\Delta T \to 0} \sum_{i=1}^{n} |P_{i-1}P_i| \\ &= \lim_{\Delta T \to 0} \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \end{aligned}$$

By Lagrange Theorem for $[x_{i-1}, x_i]$,

$$\begin{aligned} \Delta y_i &= f(x_i) - f(x_{i-1}) \\ &= f'(c_i)\Delta x_i \\ \therefore L &= \lim_{\Delta T \to 0} \sum_{i=1}^{n} \sqrt{1 + (f'(c_i))^2}\Delta x_i \\ &= \int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x \end{aligned}$$

Example 3. Find the length of the curve $y = f(x) = x^{3/2}$ on $[1, 4]$.

Solution.

$$\begin{aligned} L &= \int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x \\ \therefore L &= \int_1^4 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\,\mathrm{d}x \\ &= \int_1^4 \sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x \\ &= \frac{4}{9} \cdot \frac{\left(1 + \frac{9}{4}(4)\right)^{3/2}}{3/2} - \frac{4}{9} \cdot \frac{\left(1 + \frac{9}{4}(1)\right)^{3/2}}{3/2} \\ &= \frac{8}{27}\left(10^{3/2} - (13/4)^{3/2}\right) \end{aligned}$$

1.3 Volume of solids

Definition 1 (Volume of a cylindrical solid)**.** The volume of a cylindrical solid with a base S and a height h is equal to $V = Sh$.

For a solid situated between two planes $x = a$ and $x = b$ with a cross-sectional area $S(t)$ at any slice of the solid with the plane $x = t$, $t \in [a, b]$,

$$\begin{aligned} V &= \lim_{\Delta T \to 0} \sum_{i=1}^{n} S(c_i) \Delta x_i \\ &= \int_a^b S(x) \, \mathrm{d}x \end{aligned}$$

Example 4. Find the volume of a sphere.

Solution.

$$\begin{aligned} V &= \int_{-R}^{R} \pi (R^2 - x^2) \, \mathrm{d}x \\ &= 2\pi \int_{-R}^{R} (R^2 - x^2) \, \mathrm{d}x \\ &= 2\pi \left(R^3 - \frac{R^3}{3} \right) \\ &= \frac{4}{3} \pi R^3 \end{aligned}$$

1.3.1 Volume of solids of revolution about x-axis

If the graph of $y = f(x)$ is rotated about the x-axis,

$$\begin{aligned} V &= \int_a^b S(x) \, \mathrm{d}x \\ &= \int_a^b \pi y^2 \, \mathrm{d}x \\ &= \pi \int_a^b (f(x))^2 \, \mathrm{d}x \end{aligned}$$

Example 5. Find the volumes of solids of revolution which are obtained by rotating $y = f(x) = \sqrt{x}$, $x \in [0, 4]$ about the x-axis and the y-axis.

Solution.

$$\begin{aligned} V_x &= \pi \int_a^b (f(x))^2 \, \mathrm{d}x \\ &= \pi \int_0^4 (\sqrt{x})^2 \, \mathrm{d}x \\ &= 8\pi \end{aligned}$$

$$\begin{aligned} V_y &= \pi \int_c^d (f(y))^2 \, \mathrm{d}y \\ &= \pi \int_0^2 (y^2)^2 \, \mathrm{d}y \\ &= \pi \int_0^2 y^4 \, \mathrm{d}y \\ &= \frac{32}{5}\pi \end{aligned}$$

1.3.2 Volume of solids of revolution about y-axis

The volume of a solid of revolution which is obtained by rotating about the y-axis of a region between $y = f(x)$, $y = 0$, $x = a$, $x = b$ is

$$V = 2\pi \int_a^b x f(x) \, \mathrm{d}x$$

Example 6. Find the volumes of solids of revolution which are obtained by rotating the area under $y = f(x) = x(x-1)^2$, $x \in [0, 1]$ about the x-axis and the y-axis.

Solution.

$$
\begin{aligned}
V_x &= \pi \int_a^b (f(x))^2 \,\mathrm{d}x \\
&= \pi \int_0^1 (x(x-1)^2)^2 \,\mathrm{d}x \\
&= \pi \int_0^1 x^2(x-1)^2 \,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
V &= 2\pi \int_a^b x f(x) \,\mathrm{d}x \\
&= 2\pi \int_0^1 x \cdot x(x-1)^2 \,\mathrm{d}x
\end{aligned}
$$