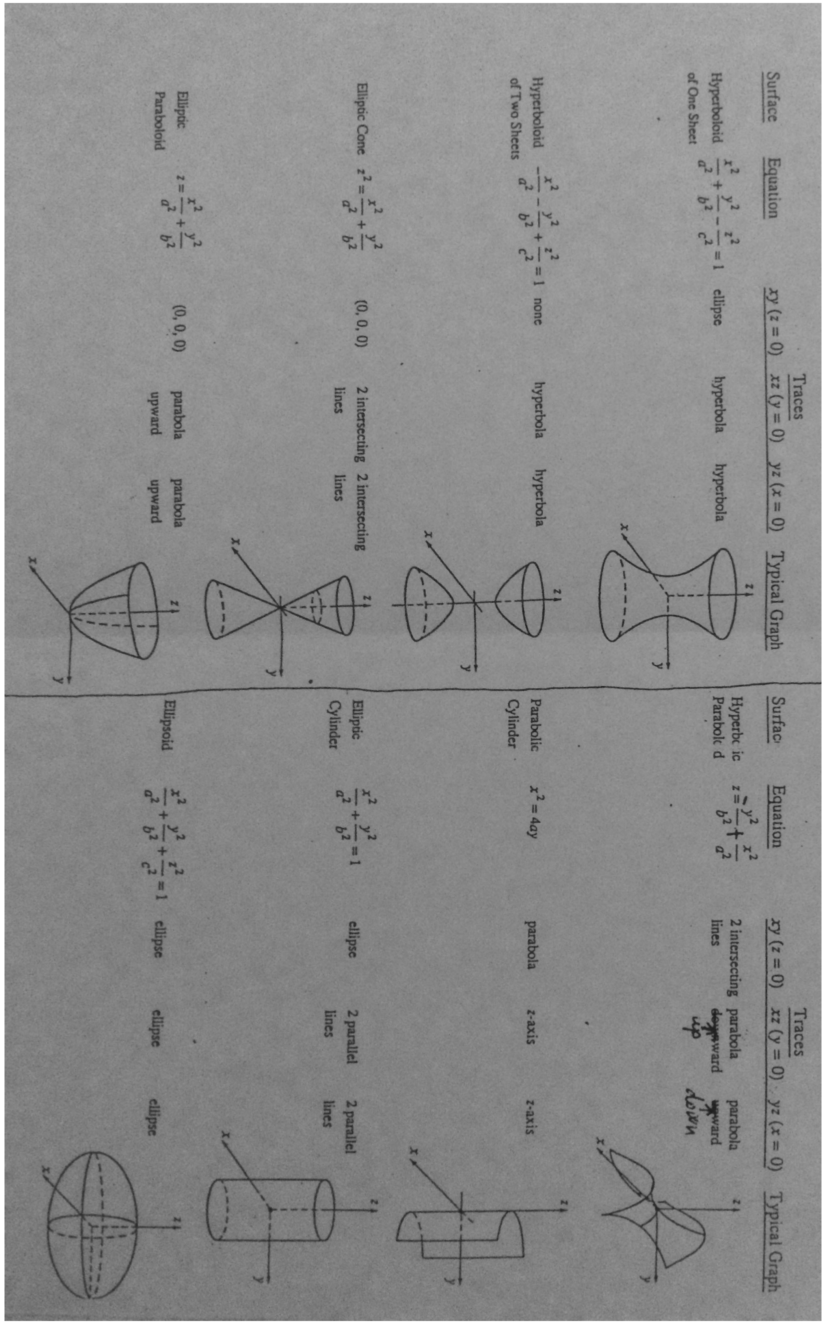


| Surface | Equation | Traces: $xy$ ($z=0$) | Traces: $xz$ ($y=0$) | Traces: $yz$ ($x=0$) |
| --- | --- | --- | --- | --- |
| Hyperboloid of One Sheet | $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ | ellipse | hyperbola | hyperbola |
| Hyperboloid of Two Sheets | $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ | none | hyperbola | hyperbola |
| Elliptic Cone | $z^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ | (0, 0, 0) | 2 intersecting lines | 2 intersecting lines |
| Elliptic Paraboloid | $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ | (0, 0, 0) | parabola upward | parabola upward |
| Hyperbolic Paraboloid | $z=\frac{y^2}{b^2}+\frac{x^2}{a^2}$ | 2 intersecting lines | parabola ~~downward~~ up | parabola ~~upward~~ down |
| Parabolic Cylinder | $x^2=4ay$ | parabola | $z$-axis | $z$-axis |
| Elliptic Cylinder | $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ | ellipse | 2 parallel lines | 2 parallel lines |
| Ellipsoid | $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ | ellipse | ellipse | ellipse |



# Differential and Integral Methods: Compendium

Aakash Jog

## Contents



---

# 1 Functions

**Definition 1** (Even function).

$$f(-x) = f(x)$$

**Definition 2** (Odd function).

$$f(-x) = -f(x)$$

## 1.1 Hyperbolic Functions

**Definition 3** (Hyperbolic functions).

$$\sinh x \doteq \frac{e^x - e^{-x}}{2} \qquad I(\sinh x) = \mathbb{R}$$

$$\cosh x \doteq \frac{e^x + e^{-x}}{2} \qquad I(\cosh x) = [1, \infty)$$

$$\tanh x \doteq \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad I(\tanh x) = (-1, 1)$$

### 1.1.1 Identities of Hyperbolic Functions

$$
\begin{aligned}
\sinh(2x) &= 2\sinh x\cosh x\\
\cosh^2 x+\sinh^2 x &= \cosh(2x)\\
\cosh^2 x-\sinh^2 x &= 1\\
\frac{\cosh(2x)-1}{2} &= \sinh^2 x\\
\frac{\cosh(2x)+1}{2} &= \cosh^2 x
\end{aligned}
$$

## 1.2 Trigonometric Identities

$$
\begin{aligned}
\sin^2 x+\cos^2 x &= 1\\
\tan^2 x+1 &= \sec^2 x\\
\cot^2 x+1 &= \csc^2 x
\end{aligned}
$$

$$
\begin{aligned}
\sin(x\pm y) &= \sin x\cos y\pm\cos x\sin y\\
\cos(x\pm y) &= \cos x\cos y\mp\sin x\sin y\\
\tan(x\pm y) &= \frac{\tan x\pm\tan y}{1\mp\tan x\tan y}
\end{aligned}
$$

$$
\begin{aligned}
\sin x\sin y &= \frac{1}{2}\left(\cos(x-y)-\cos(x+y)\right)\\
\cos x\cos y &= \frac{1}{2}\left(\cos(x-y)+\cos(x+y)\right)\\
\sin x\cos y &= \frac{1}{2}\left(\sin(x+y)+\sin(x-y)\right)\\
\cos x\sin y &= \frac{1}{2}\left(\sin(x+y)-\sin(x-y)\right)
\end{aligned}
$$

$$
\begin{aligned}
\sin x+\sin y &= 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\\
\sin x-\sin y &= 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\\
\cos x+\cos y &= 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\\
\cos x-\cos y &= -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\sin\frac{x}{2} &= \pm\sqrt{\frac{1-\cos x}{2}}\\
\cos\frac{x}{2} &= \pm\sqrt{\frac{1+\cos x}{2}}\\
\tan\frac{x}{2} &= \pm\sqrt{\frac{1-\cos x}{1+\cos x}}
\end{aligned}
$$

# 2 Limits

**Definition 4** (Cauchy's definition of a limit of a function)**.**

$\forall\epsilon>0\exists\delta>0:0<|x-a|<\delta\Rightarrow|f(x)-L|<\epsilon$

**Definition 5** (Removable discontinuity point)**.**

$\exists\lim_{x\to a}f(x)$, but either $\lim_{x\to a}f(x)\neq f(a)$ or $\nexists f(a)$

**Definition 6** (Discontinuity of first kind)**.**

$\exists\lim_{x\to a^-}f(x),\exists\lim_{x\to a^+}f(x)$, but $\lim_{x\to a^-}f(x)\neq\lim_{x\to a^+}f(x)$

**Definition 7** (Discontinuity of second kind)**.** At least one of the two one-sided limits of $f$ does not exist. (Limits are defined as finite numbers only.)

**Theorem 1** (Sandwich Theorem)**.** *Let $f(x),g(x),h(x)$ be defined on an open interval about $a$, except possibly at $a$ itself. Assume that $\forall x\neq a$ from the interval, it is satisfied that $f(x)\leq g(x)\leq h(x)$ and $\lim_{x\to a}f(x)=\lim_{x\to a}h(x)=L$. Then,*

$\lim_{x\to a}g(x)=L$

**Theorem 2.** *If $\lim_{x\to a}f(x)=0$ and $g(x)$ is bounded in an open interval about $a$, except possibly at $a$ itself, then,*

$\lim_{x\to a}(f(x)g(x))=0$

## 2.1 Useful Limits

If $\lim_{x\to x_0}g(x)=0$,

$\lim_{x\to x_0}(1+g(x))^{\frac{1}{g(x)}}=e$

$$
\begin{aligned}
\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x &= e\\
\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x &= e\\
\lim_{\theta\to 0}\frac{\sin\theta}{\theta} &= 1
\end{aligned}
$$

# 3 Derivatives

**Definition 8** (Derivative of a function)**.**

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=L$$

**Theorem 3** (Derivative of inverse functions)**.**

$$(f^{-1})'(x)=\frac{1}{f'(x)}$$

**Theorem 4** (Chain rule)**.**

$$\frac{\mathrm{d}f(g(x))}{\mathrm{d}x}=\frac{\mathrm{d}f(g(x))}{\mathrm{d}g(x)}\cdot\frac{\mathrm{d}g(x)}{\mathrm{d}x}$$

**Theorem 5** (Rolle's Theorem)**.** *Let $f(x)$ be defined on $[a,b]$, s.t.*



Therefore,

$$
\begin{aligned}
P_y &= 2e^{2x-y}\cos y+2e^{2x-y}\sin y\\
&= 2e^{2x-y}(\cos y+\sin y)\\
Q_x &= 2e^{2x-y}(\sin x+\cos y)+2y
\end{aligned}
$$

The domain is of the first kind.

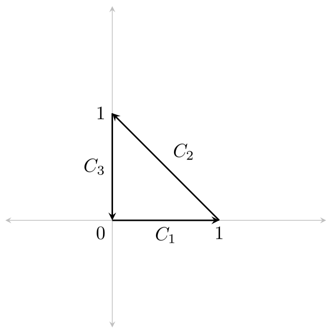
$$
\begin{aligned}
\int_C P\,\mathrm{d}x+Q\,\mathrm{d}y &= \int_C P\,\mathrm{d}x+Q\,\mathrm{d}y+\int_{C_1}P\,\mathrm{d}x+Q\,\mathrm{d}y\\
&\quad-\int_{C_1}P\,\mathrm{d}x+Q\,\mathrm{d}y\\
&= \int_{C\cup C_1}P\,\mathrm{d}x+Q\,\mathrm{d}y-\int_{C_1}P\,\mathrm{d}x+Q\,\mathrm{d}y\\
&= \iint_D(Q_x-P_y)\,\mathrm{d}A-\int_{C_1}P\,\mathrm{d}x+Q\,\mathrm{d}y
\end{aligned}
$$

**Example 11.** Find the work done by the force

$\overline{F}(x,y) = (x^4, xy)$

over the path

$C = C_1 \cup C_2 \cup C_3$




*Solution.* By Green's Theorem,

$$\begin{aligned} W &= \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y \\ &= \iint_D (Q_x - P_y)\,\mathrm{d}A \\ &= \iint_D (y-0)\,\mathrm{d}A \\ &= \int_0^1 \int_0^{1-x} y\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{1}{6} \end{aligned}$$

**Example 12.** Calculate $\int_C \overline{F} \cdot \hat{T}\,\mathrm{d}s$ when

$$\overline{F} = \left( -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

and $C$ is a simple, closed, piecewise smooth curve with positive orientation which does not pass through $(0,0)$.

*Solution.*

$$P = \frac{y}{x^2+y^2}$$
$$Q = \frac{x}{x^2+y^2}$$

Therefore,

$$\begin{aligned} P_y &= -\frac{(x^2+y^2) - y \cdot 2y}{(x^2+y^2)^2} \\ &= \frac{y^2-x^2}{(x^2+y^2)^2} \end{aligned}$$
$$Q_x = \frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2}$$

If $(0,0) \notin D$, Green's Theorem is applicable.
Therefore,

$$\begin{aligned} \int_C \overline{F} \cdot \hat{T}\,\mathrm{d}s &= \iint_D (Q_x - P_y)\,\mathrm{d}A \\ &= 0 \end{aligned}$$

If $(0,0) \in D$, Green's Theorem is not applicable as $P_y$ and $Q_x$ are not continuous in $D$.
Let $C_1$ be a circle of radius $a$, with the same orientation as $C$. Let $\widetilde{C} = C \cup (-C_1)$. Green's Theorem can be applied on the domain $D \setminus D_1$ which is enclosed by $\widetilde{C}$.

$$\begin{aligned} \int_{C\cup(-C_1)} P\,\mathrm{d}x + Q\,\mathrm{d}y &= \iint_{D\setminus D_1} (Q_x - P_y)\,\mathrm{d}A \\ &= 0 \end{aligned}$$

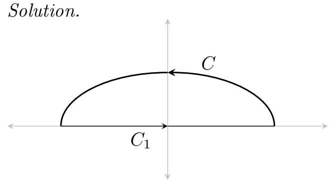
$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y + \int_{-C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y = 0$$

Therefore,

$$\begin{aligned} \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y &= \int_{C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y \\ &= \int_0^{2\pi} P\big(x(t), y(t)\big)\, x'(t)\,\mathrm{d}t \\ &\quad + \int_0^{2} Q\big(x(t), y(t)\big)\,\mathrm{d}t \\ &= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,\mathrm{d}t \\ &= 2\pi \end{aligned}$$

**Example 13.** Calculate $\int_C -2e^{2x-y}\cos y\,\mathrm{d}x + \left(e^{2x-y}(\sin y + \cos y) + 2xy\right)\mathrm{d}y$ when $C$ is the half ellipse $\left\{ \frac{x^2}{4} + y^2 = 1, y \geq 0 \right\}$ oriented from the point $(2,0)$ to the point $(-2,0)$.

*Solution.*




Let $C_1$ be the line segment as shown.

$$P = -2e^{2x-y}\cos y$$
$$Q = e^{2x-y}(\sin y + \cos y) + 2xy$$



1. *$f$ is continuous on $[a,b]$*
2. *$f$ is differentiable on $(a,b)$*
3. *$f(a) = f(b)$*

*Then, $\exists c \in (a,b)$, s.t. $f'(c) = 0$.*

**Theorem 6** (Lagrange Theorem)**.** *Let $f(x)$ be defined on $[a,b]$, s.t.*

1. *$f$ is continuous on $[a,b]$*
2. *$f$ is differentiable on $(a,b)$*

*Then,*

$$\exists c \in (a,b),\ s.t.\ f'(c) = \frac{f(b)-f(a)}{b-a}$$

# 4 Taylor's Formula

**Theorem 7** (Taylor's Formula)**.**

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i + \frac{f^{(n)}(c)}{(n+1)!}(x-a)^{n+1}$$

## 4.1 Common Derivatives

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\, x = 1 \\ &\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x \\ &\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x \\ &\frac{\mathrm{d}}{\mathrm{d}x} \tan x = \sec^2 c \\ &\frac{\mathrm{d}}{\mathrm{d}x} \sec x = \sec x \tan x \\ &\frac{\mathrm{d}}{\mathrm{d}x} \csc x = -\csc x \cot x \\ &\frac{\mathrm{d}}{\mathrm{d}x} \cot x = -\csc^2 c \\ &\frac{\mathrm{d}}{\mathrm{d}x} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \\ &\frac{\mathrm{d}}{\mathrm{d}x} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \\ &\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x = \frac{1}{1+x^2} \\ &\frac{\mathrm{d}}{\mathrm{d}x}\, a^x = a^x \ln a \\ &\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x} \\ &\frac{\mathrm{d}}{\mathrm{d}x} \log_a x = \frac{1}{x \ln a} \end{aligned}$$

# 5 Full Investigation of Functions

1. Domain of definition of $f$
2. Points of intersection of $y = f(x)$ with $x$-axis and $y$-axis
3. Symmetry and periodicity
4. Extrema points
5. Monotonicity
6. Convexity
7. Inflection points
8. Asymptotes (vertical and oblique)
9. Graph

**Definition 9** (Vertical asymptote)**.** Let $f(x)$ be defined on $(a-\delta)$ or $(a, a+\delta)$ or $(a-\delta, a+\delta) - \{a\}$ for $\delta > 0$. If atleast one of $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ is equal to $\pm\infty$, then the straight line $x = a$ is said to be a vertical asymptote of $f(x)$.

**Definition 10** (Oblique asymptote)**.** The straight line $y = ax + b$ is called an oblique asymptote of a function $y = f(x)$ at $+\infty$ (or $-\infty$), if

$$\lim_{x\to+\infty} (f(x) - (ax+b)) = 0$$
$$\left( \text{or } \lim_{x\to-\infty} (f(x) - (ax+b)) = 0 \right)$$

**Example 1.** Investigate

$$y = f(x) = \frac{(x-1)^3}{(x+1)^2}$$

*Solution.*

$$D(f) = \mathbb{R} - \{-1\}$$

$$\begin{aligned} y &= 0 &&\implies x = 1 \\ x &= 0 &&\implies y = -1 \end{aligned}$$

The function is not periodic.

$$\begin{aligned} f(-x) &\neq f(x) \\ &\neq -f(x) \end{aligned}$$

Therefore, the function is not symmetric.

$$f'(x) = \frac{(x-1)^2(x+5)}{(x+1)^3}$$

Therefore, $x = -5$ is a local maximum point.
The function is monotonically increasing in $(-\infty, -5) \cup (-1, +\infty)$ and is monotonically decreasing in $(-5, -1)$.

$$f''(x) = \frac{24(x-1)}{(x+1)^4}$$

Therefore, the function is convex upwards in $(-\infty,-1)\cup(-1,1)$ and convex downwards in $(1,\infty)$.

$$\begin{aligned}\lim_{x\to-1^-}\frac{(x-1)^3}{(x+1)^2}&=\frac{-8}{+0}\\&=-\infty\\\lim_{x\to-1^+}\frac{(x-1)^3}{(x+1)^2}&=\frac{-8}{+0}\\&=-\infty\end{aligned}$$

Therefore, $x=-1$ is a vertical asymptote of $f(x)$.

$$\begin{aligned}a_1&=\lim_{x\to+\infty}\frac{f(x)}{x}&&=1\\b_1&=\lim_{x\to+\infty}\big(f(x)-a_1x\big)&&=-5\\a_2&=\lim_{x\to-\infty}\frac{f(x)}{x}&&=1\\b_2&=\lim_{x\to-\infty}\big(f(x)-a_1x\big)&&=-5\end{aligned}$$

Therefore, $y=x-5$ is an oblique asymptote of the function, at $+\infty$ and $-\infty$.

# 6 Integration

**Definition 11** (Basic rational functions)**.** A simple rational function of one of the following forms is called a basic rational function.

$$\begin{aligned}&\frac{A}{x-\alpha}&&;A,\alpha\in\mathbb{R}\\&\frac{A}{(x-\alpha)^n}&&;A,\alpha\in\mathbb{R},n\in\mathbb{N}-\{1\}\\&\frac{Ax+B}{x^2+px+q}&&;A,B,p,q\in\mathbb{R},p^2-4q<0\\&\frac{Ax+B}{(x^2+px+q)^n}&&;A,B,p,q\in\mathbb{R},p^2-4q<0,n\in\mathbb{N}-\{1\}\end{aligned}$$

**Example 2.** Solve $\int\frac{-x+2}{x(x-1)^2}\,\mathrm{d}x$.

*Solution.*

$$\int\frac{-x+2}{x(x-1)^2}\,\mathrm{d}x=\int\left(\frac{A_1}{x}+\frac{B_1}{x-1}+\frac{B_2}{(x-1)^2}\right)\mathrm{d}x$$

$$\begin{aligned}\frac{-x+2}{x(x-1)^2}&=\frac{A_1(x-1)^2+B_1(x)(x-1)+B_2x}{x(x-1)^2}\\&=\frac{x^2(A_1+B_1)+x(-2A_1-B_1+B_2)+A_1}{x(x-1)^2}\end{aligned}$$

Therefore,

$$\begin{aligned}A_1+B_1&=0\\-2A_1-B_1+B_2&=-1\\A_1&=2\end{aligned}$$

Therefore,

$$\begin{aligned}A_1&=2\\B_1&=-2\\B_2&=1\end{aligned}$$

Therefore,

$$\begin{aligned}\int\frac{-x+2}{x(x-1)^2}\,\mathrm{d}x&=\int\left(\frac{2}{x}+\frac{-2}{x-1}+\frac{1}{(x-1)^2}\right)\mathrm{d}x\\&=2\ln|x|-2\ln|x-1|-\frac{1}{x-1}+c\\&=2\ln\left|\frac{x}{x-1}\right|-\frac{1}{x-1}+x\end{aligned}$$

**Example 3.** Solve $\int\frac{2x^2-x+4}{x^3+4x}\,\mathrm{d}x$.

*Solution.*

$$\begin{aligned}\int\frac{2x^2-x+4}{x^3+4x}\,\mathrm{d}x&=\int\frac{2x^2-x+4}{x(x^2+4)}\,\mathrm{d}x\\&=\int\left(\frac{A}{x}+\frac{Bx+C}{x^2+4}\right)\mathrm{d}x\end{aligned}$$

$$\begin{aligned}\frac{2x^2-x+4}{x^3+4x}&=\frac{A(x^2+4)+(Bx+c)x}{x(x^2+4)}\\&=\frac{x^2(A+B)+x(C)+4A}{x(x^2+4)}\end{aligned}$$

Therefore,

$$\begin{aligned}A+B&=2\\C&=-1\\4A&=4\end{aligned}$$

Therefore,

$$\begin{aligned}A&=1\\B&=1\\C&=-1\end{aligned}$$

Therefore,

$$\begin{aligned}\int\frac{2x^2-x+4}{x(x^2+4)}\,\mathrm{d}x&=\int\left(\frac{1}{x}+\frac{x-1}{x^2+4}\right)\mathrm{d}x\\&=\ln|x|\\&\quad+\int\frac{x}{x^2+4}\,\mathrm{d}x-\int\frac{1}{x^2+4}\,\mathrm{d}x+d\\&=\ln|x|\\&\quad+\frac{1}{2}\ln(x^2+4)-\frac{1}{2}\arctan\left(\frac{x}{2}\right)+d\end{aligned}$$

**Theorem 8** (Integration by Parts)**.**

$$\int uv\,\mathrm{d}x=u\int v\,\mathrm{d}x-\int u'\left(\int v\,\mathrm{d}x\right)\mathrm{d}x$$



**Theorem 21** (The Fundamental Theorem of Line Integrals)**.** *Let $C$ be a smooth curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ given parametrically by $\overline{r}(t)$, $t:a\to b$. Let $f$ be a continuous function of $(x,y)$ or $(x,y,z)$ respectively, on $C$ and $\nabla f$ be a continuous vector function in a connected domain $D$ which contains $C$. Then*

$$\begin{aligned}W&=\int_C\nabla f\cdot\hat{T}\,\mathrm{d}s\\&=f(r(b))-f(r(a))\\&=f(B)-f(A)\end{aligned}$$

**Definition 19** (Simple curve)**.** A curve $C$ is called a simple curve if it does not intersect itself.

**Definition 20** (Domain)**.** A domain $D\subset\mathbb{R}^2$ is called connected if for any two points from $D$, the is a path $C$ which connects the points and remains in $D$.

**Definition 21** (Simple connected domain)**.** A connected domain $D\subset\mathbb{R}^2$ is called simple connected if any simple closed curve from $D$ contains inside itself only points in $D$.

**Theorem 22.** *If*

$$\overline{F}(x,y)=\big(P(x,y),Q(x,y)\big)=\nabla f(x,y)$$

*is the conservative vector field in a connected domain $D$, where there exist first order partial derivatives of $P$ and $Q$ continuous in $D$, then*

$$P_y(x,y)=Q_x(x,y)\qquad\forall(x,y)\in D$$

**Theorem 23.** *Let*

$$\overline{F}(x,y)=\big(P(x,y),Q(x,y)\big)$$

*be a vector field in an open, simple connected domain $D$. If there exist first order partial derivatives of $P$ and $Q$ which are continuous in $D$, and*

$$P_y(x,y)=Q_x(x,y)\qquad\forall(x,y)\in D$$

*Then, $\exists f(x,y)$ s.t.*

$$\overline{F}(x,y)=\nabla f(x,y)$$

*i.e. $\overline{F}$ is a conservative vector field.*

**Example 10.** If

$$\overline{F}(x,y)=(3+2xy,x^2-3y^2)$$

a conservative vector field? If yes, find $f(x,y)$, s.t.

$$\overline{F}(x,y)=\nabla f(x,y)$$

and find the work done by the force $\overline{F}(x,y)$ over the curve

$$\overline{r}(t)=(e^t\sin t,e^t\cos t)\qquad t:0\to\pi$$

*Solution.*

$$\begin{aligned}P(x,y)&=3+2xy\\\therefore P_y&=2x\\Q(x,y)&=x^2-3y^2\\\therefore Q_x&=2x\\\therefore P_y&=Q_x\end{aligned}$$

Therefore, $\overline{F}(x,y)$ is a conservative vector field.

$$\begin{aligned}f_x&=P\\&=3+2xy\\\therefore f&=3x+x^2y+c(y)\\\therefore f_y&=x^2+c'(y)\end{aligned}$$

Comapring with $f_y=Q$,

$$\begin{aligned}c'(y)&=-3y^3\\\therefore c(y)&=-y^3+c\\\therefore f(x,y)&=3x+x^2y-y^3+c\end{aligned}$$

By the definition of work,

$$\begin{aligned}W&=\int_C\overline{F}\cdot\hat{T}\,\mathrm{d}s\\&=\int_a^b\big(P(\overline{r}(t))x'(t)+Q(\overline{r}(t))y'(t)\big)\,\mathrm{d}t\end{aligned}$$

Alternatively, using The Fundamental Theorem of Line Integrals,

$$\begin{aligned}W&=\int_C\overline{F}\cdot\hat{T}\,\mathrm{d}s\\&=\int_C\nabla d\cdot\hat{T}\,\mathrm{d}s\\&=f(\overline{r}(\pi))-f(\overline{r}(0))\\&=f(0,-e^\pi)-f(0,1)\\&=-(-e^\pi)^3-(-1)^3\\&=e^{3\pi}+1\end{aligned}$$

**Theorem 24** (Green's Theorem)**.** *Let $C$ be a piecewise smooth, simple, and closed curve in $\mathbb{R}^2$ with positive orientation. Let $D$ be a domain bounded by $C$. If there exist continuous first order partial derivatives of $P(x,y)$ and $Q(x,y)$ in an open domain which contains $D$, then*

$$W=\int_C\overline{F}\cdot\hat{T}\,\mathrm{d}s=\int_C P\,\mathrm{d}x+Q\,\mathrm{d}y=\iint_D(Q_x-P_y)\,\mathrm{d}A$$

*Remark* 1. Green's Theorem is also true for domains with holes.

Therefore,

$$\iiint_E x^2+y^2+z^2 \,\mathrm{d}V = \int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2+y^2+z^2 \,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

**Example 6.** Calculate $\iiint_E xe^z \,\mathrm{d}V$ where $E$ is bounded by $z = x^2+y^2$ and $z = 8-x^2-y^2$.

*Solution.* The two boundaries intersect at $x^2+y^2=4$. Therefore the projection of the volume is the circle. Therefore,

$$\iiint_E xe^z \,\mathrm{d}V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-x^2-y^2} xe^z \,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

## 10 Line Integrals of Scalar Functions

**Definition 17** (Smooth curve)**.** Let $C$ be given parametrically as

$$\overline{r}(t) = \big(x(t), y(t)\big) \quad t : a \to b$$

The curve is said to be smooth if

$$\overline{r}'(t) = \big(x'(t), y'(t)\big)$$

is a continuous function on $[a,b]$, $\overline{r}'(t) \neq \overline{0}$ on $(a,b)$ and $\overline{r}'(t)$ is also continuous on $(a,b)$.

**Theorem 20.** *If $f(x,y)$ is continuous and $C$ is smooth, then*

$$\int_C f(x,y)\,\mathrm{d}s = \int_a^b f(x(t),y(t))\sqrt{\big(x'(t)\big)^2+\big(y'(t)\big)^2}\,\mathrm{d}t$$

**Example 7.** Calculate $\int_C x^2+y^2 \,\mathrm{d}s$ where $C$ is a circle of radius 2.

*Solution.*

$$\begin{aligned}\int_C x^2+y^2\,\mathrm{d}s &= \int_0^{2\pi} \Big((2\cos t)^2 + (2\sin t)^2\Big)\cdot 2\,\mathrm{d}t \\ &= 16\pi\end{aligned}$$

## 11 Line Integrals of Vector Functions

**Definition 18** (Line integral of vector function)**.**

$$\begin{aligned}W &= \int_C \overline{F}\cdot\hat{T}\,\mathrm{d}s \\ &= \int_C \overline{F}\cdot\mathrm{d}\overline{z} \\ &= \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z\end{aligned}$$

**Example 8.** Find the work $W$ done by the force $\overline{F}(x,y) = (x, xy)$ over the curve $C : \overline{r}(t) = (2\cos t, 2\sin t), t : \pi \to 2\pi$.

*Solution.*

$$\begin{aligned}W &= \int_C \overline{F}\cdot\hat{T}\,\mathrm{d}s \\ &= \int_\pi^{2\pi} \big(2\cos t(-2\sin t) + 2\cos t\cdot 2\sin t\cdot 2\cos t\big)\,\mathrm{d}t \\ &= \int_\pi^{2\pi} \Big(-2\sin(2t) + 8\cos^2 t\sin t\Big)\,\mathrm{d}t \\ &= \cos(2t) - \frac{8}{3}\cos^3 t\Big|_\pi^{2\pi} \\ &= \left(1-\frac{8}{3}\right) - \left(1+\frac{8}{3}\right) \\ &= -\frac{16}{3}\end{aligned}$$

**Example 9.** Calculate $\int_C \frac{x}{y}\,\mathrm{d}x + \frac{y-x}{x}\,\mathrm{d}y$ where $C$ is the path over the parabola $y = x^2$ from $(2,4)$ to $(1,1)$.

*Solution.*

$$\begin{aligned}\int_C \left(\frac{x}{y}, \frac{y-x}{x}\right)\mathrm{d}r &= \int_2^1 \left(\frac{t}{t^2} + \frac{t^2-t}{t}\right)\cdot(1,2t)\,\mathrm{d}t \\ &= \int_2^1 \left(\frac{1}{t} + (t-1)\cdot 2t\right)\mathrm{d}t \\ &= \ln t + \frac{2t^3}{3} - t^2\Big|_2^1 \\ &= \ln\frac{1}{2} + \frac{2}{3} - \frac{16}{3} - 1 + 4 \\ &= 3 - \frac{14}{3} - \ln 2 \\ &= \frac{5}{3} - \ln 2\end{aligned}$$



**Theorem 9** (Fundamental Theorem of Calculus, Part 1)**.** *Let $f(x)$ be continuous on $(a,b)$ and let $c \in (a,b)$. Then, the function $F(x) = \int_c^x f(t)\,\mathrm{d}t$ is an anti-derivative function of $f(x)$ on $(a,b)$, i.e.*

$$F'(x) = f(x) \quad ; \quad \forall x \in (a,b)$$

**Theorem 10** (Fundamental Theorem of Calculus, Part 2)**.** *Let $f(x)$ be continuous on $[a,b]$ and let $G(x)$ be an arbitrary anti-derivative function of $f(x)$. Then,*

$$\int_a^b f(x)\,\mathrm{d}x = G(b) - G(a)$$

### 6.1 Common Integrals

$$\int k\,\mathrm{d}x = kx + c$$

$$\int x^n\,\mathrm{d}x = \begin{cases}\frac{1}{n+1}x^{n+1} + c & ; \quad n \neq -1 \\ \ln|x| + c & ; \quad n = -1\end{cases}$$

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + c$$

$$\int \ln x\,\mathrm{d}x = x\ln x - x + c$$

$$\int e^x\,\mathrm{d}x = e^x + c$$

$$\int \cos x\,\mathrm{d}x = \sin x + c$$

$$\int \sin x\,\mathrm{d}x = -\cos x + c$$

$$\int \sec^2 x\,\mathrm{d}x = \tan x + c$$

$$\int \csc^2 x\,\mathrm{d}x = -\cot x + c$$

$$\int \sec x\tan x\,\mathrm{d}x = \sec x + c$$

$$\int \csc x\cot x\,\mathrm{d}x = -\csc x + c$$

$$\int \tan x\,\mathrm{d}x = \ln|\sec x| + c$$

$$\int \sec x\,\mathrm{d}x = \ln|\sec x + \tan x| + c$$

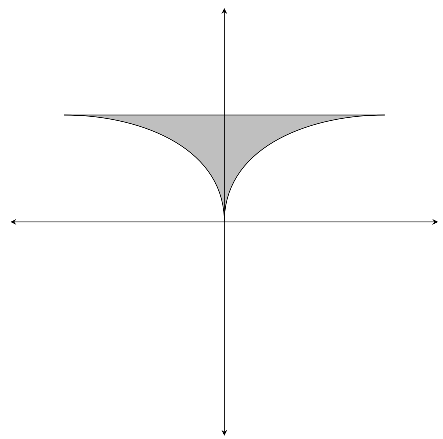
$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}\frac{x}{a} + c$$

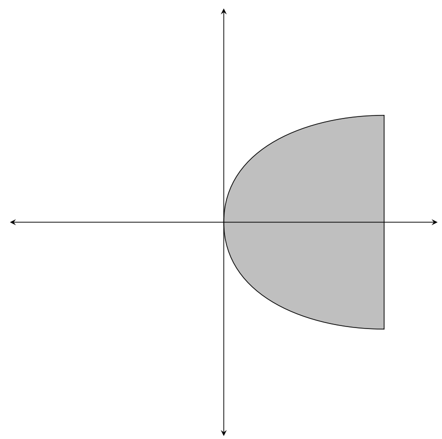
### 6.2 Length of a Curve

$$l = \int_a^b \sqrt{1+(f'(x))^2}\,\mathrm{d}x$$

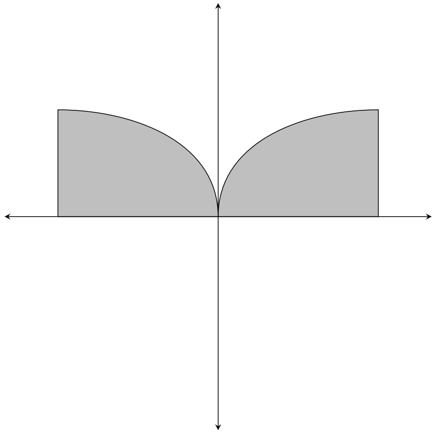
### 6.3 Volume of Solids of Rotation



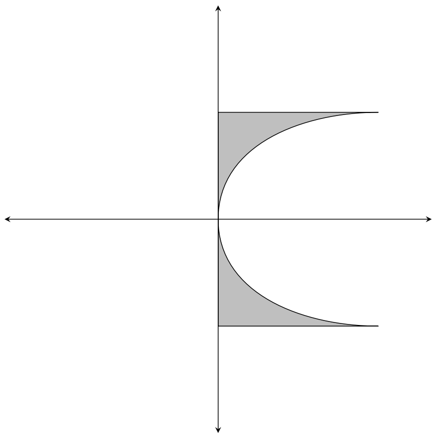
$$V = \pi\int_a^b \big(f(y)\big)^2\,\mathrm{d}y$$



$$V = \pi\int_a^b \big(f(x)\big)^2\,\mathrm{d}x$$



$$V = 2\pi \int_a^b x f(x)\,\mathrm{d}x$$



$$V = 2\pi \int_a^b y f(y)\,\mathrm{d}y$$

## 6.4 Improper Integrals

### 6.4.1 Direct Comparison Tests

**Theorem 11** (First comparison test)**.** *Let $f(x)$ and $g(x)$ be two functions defined on $[a,+\infty)$ and Riemann integrable over $[a,t], \forall t \geq a$. Assume that $\exists b \geq a$, s.t. $f(x) \geq g(x) \geq 0, \forall x \geq b$. Then,*

1. *if $\int_a^{+\infty} f(x)\,\mathrm{d}x$ converges, then $\int_a^{+\infty} g(x)\,\mathrm{d}x$ converges.*
2. *if $\int_a^{+\infty} g(x)\,\mathrm{d}x$ diverges, then $\int_a^{+\infty} f(x)\,\mathrm{d}x$ diverges.*

**Theorem 12** (Second comparison test)**.** *Assume $f(x) \geq g(x) \geq 0, \forall x \in (a,b)$. Assume that $f$, $g$ are not bounded in a neighbourhood of $b$ but integrable on intervals of the type $[a,t]$ for $a < t < b$. Assume that*

$$\lim_{x\to b^-} \frac{f(x)}{g(x)} = l > 0$$

*Then,*

$$\int_a^b f(x)\,\mathrm{d}x$$

*and*

$$\int_a^b g(x)\,\mathrm{d}x$$

*converge or diverge simultaneously.*

# 7 Multi-variable Functions

**Theorem 13** (Existence of limits)**.** *Let $\exists g(r,\theta)$, s.t. $f(x,y) = g(r,\theta)$. Then, if it exists,*

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} g(r,\theta)$$

**Definition 12** (Critical point)**.** If both of $f_x(a,b)$ and $f_y(a,b)$ are zero, or if at least one of them does not exist, then $(a,b)$ is said to be a critical point.

**Theorem 14** (A necessary condition for local extrema existence)**.** *If the function $z = f(x,y)$ has a local extrema at the point $(a,b)$ and $\exists f_x(a,b)$ and $\exists f_y(a,b)$ then $f_x(a,b) = f_y(a,b) = 0$*

**Theorem 15** (A sufficient condition for local extrema point)**.** *Assume that there exist second order partial derivates of $z = f(x,y)$, they are continuous on some open neighbourhood of $(a,b)$ and $f_x(a,b) = f_y(a,b) = 0$. Denote*

$$D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$$

1. *If $D(a,b) > 0$ and $f_{xx} < 0$ then $(a,b)$ is a local maximum point.*
2. *If $D(a,b) > 0$ and $f_{xx} > 0$ then $(a,b)$ is a local minimum point.*
3. *If $D(a,b) < 0$ then $(a,b)$ is called a saddle point.*

**Example 4.** Find all critical points of

$$z = f(x,y) = x^4 + y^4 - 4xy + 1$$

and classify them.



*Solution.*

$$f_x(x,y) = 4x^3 - 4y$$
$$f_y(x,y) = 4y^3 - 4x$$

For critical points,

$$f_x(x,y) = 0$$
$$f_y(x,y) = 0$$

Solving, $(0,0)$, $(1,1)$, $(-1,-1)$ are critical points.

$$f_{xx}(x,y) = 12x^2$$
$$f_{xy}(x,y) = -4$$
$$f_{yy}(x,y) = 12y^2$$
$$\therefore D(x,y) = 144x^2y^2 - 16$$

For $(0,0)$,

$$D = -16$$

Therefore, $(0,0)$ is a saddle point.
For $(1,1)$,

$$D = 144 - 16$$

Therefore, $(1,1)$ is a local minimum point.
For $(-1,-1)$,

$$D = 144 - 16$$

Therefore, $(-1,-1)$ is a local minimum point.

**Definition 13** (Gradient)**.**

$$\nabla f(x,y,z) = (f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)) \neq 0$$

**Theorem 16.** *If $F(x,y,z)$ is differentiable at some point $P_0(x_0,y_0,z_0)$ on the surface, then the tangent plane $\alpha$ to the surface at the point can be calculated by the formula*

$$F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0$$

## 7.1 Lagrange Multipliers

To find the extrema of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$, solve

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$
$$g(x,y,z) = k$$

# 8 Double Integrals

**Theorem 17** (Area of a surface)**.** *If $f_x(x,y)$ and $f_y(x,y)$ are continuous in $D$, then the area of the surface $\sigma : z = f(x,y)$ above $D$ is equal to*

$$S(\sigma) = \iint_D \sqrt{1 + \left(f_x(x,y)\right)^2 + \left(f_y(x,y)\right)^2}\,\mathrm{d}A$$

**Definition 14** (Centre of mass)**.** If $\rho(x,y)$ is the density function of a thin body,

$$m = \iint_D \rho(x,y)\,\mathrm{d}A$$

$$(x_{\mathrm{COM}}, y_{\mathrm{COM}}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

where

$$M_x = \iint_D y\rho(x,y)\,\mathrm{d}A$$
$$M_y = \iint_D x\rho(x,y)\,\mathrm{d}A$$

**Definition 15** (Domain of the first kind)**.** A domain $D$ is said to be the domain of the first kind if there exist continuous functions $f_1(x)$ and $f_2(x)$, s.t.

$$D_{\mathrm{I}} = \{(x,y) | a \leq x \leq b, f_1(x) \leq y \leq f_2(x)\}$$

**Theorem 18.** *If $f(x,y)$ is continuous in $D_{\mathrm{I}}$, then*

$$\iint_R f(x,y)\,\mathrm{d}A = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

**Definition 16** (Domain of the second kind)**.** A domain $D$ is said to be the domain of the second kind if there exist continuous functions $g_1(y)$ and $g_2(y)$, s.t.

$$D_{\mathrm{II}} = \{(x,y) | c \leq y \leq d, g_1(y) \leq x \leq g_2(y)\}$$

**Theorem 19.** *If $f(x,y)$ is continuous in $D_{\mathrm{II}}$, then*

$$\iint_R f(x,y)\,\mathrm{d}A = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

# 9 Triple Integrals

**Example 5.** Find $\iiint_E x^2 + y^2 + z^2\,\mathrm{d}V$ where $E$ is bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a, a > 0$.

*Solution.*