Review Session 2

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Example 1.

$$F(x,y) = \left(-\frac{y}{x^2+y^2} + 4y + 3, \frac{x}{x^2+y^2} + 4x + 4y - 2 \right)$$

Calculate $\int\limits_C F \, \mathrm{d}r$ when

1. $c = (x-10)^2 + (y-7)^2 = 1$, in negative direction
2. $c = x^2 + y^2 = 4$, in positive direction

Solution.

$$P = -\frac{y}{x^2+y^2} + 4y + 3$$
$$Q = \frac{x}{x^2+y^2} + 4x + 4y - 2$$

Therefore,

$$Q_x = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + 4$$
$$P_y = -\frac{x^2+y^2-2y^2}{(x^2+y^2)^2} + 4$$

Therefore, as $P_y = Q_x$, the field is conservative in $c = (x-10)^2 + (y-7)^2 = 1$. Therefore, the integral is over a closed curve is 0.

$$\therefore \int\limits_C F \, \mathrm{d}r = 0$$

In the second case, the function is not defined at $(0,0)$. Therefore, the above method cannot be used.
Therefore, by parametrization,

$$
\begin{aligned}
c(t) &= (2\cos t, 2\sin t) \\
\therefore c'(t) &= (-2\sin t, 2\cos t)\,\mathrm{d}t
\end{aligned}
$$

where $t : 0 \to 2\pi$.
Therefore,

$$
\begin{aligned}
\int\limits_C F\,\mathrm{d}r = \quad & \int\limits_0^{2\pi} \left(-\frac{2\sin t}{4} + 4\cdot 2\sin t + 3\right)\cdot(-2\sin t\,\mathrm{d}t) \\
&+ \int\limits_0^{2\pi} \left(\frac{2\cos t}{4} + 4\cdot 2\cos t + 4\cdot 2\sin t - 2\right)\cdot(2\cos t\,\mathrm{d}t)
\end{aligned}
$$

Example 2. Find the minimum n, such that $e = \sum_{k=1}^{n} \dfrac{1}{k!}$ with error < 0.001

Solution.

$$
\begin{aligned}
e^x &= \sum_{k=1}^{n} \frac{x^k}{k!} + \frac{e^c x^{n+1}}{(n+1)!} \\
\therefore e &= \sum_{x=1}^{n} \frac{1}{k!} + \frac{e^c}{(n+1)!}
\end{aligned}
$$

To find the minimum n, we need to check both upper and lower conditions for the error term. Therefore,

$$
\frac{1}{(n+1)!} < \frac{e^c}{(n+1)!} < \frac{e}{(n+1)!}
$$

Therefore, solving,

$$
n = 6
$$

Example 3. Find the volume of the body bounded by $x = 0$, $x + y = 8$, $z = \dfrac{3}{4}y$, $z = \dfrac{3}{2}\sqrt{y}$.

Solution. $z = \dfrac{3}{2}\sqrt{y}$ is above $z = \dfrac{3}{4}y$ for $y \in (0,4)$. $z = \dfrac{3}{4}y$ is above $z = \dfrac{3}{2}\sqrt{y}$ for $y \in (4,8)$.
Therefore,

$$\iiint\limits_E \mathrm{d}V = \int\limits_0^4 \int\limits_0^{8-y} \int\limits_{\frac{3}{4}y}^{\frac{3}{2}\sqrt{y}} \mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

Example 4. What is larger, e^π or π^e?

Solution.

$$\begin{aligned} \pi^e &\lesseqgtr e^\pi \\ \iff e\ln\pi &\lesseqgtr \pi \end{aligned}$$

Let

$$\begin{aligned} f(x) &= e\ln x - x \\ \therefore f'(x) &= \frac{e}{x} - 1 \end{aligned}$$

Therefore,

$$\begin{aligned} f'(x) > 0 &\qquad \iff \qquad & x < e \\ f'(x) < 0 &\qquad \iff \qquad & x > e \end{aligned}$$

Therefore, as $\pi > e$,

$$\begin{aligned} f(\pi) < f(e) &< 0 \\ \therefore e\ln\pi - \pi &< 0 \\ \therefore e\ln\pi &< \pi \\ \therefore \pi^e &< e^\pi \end{aligned}$$

Example 5. Find

$$\int\limits_C -\frac{y}{x^2}\sin\frac{y}{x}\,\mathrm{d}x + \frac{1}{x}\sin\frac{y}{x}\,\mathrm{d}y$$

where C is the union of $y = (x-1)^2 + 1$ in $[1,2]$ and $y = (x-3)^2 + 1$ in $[2,3]$, both directed clockwise.

Solution.

$$
\begin{aligned}
Q_x &= -\frac{1}{x^2}\sin\frac{y}{x} + \frac{1}{x}\cos\frac{y}{x}\cdot\left(-\frac{y}{x^2}\right) \\
P_y &= -\frac{1}{x^2}\sin\frac{y}{x} + \frac{1}{x}\cos\frac{y}{x}\cdot\left(-\frac{y}{x^2}\right)
\end{aligned}
$$

Therefore, the field is conservative for $x \neq 0$.
Therefore, the integral is path independent. Therefore, parametrizing and solving,

$$
\int\limits_C -\frac{y}{x^2}\sin\frac{y}{x}\,\mathrm{d}x + \frac{1}{x}\sin\frac{y}{x}\,\mathrm{d}y = -\cos\frac{1}{3} + \cos 1
$$