DIFFERENTIAL AND INTEGRAL METHODS - EXERCISE 11

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(1). Check whether the following functions are continuous at $(0,0)$:

(a)

$$f(x,y) = \begin{cases} \frac{x}{3x+5y} & (x,y) \neq (0,0) \\ 5 & (x,y) = (0,0) \end{cases}$$

$$\therefore \lim_{(x,y) \overset{y=kx}{\to} (0,0)} \frac{x}{3x+5y} = \lim_{x \to 0} \frac{x}{3x+5kx} = \frac{1}{3+5k}$$

Therefore, as the limit depends on k, the limit does not exist.
Therefore, $f(x,y)$ is not continuous at $(0,0)$.

(b)

$$f(x,y) = \begin{cases} \frac{x^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\therefore \lim_{(x,y) \overset{y=kx}{\to} (0,0)} \frac{x^2}{x^2+y^2} = \lim_{x \to 0} \frac{x^2}{x^2+k^2x^2} = \frac{1}{1+k^2}$$

Therefore, as the limit depends on k, the limit does not exist.
Therefore, $f(x,y)$ is not continuous at $(0,0)$.

(2). Find the domain of definition of the following functions:

(a). $f(x,y) = \sqrt{\frac{1+x+y}{1-x-2y}}$

$1 - x - 2y \neq 0$
$\therefore x + 2y \neq 1$

$1 + x + y \geq 0$ & $1 - x - 2y \geq 0$

or

$1 + x + y \leq 0$ & $1 - x - 2y \leq 0$

Therefore,

$D_1 = 1 - x \geq y \geq -1 - x$
$D_2 = 1 - x \leq y \leq -1 - x$
$D = D_1 \cap D_2$

(b). $f(x,y) = \dfrac{1}{\cos(x^2+y^2)}$

$$\cos(x^2+y^2) > 0$$
$$\therefore x^2+y^2 \in \left(-\frac{\pi}{2}+n\pi, \frac{\pi}{2}+n\pi\right)$$

(3). Find the following limits:

(a). $\displaystyle\lim_{(x,y)\to\left(\pi/2,0\right)} \sin x + \left(\ln\frac{x+y}{x-y}\right)\frac{x}{\sqrt{x^2+y^2}}$

$$\begin{aligned}\lim_{(x,y)\to\left(\pi/2,0\right)} \sin x + \left(\ln\frac{x+y}{x-y}\right)\frac{x}{\sqrt{x^2+y^2}} &= \sin\frac{\pi}{2} + \ln(1)\cdot 1\\ &= 1\end{aligned}$$

(b). $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{\sqrt{x^2+y^2+x^2+1}-1}{\sqrt{x^2+y^2+z^2}}$

$$\begin{aligned}\lim_{(x,y,z)\to(0,0,0)} \frac{\sqrt{x^2+y^2+x^2+1}-1}{\sqrt{x^2+y^2+z^2}} &= \lim_{(x,y,z)\to(0,0,0)} \frac{\left(\sqrt{x^2+y^2+x^2+1}-1\right)\left(\sqrt{x^2+y^2+z^2+1}+1\right)}{\left(\sqrt{x^2+y^2+z^2}\right)\left(\sqrt{x^2+y^2+z^2+1}+1\right)}\\ &= \lim_{(x,y,z)\to(0,0,0)} \frac{\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}+1}\\ &= 0\end{aligned}$$

(c). $\displaystyle\lim_{x\to\infty, y\to 4} \left(1+\frac{1}{x}\right)^{\dfrac{x^2}{x+y}}$

$$\begin{aligned}\lim_{\substack{x\to\infty\\ y\to 4}} \left(1+\frac{1}{x}\right)^{\dfrac{x^2}{x+y}} &= \lim_{\substack{x\to\infty\\ y\to 4}} \left(1+\frac{1}{x}\right)^{x\cdot\dfrac{1}{1+y/x}}\\ &= \lim_{\substack{x\to\infty\\ y\to 4}} e^{\frac{x^2}{x+y}}\\ &= e\end{aligned}$$

(d). $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$

$$\begin{aligned}\lim_{(x,y)\overset{y=kx}{\to}(0,0)} \frac{x-y}{x+y} &= \lim_{x\to 0}\frac{x-kx}{x+kx}\\ &= \frac{1-k}{1+k}\end{aligned}$$

Therefore, as the limit depends on k, the limit does not exist.
Therefore, $f(x,y)$ is not continuous at $(0,0)$.

(e). $\displaystyle\lim_{(x,y)\to(2,1)} \frac{y\sin(xy-2)}{3xy-6}$

$$\begin{aligned}\lim_{(x,y)\to(2,1)} \frac{y\sin(xy-2)}{3xy-6} &= \lim_{(x,y)\to(2,1)} \frac{y}{3}\frac{\sin(xy-2)}{xy-2}\\ &= \frac{1}{3}\end{aligned}$$

(f). $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2+(x^2-y^2)^2}$

$$\begin{aligned}\lim_{(x,y)\overset{y=kx}{\to}(0,0)} \frac{x^2y^2}{x^2y^2+(x^2-y^2)^2} &= \lim_{x\to 0} \frac{k^2x^4}{k^2x^4+(x^2(1-k^2))^2} \\ &= \frac{k^2}{k^2+(1-k^2)^2}\end{aligned}$$

Therefore, as the limit depends on k, the limit does not exist.
Therefore, $f(x,y)$ is not continuous at $(0,0)$.

(4). Find the following limits or show that they do not exist:

(a). $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

$$\begin{aligned}\lim_{(x,y)\overset{y=kx}{\to}(0,0)} \frac{xy}{x^2+y^2} &= \lim_{x\to 0} \frac{kx^2}{x^2(1+k^2)} \\ &= \frac{k}{1+k^2}\end{aligned}$$

Therefore, as the limit depends on k, the limit does not exist.
Therefore, $f(x,y)$ is not continuous at $(0,0)$.

(b). $\lim_{(x,y)\to(0,0)} \frac{3xy^2-5y^4}{x^2+2y^2}$

$$\begin{aligned}\lim_{(x,y)\overset{y=kx}{\to}(0,0)} \frac{3xy^2-5y^4}{x^2+2y^2} &= \lim_{x\to 0} \frac{3k^2x^3-5k^4x^4}{x^2+2k^2x^2} \\ &= \lim_{x\to 0} \frac{3k^2x-5k^4x^2}{1+2k^2} \\ &= 0\end{aligned}$$

(c). $\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{x^2+y^2}$
Let

$x = r\cos t$
$y = r\sin t$

Therefore,

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{x^2+y^2} &= \lim_{r\to 0} \frac{e^{r^2\sin t\cos t}-1}{r^2} \\ &= \lim_{r\to 0} \frac{e^{r^2\sin t\cos t}\cdot 2r\sin t\cos t}{2r} \\ &= \sin t\cos t\end{aligned}$$

Therefore, as the limit depends on t, the limit does not exist.

(d). $\lim_{x\to\infty,y\to\infty} \frac{2x+3y}{x^2-xy+y^2}$
Let

$x = r\cos t$
$y = r\sin t$

Therefore,

$$\begin{aligned}
\lim_{\substack{x\to\infty \\ y\to\infty}} \frac{2x+3y}{x^2-xy+y^2} &= \lim_{r\to\infty} \frac{2r\cos t+3r\sin t}{r^2\cos^2 t-r^2\sin t\cos t+r^2\sin^2 t} \\
&= \lim_{r\to\infty} \frac{2\cos t+3\sin t}{r-r\sin t\cos t} \\
&= \lim_{r\to\infty} \frac{1}{r}\cdot\frac{2\cos t+3\sin t}{1-\sin t\cos t} \\
&= 0
\end{aligned}$$

(e). $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2+y^6}$

Let

$x = r\cos t$

$y = r\sin t$

Therefore,

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} &= \lim_{r\to 0} \frac{r^4\cos t\sin^3 t}{r^2\cos^2 t+r^6\sin^6 t} \\
&= \lim_{r\to 0} \frac{r^2\cos t\sin^3 t}{\cos^2 t+r^4\sin^6 t} \\
&= 0
\end{aligned}$$

(f). $\lim\limits_{(x,y)\to(0,0)} \dfrac{3y^4+x^2y^2+3x^2}{x^2+y^4}$

Let

$x = r\cos t$

$y = r\sin t$

Therefore,

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{3y^4+x^2y^2+3x^2}{x^2+y^4} &= \lim_{r\to 0} \frac{3r^4\sin^4 t+r^4\sin^2 t\cos^2 t+3r^2\cos^2 t}{r^2\cos^2 t+r^4\sin^4 t} \\
&= \lim_{r\to 0} \frac{3r^2\sin^4 t+r^2\sin^2 t\cos^2 t+3\cos^2 t}{\cos^2 t+r^2\sin^4 t} \\
&= \frac{3\cos^2 t}{\cos^2 t} \\
&= 3
\end{aligned}$$