Review Session 1

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Example 1. Is the function

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & ; \quad x \neq 0 \\ 1 & ; \quad x = 0 \end{cases}$$

continuous at $x = 0$? Is it differentiable at $x = 0$? If yes, calculate $f'(0)$.

Solution.

$$\begin{aligned} \lim_{x \to 0} \frac{\sin x}{x} &= 1 \\ &= f(0) \end{aligned}$$

Therefore, $f(x)$ is continuous at $x = 0$.

$$\begin{aligned} f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \to 0} \frac{\frac{\sin x - x}{x}}{x} \\ &= \lim_{x \to 0} \frac{\sin x - x}{x^2} \\ &= \lim_{x \to 0} \frac{\cos x - 1}{2x} \\ &= \lim_{x \to 0} \frac{-\sin x}{2} \\ &= 0 \end{aligned}$$

Example 2. Calculate $\displaystyle\int\limits_1^{\sqrt{3}} \frac{\arctan x}{x^2} \, \mathrm{d}x$.

Solution.

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{\arctan x}{x^2}\,\mathrm{d}x &= -\frac{1}{x}\arctan x\Big|_1^{\sqrt{3}} + \int_1^{3} \frac{1}{x}\cdot\frac{1}{x^2+1}\,\mathrm{d}x \\
&= -\frac{1}{\sqrt{3}}\arctan\sqrt{3} + \arctan 1 + \int_1^{\sqrt{3}} \left(\frac{1}{x} - \frac{x}{x^2+1}\right)\mathrm{d}x \\
&= -\frac{\pi}{3\sqrt{3}} + \frac{\pi}{4} + \left(\ln x - \frac{1}{2}\cdot\ln(x^2+1)\right)\Big|_1^{\sqrt{3}} \\
&= -\frac{\pi}{3\sqrt{3}} + \frac{\pi}{4} + \ln\frac{\sqrt{6}}{2}
\end{aligned}$$

Example 3. Write the Taylor's formula for the function $f(x) = \tan(x)$ at $a = 0$ for $n = 1$ and calculate approximately $\tan 0.1$ using the formula and give the estimation of the accuracy of the accuracy.

Solution.

$$\begin{aligned}
\tan x &= \tan(0) + (\sec^2 0)(x) + 2\sec c\tan c2(x^2) \\
&= 0 + x + \sec c\tan cx^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
\tan(0.1) &= 0 + (0.1) + \sec c\tan c(0.1)^2 \\
R_{\max} &= \sec(0.1)\tan(0.1)(0.01)
\end{aligned}$$

Example 4. Find the minimum and maximum of

$$f(x, y) = \cos(2x) + \cos(2y)$$

under the constraint $x - y = \dfrac{\pi}{4}$. Find the points of minimum and maximum.

Solution. Let

$$g(x, y) = x - y$$

Therefore,

$$\begin{aligned}
\nabla f &= \lambda\nabla g \\
g &= \frac{\pi}{4}
\end{aligned}$$

$$
\begin{aligned}
f_x &= \lambda g_x \\
f_y &= \lambda g_y \\
g &= \frac{\pi}{4}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sin(x+y) &= 0 \\
\therefore x+y &= \pi k \\
x-y &= \frac{\pi}{4}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
x &= \frac{\pi}{8} + \frac{\pi k}{2} \\
y &= -\frac{\pi}{8} + \frac{\pi k}{2}
\end{aligned}
$$

For all even k, the points are points of maxima. For all odd k, the points are points of minima. The minimum value of the function is $-\sqrt{2}$ and the maximum value of the function is $\sqrt{2}$.

Example 5. Calculate the line integral

$$
\int_C \left(1 - \frac{y^2}{x^2}\cos\left(\frac{y}{x}\right)\right) \mathrm{d}x + \left(\sin\left(\frac{y}{x}\right) + \frac{y}{x}\cos\left(\frac{y}{x}\right)\right) \mathrm{d}y
$$

where C is any curve which starts at $(1,\pi)$ and ends at $(2,\pi)$ and does not intersect the y-axis.

Solution. Let

$$
\begin{aligned}
P(x,y) &= 1 - \frac{y^2}{x^2}\cos\left(\frac{y}{x}\right) \\
Q(x,y) &= \sin\left(\frac{y}{x}\right) + \frac{y}{x}\cos\left(\frac{y}{x}\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
Q_x &= \cos\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + \left(-\frac{y}{x^2}\right)\cos\left(\frac{y}{x}\right) + \frac{y}{x}\left(-\sin\left(\frac{y}{x}\right)\right)\left(-\frac{y}{x^2}\right) \\
P_y &= -\frac{2y}{x^2}\cos\left(\frac{y}{x}\right) - \frac{y^2}{x^2}\left(-\sin\left(\frac{y}{x}\right)\right)\frac{1}{x}
\end{aligned}
$$

$$
\therefore P_y = Q_x
$$

Therefore, the function is a conservative vector field. Hence, the line integral is independent of the path.
Therefore,

$$\begin{aligned}
\int\limits_C P\,\mathrm{d}x + Q\,\mathrm{d}y &= \int\limits_1^2 \left(Px' + Qy'\right)\mathrm{d}t \\
&= \int\limits_1^2 P\,\mathrm{d}t \\
&= \int\limits_1^2 \left(1 - \frac{\pi^2}{t^2}\cos\left(\frac{\pi}{t}\right)\right)\mathrm{d}t \\
&= \left(t + \pi\sin\left(\frac{\pi}{t}\right)\right)\Bigg|_1^2 \\
&= 1 + \pi
\end{aligned}$$