Lecture 7

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Contents

1 Higher Order Derivatives

Assuming $y = f(x)$ is differentiable, let $g(x) = f'(x)$. If $g(x)$ is differentiable, we say that $f(x)$ is twice differentiable.

$$f''(x) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = D^2 f$$

$f''(x)$ is called the second derivative of f.
Similarly, the n^{th} derivative of f is defined as

$$f^{(n)}(x) = \frac{\mathrm{d}^n f}{\mathrm{d}x^n} = \frac{\mathrm{d}^n y}{\mathrm{d}x^n} = D^{(n)} f$$

2 Derivative of an Implicit Function

The function $y = f(x)$ is called implicit if it is given by $F(x, y) = k$, where k is a constant.

3 Parametric Curves and Their Derivatives

If $x = f(t)$ and $y = g(t)$ are functions of t, then the set of all points $(x, y) = (f(t), g(t))$ is called a <u>curve</u> in the plane $\mathbb{R}^2$.
$x = f(t)$ and $y = g(t)$ are <u>parametric equations</u> of the curve, and t is called a <u>parameter</u>. If $a \leq t \leq b$, then, $(f(a), g(a))$ is called <u>the beginning</u> of the curve, and $(g(a), g(b))$ is called <u>the end</u> of the curve.

Theorem 1 *If $x = x(t)$ and $y = y(t)$ are differentiable, $x'(t) \neq 0$, and y as a function of x is also differentiable, then,*

$$y'(x_0) = \frac{y'(t_0)}{x'(t_0)}$$

4 Linearisation and Differential

Definition 1 *If $y = f(x)$ is differentiable at x_0, then the function $L(x) = f(x_0) + f'(x_0)(x - x_0)$, i.e. the tangent at $(x_0, f(x_0))$, is called a <u>linearisation</u> of $f(x)$ at x_0. The approximation of $f(x) \approx L(x)$ about x_0 is a <u>standard linear approximation</u> of $f(x)$ at x_0. The point x_0 is the <u>centre of the approximation</u>.*

Definition 2 *Assuming $y = f(x)$ is differentiable at x_0, $\mathrm{d}x = \Delta x$ is a <u>differential of x</u>, $\mathrm{d}y = \Delta y$ is a <u>differential of y</u>.*

$$\mathrm{d}y \neq \Delta y$$
$$\mathrm{d}y \approx \Delta y \text{ (about } x_0\text{)}$$

$$\begin{aligned} f(x_0+\Delta x)-f(x_0) &\approx f'(x_0)\Delta x \\ \therefore f(x_0+\Delta x) &\approx f(x_0)+f'(x_0)\Delta x \end{aligned}$$

4.1 Properties

Assuming that $f(x)$ and $g(x)$ are differentiable, and c is a constant,

$$\begin{aligned} \mathrm{d}c &= 0 \, \mathrm{d}x = 0 \\ \mathrm{d}(cf(x)) &= c \, \mathrm{d}f(x) \\ \mathrm{d}(f \pm g) &= \mathrm{d}f \pm \mathrm{d}g \\ \mathrm{d}(fg) &= \mathrm{d}f \cdot g + f \cdot \mathrm{d}g \\ \mathrm{d}\left(\frac{f}{g}\right) &= \frac{\mathrm{d}f \cdot g - f \cdot \mathrm{d}g}{g^2} \\ \mathrm{d}f(g(x)) &= f'(g(x)) \, \mathrm{d}g \end{aligned}$$

5 Taylor's Formula

Theorem 2 *Let $f(x)$ be differentiable $(n+1)$ times, where $n \in \mathbb{N} \cup \{0\}$ on an open interval about a, and x be an arbitrary point in this interval. Then, there exists a point c, which depends on x, between a and x, s.t.*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n)}(c)}{(n+1)!}(x-a)^{n+1}$$

is called the <u>*Lagrange remainder*</u>