Lecture 12

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Contents

1 Integration of Rational Functions

$$\frac{P(x)}{Q(x)} = M(x) + \frac{R(x)}{Q(x)}$$

Definition 1. A simple rational function of one of the following forms is called a basic rational function.

$$\begin{aligned}
&\frac{A}{x-\alpha} &&; A, \alpha \in \mathbb{R} \\
&\frac{A}{(x-\alpha)^n} &&; A, \alpha \in \mathbb{R}, n \in \mathbb{N} - \{1\} \\
&\frac{Ax+B}{x^2+px+q} &&; A, B, p, q \in \mathbb{R}, p^2 - 4q < 0 \\
&\frac{Ax+B}{(x^2+px+q)^n} &&; A, B, p, q \in \mathbb{R}, p^2 - 4q < 0, n \in \mathbb{N} - \{1\}
\end{aligned}$$

Theorem 1. *Any simple rational function* $\frac{R(x)}{Q(x)}$ *can be represented as a sum of basic rational functions.*

Remark 1.

$$\begin{aligned}
\int \frac{P(x)}{Q(x)} \, \mathrm{d}x &= \int M(x) \, \mathrm{d}x + \int \frac{R(x)}{Q(x)} \, \mathrm{d}x \\
&= \int M(x) \, \mathrm{d}x + \int \sum (\text{basic rational function}) \, \mathrm{d}x \\
&= \int M(x) \, \mathrm{d}x + \sum \int (\text{basic rational function}) \, \mathrm{d}x
\end{aligned}$$

1.1 Integrals of Basic Rational Functions

$$\int \frac{A}{x-\alpha} \, \mathrm{d}x = A \ln|x-\alpha| + c$$

$$\int \frac{A}{(x-\alpha)^n} \, \mathrm{d}x = A \frac{(x-\alpha)^{-n+1}}{-n+1} + c$$

$$\int \frac{Ax+B}{x^2+px+q} \, \mathrm{d}x = \int \frac{Ax+B}{\left(x+\frac{p}{2}\right)^2 + q - \frac{p^2}{4}}$$

Let $a = q - \dfrac{p^2}{4}$. Let $t = x + \dfrac{p}{2}$. Therefore, $\mathrm{d}t = \mathrm{d}x$.

$$
\begin{aligned}
\therefore \int \frac{Ax+B}{\left(x+\frac{p}{2}\right)^2 + q - \frac{p^2}{4}} &= \frac{A\left(t-\frac{p}{2}\right)+B}{t^2+a^2}\,\mathrm{d}t \\
&= \frac{A}{2}\int \frac{2t}{t^2+a^2}\,\mathrm{d}t + \left(B - \frac{Ap}{2}\right)\int \frac{1}{t^2+a^2}\,\mathrm{d}t \\
&= \frac{A}{2}\ln(t^2+a^2) + \frac{B-\frac{AP}{2}}{a}\arctan\left(\frac{t}{a}\right) + c \\
&= \frac{A}{2}\ln(x^2+px+q) + \frac{B-\frac{Ap}{2}}{a}\arctan\left(\frac{x+\frac{p}{2}}{a}\right) + c
\end{aligned}
$$

$$
\int \frac{Ax+B}{(x^2+px+q)^n}\,\mathrm{d}x = \int \frac{Ax+B}{\left(\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}\right)^n}
$$

Let $a = q - \dfrac{p^2}{4}$. Let $t = x + \dfrac{p}{2}$. Therefore, $\mathrm{d}t = \mathrm{d}x$.

$$
\begin{aligned}
\therefore \int \frac{Ax+B}{\left(\left(x+\frac{p}{2}\right)^2 + q - \frac{p^2}{4}\right)^n} &= \frac{A\left(t-\frac{p}{2}\right)+B}{(t^2+a^2)^n}\,\mathrm{d}t \\
&= \frac{A}{2}\int \frac{2t}{(t^2+a^2)^n}\,\mathrm{d}t + \left(B - \frac{Ap}{2}\right)\int \frac{1}{(t^2+a^2)^n}\,\mathrm{d}t \\
&= \frac{A}{2}\frac{(t^2+a^2)^{-n+1}}{-n+1} + \left(B - \frac{Ap}{2}\right)\int \frac{1}{(t^2+a^2)^n}\,\mathrm{d}t + c
\end{aligned}
$$

1.2 Finding Basic Rational Functions

1.2.1 Type 1

If

$$Q(x) = (a_1x+b_1)\dots(a_kx+b_k)$$

and the multipliers are different from each other, then,

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \cdots + \frac{A_2}{a_kx+b_k}$$

Example 1.

$$
\begin{aligned}
\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,\mathrm{d}x &= \int \frac{x^2+2x-1}{x(2x^2+3x-2)}\,\mathrm{d}x \\
&= \int \frac{x^2+2x-1}{(x)(2x-1)(x+2)} \\
\frac{x^2+2x-1}{(x)(2x-1)(x+2)} &= \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2} \\
&= \frac{A(2x-1)(x+2) + B(x)(x+2) + C(x)(2x-1)}{(x)(2x-1)(x+2)} \\
&= \frac{x^2(2A+B+C) + x(3A+2B-C) - 2A}{(x)(2x-1)(x+2)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
2A+B+C &= 1 \\
3A+2B-C &= 2 \\
-2A &= -1
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A &= \frac{1}{2} \\
B &= \frac{1}{5} \\
C &= -\frac{1}{10}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int \frac{x^2+2x-1}{(x)(2x-1)(x+2)} &= \int \frac{\frac{1}{2}}{x} + \frac{\frac{1}{5}}{2x-1} + \frac{-\frac{1}{10}}{x+2} \\
&= \frac{1}{2}\ln|x| + \frac{1}{5}\frac{\ln|2x-1|}{2} - \frac{1}{10}\ln|x+2| + d \\
&= \frac{1}{2}\ln|x| + \frac{1}{10}\ln\left|\frac{2x-1}{x+2}\right| + d
\end{aligned}
$$

1.2.2 Type 2

If

$$
Q(x) = (a_1x+b_1)^m(a_2x+b_2)\dots(a_kx+b_k)
$$

and the multipliers are different from each other, $m \in \mathbb{N} - \{1\}$, then,

$$
\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \dots + \frac{A_m}{(a_1x+b_1)^m} + \frac{B_2}{a_2x+b_2} + \dots + \frac{B_k}{a_kx+b_k}
$$

Example 2.

$$\begin{aligned}
\int \frac{-x+2}{x(x-1)^2}\,\mathrm{d}x &= \int \left(\frac{A_1}{x} + \frac{B_1}{x-1} + \frac{B_2}{(x-1)^2} \right) \mathrm{d}x \\
\frac{-x+2}{x(x-1)^2} &= \frac{A_1(x-1)^2 + B_1(x)(x-1) + B_2 x}{x(x-1)^2} \\
&= \frac{x^2(A_1+B_1) + x(-2A_1 - B_1 + B_2) + A_1}{x(x-1)^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
A_1 + B_1 &= 0 \\
-2A_1 - B_1 + B_2 &= -1 \\
A_1 &= 2
\end{aligned}$$

Therefore,

$$\begin{aligned}
&A_1 = 2 \\
&B_1 = -2 \\
&B_2 = 1
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int \frac{-x+2}{x(x-1)^2}\,\mathrm{d}x &= \int \left(\frac{2}{x} + \frac{-2}{x-1} + \frac{1}{(x-1)^2} \right) \mathrm{d}x \\
&= 2\ln|x| - 2\ln|x-1| - \frac{1}{x-1} + c \\
&= 2\ln\left|\frac{x}{x-1}\right| - \frac{1}{x-1} + x
\end{aligned}$$

1.2.3 Type 3

If

$$Q(x) = (ax^2 + bx + c)(a_2 x + b_2)\ldots(a_k x + b_k)$$

and the multipliers are different from each other, $b^2 - 4ac < 0$, then,

$$\frac{R(x)}{Q(x)} = \frac{Ax+B}{ax^2+bx+c} + \frac{A_2}{a_2 x + b_2} + \cdots + \frac{A_k}{a_k x + b_k}$$

Example 3.

$$\begin{aligned}
\int \frac{2x^2 - x + 4}{x^3 + 4x}\,\mathrm{d}x &= \int \frac{2x^2 - x + 4}{x(x^2 + 4)}\,\mathrm{d}x \\
&= \int \left(\frac{A}{x} + \frac{Bx + C}{x^2 + 4}\right)\mathrm{d}x \\
\frac{2x^2 - x + 4}{x^3 + 4x} &= \frac{A(x^2 + 4) + (Bx + c)x}{x(x^2 + 4)} \\
&= \frac{x^2(A + B) + x(C) + 4A}{x(x^2 + 4)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
A + B &= 2 \\
C &= -1 \\
4A &= 4
\end{aligned}$$

Therefore,

$$\begin{aligned}
A &= 1 \\
B &= 1 \\
C &= -1
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int \frac{2x^2 - x + 4}{x(x^2 + 4)}\,\mathrm{d}x &= \int \left(\frac{1}{x} + \frac{x - 1}{x^2 + 4}\right)\mathrm{d}x \\
&= \ln|x| + \int \frac{x}{x^2 + 4}\,\mathrm{d}x - \int \frac{1}{x^2 + 4}\,\mathrm{d}x + d \\
&= \ln|x| + \frac{1}{2}\ln(x^2 + 4) - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + d
\end{aligned}$$

1.2.4 Type 4

If

$$Q(x) = (ax^2 + bx + c)^m (a_2 x + b_2) \dots (a_k x + b_k)$$

and the multipliers are different from each other, $m \in \mathbb{N} - \{1\}$, $b^2 - 4ac < 0$, then,

$$\frac{R(x)}{Q(x)} = \frac{A_1 x + B_1}{ax^2 + bx + c} + \dots + \frac{A_m x + B_m}{(ax^2 + bx + c)^m} + \frac{C_2}{a_2 x + b_2} + \dots + \frac{C_k}{a_k x + b_k}$$