Lecture 6

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Contents

1 Theorem: Derivative of Inverse Functions

Let $f(x)$ be invertible and continuous in an open interval about x_0. If $\exists f'(x_0) \neq 0$, then, the inverse function $x = g(y)$ is differentiable at $y = f(x_0)$ and

$$g'(y_0) = \frac{1}{f'(x_0)}$$

1.1 Examples

1.1.1 Example 1

$$\begin{aligned} y = f(x) &= \tan x \\ \therefore (\tan^{-1})'y &= \frac{1}{\tan' x} \\ &= \frac{1}{\frac{1}{\cos^2 x}} \\ &= \frac{1}{1+\tan^2 x} \\ &= \frac{1}{1+y^2} \end{aligned}$$

Similarly,

$$(\cot^{-1})'x = -\frac{1}{1+x^2}$$

2 Chain Rule

Let $y = f(u)$ be differentiable at u_0, and $u = g(x)$ be differentiable at x_0, s.t. $u_o = g(x_0)$. Then, $y = f(g(x))$ is differentiable at x_0, and,

$$y'(x_0) = f'(u_0) \cdot g'(x_0)$$

2.1 Proof

$$g'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

Therefore, by Theorem 2,

$$\begin{aligned} \therefore \frac{\Delta u}{\Delta x} &= g'(x_0) + \alpha_1(\Delta x); \alpha_1(\Delta x) \to 0 \text{ if } \Delta x \to 0 \\ \therefore \frac{\Delta y}{\Delta u} &= f'(u_0) + \alpha_2(\Delta u); \alpha_1(\Delta u) \to 0 \text{ if } \Delta u \to 0 \end{aligned}$$

Therefore,

$$\begin{aligned}
\Delta u &= (g'(x_0) + \alpha_1)\Delta x \\
\Delta y &= (f'(u_0) + \alpha_2)\Delta u \\
\therefore \Delta y &= (f'(u_0) + \alpha_2)(g'(x_0) + \alpha_1)\Delta x \\
\therefore \frac{\Delta y}{\Delta x} &= (f'(u_0) + \alpha_2)(g'(x_0) + \alpha_1)
\end{aligned}$$

$$\begin{aligned}
\Delta x \to 0 &\Rightarrow \Delta u \to 0, \alpha_1 \to 0 \\
&\Rightarrow \alpha_2 \to 0
\end{aligned}$$

Substituting,

$$y'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \big(f'(u_0) + a_2\big)\big(g'(x_0) + \alpha_1\big) = f'(u_0) \cdot g'(x_0)$$

3 Fermat Theorem

Let $f(x)$ be defined on an open interval (a, b) and differentiable at $x_0 \in (a, b)$. If $f(x)$ has its extremum at x_0, then, $f'(x_0) = 0$

3.1 Proof

Assume that $f(x_0)$ is the maximum value of $f(x)$ on (a, b). Then, $\forall \Delta x, f(x_0 + \Delta x) \leq f(x_0)$.

Case I: $\Delta x > 0$

$$\begin{aligned}
\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} &\leq 0 \\
\therefore \text{RHD} = \lim_{\Delta x \to 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} &\leq 0
\end{aligned}$$

Case II: $\Delta x < 0$

$$\begin{aligned}
\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} &\geq 0 \\
\therefore \text{LHD} = \lim_{\Delta x \to 0^-} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} &\geq 0
\end{aligned}$$

$$\begin{aligned}
\exists f'(x_0) &\Rightarrow \text{LHD} = \text{RHD} \\
\therefore 0 &\leq f'(x_0) \leq 0 \\
\therefore f'(x_0) &= 0
\end{aligned}$$

4 Rolle Theorem

Let $f(x)$ be defined on $[a, b]$, s.t.

(1) f is continuous on $[a, b]$

(2) f is differentiable on (a, b)

(3) $f(a) = f(b)$

Then, $\exists c \in (a, b)$, s.t. $f'(c) = 0$.

4.1 Proof

By Weirstrauss Theorem, as $f(x)$ is continuous on $[a, b]$, $f(x)$ has its maximum M and minimum m on $[a, b]$.

Case I: $m = M$

$$\begin{aligned} f(x) &= \text{constant} \\ \therefore f'(x) &= 0 \text{ on } [a, b] \end{aligned}$$

Case II: $m < M$

Atleast one of m and M must be in (a, b), otherwise $f(a) \neq f(b)$, which contradicts (3).
Let $M = c \in (a, b)$. Therefore, by Theorem 3, $f'(c) = 0$

5 Lagrange Theorem

Let $f(x)$ be defined on $[a, b]$, s.t.

(1) f is continuous on $[a, b]$

(2) f is differentiable on (a, b)

Then, $\exists c \in (a, b)$, s.t. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

6 Theorem

Let $f(x)$ be continuous on $(x_0 - \delta, x_0 + \delta)$ and differentiable on $(x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$.
If $\lim\limits_{x \to x_0^+} f'(x) = \lim\limits_{x \to x_0^-} f'(x) = L$, then, $\exists f'(x_0) = L$.