# Lecture 15

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## Contents

# 1 Applications of Definite Integrals

## 1.1 Centre of Mass

**Example 1.** Find the centre of mass of a metal rod $[a, b]$ with density $\rho(x)$ and mass $m$.

*Solution.* Dividing the rod into $n$ parts, from $x_0 = a$ to $x_n = b$, and assuming that the mass $\Delta m_i$ is concentrated at $c_i$, we have a system of masses $\Delta m_1, \ldots, \Delta m_n$.

$$\therefore x_{\text{COM}} \approx \frac{\sum\limits_{i=1}^{n} c_i \Delta m_i}{\sum\limits_{i=1}^{n} \Delta m_i}$$

$$\therefore x_{\text{COM}} = \lim_{\Delta T \to 0} \frac{\sum\limits_{i=1}^{n} c_i \rho(c_i) \Delta x_i}{\sum\limits_{i=1}^{n} \rho(c_i) \Delta x_i} = \frac{\int\limits_a^b x\rho(x)\,\mathrm{d}x}{\int\limits_a^b \rho(x)\,\mathrm{d}x}$$

# 2 Improper Integrals

**Definition 1** (Improper integral of the first kind)**.** Assume that there exists the integral

$$I(t) = \int\limits_a^t f(x)\,\mathrm{d}x \qquad ; \quad t \geq a$$

If $\exists I = \lim\limits_{t\to\infty} I(t)$, then $I$ is called an improper integral of the first kind. The improper integral is said to converge. $I$ is denoted as

$$I = \int\limits_a^{+\infty} f(x)\,\mathrm{d}x = \lim_{t\to+\infty} \int\limits_a^t f(x)\,\mathrm{d}x$$

Otherwise, the improper integral is said to diverge.
Similarly for

$$I = \int\limits_{-\infty}^{a} f(x)\,\mathrm{d}x = \lim_{t \to -\infty} \int\limits_{t}^{a} f(x)\,\mathrm{d}x$$

If both

$$\int\limits_{-\infty}^{a} f(x)\,\mathrm{d}x$$

and

$$\int\limits_{a}^{+\infty} f(x)\,\mathrm{d}x$$

converge, then

$$\int\limits_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = \int\limits_{-\infty}^{a} f(x)\,\mathrm{d}x + \int\limits_{a}^{+\infty} f(x)\,\mathrm{d}x$$

**Example 2.**

$$\int\limits_{1}^{+\infty} \frac{1}{x^2}\,\mathrm{d}x$$

*Solution.*

$$\begin{aligned}
\int\limits_{1}^{+\infty} \frac{1}{x^2}\,\mathrm{d}x &= \lim_{t \to +\infty} \int\limits_{1}^{t} \frac{1}{x^2}\,\mathrm{d}x \\
&= \lim_{t \to +\infty} \left(-\frac{1}{x}\right)\Bigg|_{1}^{t} \\
&= \lim_{t \to +\infty} \left(-\frac{1}{t} + 1\right) \\
&= 1
\end{aligned}$$

Hence, the integral converges.

**Example 3.**

$$\int_{1}^{+\infty} \frac{1}{x}\,\mathrm{d}x$$

*Solution.*

$$\begin{aligned}\int_{1}^{+\infty} \frac{1}{x}\,\mathrm{d}x &= \lim_{t\to+\infty}\int_{1}^{t} \frac{1}{x}\,\mathrm{d}x\\ &= \lim_{t\to+\infty} (\ln x)\Big|_{1}^{t}\\ &= \lim_{t\to+\infty}(\ln t - \ln 1)\\ &= +\infty\end{aligned}$$

Hence, the integral diverges.

**Theorem 1** (Direct comparison test)**.** *Let $f(x)$ and $g(x)$ be two functions defined on $[a, +\infty)$ and Riemann integrable over $[a, t], \forall t \geq a$. Assume that $\exists b \geq a$, s.t. $f(x) \geq g(x) \geq 0, \forall x \geq b$. Then,*

1. *if $\int_{a}^{+\infty} f(x)\,\mathrm{d}x$ converges, then $\int_{a}^{+\infty} g(x)\,\mathrm{d}x$ converges.*
2. *if $\int_{a}^{+\infty} g(x)\,\mathrm{d}x$ diverges, then $\int_{a}^{+\infty} f(x)\,\mathrm{d}x$ diverges.*

**Example 4.** Show that

$$\int_{0}^{+\infty} e^{-x^2}\,\mathrm{d}x$$

converges.

*Solution.* If

$$\begin{aligned} x &\geq 1\\ \implies x^2 &\geq x\\ \implies -x^2 &\leq -x\end{aligned}$$

Therefore, $\forall x \geq 1$

$$0 \leq e^{-x^2} \leq e^{-x}$$

$$
\begin{aligned}
\int_0^{+\infty} e^{-x}\,\mathrm{d}x &= \lim_{t\to+\infty}\int_0^{t} e^{-x}\,\mathrm{d}x \\
&= \lim_{t\to+\infty}\left(-e^{-t}+1\right) \\
&= 1
\end{aligned}
$$

Hence, the integral converges.
Therefore, by the Direct comparison test,

$$\int_0^{+\infty} e^{-x^2}\,\mathrm{d}x$$

also converges.

**Example 5.** Show that

$$\int_1^{+\infty} \frac{1+e^{-x}}{x}\,\mathrm{d}x$$

diverges.

*Solution.* $\forall x \geq 1$,

$$\frac{1+e^{-x}}{x} \geq \frac{1}{x} \geq 0$$

and

$$\int_1^{+\infty} \frac{1}{x}\,\mathrm{d}x$$

diverges.
Therefore, by the Direct comparison test,

$$\int_1^{+\infty} \frac{1+e^{-x}}{x}\,\mathrm{d}x$$

also diverges.

**Definition 2** (Absolute integrability)**.** $f(x)$ is said to be absolutely integrable over $[a, +\infty)$ if the improper integral

$$\int\limits_a^{+\infty} |f(x)| \, \mathrm{d}x$$

converges.

**Theorem 2.** *If there exists the definite integral*

$$\int\limits_a^t f(x) \, \mathrm{d}x, \forall t \geq a$$

*and $f(x)$ is absolutely integrable over $[a, +\infty)$, then*

$$\int\limits_a^{+\infty} f(x) \, \mathrm{d}x$$

*converges.*

**Definition 3** (Improper integral of the second kind)**.** Assume that $f(x)$ is defined and is not bounded on $[a, b)$ and assume that $\forall a \leq t \leq b$, there exists the definite integral

$$I(t) = \int\limits_a^t f(x) \, \mathrm{d}x$$

If there exists the limit

$$I = \lim_{t \to b^-} I(t)$$

then $I$ is called an improper integral of the second kind of $f(x)$ on $[a, b)$ and is denoted as

$$I = \lim_{t \to b^-} \int\limits_a^t f(x) \, \mathrm{d}x = \int\limits_a^b f(x) \, \mathrm{d}x$$

Similarly if $f(x)$ is not bounded at $a$,

$$\lim_{t \to a^+} \int\limits_t^a f(x) \, \mathrm{d}x = \int\limits_a^b f(x) \, \mathrm{d}x$$

If $f(x)$ is not bounded at $c$, s.t. $a < c < b$, and

$$\exists \int\limits_a^c f(x)\,\mathrm{d}x$$

and

$$\exists \int\limits_c^b f(x)\,\mathrm{d}x$$

then,

$$\exists \int\limits_a^b f(x)\,\mathrm{d}x = \int\limits_a^c f(x)\,\mathrm{d}x + \int\limits_c^b f(x)\,\mathrm{d}x$$

**Example 6.**

$$\int\limits_0^3 \frac{\mathrm{d}x}{x-1}$$

*Solution.*

$$\begin{aligned}
\int\limits_0^1 \frac{\mathrm{d}x}{x-1}\,\mathrm{d}x &= \lim_{t\to 1^-} \int\limits_0^t \frac{\mathrm{d}x}{x-1} \\
&= \lim_{t\to 1^-} \ln|x-1|\Big|_0^1 \\
&= \lim_{t\to 1^-} (\ln|t-1| - \ln 1) \\
&= -\infty
\end{aligned}$$

Therefore, as

$$\int\limits_0^1 \frac{\mathrm{d}x}{x-1}$$

diverges,

$$\int\limits_0^3 \frac{\mathrm{d}x}{x-1}$$

also diverges.

# 3 Definite Integral Calculations using Taylor's Formula

$$\begin{aligned}
f(x) = f(a) \\
&+ \frac{f'(a)}{1!}(x-a) \\
&+ \dots \\
&+ \frac{f^{(n)}(a)}{n!}(x-a)^n \\
&+ \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}
\end{aligned}$$

$$\begin{aligned}
\therefore \int\limits_a^b f(x)\,\mathrm{d}x = f(a)(b-a) \\
&+ \frac{f'(a)}{1!}\frac{(b-a)^2}{2} \\
&+ \frac{f''(a)}{2!}\frac{(b-a)^3}{3} \\
&+ \dots \\
&+ \frac{f^{(n)}(a)}{n!}\frac{(b-a)^{n+1}}{n+1} \\
&+ \frac{1}{(n+1)!}\int\limits_a^b f^{(n+1)}(c)(x-a)^{n+1}\,\mathrm{d}x
\end{aligned}$$

$$R_n(I) = \frac{1}{(n+1)!}\int\limits_a^b f^{(n+1)}(c)(x-a)^{n+1}\,\mathrm{d}x$$

is called the integral Lagrange remainder.

**Example 7.** Calculate

$$\int\limits_0^1 e^{x^2}\,\mathrm{d}x$$

with accuracy 0.01.

*Solution.* For $0 < c < x^2 \le 1$,

$$e^{x^2} = 1 + \frac{x^2}{1!} + \cdots + \frac{x^2 n}{n!} + \frac{x^{2n+2}}{(n+1)!} e^c$$

$$\therefore \int\limits_0^1 e^{x^2}\,\mathrm{d}x = 1 + \frac{1}{1!}\cdot\frac{1}{3} + \frac{1}{2!}\cdot\frac{1}{5} + \cdots + \frac{1}{n!}\cdot\frac{1}{2n+1}$$

$$+ \frac{1}{(n+1)!}\int\limits_0^1 e^c x^{2n+2}\,\mathrm{d}x$$

For $n = 4$,

$$|R_n(I)| \le 0.01$$

Therefore,

$$\int\limits_0^1 e^{x^2}\,\mathrm{d}x \approx 1 + \frac{1}{1!}\cdot\frac{1}{3} + \frac{1}{2!}\cdot\frac{1}{5} + \frac{1}{3!}\cdot\frac{1}{7} + \frac{1}{4!}\cdot\frac{1}{9}$$