Lecture 16

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Contents

1 Functions of Multiple Variables

Definition 1 (Function of multiple variables)**.** Let D be a set of points from $\mathbb{R}^2$. A function f of two variables is a law which corresponds to each pair $(x, y) \in D$ a unique real number z. It is denoted as $f(x, y) = z$. The set D is called the domain of definition of $f(x, y)$ and a set of all values of $f(x, y)$ is called the image of f.

Definition 2 (Graph)**.** If f is a function of two variables with the domain of definition D, then the graph of f is the set

$$S = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y), (x, y) \in D\}$$

Definition 3 (Limit)**.** Assume that $f(x, y)$ is defined on some open neighbourhood of (a, b), except perhaps at (a, b) itself. A number L is called the limit of $f(x, y)$ at (a, b) if, as $(x, y) \to (a, b)$, $f(x, y) \to L$ over any curve which ends at (a, b).

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

Definition 4 (Continuous function)**.** A function $f(x, y)$ is defined in an open neighbourhood of (a, b) is caed continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

Remark 1. If there exist two curves C_1 and C_1, s.t.

$$\lim_{(x,y)\overset{C_1}{\to}(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = L_1 \quad \lim_{(x,y)\overset{C_2}{\to}(0,0)} \frac{x^2 - y^2}{x^2 + y^2} \qquad = L_2$$

and $L_1 \neq L_2$, then, $\nexists \lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$.

Example 1. Does $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ exist?

Solution. If $y = 0$, $x \to 0^+$,

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} &= \frac{x^2}{x^2} \\ &= 1 \end{aligned}$$

If $x = 0$, $y \to 0^+$,

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} &= \frac{-y^2}{y^2} \\ &= -1 \end{aligned}$$

Hence, the limit does not exist.

2 Partial Derivatives of Functions of Multiple Variables

Definition 5 (Partial derivative of $f(x,y)$)**.** The limit

$$f_x(a,b) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x, b) - f(a,b)}{\Delta x}$$

if it exists, is called the partial derivative of $f(x,y)$ with respect to x at (a,b). It can be denoted as f_x, f'_x, $\dfrac{\partial f}{\partial x}$ or $D_x f$. Similarly for the partial derivative of $f(x,y)$ with respect to y.

2.1 Geometrical Interpretation

Let C_1 be the intersection line of $z = f(x,y)$ and the plane $y = b$. Let T_1 be the tangent line to C_1, in the plane $y = b$ at P. Then, $f_x(a,b)$ is the slope of T_1.

Definition 6. If $\exists f_x(x,y)$ and $\exists f_y(x,y)$ in some open neighbourhood of (a,b) and $f_x(x,y)$ and $f_y(x,y)$ are continuous at (a,b), then the tangent plane to $z = f(x,y)$ at the point $P(a,b,f(a,b))$ is a plane which passes through P and contains the straight lines T_1 and T_2. It is given by

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$