Lecture 23

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Contents

1 Green's Theorem

Theorem 1 (Green's Theorem)**.** *Let C ne a piecewise smooth, simple, and closed curve in $\mathbb{R}^2$ with positive orientation. Let D be a domain bounded by C. If there exist continuous first order partial derivatives of $P(x,y)$ and $Q(x,y)$ in an open domain which contains D, then*

$$W = \int\limits_C \overline{F} \cdot \hat{T} \, \mathrm{d}s = \int\limits_C P \, \mathrm{d}x + Q \, \mathrm{d}y = \iint\limits_D (Q_x - P_y) \, \mathrm{d}A$$

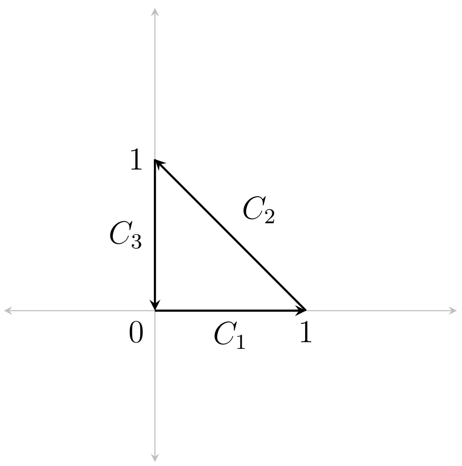
Remark 1. Green's Theorem is also true for domains with holes.

Example 1. Find the work done by the force

$$\overline{F}(x,y) = (x^4, xy)$$

over the path

$$C = C_1 \cup C_2 \cup C_3$$



Solution. By Green's Theorem,

$$\begin{aligned}
W &= \int\limits_C P\,\mathrm{d}x + Q\,\mathrm{d}y \\
&= \iint\limits_D (Q_x - P_y)\,\mathrm{d}A \\
&= \iint\limits_D (y - 0)\,\mathrm{d}A \\
&= \int\limits_0^1 \int\limits_0^{1-x} y\,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{1}{6}
\end{aligned}$$

Example 2. Calculate $\int\limits_C \overline{F} \cdot \hat{T}\,\mathrm{d}s$ when

$$\overline{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

and C is a simple, closed, piecewise smooth curve with positive orientation which does not pass through $(0,0)$.

Solution.

$$\begin{aligned}
P &= \frac{y}{x^2+y^2} \\
Q &= \frac{x}{x^2+y^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
P_y &= -\frac{(x^2+y^2) - y \cdot 2y}{(x^2+y^2)^2} \\
&= \frac{y^2 - x^2}{(x^2+y^2)^2} \\
Q_x &= \frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2}
\end{aligned}$$

If $(0,0) \notin D$, Green's Theorem is applicable.
Therefore,

$$\int_C \overline{F} \cdot \hat{T} \, \mathrm{d}s = \iint_D (Q_x - P_y) \, \mathrm{d}A$$
$$= 0$$

If $(0,0) \in D$, Green's Theorem is not applicable as P_y and Q_x are not continuous in D.
Let C_1 be a circle of radius a, with the same orientation as C. Let $\widetilde{C} = C \cup (-C_1)$. Green's Theorem can be applied on the domain $D \setminus D_1$ which is enclosed by $\widetilde{C}$.

$$\int_{C \cup (-C_1)} P \, \mathrm{d}x + Q \, \mathrm{d}y = \iint_{D \setminus D_1} (Q_x - P_y) \, \mathrm{d}A$$
$$= 0$$

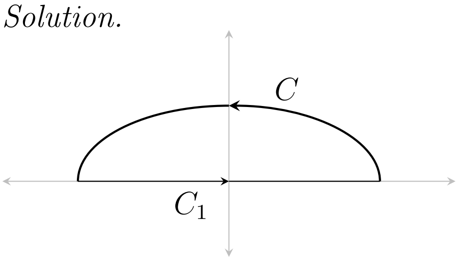
$$\int_C P \, \mathrm{d}x + Q \, \mathrm{d}y + \int_{-C_1} P \, \mathrm{d}x + Q \, \mathrm{d}y = 0$$

Therefore,

$$\begin{aligned}
\int_C P \, \mathrm{d}x + Q \, \mathrm{d}y &= \int_{C_1} P \, \mathrm{d}x + Q \, \mathrm{d}y \\
&= \int_0^{2\pi} \Big(P\left(x(t), y(t)\right) x'(t) + Q\left(x(t), y(t)\right) \Big) \\
&= \int_0^{2\pi} (\sin^2 t + \cos^2 t) \, \mathrm{d}t \\
&= 2\pi
\end{aligned}$$

Example 3. Calculate $\displaystyle\int_C -2e^{2x-y} \cos y \, \mathrm{d}x + \left(e^{2x-y}(\sin y + \cos y) + 2xy\right) \mathrm{d}y$ when C is the half ellipse $\left\{ \dfrac{x^2}{4} + y^2 = 1, y \geq 0 \right\}$ oriented from the point $(2,0)$ to the point $(-2,0)$.

Solution.



Let C_1 be the line segment as shown.

$$
\begin{aligned}
P &= -2e^{2x-y}\cos y \\
Q &= e^{2x-y}(\sin y + \cos y) + 2xy
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
P_y &= 2e^{2x-y}\cos y + 2e^{2x-y}\sin y \\
&= 2e^{2x-y}(\cos y + \sin y) \\
Q_x &= 2e^{2x-y}(\sin x + \cos y) + 2y
\end{aligned}
$$

The domain is of the first kind.

$$
\begin{aligned}
\int\limits_C P\,\mathrm{d}x + Q\,\mathrm{d}y &= \int\limits_C P\,\mathrm{d}x + Q\,\mathrm{d}y + \int\limits_{C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y - \int\limits_{C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y \\
&= \int\limits_{C\cup C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y - \int\limits_{C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y \\
&= \iint\limits_D (Q_x - P_y)\,\mathrm{d}A - \int\limits_{C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y
\end{aligned}
$$

2 Surface Integrals of Scalar Functions

Theorem 2. *Let S be a surface given by $z = g(x,y)$, $(x,y) \in D$. Then the surface integral of a scalar function $f(x,y,z)$ over S is equal to*

$$
\iint\limits_S f(x,y,z)\,\mathrm{d}S = \iint\limits_S f\left(x,y,g(x,y)\right)\sqrt{1 + \left(g_x(x,y)\right)^2 + \left(g_y(x,y)\right)^2}\,\mathrm{d}A
$$

Example 4. Calculate $\iint\limits_S (x,y,z)\,\mathrm{d}S$ when S is $z = 1 - x - y$ above the domain

$$
D = \{(x,y) | 0 \le x \le 1, 0 \le y \le 1 - x\}
$$

Solution.

$$\begin{aligned}\iint\limits_S (xy+z)\,\mathrm{d}S &= \iint\limits_D (xy+z)\sqrt{1+1+1}\,\mathrm{d}A \\ &= \sqrt{3}\int\limits_0^1\int\limits_0^{1-x} \big((x-1)y+(1-x)\big)\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{5\sqrt{3}}{24}\end{aligned}$$

3 Surface Integrals of Vector Functions

Definition 1 (Positive and negatively oriented surfaces)**.** Let S be a surface given by $z=g(x,y)$, $(x,y)\in D$. Then the surface is called positively oriented if, on S, the normal $\overline{n}=(n_1,n_2,n_3)$ is given with $n_3>0$, and negatively oriented if $n_3<0$.
If S is closed, then the surface is called positively oriented if the normal is outwards, and negatively oriented if the normal is inwards.

Definition 2. If

$$\overline{F}(x,y,z)=\big(P(x,y,z),Q(x,y,z),R(x,y,z)\big)$$

is a vector function defined on S with the normal $\hat{n}$ then the surface integral of the vector function is

$$\iint\limits_S \overline{F}\cdot\mathrm{d}\overline{S} = \int\limits_S \overline{F}(x,y,z)\cdot\hat{n}(x,y,z)\,\mathrm{d}S$$

Theorem 3. *Let $z=g(x,y)$, $(x,y)\in D$ and S be positively oriented. Then*

$$\iint\limits_S \overline{F}\cdot\mathrm{d}\overline{S} = \iint\limits_S (-Pg_x-Qg_y+R)\,\mathrm{d}A$$

Example 5. Find $\iint\limits_S \overline{F}\cdot\mathrm{d}\overline{S}$ when $\overline{F}=(x,y,z)$ and S is a lateral surface of a solid bounded by the elliptical paraboloid $z=2-x^2-y^2$ and the plane $z=1$.

Solution. Let

$$S_1 : z = 2-x^2-y^2$$
$$S_2 : z = 1$$

Therefore, $S = S_1 \cup S_2$.
The normals to S_1 and S_2 are directed outwards with respect to the solid enclosed by S_1 and S_2.[1]

$$\begin{aligned}
\iint\limits_{S} \overline{F} \cdot \mathrm{d}\overline{S} &= \iint\limits_{S_1} \overline{F} \cdot \mathrm{d}\overline{S} + \iint\limits_{S_2} \overline{F} \cdot \mathrm{d}\overline{S} \\
&= \iint\limits_{D} \big(-P(g_1)_x - Q(g_1)_y \quad + R\big)\,\mathrm{d}A \\
&\quad + \iint\limits_{D} \big(-P(g_2)_x - Q(g_2)_y + R\big)\,\mathrm{d}A \\
&= \iint\limits_{D} \big(-y(-2x) - x(-2y) + 2 - x^2 - y^2\big)\,\mathrm{d}A \\
&\quad + \iint\limits_{D} \big(y(0) + x(0) - 1\big)\,\mathrm{d}A \\
&= \iint\limits_{D} (4xy + 2 - x^2 - y^2)\,\mathrm{d}A - \iint\limits_{D} \mathrm{d}A \\
&= \frac{\pi}{2}
\end{aligned}$$

[1] If the orientation of a surface is not given, it can be assumed to be positive.