Lecture 8

Aakash Jog

Thursday 20th November, 2014

Contents

1 Taylor's Formula

Theorem 1 (Taylor's Formula)**.** *Let $f(x)$ be differentiable $(n+1)$ times, where $n \in \mathbb{N} \cup \{0\}$ on an open interval about a, and x be an arbitrary point in this interval. Then, there exists a point c, which depends on x, between a and x, s.t.*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

is called the Lagrange remainder

Proof. Let

$$\begin{aligned} T_n(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ R_n^0(x) &= f(x) - T_n(x) \end{aligned}$$

Therefore, it is ETPT that

$$R_n^0(x) = R_n(x)$$

Let $x \neq a$ be a fixed point.
Let

$$g(t) = f(x) - f(t) - \frac{f'(t)}{1!}(x-t) + \cdots + \frac{f^{(n)}(t)}{n!}(x-t)^n - R_n^0(x)\frac{(x-t)^{n+1}}{(x-a)^{n+1}}$$

$$\begin{aligned} g(x) &= 0 \\ g(a) &= f(x) - T_n(x) - R_n^0(x) = 0 \\ \therefore g(x) &= g(a) \end{aligned}$$

Also, $g(x)$ is continuous and differentiable on the interval. Hence we can apply Rolle's Theorem.
Therefore, $\exists c$ between a and x, s.t. $f'(c) = 0$.

$$\begin{aligned} g'(t) = -f'(t) - \frac{f''(t)}{1!}(x-t) + \frac{f'(t)}{1!} - \frac{f'''(t)}{2!}(x-t)^2 + \frac{f''(t)}{2!}2(x-t) - \ldots \\ - \frac{f^{(n+1)}(t)}{n!}(x-t)^n + \frac{f^{(n)}(t)}{n!}(x-t)^{n-1} + R_n^0(x)\frac{(n+1)(x-t)^n}{(x-a)^{n+1}} \\ \therefore g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + R_n^0(x)\frac{(n+1)(x-t)^n}{(x-a)^{n+1}} \end{aligned}$$

$f'(c) = 0$

$$\therefore R_n^0(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$$
$$\therefore R_n^0(x) = R_n(x)$$

$\square$

Remark 1. The Lagrange Theorem is a particular case of Taylor's Formula, with $n = 0, x = b$.

Remark 2. If $f(x)$ is infinitely differentiable on an open interval about a, and $\lim_{n\to\infty} R_n(x) = 0$, for any x in the interval, then

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

This infinite sum is called the Taylor's series.
If $a = 0$,

$$f(x) = f(0) + \frac{f'(0)}{1!}(x) + \dots + \frac{f^{(n)}(0)}{n!}(x)^n + \dots$$

This infinite sum is called the Maclaurin series.

Example 1. Calculate $e^{-0.05}$ with accuracy 0.0001.

Solution. Substitute $x = -0.05$ in

$$e^x = 1 + \frac{x}{1!} + \frac{x^n}{n!} + \frac{e^c x^{n+1}}{(n+1)!}$$

where c is between 0 and x. Then, $\exists c \in (-0.05, 0)$, and

$$\begin{aligned} |R_n(-0.05)| &= \left| \frac{e^c(-0.05)^{n+1}}{(n+1)!} \right| \\ &= \frac{e^c(0.05)^{n+1}}{(n+1)!} \\ &< \frac{(0.05)^{n+1}}{(n+1)!} \end{aligned}$$

If $n = 2$

$$\frac{(0.05)^{n+1}}{(n+1)!} \leq 0.0001$$
$$\therefore e^{-0.05} \approx 1 + \frac{(-0.05)}{1!} + \frac{(-0.05)^2}{2!} = 0.95125$$

2 L'Hospital's Rule

Theorem 2 (L'Hospital's Rule)**.** *Let $f(x)$ and $g(x)$ be differentiable on an open interval about a, except possibly at a itself. Assume*

i. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ *or* $\lim_{x\to a} f(x) = \pm\infty, \lim_{x\to a} g(x) = \pm\infty$

ii. $g(x) \neq 0, \forall x \neq a$ *from the interval*

iii. $\exists \lim_{x\to a} \dfrac{f'(x)}{g'(x)}$

Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Proof. For a particular case where f and g are differentiable at a, and the derivatives are continuous at a, $g'(a) \neq 0$, consider

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$$

Using Taylor's Formula with $n = 0$,

$$\begin{aligned}\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f(a) + \dfrac{f'(c_1)}{1!}(x-a)}{g(a) + \dfrac{g'(c_2)}{1!}(x-a)} \\ &= \lim_{x\to a} \frac{f'(c_1)}{g'(c_2)}\end{aligned}$$

c_1 and c_2 are constrained to lie between x and a. Therefore, as $x \to a$, $c_1 \to a$ and $c_2 \to a$.

$$\begin{aligned} &= \frac{f'(a)}{g'(a)} \\ &= \lim_{x\to a} \frac{f'(x)}{g'(x)}\end{aligned}$$

□

Remark 3. L'Hospital's Rule can be applied repeatedly to $\lim_{x\to a} \dfrac{f(x)}{g(x)}$.

Remark 4. L'Hospital's Rule is also true for one sided limits.

Example 2. Find

$$\lim_{x\to 0^+} x^2 \ln x$$

Solution.

$$\begin{aligned}
\lim_{x\to 0^+} x^2 \ln x &= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^2}} \\
&= \frac{\frac{1}{x}}{-2x^{-3}} \\
&= \lim_{x\to 0^+} \frac{x^2}{-2} \\
&= 0
\end{aligned}$$

Example 3. Find

$$\lim_{x\to +\infty} x^{\frac{1}{x}}$$

Solution.

$$\begin{aligned}
\lim_{x\to +\infty} x^{\frac{1}{x}} &= \lim_{x\to +\infty} e^{\ln x \frac{1}{x}} \\
&= e^{\lim\limits_{x\to +\infty} \frac{\ln x}{x}} \\
&= e^{\lim\limits_{x\to +\infty} \frac{\frac{1}{x}}{1}} \\
&= e^0 \\
&= 1
\end{aligned}$$

3 Useful Properties of Derivatives

1. $f'(x) = 0$ on (a, b) iff $f(x)$ is constant on (a, b).
2. $f'(x) = g'(x)$ on (a, b) iff $f(x) = g(x) + \text{constant}$ on (a, b).
3. $\exists f'(x)$ and $f(x)$ is monotonically increasing on (a, b) iff $f'(x) \geq 0$.
4. $\exists f'(x)$ and $f(x)$ is monotonically decreasing on (a, b) iff $f'(x) \leq 0$.
5. If $f'(x) > 0$ on (a, b), then $f(x)$ is monotonically strongly increasing on (a, b).
6. If $f'(x) < 0$ on (a, b), then $f(x)$ is monotonically strongly decreasing on (a, b).