Recitation 10

Wednesday 31$^{\text{st}}$ December, 2014

Contents

1 Improper Integrals

Example 1. Does

$$\int_0^\infty e^{-x}\,\mathrm{d}x$$

converge?

Solution.

$$\begin{aligned}\lim_{t\to 0}\int_0^t e^{-x}\,\mathrm{d}x &= \lim_{t\to\infty} e^{-x}\Big|_0^t \\ &= \lim_{t\to\infty} -e^{-t}+e^0 \\ &= 1\end{aligned}$$

Therefore, the integral converges.

Theorem 1 (First comparison test)**.** *Let $f(x)$ and $g(x)$ be two functions defined on $[a,+\infty)$ and Riemann integrable over $[a,t], \forall t \geq a$. Assume that $\exists b \geq a$, s.t. $f(x) \geq g(x) \geq 0, \forall x \geq b$. Then,*

1. *if $\int\limits_a^{+\infty} f(x)\,\mathrm{d}x$ converges, then $\int\limits_a^{+\infty} g(x)\,\mathrm{d}x$ converges.*
2. *if $\int\limits_a^{+\infty} g(x)\,\mathrm{d}x$ diverges, then $\int\limits_a^{+\infty} f(x)\,\mathrm{d}x$ diverges.*

Theorem 2 (Second comparison test)**.** *Assume $f(x) \geq g(x) \geq 0, \forall x \in (a,b)$. Assume that f, g are not bounded in a neighbourhood of b but integrable on intervals of the type $[a,t]$ for $a < t < b$. Assume that*

$$\lim_{x\to b^-}\frac{f(x)}{g(x)} = l > 0$$

Then,

$$\int_a^b f(x)\,\mathrm{d}x$$

and

$$\int_a^b g(x)\,\mathrm{d}x$$

converge or diverge simultaneously.

Example 2. Does the following integral converge?

$$\int_0^1 \frac{\arctan(x-1)}{(x-1)\sqrt{x+x^3}}\,\mathrm{d}x$$

Solution.

$$\begin{aligned}\lim_{x\to 1}\frac{\arctan(x-1)}{(x-1)} &= \lim_{x\to 1}\frac{\frac{1}{1+(x-1)^2}}{1}\\ &= 1 \therefore \frac{\arctan(x-1)}{(x-1)\sqrt{x+x^3}} &&= \frac{1}{\sqrt{2}}\end{aligned}$$

Therefore, the function can be extended to a continuous function at 1 by defining $f(1) = \frac{1}{\sqrt{2}}$.
Therefore, f is Riemann integrable in $[{}^1\!/\!_2, 1]$.

$$\begin{aligned}\lim_{x\to 0}\frac{\arctan(x-1)}{(x-1)} &= \frac{\pi}{4}\\ \therefore \lim_{x\to 0}\frac{\arctan(x-1)}{(x-1)\sqrt{x+x^3}} &\le \frac{\pi}{4}\cdot\frac{1}{\sqrt{x}}\end{aligned}$$

As

$$\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$$

converges, by the first comparison test,

$$\int_0^1 \frac{\arctan(x-1)}{(x-1)\sqrt{x+x^3}}\,\mathrm{d}x$$

Example 3. Check the convergence of

$$\int_0^\infty \frac{\sin x}{x^2}\,\mathrm{d}x$$

Solution.

$$\left|\frac{\sin x}{x^2}\right| \leq \frac{1}{x^2}$$

Therefore, by first comparison test,

$$\left|\frac{\sin x}{x^2}\right|$$

converges. Therefore,

$$\frac{\sin x}{x^2}$$

also converges.

Example 4. Check the convergence of

$$\int_0^\infty \frac{\sin x}{x^{3/2}+x^2}\,\mathrm{d}x$$

Solution. For $[0,1]$, $\dfrac{\sin x}{x^{3/2}+x^2}$ is non-negative.

$$\frac{\sin x}{x^{3/2}+x^2} \leq \frac{\sin x}{x^{3/2}}$$

$$\begin{aligned}\lim_{x\to 0^+} \frac{\frac{\sin x}{x^{3/2}}}{\frac{1}{x^{1/2}}} &= \lim_{x\to 0^+} \frac{\sin x}{x} \\ &= 1\end{aligned}$$

Therefore, by the second comparision test,

$$\int_0^1 \frac{\sin x}{x^{3/2} + x^2}$$

and

$$\int_0^1 \frac{\sin x}{x^{1/2}}$$

converge simultaneously.
Therefore, by the first comparison test,

$$\int_0^1 \frac{\sin x}{x^{3/2} + x^2}$$

converges.

Example 5. For which values of α does the integral

$$\int_1^\infty \left(\sqrt{x+1} - \sqrt{x}\right)^\alpha \mathrm{d}x$$

converge?

Solution.

$$\begin{aligned} \left(\sqrt{x+1} - \sqrt{x}\right)^\alpha &= \left(\frac{x+1-x}{\sqrt{x+1}+\sqrt{x}}\right)^\alpha \\ &= \left(\frac{1}{\sqrt{x+1}+\sqrt{x}}\right)^\alpha \end{aligned}$$

$$\begin{aligned} \lim_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x+1}+\sqrt{x}}\right)^\alpha}{\left(\frac{1}{\sqrt{x}}\right)^\alpha} &= \lim_{x\to\infty} \left(\frac{\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}\right)^\alpha \\ &= \left(\frac{1}{2}\right)^\alpha \\ &> 0 \end{aligned}$$

Therefore,

$$\int_1^{\infty} \left(\frac{1}{\sqrt{x}}\right)^{\alpha} \mathrm{d}x$$

and

$$\int_1^{\infty} \left(\sqrt{x+1} - \sqrt{x}\right)^{\alpha} \mathrm{d}x$$

converge simultaneously.
Therefore,

$$\int_1^{\infty} \left(\sqrt{x+1} - \sqrt{x}\right)^{\alpha} \mathrm{d}x$$

converges if and only if $\alpha > 2$.