Lecture 4

Thursday 6th November, 2014

Contents

1 A Classification of Discontinuity Points

Let $f(x)$ be defined on an open interval about a, except possibly at a itself.

1.1 Removable Discontinuity Point

The point a is a removable discontinuity point of f if, $\lim_{x\to a} f(x)$ exists, but either $\lim_{x\to a} f(x) \neq f(a)$ or $f(a)$ does not exist.

1.2 Discontinuity of First Kind

The point a is a discontinuity point of the first kind if both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist, but $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

1.3 Discontinuity of Second Kind

The point a is a discontinuity point of the second kind if atleast one of the two one-sided limits of f does not exist.
Note that the limits are defined as finite numbers only.

2 Sandwich Theorem

Let $f(x), g(x), h(x)$ be defined on an open interval about a, except possibly at a itself. Assume that $\forall x \neq a$ from the interval, it is satified that $f(x) \leq g(x) \leq h(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then, $\lim_{x\to a} g(x) = L$.

Proof

$\forall \varepsilon > 0, \exists \delta > 0 : 0 < |x-a| < \delta \Rightarrow |g(x) - L| < \varepsilon$, i.e., $L - \varepsilon < g(x) < L + \varepsilon$
Given $\exists \delta_1 > 0 : 0 < |x-a| < \delta_1 \Rightarrow f(x) \leq g(x) \leq h(x)$
For this $\varepsilon > 0, \exists \delta_2 > 0 : 0 < [x-a] < \delta_2 \Rightarrow |f(x) - L| < \varepsilon$, i.e., $L - \varepsilon < f(x) < L + \varepsilon$
$\varepsilon > 0, \exists \delta_3 > 0 : 0 < [x-a] < \delta_3 \Rightarrow |h(x) - L| < \varepsilon$, i.e., $L - \varepsilon < h(x) < L + \varepsilon$
So, $\forall \varepsilon > 0, \exists \delta = \min \delta_1, \delta_2, \delta_3 > 0 : 0 < |x-a| < \delta \Rightarrow L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon$

3 Theorem 5: If $\lim_{x \to a} f(x) = 0$ and $g(x)$ is bounded in an open interval about a, except possibly at a itself, then, $\lim_{x \to a}(f(x)g(x)) = 0$.

Proof

We have to prove that

$\forall \varepsilon > 0, \exists \delta > 0 : 0 < |x - a| < \delta \Rightarrow |f(x)g(x) - 0| < \varepsilon$

Given $\lim_{x \to a} f(x) = 0$,

$\forall \varepsilon_1 > 0, \exists \delta_1 > 0 : 0 < |x - a| < \delta_1 \Rightarrow |f(x) - 0| < \varepsilon_1$

As $g(x)$ is bounded, in an open interval about a, except possibly at a itself,

$\exists \delta_2 > 0, \exists M > 0 : 0 < |x - a| < \delta_2 \Rightarrow |g(x)| \leq M$

So, if we choose $\varepsilon = \dfrac{\varepsilon}{M}$,

$\forall \varepsilon > 0, \exists \delta = \min\{\delta_1, \delta_2\} > 0 : 0 < |x - a|\, \delta \Rightarrow |f(x)g(x) - 0| = |f(x)||g(x)| < \varepsilon_1 M = \varepsilon$

4 Infinite Limits

$$\lim_{x \to a} f(x) = +\infty \Leftrightarrow \forall M > 0, \exists \delta > 0 : 0 < |x - a| < \delta \Rightarrow f(x) > M$$
$$\lim_{x \to a} f(x) = -\infty \Leftrightarrow \forall M < 0, \exists \delta > 0 : 0 < |x - a| < \delta \Rightarrow f(x) < M$$
$$\lim_{x \to +\infty} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists M > 0 : x > M \Rightarrow |f(x) - L| < \varepsilon$$
$$\lim_{x \to -\infty} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists M > 0 : x > M \Rightarrow |f(x) - L| < \varepsilon$$

5 Known Limits

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$
$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$$
$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

5.1 Proof of $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

6 Excercise

6.1 $\lim_{x \to 0} \frac{\tan 2x}{x}$

$$
\begin{aligned}
\lim_{x \to 0} \frac{\tan 2x}{x} &= \lim_{x \to 0} \frac{\frac{\sin 2x}{\cos 2x}}{x} \\
&= \lim_{x \to 0} \frac{\sin 2x}{2x} \frac{2}{\cos x} \\
&= \lim_{x \to 0} \frac{\sin 2x}{2x} \lim_{x \to 0} \frac{2}{\cos x} \\
&= 1 \cdot 2
\end{aligned}
$$

6.2 $\lim_{x \to 0} \frac{\cos x - 1}{x}$

$$
\lim_{x \to 0^-} \frac{\cos x - 1}{x} = \lim_{h \to 0} \frac{\cos h - 1}{h}
$$