Lecture 11

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Contents

1 Indefinite Integrals

Definition 1 (Anti-derivative of a function)**.** A function $F(x)$ is called an anti-derivative of $f(x)$ in the domain D, if

$$F'(x) = f(x) \quad ; \forall x \in D$$

If $F(x)$ is an anti-derivative of $f(x)$, then $F(x) + c$ is also an anti-derivative of $f(x)$.

Definition 2 (Indefinite integral of a function)**.** Let $F(x)$ be an anti-derivative of $f(x)$. The set of all anti-derivatives of $f(x)$, i.e., $F(x) + c$ is called the indefinite integral of $f(x)$, and is denoted as

$$\int f(x)\,\mathrm{d}x = F(x)$$

$$\int x^a\,\mathrm{d}x = \begin{cases} \dfrac{x^{a+1}}{a+1} + c & ; a \neq -1 \\ \ln|x| + c & ; a = -1 \end{cases} \tag{1}$$

$$\int \cos x\,\mathrm{d}x = \sin x + c \tag{2}$$

$$\int \sin x\,\mathrm{d}x = -\cos x + c \tag{3}$$

$$\int \frac{\mathrm{d}x}{1+x^2} = \arctan x + c \tag{4}$$

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + c \tag{5}$$

1.1 Integration by Parts

$$\begin{aligned} \int uv'\,\mathrm{d}x &= uv - \int u'v\,\mathrm{d}x \\ \int u\,\mathrm{d}v &= uv - \int v\,\mathrm{d}u \\ \int uv\,\mathrm{d}x &= u\int v\,\mathrm{d}x - \int u'\left(\int v\,\mathrm{d}x\right)\mathrm{d}x \end{aligned}$$

1.2 Integration of Trigonometric Functions

Example 1.

$$\begin{aligned} \int \sin(3x)\cos(7x)\,\mathrm{d}x &= \frac{1}{2}\int(\sin(10x) - \sin(4x))\,\mathrm{d}x \\ &= \frac{1}{2}\left(-\frac{\cos(10x)}{10} + \frac{\cos(4x)}{4}\right) + c \end{aligned}$$

Example 2.

$$\begin{aligned}\int \sin^{2k+1} x \cos^m x \,\mathrm{d}x &= \int \sin^2 kx \cos^m x \sin x \,\mathrm{d}x \\ &= \int (1-\cos^2 x)^k \cos^m x \sin x \,\mathrm{d}x\end{aligned}$$

Let $t = \cos x$. Therefore, $\mathrm{d}t = -\sin x \,\mathrm{d}x$.

$$\therefore \int (1-\cos^2 x)^k \cos^m x \sin x \,\mathrm{d}x = \int (1-t^2)^k t^m \,\mathrm{d}t$$

1.3 Special Trigonometric Substitutions

$\sqrt{a^2-x^2}$	$x = a\sin\theta \quad ; -\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta \quad ; -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$	$1+\tan^2\theta = \sec\theta$
$\sqrt{x^2-a^2}$	$x = \sec\theta \quad ; 0 \leq \theta \leq \dfrac{\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Example 3. Find

$$\int \frac{\sqrt{9-x^2}}{x^2} \,\mathrm{d}x$$

Solution.

Let $x = 3\sin\theta$. Therefore, $\mathrm{d}x = 3\cos\theta \,\mathrm{d}\theta$

$$\begin{aligned}\int \frac{\sqrt{9-x^2}}{x^2} \,\mathrm{d}x &= \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta} 3\cos\theta \,\mathrm{d}\theta \\ &= \int \frac{\cos^2\theta}{\sin^2\theta} \,\mathrm{d}\theta \\ &= \left(\frac{1}{\sin^2\theta} - 1\right) \mathrm{d}\theta \\ &= -\cot\theta - \theta + c \\ &= -\frac{9-x^2}{x} - \arcsin\frac{x}{3} + c\end{aligned}$$

2 Integration of Rational Functions

Definition 3 (Simple rational function)**.** A rational function $\dfrac{P(x)}{Q(x)}$ is called simple if the degree of the polynomial $P(x)$ is less than the degree of the polynomial $Q(x)$.

Remark 1. If a rational function $\dfrac{P(x)}{Q(x)}$ is not simple, then it can be written as a sum of a polynomial $M(x)$ and a simple rational function $\dfrac{R(x)}{Q(x)}$.

$$\frac{P(x)}{Q(x)} = M(x) + \frac{R(x)}{Q(x)}$$