Lecture 18

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Contents

1 Local Extrema

Theorem 1 (A necessary condition for local extrema existence)**.** *If the function $z = f(x, y)$ has a local extrema at the point (a, b) and $\exists f_x(a, b)$ and $\exists f_y(a, b)$ then $f_x(a, b) = f_y(a, b) = 0$*

Example 1.

$$z = x^2 + y^2$$

Solution.

$$f(x, y) \geq f(0, 0)$$

Therefore, $(0, 0)$ is a point of local minimum.

$$\begin{aligned} f_x &= 2x \\ f_y &= 2y \end{aligned}$$

Therefore,

$$f_x(0, 0) = f_y(0, 0) = 0$$

Example 2.

$$z = \sqrt{x^2 + y^2}$$

Solution.

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\sqrt{(\Delta x)^2}}{\Delta x} \\ &= \pm 1 \end{aligned}$$

Therefore, the limit does not exist.

Definition 1 (Critical point)**.** Let the function $z = f(x, y)$ be defined on some open neighbourhood of (a, b). The point (a, b) is called a critical point of $z = f(x, y)$ if $f_x(a, b) = f_y(a, b) = 0$ or at least one of the partial derivative $f_x(a, b)$ and $f_y(a, b)$ does not exist.

Remark 1. Every extremum point is a critical point but the converse is not true.

Example 3. Is $(0, 0)$ an local extremum point of

$$z = f(x, y) = y^2 - z^2$$

?

Solution.

$$f_x(0,0) = 0$$
$$f_y(0,0) = 0$$

Therefore, $(0,0)$ is a critical point.
If possible let $(0,0)$ be a local minimum point.
Then, $f(x,y) \geq f(0,0)$ in some neighbourhood of $(0,0)$.
Therefore,

$$y^2 - x^2 \geq 0$$

For any point of the form $(x,0)$, this is a contradiction.
Therefore $(0,0)$ is not a local minimum point.
Similarly, $(0,0)$ is not a local maximum point.

Theorem 2 (A sufficient condition for local extrema point)**.** *Assume that there exist second order partial derivates of $z = f(x,y)$, they are continuous on some open neighbourhood of (a,b) and $f_x(a,b) = f_y(a,b) = 0$. Denote*

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - \big(f_{xy}(a,b)\big)^2$$

1. *If $D(a,b) > 0$ and $f_{xx} < 0$ then (a,b) is a local maximum point.*
2. *If $D(a,b) > 0$ and $f_{xx} > 0$ then (a,b) is a local minimum point.*
3. *If $D(a,b) < 0$ then (a,b) is called a saddle point.*

Example 4. Find all critical points of

$$z = f(x,y) = x^4 + y^4 - 4xy + 1$$

and classify them.

Solution.

$$f_x(x,y) = 4x^3 - 4y$$
$$f_y(x,y) = 4y^3 - 4x$$

For critical points,

$$f_x(x,y) = 0$$
$$f_y(x,y) = 0$$

Solving, $(0,0)$, $(1,1)$, $(-1,-1)$ are critical points.

$$\begin{aligned} f_{xx}(x,y) &= 12x^2 \\ f_{xy}(x,y) &= -4 \\ f_{yy}(x,y) &= 12y^2 \\ \therefore D(x,y) &= 144x^2y^2 - 16 \end{aligned}$$

For $(0,0)$,

$$D = -16$$

Therefore, $(0,0)$ is a saddle point.
For $(1,1)$,

$$D = 144 - 16$$

Therefore, $(1,1)$ is a local minimum point.
For $(-1,-1)$,

$$D = 144 - 16$$

Therefore, $(-1,-1)$ is a local minimum point.

2 Global Extrema

2.1 Algorithm for Finding Maxima and Minima of a Function

Step 1 Find all critical points of $f(x,y)$ on the domain, excluding the end points.

Step 2 Calculate the values of $f(x,y)$ at the critical points.

Step 3 Calculate the values of $f(x,y)$ at the end points of the domain.

Step 4 Select the maximum and minimum values from Step 2 and Step 3

Example 5. Find the global maxima and minima of

$$z = x^2 - 2xy + 2y$$

in the domain

$$D = \left\{ (x,y) \middle| 0 \le x \le 3, 0 \le y \le -\frac{2}{3}x + 2 \right\}$$

Solution.

$$\begin{aligned}
f_x(x,y) &= 0\\
\therefore 2x - 2y &= 0\\
f_y(x,y) &= 0\\
\therefore -2x + 2 &= 0
\end{aligned}$$

Therefore, $(1,1)$ is a critical point in D.
The boundary of D is $L_1 \cup L_2 \cup L_3$, where

$$\begin{aligned}
&L_1 : y = 0, 0 \leq x \leq 3\\
&L_2 : x = 0, 0 \leq y \leq 2\\
&L_3 :
\end{aligned}$$

Therefore,
over L_1,

$$\begin{aligned}
f(x,y) &= x^2\\
\therefore \min_{L_1} f &= f(0,0) = 0\\
\therefore \max_{L_1} f &= f(3,0) = 9
\end{aligned}$$

over L_2,

$$\begin{aligned}
f(x,y) &= 2y\\
\therefore \min_{L_2} f &= f(0,0) = 0\\
\therefore \max_{L_2} f &= f(0,2) = 4
\end{aligned}$$

over L_3,

$$\begin{aligned}
f(x,y) &= x^2 - 2x\left(-\frac{2}{3}x + 2\right) + 2\left(-\frac{2}{3}x + 2\right)\\
&= \frac{7}{3}x^2 - \frac{16}{3}x + 4\\
\therefore f' &= \frac{14}{3}x - \frac{16}{3}\\
\therefore f'\left(\frac{8}{7}\right) &= 0\\
\therefore f\left(\frac{8}{7}, \frac{26}{21}\right) &= 0.952\\
\therefore \min_{L_3} f &= f\left(\frac{8}{7}, \frac{26}{21}\right) = 0.952\\
\therefore \max_{L_3} f &= f(3,0) = 9
\end{aligned}$$

Therefore,

$$\therefore \min_{D} f = f(0,0) = 0$$
$$\therefore \max_{D} f = f(3,0) = 9$$