Lecture 17

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Contents

1 Partial Derivatives of Higher Order

1.1 Definition

Definition 1 (Partial Derivatives of Higher Order)**.**

$$
\begin{aligned}
(f_x)_x = f_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = D_{x^2} f \\
(f_x)_y = f_{xy} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = D_{xy} f \\
(f_y)_x = f_{yx} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial x} = D_{yx} f \\
(f_y)_x = f_{yy} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = D_{y^2} f
\end{aligned}
$$

Theorem 1. *Let $x = f(x, y)$ be continuous in some open neighbourhood of (a, b). Assuming $\exists f_{xy}(x, y)$ and $\exists f_{yx}(x, y)$ and are continuous in this neighbourhood, $f_{xy}(a, b) = f_{yx}(a, b)$.*

1.2 Differential

Definition 2 (Differentials of $f(x, y)$)**.** Consider $z = f(x, y)$. Then,

$$
\begin{aligned}
\mathrm{d}x &= \Delta x \\
\mathrm{d}y &= \Delta y \\
\mathrm{d}z &= f_x(a, b)\,\mathrm{d}x + f_y(a, b)\,\mathrm{d}y
\end{aligned}
$$

are the differentials of $f(x, y)$ at (a, b).

1.3 Differentiability

Theorem 2. *$f(x, y)$ is differentiable at (a, b) if $\exists \varepsilon_1(\Delta x, \Delta y)$ and $\exists \varepsilon_2(\Delta x, \Delta y)$, with*

$$
\begin{aligned}
\lim_{(\Delta x, \Delta y) \to (0,0)} \varepsilon_1(\Delta x, \Delta y) &= 0 \\
\lim_{(\Delta x, \Delta y) \to (0,0)} \varepsilon_2(\Delta x, \Delta y) &= 0
\end{aligned}
$$

such that

$$
\Delta x = \mathrm{d}z + \varepsilon_1(\Delta x, \Delta y)\Delta x + \varepsilon_2(\Delta x, \Delta y)\Delta y
$$

Theorem 3. *If $f(x, y)$ is differentiable at (a, b), then $f(x, y)$ is continuous at (a, b).*

Proof. If $f(x, y)$, then

$$\begin{aligned}
\Delta z &= f(a + \Delta x, b + \Delta y) - f(a, b) \\
\Delta z &= \mathrm{d}z + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \\
&= f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \\
\therefore f(a + \Delta x, b + \Delta y) &= f(a, b) \\
&\quad + f_x(a, b)\Delta x + f_y(a, b)\Delta y \\
&\quad + \varepsilon \Delta x + \varepsilon_2 \Delta y
\end{aligned}$$

Therefore, taking limits on both sides,

$$\lim_{(\Delta x, \Delta y) \to (0,0)} f(a + \Delta x, b + \Delta y) = f(a, b)$$

$\square$

Theorem 4. *If $\exists f_x(x, y)$ and $\exists f_y(x, y)$ in some open neighbourhood of (a, b) and the partial derivatives are continuous at (a, b) then $f(x, y)$ is differentiable at (a, b).*

Example 1. Given

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & ; \quad (x, y) \neq (0, 0) \\ 0 & ; \quad (x, y) = (0, 0) \end{cases}$$

1. Calculate $f_x(0, 0)$ and $f_y(0, 0)$.
2. Prove: $\nexists \lim\limits_{(x,y) \to (0,0)} f(x, y)$
3. Is the function continuous at $(0, 0)$?
4. Is the function differentiable at $(0, 0)$?

Solution.

$$\begin{aligned}
f_x(0, 0) &= \lim_{\Delta x \to 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{\Delta x \cdot 0}{(\Delta x)^2 + 0}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{0}{\Delta x} \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
f_y(0,0) &= \lim_{\Delta y \to 0} \frac{f(0, 0+\Delta y) - f(0,0)}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{0}{\Delta y} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\lim_{(x,y) \overset{y=kx}{\to} (0,0)} f(x,y) &= \lim_{x \to 0} \frac{x \cdot kx}{x^2 + (kx)^2} \\
&= \lim_{x \to 0} \frac{kx^2}{x^2(1+k)} \\
&= \lim_{x \to 0} \frac{k}{1+k^2} \\
&= \frac{k}{1+k^2}
\end{aligned}
$$

As the limit depends on k, $\nexists \lim_{(x,y) \to (0,0)} f(x,y)$.
Hence, $f(x,y)$ is not continuous at (a,b).
Hence, $f(x,y)$ is not differentiable at (a,b).

1.4 Chain Rule

Theorem 5. *Let $z = f(x,y)$ be differentiable and let there exist derivatives of $x = g(t)$ and $y = h(t)$. Then $\exists z'(t)$ and*

$$
z'(t) = z_x \cdot g'(t) + z_y \cdot h'(t)
$$

Theorem 6. *Let $z = f(x,y)$ be differentiable and $x = g(s,t)$, $y = h(s,t)$. Assuming there exist g_s, g_t, h_s, h_t,*

$$
\begin{aligned}
z_s &= z_x \cdot x_s + z_y \cdot y_s \\
z_t &= z_x \cdot x_t + z_y \cdot y_t
\end{aligned}
$$

Example 2. Given

$$
\begin{aligned}
z &= e^x \sin y \\
x &= st^2 \\
y &= s^2 \cos t
\end{aligned}
$$

Find z_s.

Solution.

$$\begin{aligned}
z_s &= z_x \cdot x_s + z_y \cdot y_s \\
&= e^x \sin y \cdot t^2 + e^x \cos y \cdot 2s \cos t \\
&= e^{st^2} \sin(s^2 \cos t) \cdot t^2 + e^{st^2} \cos(s^2 \cos t) \cdot 2s \cos t
\end{aligned}$$

1.5 Derivative of an Implicit Function

Example 3. Given $F(x, y) = k$ and $y = y(x)$, find $y'(x)$.

Solution. Differentiating both sides,

$$\begin{aligned}
F_x x_x + F_y y_x = 0 \\
\therefore y_x = -\frac{F_x}{F_y}
\end{aligned}$$

Example 4. Given $F(x, y, z) = k$ and $z = f(x, y)$, find z_x and z_y.

Solution. Differentiating both sides,

$$\begin{aligned}
F_x x_x + F_y y_x + F_z z_x = 0 \\
\therefore z_x = -\frac{F_x}{F_z}
\end{aligned}$$

Similarly,

$$z_y = -\frac{F_y}{F_z}$$

2 Gradient and the Tangent Plane to a Surface.

Definition 3 (Gradient of $f(x, y)$)**.** If $\exists f_x(x, y)$ and $\exists f_y(x, y)$ then the vector

$$\nabla f(x, y) = (f_x(x, y), f_y(x, y)) \neq 0$$

is called the gradient of $f(x, y)$.
$\nabla f : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector function.

Definition 4 (Gradient of $f(x, y, z)$)**.** Given $f(x, y, z)$,

$$\nabla f(x, y, z) = (f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)) \neq 0$$

is called the gradient of $f(x, y, z)$.
$\nabla f : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector function.

Theorem 7. *If $F(x, y, z)$ is differentiable at some point $P_0(x_0, y_0, z_0)$ on the surface, then the tangent plane α to the surface at the point can be calculated by the formula*

$$F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0$$

Example 5. Find the tangent plane α to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

at the point $P_0(-2, 1, 3)$.

3 Extrema of $f(x, y)$

Definition 5 (Local maximum of $f(x, y)$)**.** The function $z = f(x, y)$ is said to have a local maximum at (a, b), if there exists an open neighbourhood of (a, b) in which $f(x, y) \leq f(a, b)$

Definition 6 (Local minimum of $f(x, y)$)**.** The function $z = f(x, y)$ is said to have a local minimum at (a, b), if there exists an open neighbourhood of (a, b) in which $f(x, y) \geq f(a, b)$

Definition 7 (Global maximum of $f(x, y)$)**.** The function $z = f(x, y)$ is said to have a global maximum at (a, b), if $f(x, y) \leq f(a, b)$, $\forall (x, y) \in D$.

Definition 8 (Global minimum of $f(x, y)$)**.** The function $z = f(x, y)$ is said to have a global minimum at (a, b), if $f(x, y) \geq f(a, b)$, $\forall (x, y) \in D$.