Recitation 1

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Contents

1 General Information

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2 Domain of Definition of a Function

2.1 Examples

2.1.1 Find the domain of definition of the following functions.

2.1.1.1 $\ln(1-|x|)+\dfrac{1}{\sin x}$

For $\ln x$ to be defined, it is necessary that $1-|x|>0$

$\therefore |x|<1 \Rightarrow -1<x<1$

For $\dfrac{1}{\sin x}$ to be defined, $\sin x \neq 0$

$\therefore x \neq k\pi, k \in \mathbb{Z}$

Therefore, the domain of definition is $(-1,1)-\{0\}$

2.1.1.2 $\sqrt{\dfrac{x+5}{|x^4-16|}}$

For the square root to be defined,

$$\frac{x+5}{|x^4-16|}>0$$

For the ratio to be defined,

$x^4-16/\neq 0$

Therefore, the domain is $[-5,\infty)-\{-2,2\}$

3 Odd and Even Functions

3.1 Even function

If $f(-x) = f(x); (x, -x \in D)$ then, f is called an even function.
Each even function is symmeteric about the y-axis.

3.2 Odd function

If $f(-x) = -f(x); (x, -x \in D)$ then, f is called an odd function.
Each odd function is symmeteric about the origin.

3.3 Examples

3.3.1 Prove that if f is even and g is odd, then $f \cdot g$ is odd

Let $x \in D(f) \bigcap D(g)$. Then, $-x \in D(f) \bigcap D(g)$, because f and g are even and odd respectively.

$$f(-x)g(-x) = f(x)(-g(x)) = -f(x)g(x)$$
$$\Rightarrow f \cdot g \text{ is odd.}$$

3.3.2 Check if $f(x) = x^5 + x^3 - x$ is odd or even

$$\begin{aligned} f(-x) &= (-x)^5 + (-x)^3 - (-x) \\ &= -x^5 - x^3 + x \\ &= -f(x) \end{aligned}$$

Therefore, f is odd.

3.3.3 Check if $f(x) = 2^{x^2+x}$ is odd, even or neither

$$\begin{aligned} f(1) &= 2^{1^2+1} = 4 \\ f(-1) &= 2^{(-1)^2+1} = 2^0 = 1 \end{aligned}$$

Therefore, f is neither odd nor even.

4 Image and Range of a Function

4.1 Examples

4.1.1 What is the image of

4.1.1.1 $f(x) = \ln x$ in the domain $(0, 4]$

As the function is monotonic, it is easy to observe that the image is $(-\infty, \ln 4]$

4.1.1.2 $f(x) = \cos x$ in the domain $(0, 4]$

From the graph of the function, it is evident that the image is $[-1, 1)$

5 Graphs

5.1 Shifting with respect to the axes

$f(x+a)$ is the graph of $f(x)$, shifted by a, in the direction of the x-axis, opposite to the sign of a.
$f(x) + a$ is the graph of $f(x)$, shifted by a, in the direction of the y-axis, according to the sign of a.

5.2 Mirror Images with respect to the axes

$f(-x)$ is the mirror image of $f(x)$ w.r.t. the y-axis.
$-f(x)$ is the mirror image of $f(x)$ w.r.t. the x-axis.

6 Monotonic Functions

6.1 Examples

6.1.1 Are the following functions monotonic?

6.1.1.1 e^{e^x}

$e^{e^x} = e^{(e^x)}$ is a composition of two monotonically increasing functions. Therefore, it is monotonically increasing.

6.1.1.2 $x^2 - 1$

The function is not monotonic over $\mathbb{R}$.

7 Inequalities

7.1 Examples

7.1.1 Solve the following

7.1.1.1 $|x+6| < |x-2|$

Solving by dividing the domain into regions
The regions are

$$\begin{aligned} x &\leq -6 \\ -6 < x &\leq 2 \\ 2 &< x \end{aligned}$$

Solving by squaring both sides

$$\begin{aligned} |x+6| &< |x-2| \\ \Leftrightarrow (x+6)^2 &< (x-2)^2 \\ \Leftrightarrow 12x + 36 &< -4x + 4 \\ \Leftrightarrow 16x &< -32 \\ \Leftrightarrow x &< -2 \end{aligned}$$

7.1.1.2 $\dfrac{x-1}{x-3} > \dfrac{x+3}{x+1}$

We multiply both sides by $(x-3)^2$ and $(x+1)^2$, rather than $(x-3)$ and $(x+1)$, to avoid dealing with flipping of the direction of the inequality.

$$\begin{aligned} \therefore (x-1)(x-3)(x+1)^2 &> (x+3)(x+1)(x-3)^2 \\ \Leftrightarrow (x-3)(x+1)\left((x-1)(x+1-(x+3)(x-3)\right) &> 0 \\ \Leftrightarrow 8(x-3)(x+1) &> 0 \end{aligned}$$

Therefore the inequality holds iff $x > 3$ or $x < -1$

8 Periodic Functions

f is periodic iff $\exists T > 0$ s.t. $f(x+T) = f(x), \forall x \in D(f)$
The smallest T, if it exists, for which the above equality holds true, is called the period of f.
Note that if f has period T, and g has a period which is a rational multiple of T, say $\dfrac{m}{n}T$, then f and g have a mutual period mT.