

# Differential and Integral Methods

Aakash Jog

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## **Part I**

# **General Information**

## **1 Contact Information**

**Dr. Yakov Yakubov**  
yakubov@post.tau.ac.il

## **2 Office Hours**

Monday  
16:15 - 17:15  
Room 233, Schreiber Building  
Tel: 03-6405357

## Part II

# Functions

### 3 Notation

$\mathbb{N}$  = Set of all natural numbers

$\mathbb{Z}$  = Set of all integers

$$\mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N} \right\}$$

### 4 Basic Definitions

**Definition 1** (Domain, Range and Variables). Let  $D$  and  $E$  be two sets of real numbers. A function  $f$  from  $D$  into  $E$  is a well defined law which, to each  $x \in D$  corresponds to a unique number  $y \in E$ . The set  $D$  is called the domain of  $f$  and the set  $E$  is called the range of  $f$ .

Denote  $f : D \rightarrow E$  or  $y = f(x)$ .

The variable  $x$  is called independent variable and the variable  $y$  is called dependent variable.

The variable  $x$  is also called the origin of  $y$  and  $y$  is also called the image of  $x$ .

**Definition 2** (Image of a function). Given  $f : D \rightarrow E$ . Then the image of  $f$  is a set of all  $y \in E$  s.t.  $\exists x \in D, y = f(x) : I(f) = \{y \in E : \exists x \in D, y = f(x)\}$

**Definition 3** (Existence domain). The biggest possible domain of a function  $f$  is called the existence domain of  $f$ .

**Definition 4** (Graph of a function). A set of point  $\{(x, f(x)) : x \in D\}$  in the plane  $\mathbb{R}^2$  is called a graph of a function  $y = f(x)$ .

**Definition 5** (Even function). If  $f(-x) = f(x); (x, -x \in D)$  then,  $f$  is called an even function.

Each even function is symmetric about the  $y$ -axis.

**Definition 6** (Odd function). If  $f(-x) = -f(x); (x, -x \in D)$  then,  $f$  is called an odd function.

Each odd function is symmetric about the origin.

**Definition 7** (Periodical function). A function  $y = f(x)$  which is defined on  $D$  is called periodical if  $\exists T \neq 0$  which is called a period of  $f$  s.t.  $\forall x \in D \Rightarrow$

$x + T \in D$  and  $f(x + T) = f(x)$ .

The smallest such  $T > 0$  (if it exists) is called the minimal period.

**Definition 8** (Shifting with respect to  $y$ -axis).  $f(x + a)$  is the graph of  $f(x)$ , shifted by  $a$ , in the direction of the  $x$ -axis, opposite to the sign of  $a$ .

**Definition 9** (Shifting with respect to  $x$ -axis).  $f(x) + a$  is the graph of  $f(x)$ , shifted by  $a$ , in the direction of the  $y$ -axis, according to the sign of  $a$ .

**Definition 10** (Monotonic function). A function  $y = f(x)$  is called monotonic increasing (strongly increasing) in  $D$  if  $\forall x_1, x_2 \in D, x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$  ( $f(x_1) < f(x_2)$ ).

A function  $y = f(x)$  is called monotonic decreasing (strongly decreasing) in  $D$  if  $\forall x_1, x_2 \in D, x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2)$  ( $f(x_1) > f(x_2)$ ).

**Definition 11** (One-to-one function). A function  $f : D(f) \rightarrow E$  is called one-to-one if  $\forall y \in I(f) \Rightarrow \exists! x \in D(f)$  s.t.  $y = f(x)$ .

Equivalently,  $\forall x_1, x_2 \in D(f)$ , if  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ .

## 5 Inverse of a Function

**Definition 12** (Inverse function). If  $f : D(f) \rightarrow I(f)$  is one-to-one and onto. Then, we can define  $g : I(f) \rightarrow D(f)$ , which is one-to-one and onto, by  $g(y) = x$ , where  $y = f(x)$ . Therefore,  $g(f(x)) = x$ .  $g$  is called the inverse function of  $f$ .

The inverse function is denoted as  $g = f^{-1}$  (Note:  $f^{-1} \neq \frac{1}{f}$ )

$$D(f) = I(f^{-1}) \quad (1)$$

$$I(f) = D(f^{-1}) \quad (2)$$

The graphs of a  $f$  and  $f^{-1}$  are symmetric about  $y = x$ .

## 6 Elementary Operations between Functions

**Definition 13** (Addition and subtraction of functions).

$$h(x) = f(x) \pm g(x) \quad (3)$$

$$D(h) = D(f) \cap D(g) \quad (4)$$

**Definition 14** (Multiplication of a function by a constant).

$$h(x) = kf(x) \quad (5)$$

$$D(h) = D(f) \quad (6)$$

**Definition 15** (Multiplication of functions).

$$h(x) = f(x)g(x) \quad (7)$$

$$D(h) = D(f) \cap D(g) \quad (8)$$

**Definition 16** (Division of functions).

$$h(x) = \frac{f(x)}{g(x)} \quad (9)$$

$$D(h) = \{x \in D(f) \cap D(g) : g(x) \neq 0\} \quad (10)$$

## 7 Composite Functions

**Definition 17** (Composition of Functions). Let  $f : D(f) \rightarrow E$  and  $g : D(g) \rightarrow F$  be two functions. A composition of  $f$  with  $g$  is a function  $h : D(h) \rightarrow F$  where  $h(x) = g(f(x))$ . It is denoted as  $g \circ f$

$$D(h) = \{x \in D(f) : f(x) \in D(g)\} \quad (11)$$

## 8 Elementary Functions

### 8.1 Polynomial

$$y = f(x) = a_0 + a_1x + \cdots + a_nx^n; a_0, \dots, a_n \in \mathbb{R} \quad (12)$$

$$D(f) = \mathbb{R} \quad (13)$$

1. If  $n = 0, y = f(x) = a_0$  represents a constant function.
2. If  $n = 1, y = f(x) = a_0 + a_1x$  represents a straight line in the  $X - Y$  plane.
3. If  $n = 2, y = f(x) = a_0 + a_1x + a_2x^2$  represents a parabola in the  $X - Y$  plane.

### 8.2 Power Function

$$y = f(x) = x^a; a \in \mathbb{R} \quad (14)$$

$$D(f) \text{ depends on } a \quad (15)$$

### 8.3 Exponential Function

$$y = f(x) = a^x; a > 0, a \neq 1 \quad (16)$$

$$D(f) = \mathbb{R} \quad (17)$$

$$I(f) = (0, \infty) \quad (18)$$

## 8.4 Logarithmic Function

$$y = f(x) = \log_a x \quad (19)$$

$$D(f) = (0, \infty) \quad (20)$$

$$I(f) = \mathbb{R} \quad (21)$$

## 8.5 Trigonometric Functions

$$y = f(x) = \sin x \quad (22)$$

$$y = f(x) = \cos x \quad (23)$$

$$y = f(x) = \tan x = \frac{\sin x}{\cos x} \quad (24)$$

$$y = f(x) = \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x} \quad (25)$$

$$y = f(x) = \csc x = \frac{1}{\sin x} \quad (26)$$

$$y = f(x) = \sec x = \frac{1}{\cos x} \quad (27)$$

## 8.6 Inverse Trigonometric Functions

$$y = f^{-1}(x) = \sin^{-1} x = \arcsin x \quad (28)$$

$$D(\arcsin x) = [-1, 1] \quad (29)$$

$$I(\arcsin x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad (30)$$

$$y = f^{-1}(x) = \cos^{-1} x = \arccos x \quad (31)$$

$$D(\arccos x) = [-1, 1] \quad (32)$$

$$I(\arccos x) = [0, \pi] \quad (33)$$

$$y = f^{-1}(x) = \tan^{-1} x = \arctan x \quad (34)$$

$$D(\arctan x) = \mathbb{R} \quad (35)$$

$$I(\arctan x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad (36)$$

## 8.7 Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad (37)$$

$$D(\sinh x) = \mathbb{R} \quad (38)$$

$$I(\sinh x) = \mathbb{R} \quad (39)$$

$$\cosh x \doteq \frac{e^x + e^{-x}}{2} \quad (40)$$

$$D(\cosh x) = \mathbb{R} \quad (41)$$

$$I(\cosh x) = [1, \infty) \quad (42)$$

$$\tanh x \doteq \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad (43)$$

$$D(\tanh x) = \mathbb{R} \quad (44)$$

$$I(\tanh x) = (-1, 1) \quad (45)$$

### 8.7.1 Identities of Hyperbolic Functions

$$\sinh(2x) = 2 \sinh x \cosh x \quad (46)$$

$$\cosh^2 x + \sinh^2 x = \cosh(2x) \quad (47)$$

$$\cosh^2 x - \sinh^2 x = 1 \quad (48)$$

$$\frac{\cosh(2x) - 1}{2} = \sinh^2 x \quad (49)$$

$$\frac{\cosh(2x) + 1}{2} = \cosh^2 x \quad (50)$$

## 8.8 Absolute Value

$$y = f(x) = \begin{cases} x; & x > 0 \\ 0; & x = 0 \\ -x; & x < 0 \end{cases} \quad (51)$$

## 8.9 Floor Function

$$y = f(x) = \lfloor x \rfloor = \text{the largest integer less than or equal to } x \quad (52)$$



## Part III

# Limits and Continuity

**Definition 18** (Existence of a limit). If  $x \rightarrow a^+$  then  $f(x) \rightarrow L_2$ , and if  $x \rightarrow a^-$  then  $f(x) \rightarrow L_1$ .

We say that  $\exists \lim_{x \rightarrow a} f(x)$  iff  $L_1 = L_2$

**Definition 19** (Continuity of a function). If  $x \rightarrow a$  then  $f(x) \rightarrow L$ , we say that  $L$  is the limit of  $f(x)$  at  $x = a$ .

$$\lim_{x \rightarrow a} f(x) = L$$

We say that  $f(x)$  is continuous at  $x = a$ , iff

$$\lim_{x \rightarrow a} f(x) = L = f(a)$$

**Definition 20** (Cauchy's definition of a limit of a function). Let  $f(x)$  be defined on an open interval about  $a$ , except possibly at  $a$  itself.

A number  $L$  is called the limit of  $f(x)$  at  $a$  if

$$\forall \epsilon > 0 \exists \delta > 0 : 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon \quad (53)$$

## 9 A Classification of Discontinuity Points

Let  $f(x)$  be defined on an open interval about  $a$ , except possibly at  $a$  itself.

**Definition 21** (Removable Discontinuity point). The point  $a$  is a removable discontinuity point of  $f$  if,  $\lim_{x \rightarrow a} f(x)$  exists, but either  $\lim_{x \rightarrow a} f(x) \neq f(a)$  or  $f(a)$  does not exist.

**Definition 22** (Discontinuity of First Kind). The point  $a$  is a discontinuity point of the first kind if both  $\lim_{x \rightarrow a^-} f(x)$  and  $\lim_{x \rightarrow a^+} f(x)$  exist, but  $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$

**Definition 23** (Discontinuity of Second Kind). The point  $a$  is a discontinuity point of the second kind if atleast one of the two one-sided limits of  $f$  does not exist.

Note that the limits are defined as finite numbers only.

## Theorems

**Theorem 1** (Sandwich Theorem). Let  $f(x), g(x), h(x)$  be defined on an open interval about  $a$ , except possibly at  $a$  itself. Assume that  $\forall x \neq a$  from the interval, it is satisfied that  $f(x) \leq g(x) \leq h(x)$  and  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$ . Then,  $\lim_{x \rightarrow a} g(x) = L$ .

*Proof.*

$$\forall \varepsilon > 0, \exists \delta > 0 : 0 < |x - a| < \delta$$

$$\text{therefore } |g(x) - L| < \varepsilon$$

i.e.,

$$L - \varepsilon < g(x) < L + \varepsilon$$

Given  $\exists \delta_1 > 0 : 0 < |x - a| < \delta_1$

$$f(x) \leq g(x) \leq h(x)$$

For this  $\varepsilon > 0$ ,

$$\exists \delta_2 > 0 : 0 < |x - a| < \delta_2$$

$$\therefore |f(x) - L| < \varepsilon$$

i.e.,

$$L - \varepsilon < f(x) < L + \varepsilon$$

$$\varepsilon > 0, \exists \delta_3 > 0 : 0 < |x - a| < \delta_3$$

$$\therefore |h(x) - L| < \varepsilon$$

i.e.,

$$L - \varepsilon < h(x) < L + \varepsilon$$

So,  $\forall \varepsilon > 0$ ,

$$\exists \delta = \min\{\delta_1, \delta_2, \delta_3\} > 0 : 0 < |x - a| < \delta$$

$$\therefore L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon$$

□

**Theorem 2** (Theorem 5). *If  $\lim_{x \rightarrow a} f(x) = 0$  and  $g(x)$  is bounded in an open interval about  $a$ , except possibly at  $a$  itself, then,  $\lim_{x \rightarrow a} (f(x)g(x)) = 0$ .*

*Proof.* We have to prove that

$$\forall \varepsilon > 0, \exists \delta > 0 : 0 < |x - a| < \delta \Rightarrow |f(x)g(x) - 0| < \varepsilon$$

Given  $\lim_{x \rightarrow a} f(x) = 0$ ,

$$\forall \varepsilon_1 > 0, \exists \delta_1 > 0 : 0 < |x - a| < \delta_1 \Rightarrow |f(x) - 0| < \varepsilon_1$$

As  $g(x)$  is bounded, in an open interval about  $a$ , except possibly at  $a$  itself,

$$\exists \delta_2 > 0, \exists M > 0 : 0 < |x - a| < \delta_2 \Rightarrow |g(x)| \leq M$$

So, if we choose  $\varepsilon = \frac{\varepsilon}{M}$ ,

$$\forall \varepsilon > 0, \exists \delta = \min\{\delta_1, \delta_2\} > 0 : 0 < |x - a| < \delta \Rightarrow |f(x)g(x) - 0| = |f(x)||g(x)| < \varepsilon_1 M = \varepsilon$$

□

## 10 Infinite Limits

$$\lim_{x \rightarrow a} f(x) = +\infty \Leftrightarrow \forall M > 0, \exists \delta > 0 : 0 < |x - a| < \delta \Rightarrow f(x) > M$$

$$\lim_{x \rightarrow a} f(x) = -\infty \Leftrightarrow \forall M < 0, \exists \delta > 0 : 0 < |x - a| < \delta \Rightarrow f(x) < M$$

$$\lim_{x \rightarrow +\infty} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists M > 0 : x > M \Rightarrow |f(x) - L| < \varepsilon$$

$$\lim_{x \rightarrow -\infty} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists M > 0 : x < -M \Rightarrow |f(x) - L| < \varepsilon$$

## 11 Known Limits

$$\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

## 12 Examples

**Example 1.** Find

$$\lim_{x \rightarrow 0} \frac{\tan 2x}{x}$$

*Solution.*

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan 2x}{x} &= \lim_{x \rightarrow 0} \frac{\frac{\sin 2x}{\cos 2x}}{x} \\ &= \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \cdot \frac{2}{\cos x} \\ &= \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \lim_{x \rightarrow 0} \frac{2}{\cos x} \\ &= 1 \cdot 2 = 2 \end{aligned}$$

## Part IV

# Derivatives

## 13 Derivative of a Function

### 13.1 Algebraic Definition

**Definition 24.** If there exists

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = L$$

then the limit is called the derivative of  $f$  at  $x_0$ .

### 13.2 Geometrical Interpretation

The slope of  $\overleftrightarrow{AB}$  is

$$\tan \alpha = \frac{\Delta y}{\Delta x}$$

When  $\Delta x \rightarrow 0$ ,  $\overleftrightarrow{AB}$  tends to a straight line which is called the tangent to  $y = f(x)$  at  $(x_0, f(x_0))$ .

The derivative is the slope of the tangent to  $y = f(x)$  at  $(x_0, f(x_0))$ .

### 13.3 Notation

$$L = f'(x_0) = \frac{df(x_0)}{dx} = \frac{dy(x_0)}{dx} = Df(x_0)$$

## 14 Derivative Function

If we calculate the derivative of  $f(x)$  at any possible  $x$ , we get the derivative function.

$$f'(x) = \frac{df}{dx} = \frac{dy}{dx} = Df$$

## 15 The Tangent Line

$$y - f(x_0) = f'(x_0)(x - x_0)$$

## 16 The Normal Line

$$y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0) \quad ; f'(x_0) \neq 0$$
$$x = x_0 \quad ; f'(x_0) = 0$$

## 16.1 Proofs of Standard Derivatives

16.1.1  $y = f(x) = c$

$$\begin{aligned} f'(x) &= \lim_{x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{x \rightarrow 0} \frac{c - c}{\Delta x} \\ &= 0 \end{aligned}$$

16.1.2  $y = f(x) = x^n; n \in \mathbb{N}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} \\ a^n - b^n &= (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) \\ \therefore f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x((x + \Delta x)^{n-1} + (x + \Delta x)^{n-2}x + \cdots + (x + \Delta x)x^{n-2} + x^{n-1})}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (x + \Delta x)^{n-1} + (x + \Delta x)^{n-2}x + \cdots + (x + \Delta x)x^{n-2} + x^{n-1} \\ &= x^{n-1} + x \cdot x^{n-2} + \cdots + x^{n-2} \cdot x + x^{n-1} \\ &= nx^{n-1} \end{aligned}$$

16.1.3  $y = f(x) = x^{-n}; n \in \mathbb{N}, x \neq 0$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^{-n} - x^{-n}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{(x + \Delta x)^n} - \frac{1}{x^n}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^n - (x + \Delta x)^n}{\Delta x(x^n(x + \Delta x)^n)} \\ &= \frac{-nx^{n-1}}{x^n x^n} \end{aligned}$$

**16.1.4**  $y = f(x) = \sin x$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{2 \sin \frac{\Delta x}{2} \cos(x + \frac{\Delta x}{2})}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cos\left(x + \frac{\Delta x}{2}\right) \\
 &= \cos x
 \end{aligned}$$

**Theorem 3.** *Theorem: If  $\exists f'(x)$ ,  $\exists g'(x)$  and  $c$  is a constant, then,*

$$\begin{aligned}
 (cf(x))' &= cf'(x) \\
 (f(x) \pm g(x))' &= f'(x) \pm g'(x) \\
 (f(x)g(x))' &= f'(x)g(x) + f(x)g'(x) \\
 \left(\frac{f(x)}{g(x)}\right)' &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}
 \end{aligned}$$

*Proof.*

$$\begin{aligned}
 (f(x)g(x))' &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x)g(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(f(x + \Delta x)g(x + \Delta x) - f(x)g(x + \Delta x)) + (f(x)g(x + \Delta x) - f(x)g(x))}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(f(x + \Delta x) - f(x))g(x + \Delta x)}{\Delta x} + \frac{f(x)(g(x + \Delta x) - g(x))}{\Delta x} \\
 &= f'(x)g(x) + f(x)g'(x)
 \end{aligned}$$

□

**Theorem 4.** *Let  $f$  be defined on an open interval about  $x_0$ . Then,  $f(x)$  is differentiable at  $x_0$ , iff  $\exists A \in \mathbb{R}$  and  $\exists \alpha(\Delta x)$ , with  $\lim_{\Delta x \rightarrow 0} \alpha(\Delta x) = 0$ , s.t.  $\frac{\Delta y}{\Delta x} = A + \alpha(\Delta x)$ ;  $A = f'(x_0)$*

**Theorem 5.** *If  $y = f(x)$  is differentiable at  $x_0$ , then,  $f(x)$  is continuous at  $x_0$ .*

*Proof.*

$$\begin{aligned}
 \exists f'(x_0) &\Rightarrow \frac{\Delta y}{\Delta x} = f'(x_0) + \alpha(\Delta x) \\
 \therefore f(x_0 + \Delta x) - f(x_0) &= \Delta y = \Delta x(f'(x_0) + \alpha(\Delta x)) \\
 \therefore f(x_0 + \Delta x) &= f(x_0) + \Delta x(f'(x_0) + \alpha(\Delta x)) \\
 \therefore \lim_{\Delta x \rightarrow 0} f(x_0 + \Delta x) &= \lim_{\Delta x \rightarrow 0} f(x_0) + \Delta x(f'(x_0) + \alpha(\Delta x)) \\
 &= x_0
 \end{aligned}$$

*Remark 1.* The converse of this theorem is not true. □

**Theorem 6** (Derivative of Inverse Functions). *Let  $f(x)$  be invertible and continuous in an open interval about  $x_0$ . If  $\exists f'(x_0) \neq 0$ , then, the inverse function  $x = g(y)$  is differentiable at  $y = f(x_0)$  and*

$$g'(y_0) = \frac{1}{f'(x_0)}$$

## 16.2 Examples

**Example 2.** Find the derivative of

$$y = f(x) = \tan x$$

*Solution.*

$$\begin{aligned} y = f(x) &= \tan x \\ \therefore (\tan^{-1})'y &= \frac{1}{\tan' x} \\ &= \frac{1}{\frac{1}{\cos^2 x}} \\ &= \frac{1}{1 + \tan^2 x} \\ &= \frac{1}{1 + y^2} \end{aligned}$$

Similarly,

$$(\cot^{-1})'x = -\frac{1}{1 + x^2}$$

**Theorem 7** (Chain Rule). *Let  $y = f(u)$  be differentiable at  $u_0$ , and  $u = g(x)$  be differentiable at  $x_0$ , s.t.  $u_0 = g(x_0)$ . Then,  $y = f(g(x))$  is differentiable at  $x_0$ , and,*

$$y'(x_0) = f'(u_0) \cdot g'(x_0)$$

*Proof.*

$$g'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x}$$

Therefore, by Theorem 2,

$$\begin{aligned} \therefore \frac{\Delta u}{\Delta x} &= g'(x_0) + \alpha_1(\Delta x); \alpha_1(\Delta x) \rightarrow 0 \text{ if } \Delta x \rightarrow 0 \\ \therefore \frac{\Delta y}{\Delta u} &= f'(u_0) + \alpha_2(\Delta u); \alpha_2(\Delta u) \rightarrow 0 \text{ if } \Delta u \rightarrow 0 \end{aligned}$$

Therefore,

$$\begin{aligned}
\Delta u &= (g'(x_0) + \alpha_1)\Delta x \\
\Delta y &= (f'(u_0) + \alpha_2)\Delta u \\
\therefore \Delta y &= (f'(u_0) + \alpha_2)(g'(x_0) + \alpha_1)\Delta x \\
\therefore \frac{\Delta y}{\Delta x} &= (f'(u_0) + \alpha_2)(g'(x_0) + \alpha_1)
\end{aligned}$$

$$\begin{aligned}
\Delta x \rightarrow 0 &\Rightarrow \Delta u \rightarrow 0, \alpha_1 \rightarrow 0 \\
&\Rightarrow \alpha_2 \rightarrow 0
\end{aligned}$$

Substituting,

$$y'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} (f'(u_0) + \alpha_2)(g'(x_0) + \alpha_1) = f'(u_0) \cdot g'(x_0)$$

□

**Theorem 8** (Fermat Theorem). *Let  $f(x)$  be defined on an open interval  $(a, b)$  and differentiable at  $x_0 \in (a, b)$ . If  $f(x)$  has its extremum at  $x_0$ , then,  $f'(x_0) = 0$*

*Proof.* Assume that  $f(x_0)$  is the maximum value of  $f(x)$  on  $(a, b)$ . Then,  $\forall \Delta x, f(x_0 + \Delta x) \leq f(x_0)$ .

Case 1.  $\Delta x > 0$

$$\begin{aligned}
&\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \leq 0 \\
\therefore \text{RHD} &= \lim_{\Delta x \rightarrow 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \leq 0
\end{aligned}$$

Case 2.  $\Delta x < 0$

$$\begin{aligned}
&\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \geq 0 \\
\therefore \text{LHD} &= \lim_{\Delta x \rightarrow 0^-} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \geq 0
\end{aligned}$$

$$\begin{aligned}
\exists f'(x_0) &\Rightarrow \text{LHD} = \text{RHD} \\
\therefore 0 &\leq f'(x_0) \leq 0 \\
\therefore f'(x_0) &= 0
\end{aligned}$$

□

**Theorem 9** (Rolle's Theorem). *Let  $f(x)$  be defined on  $[a, b]$ , s.t.*

1.  $f$  is continuous on  $[a, b]$



2.  $f$  is differentiable on  $(a, b)$

3.  $f(a) = f(b)$

Then,  $\exists c \in (a, b)$ , s.t.  $f'(c) = 0$ .

*Proof.* By Weierstrauss Theorem, as  $f(x)$  is continuous on  $[a, b]$ ,  $f(x)$  has its maximum  $M$  and minimum  $m$  on  $[a, b]$ .

Case 3.  $m = M$

$$\begin{aligned} f(x) &= \text{constant} \\ \therefore f'(x) &= 0 \text{ on } [a, b] \end{aligned}$$

Case 4.  $m < M$

Atleast one of  $m$  and  $M$  must be in  $(a, b)$ , otherwise  $f(a) \neq f(b)$ , which contradicts (3).

Let  $M = c \in (a, b)$ . Therefore, by Theorem 8,  $f'(c) = 0$

□

## 17 Lagrange Theorem

Let  $f(x)$  be defined on  $[a, b]$ , s.t.

1.  $f$  is continuous on  $[a, b]$
2.  $f$  is differentiable on  $(a, b)$

Then,  $\exists c \in (a, b)$ , s.t.  $f'(c) = \frac{f(b) - f(a)}{b - a}$

## 18 Theorem

Let  $f(x)$  be continuous on  $(x_0 - \delta, x_0 + \delta)$  and differentiable on  $(x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$ .

If  $\lim_{x \rightarrow x_0^+} f'(x) = \lim_{x \rightarrow x_0^-} f'(x) = L$ , then,  $\exists f'(x_0) = L$ .