Lecture 21

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Contents

1 Triple Integrals

Definition 1 (Solid of the second kind)**.** A solid E in $\mathbb{R}^3$ is called a solid of the second kind if there exist continuous functions $\varphi_1(x,z)$ and $\varphi_2(x,z)$, s.t.

$$E_{\mathrm{II}} = \{(x,y,z)|(x,z) \in D, \varphi_1(x,z) \leq y \leq \varphi_2(x,z)\}$$

Theorem 1. *If $f(x,y,z)$ is continuous on E_{II},*

$$\iiint\limits_{E_{\mathrm{II}}} f(x,y,z)\,\mathrm{d}V = \iint\limits_{D} \left(\int\limits_{\varphi_1(x,z)}^{\varphi_2(x,z)} f(x,y,z)\,\mathrm{d}y \right) \mathrm{d}A$$

Definition 2 (Solid of the third kind)**.** A solid E in $\mathbb{R}^3$ is called a solid of the third kind if there exist continuous functions $\varphi_1(y,z)$ and $\varphi_2(y,z)$, s.t.

$$E_{\mathrm{III}} = \{(x,y,z)|(y,z) \in D, \varphi_1(y,z) \leq y \leq \varphi_2(y,z)\}$$

Theorem 2. *If $f(x,y,z)$ is continuous on E_{III},*

$$\iiint\limits_{E_{\mathrm{III}}} f(x,y,z)\,\mathrm{d}V = \iint\limits_{D} \left(\int\limits_{\varphi_1(y,z)}^{\varphi_2(y,z)} f(x,y,z)\,\mathrm{d}z \right) \mathrm{d}A$$

Example 1. Calculate $\iiint\limits_{E} xz\,\mathrm{d}V$ when E is a solid bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

Solution.

$$
\begin{aligned}
\iiint\limits_{E_1} xz \,\mathrm{d}V &= \iint\limits_{D} \left(\int\limits_{\varphi_1(x,y)}^{\varphi_2(x,y)} xz \,\mathrm{d}z \right) \mathrm{d}A \\
&= \iint\limits_{D} x \left. \frac{z^2}{2} \right|_{z=0}^{z=1-x-y} \mathrm{d}A \\
&= \frac{1}{2} \iint\limits_{D} x(1-x-y)^2 \,\mathrm{d}A \\
&= \frac{1}{2} \int\limits_{0}^{1} \int\limits_{0}^{1-x} x(1-x-y)^2 \,\mathrm{d}y \,\mathrm{d}x \\
&= \frac{1}{2} \int\limits_{0}^{1} \left. -x \frac{(1-x-y)^3}{3} \right|_{y=0}^{y=1-x} \mathrm{d}x \\
&= \frac{1}{6} \int\limits_{0}^{1} x(1-x)^3 \,\mathrm{d}x \\
&= \frac{1}{120}
\end{aligned}
$$

2 Line Integrals of Scalar Functions

Definition 3 (Line integral of scalar function)**.** Let a curve from A to B be divided into n parts, s.t. $A = P_0$, $B = P_n$.
Let Δs_i be the length of the curve $P_{i-1}P_i$.
Let

$$\Delta T = \max\{\Delta s_i\}$$

Then,

$$\int\limits_{C} f(x,y) \,\mathrm{d}s = \lim_{\Delta T \to 0} \sum_{i=1}^{n} f({x_i}^*, {y_i}^*) \Delta s_i$$

is called a line integral of the first kind or line integral of scalar function.
This integral does not depend on the direction of the curve.
Geometrically, the line integral is the area under the curve $z = f(x,y)$ above the line C.

Definition 4 (Smooth curve)**.** Let C be given parametrically as

$$\overline{r}(t) = \big(x(t), y(t)\big) \quad t : a \to b$$

The curve is said to be smooth if

$$\overline{r}'(t) = \big(x'(t), y'(t)\big)$$

is a continuous function on $[a, b]$, $\overline{r}'(t) \neq \overline{0}$ on (a, b) and $\overline{r}'(t)$ is also continuous on (a, b).

Theorem 3. *If $f(x, y)$ is continuous and C is smooth, then*

$$\int\limits_C f(x, y)\,\mathrm{d}s = \int\limits_a^b f(x(t), y(t))\sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}\,\mathrm{d}t$$

Example 2. Find $\displaystyle\int\limits_C (2 + x^2 y)\,\mathrm{d}s$ when $C : y = \sqrt{1 - x^2}, -1 \leq x \leq 1$.

Solution.

$$\begin{aligned}\int\limits_C (2 + x^2 y)\,\mathrm{d}s &= \int\limits_0^1 (2 + \cos^2 t \sin t)\sqrt{-(\sin t)^2 + (\cos t)^2}\,\mathrm{d}t \\ &= 2\pi + \frac{2}{3}\end{aligned}$$

Theorem 4. *If the curve C is a piecewise smooth curve, i.e. $C = C_1 \cup \cdots \cup \ldots C_n$, where each C_i is smooth, and the length of all $C_i \cap C_j$ is zero, then*

$$\int\limits_C f(x, y)\,\mathrm{d}s = \int\limits_{C_1} f(x, y)\,\mathrm{d}s + \cdots + \int\limits_{C_n} f(x, y)\,\mathrm{d}s$$

Definition 5 (Line integral with respect to x and y)**.** A line integral of $f(x, y)$ over C with respect to x is

$$\int\limits_C f(x, y)\,\mathrm{d}x = \lim_{\Delta T \to 0} \sum_{i=1}^{n} f({x_i}^*, {y_i}^*)\Delta x_i$$

and with respect to y is

$$\int\limits_C f(x, y)\,\mathrm{d}y = \lim_{\Delta T \to 0} \sum_{i=1}^{n} f({x_i}^*, {y_i}^*)\Delta y_i$$

These integrals depend on the direction of C.

Theorem 5. *If $f(x, y)$ is continuous on C and C is smooth, then*

$$\int_C f(x, y)\,\mathrm{d}x = \int_a^b f(x(t), y(t))x'(t)\,\mathrm{d}t$$
$$\int_C f(x, y)\,\mathrm{d}y = \int_a^b f(x(t), y(t))y'(t)\,\mathrm{d}t$$

Example 3. Calculate by definition $\int_C \mathrm{d}s$, $\int_C \mathrm{d}x$, $\int_C \mathrm{d}y$ for the curve C given by the union of the line segments from $(0, 0, 0)$ to $(1, 1, 0)$ and $(1, 1, 0)$ to $(2, 0, 0)$.

Solution.

$$\begin{aligned}\int_C \mathrm{d}s &= \lim_{\Delta T \to 0} \sum_{i=1}^{n} \Delta s_i \\ &= \text{ length of line segment} \\ &= 2\sqrt{2}\end{aligned}$$

$$\begin{aligned}\int_C \mathrm{d}x &= \lim_{\Delta T \to 0} \sum_{i=1}^{n} \Delta x_i \\ &= \Delta x \\ &= 2\end{aligned}$$

$$\begin{aligned}\int_C \mathrm{d}y &= \lim_{\Delta T \to 0} \sum_{i=1}^{n} \Delta x_i \\ &= \Delta y \\ &= 0\end{aligned}$$

3 Line Integrals of Vector Functions

Definition 6 (Line integral of vector function)**.**

$$\begin{aligned} W &= \int\limits_C \overline{F} \cdot \hat{T} \, \mathrm{d}s \\ &= \int\limits_C \overline{F} \cdot \mathrm{d}\overline{z} \\ &= \int\limits_C P \, \mathrm{d}x + Q \, \mathrm{d}y + R \, \mathrm{d}z \end{aligned}$$

Example 4. Find the work W done by the force $\overline{F}(x, y) = (x, xy)$ over the curve $C : \overline{r}(t) = (2\cos t, 2\sin t), t : \pi \to 2\pi$.

Solution.

$$\begin{aligned} W &= \int\limits_C \overline{F} \cdot \hat{T} \, \mathrm{d}s \\ &= \int\limits_{\pi}^{2\pi} \big(2\cos t(-2\sin t) + 2\cos t \cdot 2\sin t \cdot 2\cos t\big) \, \mathrm{d}t \\ &= \int\limits_{\pi}^{2\pi} \big(-2\sin(2t) + 8\cos^2 t \sin t\big) \, \mathrm{d}t \\ &= \cos(2t) - \frac{8}{3}\cos^3 t \Big|_{\pi}^{2\pi} \\ &= \left(1 - \frac{8}{3}\right) - \left(1 + \frac{8}{3}\right) \\ &= -\frac{16}{3} \end{aligned}$$