Recitation 6

Wednesday 3rd December, 2014

Contents

1 Taylor's Series

Example 1. Calculate $\sqrt[3]{29}$ with an accuracy of 10^{-3}.

Solution.

$$\begin{aligned}
f(x) &= x^{1/3} \\
\therefore f'(x) &= \frac{1}{3} \cdot x^{-2/3} \\
\therefore f''(x) &= -\frac{1}{3} \cdot \frac{2}{3} x^{-5/3} \\
\therefore f'''(x) &= \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{5}{3} x^{-8/3} \vdots \\
&= f^{(n)} \frac{1 \cdot 2 \cdot 5 \cdot 8 \cdot \dots \cdot (3(n-1)-1)}{3^n} (-1)^{n+1} x^{-\frac{3n-1}{3}}
\end{aligned}$$

$$\begin{aligned}
f(x) &= \sum_{k=0}^{n} \frac{f^{(n)}(x_0)}{k!}(x-x_0)^k + R_n(x) \\
\therefore \sqrt[3]{29} &= \sum_{k=0}^{n} \frac{f^{(n)}}{k!}(29-27)^k + R_n(x) \\
R_n(29) &= \frac{f^{(n+1)}(c)}{(n+1)!}(29-27)^{n+1} \qquad ; \quad 27 \leq c \leq 29
\end{aligned}$$

According to the given accuracy,

$$|R_n(x)| < 10^{-3}$$

$$\therefore \frac{1 \cdot 2 \cdot 5 \cdot 8 \cdot (3n-1)}{2^{n+1}} c^{-\frac{3n+2}{3}} \frac{2^{n+1}}{(n+1)!} < 10^{-3}$$

At $c = 27$, $R_n(x)$ is maximum.

$$\therefore (R_n(x))_{\max} = \frac{1 \cdot 2 \cdot 5 \cdot 8 \cdot (3n-1)}{2^{n+1}} 27^{-\frac{3n+2}{3}} \frac{2^{n+1}}{(n+1)!}$$

For $n = 2$,

$$(R_n(x))_{\max} < 10^{-3}$$

Therefore,

$$\sqrt[3]{29} \approx \sqrt[3]{27} + \frac{1}{3} \cdot 27^{-2/3}(29-27) - \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{27^{-5/3}(29-27)^2}{2!}$$

Example 2. Calculate e^x with accuracy of 10^{-5} for $0 \leq x \leq 1$.

Solution.

$$
\begin{aligned}
e^x &= \sum_{k=0}^{n} \frac{x^k}{k!} + R_n(x) \\
R_n(x) &= \frac{e^c}{(n+1)!} x^{n+1}
\end{aligned}
$$

According to the given accuracy,

$$
\begin{aligned}
|R_n(x)| \leq 10^{-5} \frac{e^x}{(n+1)!} x^{n+1} & & \leq \frac{e}{(n+1)!} \\
< \frac{3}{(n+1)!} & & < 10^{-5}
\end{aligned}
$$

Therefore, $n = 10$ satisfies the required accuracy.

$$
\therefore e^x \approx \sum_{k=0}^{10} \frac{x^k}{k!}
$$