Lecture 13

Aakash Jog

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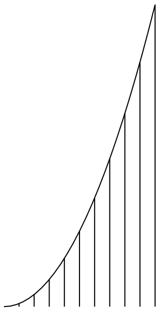
Contents

1 Definite Integrals

1.1 Definition

Example 1. Calculate an area S which is situated between $y = f(x) = x^2$ and the x-axis on $[0, b]$.

Solution. Divide $[0, b]$ on n sub-intervals.



Therefore, for $i = 1, \dots, n$,

$$\begin{aligned} \Delta x_i &= x_i - x_{i-1} \\ &= \frac{b}{n} \end{aligned}$$

Therefore, for $i = 0, \dots, n$,

$$x_i = \frac{ib}{n}$$

Let $\underline{S_n}$ be the sum of areas of rectangles below the function.
Let $\overline{S_n}$ be the sum of areas of rectangles above the function.

$$\sum_{k=1}^{n} k^2 = \frac{(n)(n+1)(2n+1)}{6}$$

$$\begin{aligned}
\therefore \underline{S_n} &= \sum_{i=1}^{n} f(x_{i-1})\Delta x_i \\
&= \sum_{i=1}^{n} x_{i-1}^2 \Delta x_0 \\
&= \sum_{i=1}^{n} \frac{(i-1)^2 b^2}{n^2} \cdot \frac{b}{n} \\
&= \frac{b^3}{n^3} \cdot \frac{(n)(n-1)(2n+1)}{6} \\
\therefore \overline{S_n} &= \sum_{i=1}^{n} f(x_i)\Delta x_i \\
&= \sum_{i=1}^{n} x_i^2 \Delta x_0 \\
&= \sum_{i=1}^{n} \frac{i^2 b^2}{n^2} \cdot \frac{b}{n} \\
&= \frac{b^3}{n^3} \cdot \frac{(n)(n-1)(2n+1)}{6}
\end{aligned}$$

$\forall n \in \mathbb{N}$,

$$S_n \leq S \leq \overline{S_n}$$

Also,

$$\lim_{n\to\infty} \overline{S_n} = \frac{b^3}{3} = \lim_{n\to\infty} \underline{S_n}$$

Therefore, by the Sandwich Rule,

$$\lim_{n\to\infty} S = S = \frac{b^3}{3}$$

Definition 1 (Riemann integral sum)**.** Let $f(x)$ be defined on the closed interval $[a, b]$. Let $n \in \mathbb{N}$ and T be a partition of $[a, b]$ on n sub-intervals, i.e.

$$a = x_0 < x_1 < \cdots < x_i < \cdots < x_{n-1} < x_n = b$$

Let $\Delta x_i = x_i - x_{i-1}$.
$\Delta T = \max\{\Delta x_i\}$ is called the norm of the partition T.
For $c_i \in [x_{i-1,x_i}]$, $\sum_{i=1}^{n} f(c_i)\Delta x_i$ is called the Riemann integral sum corresponding to T.

Definition 2 (Definite integral)**.** A function $f(x)$ is called integrable by Riemann over $[a, b]$ if $\exists I = \lim\limits_{\Delta T \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$ and the limit does not depend on the choice of T and c_i. This limit is denoted by

$$I = \int\limits_a^b f(x)\,\mathrm{d}x$$

and is called the definite integral or the Riemann integral of $f(x)$ over $[a, b]$.

1.2 Integrability

Theorem 1. *If $f(x)$ is not bounded on $[a, b]$, then $f(x)$ is not integrable by Riemann over $[a, b]$.*

Corollary 1. *If $f(x)$ is integrable over $[a, b]$ then $f(x)$ is bounded on $[a, b]$.*

Definition 3. A function $f(x)$ which is defined and bounded on $[a, b]$ is called piecewise continuous if it has at most a finite number of discontinuous point and all of these are removable discontinuity points or discontinuous points of the first kind.

Theorem 2. *If $f(x)$ is piecewise continuous on $[a, b]$, then $f(x)$ is Riemann integrable over $[a, b]$.*

Theorem 3. *If $f(x)$ is bounded and monotonic on $[a, b]$, then $f(x)$ is Riemann integrable over $[a, b]$.*

1.3 Properties

If all following integrals exist, and $a, b, c \in \mathbb{R}$

$$\int_a^a f(x)\,\mathrm{d}x = 0$$

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$

$$\int_a^b x\,\mathrm{d}x = c(b-a)$$

$$\int_a^b (f(x) \pm g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$$

$$\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$$

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x$$

$$f(x) \geq 0 \forall x \in [a,b] \implies \int_a^b f(x)\,\mathrm{d}x \geq 0$$

$$f(x) \geq g(x) \forall x \in [a,b] \implies \int_a^b f(x)\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x$$

$$m \leq f(x) \leq M \forall x \in [a,b] \implies m(b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M(b-a)$$

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b \left|f(x)\right|\,\mathrm{d}x$$

Theorem 4 (Integral Intermediate Value Theorem)**.** *If $f(x)$ is continuous* *on* $[a,b]$*, then* $\exists c \in [a,b]$ *s.t.*

$$\int\limits_a^b f(x)\,\mathrm{d}x = f(c)(b-a)$$

Theorem 5 (Fundamental Theorem of Calculus, Part 1)**.** *Let* $f(x)$ *be continuous on* (a,b) *and let* $c \in (a,b)$*. Then, the function* $F(x) = \int\limits_c^x f(t)\,\mathrm{d}t$ *is an anti-derivative function of* $f(x)$ *on* (a,b)*, i.e.*

$$F'(x) = f(x) \quad ; \quad \forall x \in (a,b)$$

Proof. For any $x \in (a,b)$ and $x + \Delta x \in (a,b)$,

$$\begin{aligned} F(x+\Delta x) - F(x) &= \int\limits_c^{x+\Delta x} f(t)\,\mathrm{d}t - \int\limits_c^x f(t)\,\mathrm{d}t \\ &= \int\limits_c^x f(t)\,\mathrm{d}t + \int\limits_x^{x+\Delta x} f(t)\,\mathrm{d}t - \int\limits_c^x f(t)\,\mathrm{d}t \\ &= \int\limits_x^{x+\Delta x} f(t)\,\mathrm{d}t \end{aligned}$$

By Integral Intermediate Value Theorem, $\exists d \in [x, x+\Delta x]$

$$\begin{aligned} \therefore \int\limits_x^{x+\Delta x} f(t)\,\mathrm{d}t &= f(d)\Delta x \\ \therefore \frac{F(x+\Delta x) - F(x)}{\Delta x} &= f(d) \\ \therefore F'(x) &= \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} f(d) \\ &= f(x) \end{aligned}$$

$\square$

Theorem 6 (Fundamental Theorem of Calculus, Part 2)**.** *Let $f(x)$ be continuous on $[a, b]$ and let $G(x)$ be an arbitrary anti-derivative function of $f(x)$. Then,*

$$\int_a^b f(x)\,\mathrm{d}x = G(b) - G(a)$$

Proof. Let d be an arbitrary point in (a, b).
By Fundamental Theorem of Calculus, Part 1,

$$F(x) = \int_d^x f(t)\,\mathrm{d}t$$

is an anti-derivative function of $f(x)$ on $[a, b]$.
Therefore, for any other anti-derivative function $G(x) = F(x) + c$,

$$\begin{aligned}
G(b) - G(a) &= (F(b) + c)0(F(a) + c) \\
&= F(b) - F(a) \\
&= \int_d^b f(t)\,\mathrm{d}t - \int_d^a f(t)\,\mathrm{d}t \\
&= \int_d^a f(t)\,\mathrm{d}t + \int_a^b f(t)\,\mathrm{d}t - \int_d^a f(t)\,\mathrm{d}t \\
&= \int_a^b f(t)\,\mathrm{d}t
\end{aligned}$$

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