

Lecture 20

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1 Double Integrals

Example 1. Calculate $\iint_D \sin(y^2) \, dA$ for D given by the triangle enclosed by $(0, 0)$, $(0, 1)$, $(1, 1)$.

Solution. The domain D is of the first and second kind.

$$\begin{aligned} D_I : 0 \leq x \leq 1 \quad \& \quad x \leq y \leq 1 \\ D_{II} : 0 \leq y \leq 1 \quad \& \quad 0 \leq x \leq y \end{aligned}$$

Therefore,

$$\iint_{D_I} \sin(y^2) \, dA = \int_0^1 \int_x^1 \sin(y^2) \, dy \, dx$$

This form is unsolvable.

$$\begin{aligned} \iint_{D_{II}} \sin(y^2) \, dA &= \int_0^1 \int_0^y \sin(y^2) \, dx \, dy \\ &= \int_0^1 \sin(y^2) x \Big|_{x=0}^{x=y} \, dy \\ &= \int_0^1 y \sin(y^2) \, dy \\ &= -\frac{1}{2} \cos(y^2) \Big|_0^1 \\ &= -\frac{1}{2} \cos 1 + \frac{1}{2} \end{aligned}$$

Theorem 1. If D is not a domain of both kinds, but $D = D_I \cup D_{II}$, s.t. the area of $D_I \cap D_{II}$ is zero, then

$$\iint_D f(x, y) \, dA = \iint_{D_I} f(x, y) \, dA + \iint_{D_{II}} f(x, y) \, dA$$

1.1 Applications

1. If $f_x(x, y)$ and $f_y(x, y)$ are continuous in D , then the area of the surface $\sigma : z = f(x, y)$ above D is equal to

$$S(\Sigma) = \iint_D \sqrt{1 + (f_x(x, y))^2 + (f_y(x, y))^2} \, dA$$

2. If $\rho(x, y)$ is the density function of a thin body,

$$m = \iint_D \rho(x, y) \, dA$$

$$(x_{\text{COM}}, y_{\text{COM}}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$$

where

$$M_x = \iint_D y \rho(x, y) \, dA$$
$$M_y = \iint_D x \rho(x, y) \, dA$$

2 Triple Integrals

2.1 Triple Integrals on Parallelepiped Domains

Definition 1 (Triple integral on parallelepiped domain). Consider a parallelepiped

$$B = \{(x, y, z) | a \leq x \leq b, c \leq y \leq d, e \leq z \leq g\}$$

Consider a partition T

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$
$$c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d$$
$$e = z_0 < z_1 < \cdots < z_{n-1} < z_n = g$$

Consider a parallelepiped

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

Let

$$P_{ijk}^* = (x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$$

be a point in B_{ijk} . Let

$$\Delta x_i = x_i - x_{i-1}$$

$$\Delta y_j = y_j - y_{j-1}$$

$$\Delta z_k = z_k - z_{k-1}$$

Therefore,

$$\Delta V_{ijk} = \Delta x_i \cdot \Delta y_j \cdot \Delta z_k$$

Let

$$\Delta T = \max\{\Delta x_i, \Delta y_j, \Delta z_k\}$$

$\sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^l f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V_{ijk}$ is called a Riemann triple integral sum.

The double integral of $f(x, y, z)$ over the domain of definition B is the limit, if it exists, of the Riemann double integral sum, as $\Delta T \rightarrow 0$.

$$\iiint_B f(x, y, z) \, dV = \lim_{\Delta T \rightarrow 0} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^l f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V_{ijk}$$

Theorem 2 (Fubini Theorem). *If $f(x, y, z)$ is continuous on $B = [a, b] \times [c, d] \times [e, g]$, then the triple integral and six iterated integrals exist, and they are equal.*

$$\iiint_B f(x, y, z) \, dV = \int_a^b \int_c^d \int_e^g f(x, y, z) \, dz \, dy \, dx = \dots$$

Example 2. Calculate

$$\iiint_{[0,1] \times [-1,2] \times [1,3]} z^2 \sin(x+y) \, dV$$

Solution.

$$\begin{aligned}
\iiint_{[0,1] \times [-1,2] \times [1,3]} z^2 \sin(x+y) \, dV &= \int_0^1 \int_{-1}^2 \int_1^3 z^2 \sin(x+y) \, dz \, dy \, dx \\
&= \int_0^1 \int_{-1}^2 \left. \frac{z^3}{3} \sin(x+y) \right|_{z=0}^{z=3} dy \, dx \\
&= \frac{26}{3} \int_0^1 \int_{-1}^2 \sin(x+y) \, dy \, dx \\
&= \frac{26}{3} \int_0^1 -\cos(x+y) \Big|_{y=-1}^{y=2} dx \\
&= \frac{26}{3} \int_0^1 (-\cos(x+2) + \cos(x-1)) \, dx \\
&= \frac{26}{3} (-\sin(x+2) + \sin(x-1)) \Big|_0^1 \\
&= \frac{26}{3} (-\sin 3 + 0 - (-\sin 2 + \sin(-1))) \\
&= \frac{26}{3} (-\sin 3 + \sin 2 + \sin 1)
\end{aligned}$$

2.2 Triple Integrals on Arbitrary Domains

Definition 2 (Triple integral on arbitrary domain). Let E be an arbitrary solid which is bounded and closed in \mathbb{R}^3 and $f(x, y, z)$ be defined on E . Consider a parallelepiped B , s.t. $E \subset B$ and construct the function $F(x, y, z)$, s.t.

$$F(x, y, z) = \begin{cases} f(x, y, z) & ; \quad (x, y, z) \in E \\ 0 & ; \quad (x, y, z) \notin E \end{cases}$$

If $\exists \iiint_B F(x, y, z) \, dV$, then $\exists \iiint_E f(x, y, z) \, dV$, and

$$\iiint_E f(x, y, z) \, dV = \iiint_B F(x, y, z) \, dV$$

Definition 3 (Solid of the first kind). A solid E in \mathbb{R}^3 is called a solid of the first kind if there exist continuous functions $\varphi_1(x, y)$ and $\varphi_2(x, y)$, s.t.

$$E_1 = \{(x, y, z) | (x, y) \in D, \varphi_1(x, y) \leq z \leq \varphi_2(x, y)\}$$

Theorem 3. If $f(x, y, z)$ is continuous on E_1 ,

$$\iiint_{E_1} f(x, y, z) \, dV = \iint_D \left(\int_{\varphi_1(x, y)}^{\varphi_2(x, y)} f(x, y, z) \, dz \right) \, dA$$