Recitation 5

Wednesday 19^{th} November, 2014

Contents

1 Exercises

Example 1. Suppose $f : \mathbb{R} \to \mathbb{R}$ s.t. $f'(x) \neq 0, \forall x \in \mathbb{R}$. Show that $f(x) = 10$ has atmost one solution.

Solution. If possible, let $\exists x_1, x_2$, s.t. $x_1 \neq x_2, f(x_1) = f(x_2) = 10$.
WLG, let $x_1 < x_2$.
Therefore, by Rolle's Theorem, $\exists x \in (x_1, x_2)$, s.t. $f'(c) = 0$. This contradicts the given condition $f'(x) \neq 0$.
Therefore, $f(x) = 10$ has atmost one solution.

Example 2. Let $f : (a, b) \to \mathbb{R}$, s.t. $f'(x)$ and $f''(x)$ exist. Suppose $\exists a < x_1 < x_2 < x_3 < b$, s.t. $f(x_1) = f(x_2) = f(x_3)$. Show that $\exists c$, s.t. $f''(c) = 0$.

Solution. Applying Rolle's Theorem to $[x_1, x_2]$ and $[x_2, x_3]$, $\exists c_1, c_2$, s.t. $c_1 \in (x_1, x_2), x_2 \in (x_2, x_3)$ and $f'(c_1) = f'(c_2) = 0$.
Therefore, by Rolle's Theorem, $\exists c \in (c_1, c_2)$, s.t. $f''(c) = 0$.

Example 3. Show that $\forall 0 < a < b < \dfrac{\pi}{2}$

$$\frac{1}{\cos^2 a} < \frac{\tan b - \tan a}{b - a} < \frac{1}{\cos^2 b}$$

Solution. Let

$$\begin{aligned} f(x) &= \tan x \\ \therefore f'(x) &= \frac{1}{\cos^2 x} \end{aligned}$$

By LMVT,

$$\frac{\tan b - \tan a}{b - a} = \frac{1}{\cos^2 c}$$

$$\begin{aligned} &\frac{1}{\cos^2 a} < \frac{1}{\cos^2 b} < \frac{1}{\cos^2 c} \\ &\Leftrightarrow \cos^2 a > \cos^2 b > \cos^2 c \end{aligned}$$

Example 4. Show that

$$x - \ln x - 2 = 0$$

has exactly two solutions.

Solution. Let

$$\begin{aligned} f(x) &= x - \ln x - 2 \\ \therefore f'(x) &= 1 - \frac{1}{x} \end{aligned}$$

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow x = 1 \\ f(1) &= -1 \\ f(e^{-4}) &= e^{-4} - \ln e^{-4} - 2 \\ &= e^{-4} + 4 - 2 \\ &> 0 \\ f(e^4) &= e^4 - 4 - 2 \\ &> 0 \end{aligned}$$

By mean value theorem, $\exists c_1 \in (e^{-4}, 1)$ and $\exists c_2 \in (1, e^{-4})$, s.t. $f(c_1) = f(c_2) = 0$.
If there are 3 solutions to $f(x) = 0$, then there are 2 solutions to $f'(x) = 0$. This contradicts the fact that $f'(x)$ has exactly 1 solution.
Therefore, $f(x)$ has exactly 2 solutions.

Example 5. FInd y' where $(x - y)^2 - x - y = -1$.

Solution.

$$\begin{aligned} (x-y)^2 - x - y &= -1 \\ \therefore 2(x-y)(1-y') - 1 - y' &= 0 \\ \therefore 2(x-y)(1-y') + 1 - y' &= 2 \\ \therefore (1-y')(2x - 2y + 1) &= 2 \\ \therefore 1 - y' &= \frac{2}{2x - 2y + 1} \\ \therefore y' &= 1 - \frac{2}{2x - 2y + 1} \end{aligned}$$

Example 6. Find $(\arcsin x)'$ for $\arcsin x \in \left[0, \frac{\pi}{2}\right]$

Solution. Let

$$\begin{aligned} f(x) &= \sin x \\ \therefore f^{-1}(x) &= \arcsin x \end{aligned}$$

$$
\begin{aligned}
(f^{-1}(x))' &= \frac{1}{f'(f^{-1}(x))} \\
&= \frac{1}{\cos(\arcsin x)} \\
&= \frac{1}{\sqrt{1-x^2}}
\end{aligned}
$$

Example 7. Find the tangent line to $(x^2+y^2)^3 = 8x^2y^2$ at the point $(-1,1)$.

Solution.

$$(x^2+y^2)^3 = 8x^2y^2$$

Differentiating,

$$
\begin{aligned}
3(x^2+y^2)^2(2x+2yy') &= 8(2x)(y^2) + 8(x^2)(2yy') \\
\therefore 3(x^2+y^2)^2(2x+2yy') &= 16xy^2 + 16x^2yy'
\end{aligned}
$$

$x=-1, y=1$

$$
\begin{aligned}
\therefore 12(-2+2y') &= -16+16y' \\
\therefore y'(-1) &= 1
\end{aligned}
$$

Therefore the tangent at $(-1,1)$ is $y = x+2$.