

# Lecture 5

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# 1 Derivative of a Function

## 1.1 Definition

If there exists

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = L$$

then the limit is called the derivative of  $f$  at  $x_0$ .

### 1.1.1 Geometrical Interpretation

The slope of  $\overleftrightarrow{AB}$  is

$$\tan \alpha = \frac{\Delta y}{\Delta x}$$

When  $\Delta x \rightarrow 0$ ,  $\overleftrightarrow{AB}$  tends to a straight line which is called the tangent to  $y = f(x)$  at  $(x_0, f(x_0))$ .

The derivative is the slope of the tangent to  $y = f(x)$  at  $(x_0, f(x_0))$ .

## 1.2 Notation

$$L = f'(x_0) = \frac{df(x_0)}{dx} = \frac{dy(x_0)}{dx} = Df(x_0)$$

## 1.3 Derivative Function

If we calculate the derivative of  $f(x)$  at any possible  $x$ , we get the derivative function.

$$f'(x) = \frac{df}{dx} = \frac{dy}{dx} = Df$$

## 1.4 The Tangent Line

$$y - f(x_0) = f'(x_0)(x - x_0)$$

## 1.5 The Normal Line

$$y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0) \quad ; f'(x_0) \neq 0$$
$$x = x_0 \quad ; f'(x_0) = 0$$

## 1.6 Proofs of Standard Derivatives

**1.6.1**  $y = f(x) = c$

$$\begin{aligned} f'(x) &= \lim_{x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{x \rightarrow 0} \frac{c - c}{\Delta x} \\ &= 0 \end{aligned}$$

**1.6.2**  $y = f(x) = x^n; n \in \mathbb{N}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} \\ a^n - b^n &= (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) \\ \therefore f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x((x + \Delta x)^{n-1} + (x + \Delta x)^{n-2}x + \cdots + (x + \Delta x)x^{n-2} + x^{n-1})}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (x + \Delta x)^{n-1} + (x + \Delta x)^{n-2}x + \cdots + (x + \Delta x)x^{n-2} + x^{n-1} \\ &= x^{n-1} + x \cdot x^{n-2} + \cdots + x^{n-2} \cdot x + x^{n-1} \\ &= nx^{n-1} \end{aligned}$$

**1.6.3**  $y = f(x) = x^{-n}; n \in \mathbb{N}, x \neq 0$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^{-n} - x^{-n}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{(x + \Delta x)^n} - \frac{1}{x^n}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^n - (x + \Delta x)^n}{\Delta x(x^n(x + \Delta x)^n)} \\ &= \frac{-nx^{n-1}}{x^n x^n} \end{aligned}$$

**1.6.4**  $y = f(x) = \sin x$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{2 \sin \frac{\Delta x}{2} \cos\left(x + \frac{\Delta x}{2}\right)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cos\left(x + \frac{\Delta x}{2}\right) \\
 &= \cos x
 \end{aligned}$$

**1.7 Theorem:** If  $\exists f'(x)$ ,  $\exists g'(x)$  and  $c$  is a constant, then,  
 $(cf(x))' = cf'(x)$ ,  $(f(x) \pm g(x))' = f'(x) \pm g'(x)$ ,  $(f(x)g(x))' =$   
 $f'(x)g(x) + f(x)g'(x)$ ,  $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

$$\begin{aligned}
 (f(x)g(x))' &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x)g(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(f(x + \Delta x)g(x + \Delta x) - f(x)g(x + \Delta x)) + (f(x)g(x + \Delta x) - f(x)g(x))}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(f(x + \Delta x) - f(x))g(x + \Delta x)}{\Delta x} + \frac{f(x)(g(x + \Delta x) - g(x))}{\Delta x} \\
 &= f'(x)g(x) + f(x)g'(x)
 \end{aligned}$$

**1.8 Theorem:** Let  $f$  be defined on an open interval about  $x_0$ . Then,  $f(x)$  is differentiable at  $x_0$ , iff  $\exists A \in \mathbb{R}$  and  $\exists \alpha(\Delta x)$ , with  $\lim_{\Delta x \rightarrow 0} \alpha(\Delta x) = 0$ , s.t.  $\frac{\Delta y}{\Delta x} = A + \alpha(\Delta x)$ ;  $A = f'(x_0)$

**1.9** If  $y = f(x)$  is differentiable at  $x_0$ , then,  $f(x)$  is continuous at  $x_0$ .

$$\begin{aligned}
 \exists f'(x_0) &\Rightarrow \frac{\Delta y}{\Delta x} = f'(x_0) + \alpha(\Delta x) \\
 \therefore f(x_0 + \Delta x) - f(x_0) &= \Delta y = \Delta x(f'(x_0) + \alpha(\Delta x)) \\
 \therefore f(x_0 + \Delta x) &= f(x_0) + \Delta x(f'(x_0) + \alpha(\Delta x)) \\
 \therefore \lim_{\Delta x \rightarrow 0} f(x_0 + \Delta x) &= \lim_{\Delta x \rightarrow 0} f(x_0) + \Delta x(f'(x_0) + \alpha(\Delta x)) \\
 &= x_0
 \end{aligned}$$

The converse of this theorem is not true.