

# Differential and Integral Methods - Exercise 1

Aakash Jog

5<sup>th</sup> November, 2014

## (1) Find the domain of the following functions

(a)  $f(x) = 2x - 3 \sin x$

There is no restriction on the values of  $x$  in  $f(x)$ .

$$\boxed{\therefore D(f) = \mathbb{R}}$$

(b)  $f(x) = \frac{1}{\sqrt{|x|} - x}$

For the square root to exist,  $|x| - x \geq 0$ , and for the fraction to exist,  $\sqrt{|x|} - x \neq 0$ .

$$\therefore |x| - x > 0$$

$$\therefore x \in (-\infty, 0)$$

$$\boxed{\therefore D(f) = (-\infty, 0)}$$

(c)  $f(x) = \sqrt{\frac{x+3}{|x^2-4|}}$

Conditions on  $x$ :

$$\frac{x+3}{|x^2-4|} \geq 0 \tag{1}$$

$$|x^2-4| \neq 0 \tag{2}$$

$$(1) \Rightarrow x+3 \geq 0$$

$$\therefore x \geq -3$$

$$(2) \Rightarrow x^2 \neq 4$$

$$\therefore x \notin \{-2, 2\}$$

$$\boxed{\therefore D(f) = [-3, \infty) - \{-2, 2\}}$$

(d)  $f(x) = \ln(x+2)$

Conditions on  $x$ :

$$x+2 > 0 \tag{3}$$

$$\therefore x > -2$$

$$\boxed{\therefore D(f) = (-2, \infty)}$$

$$(e) \quad f(x) = \ln(|x + 2|)$$

Conditions on  $x$ :

$$|x + 2| > 0 \tag{4}$$

$$\therefore x \neq -2$$

$$\boxed{\therefore D(f) = \mathbb{R} - \{-2\}}$$

$$(f) \quad f(x) = \log_2(\log_2 x)$$

Conditions on  $x$ :

$$x > 0 \tag{5}$$

$$\log_2 x > 0 \tag{6}$$

$$\therefore x > 1$$

$$\boxed{\therefore D(f) = (1, \infty)}$$

$$(g) \quad \frac{1}{1 - \cos x}$$

Conditions on  $x$ :

$$1 - \cos x \neq 0 \tag{7}$$

$$\tag{8}$$

$$\therefore \cos x \neq 1$$

$$\therefore x \neq 2n\pi; n \in \mathbb{Z}$$

$$\boxed{\therefore D(f) = \mathbb{R} - \{2n\pi : n \in \mathbb{Z}\}}$$

$$(h) \quad f(x) = \frac{1 + \sin x}{\sqrt{1 - \sin^2 x}}$$

$$\begin{aligned} f(x) &= \frac{1 + \sin x}{\sqrt{1 - \sin^2 x}} \\ &= \frac{1 + \sin x}{\sqrt{\cos^2 x}} \\ &= \frac{1 + \sin x}{\cos x} \end{aligned}$$

Therefore, conditions on  $x$ :

$$\cos x \neq 0 \tag{9}$$

$$\therefore x \neq (2n + 1)\frac{\pi}{2}$$

$$\boxed{\therefore x \in \mathbb{R} - \{(2n + 1)\frac{\pi}{2} : n \in \mathbb{Z}\}}$$

$$(i) \quad f(x) = \sqrt{\frac{x-2}{x+2}} + \sqrt{\frac{1+x}{1-x}}$$

Conditions on  $x$ :

$$\frac{x-2}{x+2} \geq 0 \tag{10}$$

$$x+2 \neq 0 \tag{11}$$

$$\frac{1+x}{1-x} \geq 0 \tag{12}$$

$$1-x \neq 0 \tag{13}$$

$$\therefore (10) \Rightarrow x \in (\infty, -2] \cup [2, \infty)$$

$$\therefore (11) \Rightarrow x \neq -2$$

$$\therefore (10) \Rightarrow x \in [-1, 1]$$

$$\therefore (11) \Rightarrow x \neq 1$$

$$\boxed{D(f) = \{\}} \tag{14}$$

$$(j) \quad f(x) = \tan\left(x + \frac{\pi}{4}\right)$$

Conditions on  $x$ :

$$x + \frac{\pi}{4} \neq (2n+1)\frac{\pi}{2}; n \in \mathbb{Z} \tag{14}$$

$$\therefore x \neq (2n+1)\frac{\pi}{2} - \frac{\pi}{4}; n \in \mathbb{Z}$$

$$\boxed{\therefore D(f) = \mathbb{R} - \{(2n+1)\frac{\pi}{2} - \frac{\pi}{4} : n \in \mathbb{Z}\}}$$

**(2) Write which of the following functions are even, odd or neither odd nor even:**

**(a)**  $f(x) = x - x^3 + x^5$

$$\begin{aligned} f(-x) &= (-x) - (-x)^3 + (-x)^5 \\ &= (-x) - (-x^3) + (-x^5) \\ &= -x + x^3 - x^5 \\ &= -f(x) \end{aligned}$$

Therefore,  $f(x)$  is odd.

**(b)**  $f(x) = x^2 - x^3 + x^6$

$$\begin{aligned} f(-x) &= (-x)^2 - (-x)^3 + (-x)^6 \\ &= x^2 - (-x^3) + x^6 \\ &= x^2 + x^3 + x^6 \end{aligned}$$

Therefore,  $f(x)$  is neither odd nor even.

**(c)**  $f(x) = 5^x$

$$\begin{aligned} f(-x) &= 5^{-x} \\ &= \frac{1}{5^x} \end{aligned}$$

Therefore,  $f(x)$  is neither odd nor even.

**(d)**  $f(x) = \sin(\sin x)$

$$\begin{aligned} f(-x) &= \sin(\sin(-x)) \\ &= \sin(-\sin x) \\ &= -\sin(\sin x) \\ &= -f(x) \end{aligned}$$

Therefore,  $f(x)$  is odd.

**(e)**  $f(x) = \cos(\sin x)$

$$\begin{aligned} f(-x) &= \cos(\sin(-x)) \\ &= \cos(-\sin x) \\ &= \cos(\sin x) \\ &= f(x) \end{aligned}$$

Therefore,  $f(x)$  is even.

**(f)**  $f(x) = \sin(\cos x)$

$$\begin{aligned} f(-x) &= \sin(\cos(-x)) \\ &= \sin(\cos x) \\ &= f(x) \end{aligned}$$

Therefore,  $f(x)$  is even.

$$\textbf{(g)} \quad f(x) = x \sin x$$

$$\begin{aligned} f(-x) &= (-x) \sin(-x) \\ &= (-x)(-\sin x) \\ &= x \sin x \end{aligned} \qquad = f(x)$$

Therefore,  $f(x)$  is even.

$$\textbf{(h)} \quad f(x) = \frac{1-x}{1+x^2}$$

$$\begin{aligned} f(-x) &= \frac{1-(-x)}{1+(-x)^2} \\ &= \frac{1+x}{1+x^2} \end{aligned}$$

Therefore,  $f(x)$  is neither odd nor even.

**(3) Find the image of the following functions in the specified domains:**

**(a)  $f(x) = \sin x + 3$  in the interval  $\left[\pi, \frac{3\pi}{2}\right]$**

The image of  $\sin x$  in  $\left[\pi, \frac{3\pi}{2}\right]$  is  $[-1, 0]$ .

$\therefore$  the image of  $f(x)$  is  $[(-1 + 3), (0 + 3)]$ , i.e.  $[2, 3]$ .

**(b)  $f(x) = -x^2 + 8$  in the interval  $(0, 7]$**

The image of  $x^2$  in  $(0, 7]$  is  $(0, 49]$ .

$\therefore$  the image of  $-x^2$  in  $(0, 7]$  is  $[-49, 0)$ .

$\therefore$  the image of  $f(x)$  in  $(0, 7]$  is  $[-41, 8)$ .

**(4) Let  $f(x) = x^3$ . Find the function obtained from performing the following actions on  $f$ :**

**(a) Translation downwards by 3**

$$g(x) = f(x) - 3$$

$$\therefore g(x) = x^3 - 3$$

**(b) Right translation by 4**

$$g(x) = f(x - 4)$$

$$\therefore g(x) = (x - 4)^3$$

**(c) Reflection around the  $x$ -axis.**

$$g(x) = -f(x)$$

$$\therefore g(x) = -x^3$$

**(d) Reflection around the  $y$ -axis.**

$$g(x) = f(-x)$$

$$\therefore g(x) = (-x)^3 = -x^3$$

**(e) Left translation by 1**

$$g(x) = f(x + 1)$$

$$\therefore g(x) = (x + 1)^3$$

$$\therefore g(x) = x^3 + 3x^2 + 3x + 1$$

- (5) Draw the graph of the function  $f(x) = x - [x]$ , where  $[x]$  is the biggest integer smaller or equal to  $x$ .

- (6) Solve the following inequalities:

(a)  $x^2 - 5x + 6 \leq 0$

$$\begin{aligned}x^2 - 5x + 6 &\leq 0 \\ \therefore (x - 2)(x - 3) &\leq 0 \\ \therefore x &\in [2, 3]\end{aligned}$$

(b)  $x^2 - 4x > 21$

$$\begin{aligned}x^2 - 4x &> 21 \\ \therefore x^2 - 4x - 21 &> 0 \\ \therefore (x - 7)(x + 3) &> 0 \\ \therefore x &\in (-\infty, -3) \cup (7, \infty)\end{aligned}$$

(c)  $|-x + 3| < 7$

$$\begin{aligned}|-x + 3| &< 7 \\ \therefore -7 &< -x + 3 < 7 \\ \therefore -10 &< -x < 4 \\ \therefore 10 &> x > -4 \\ \therefore x &\in (-4, 10)\end{aligned}$$

(d)  $|2x - 3| \leq x + 3$

$$\begin{aligned}|2x - 3| &\leq x + 3 \\ \therefore -x - 3 &\leq 2x - 3 \leq x + 3 \\ \therefore 0 &\leq x \leq 6\end{aligned}$$

$$(e) \quad |x+1| - |2-x| \geq 0$$

$$\begin{aligned} |x+1| &\geq |2-x| \\ \therefore (x+1)^2 &\geq (2-x)^2 \\ \therefore x^2 + 2x + 1 &\geq 4 - 4x + x^2 \\ \therefore 6x &\geq 3 \\ \therefore x &\geq \frac{1}{2} \end{aligned}$$

$$\therefore x \in \left[ \frac{1}{2}, \infty \right)$$

$$(f) \quad |x-1| + |x-2| - |x-3| \geq x$$

**Case I:**  $x > 3$

$$\therefore (x-1) + (x-2) - (x-3) - x \geq 0$$

No contradiction

$$\therefore x \in (3, \infty)$$

**Case II:**  $2 < x \leq 3$

$$\begin{aligned} (x-1) + (x-2) + (x-3) - x &\geq 0 \\ \therefore x &\geq 3 \end{aligned}$$

$$\therefore x = 3$$

**Case III:**  $1 < x \leq 2$

$$\begin{aligned} (x-1) - (x-2) + (x-3) - x &\geq 0 \\ \therefore -2 &\geq 0 \end{aligned}$$

$\therefore$  no values of  $x$  exist.

**Case IV:**  $x \leq 1$

$$\begin{aligned} -(x-1) - (x-2) + (x-3) - x &\geq 0 \\ \therefore x &\leq 0 \end{aligned}$$

$$\therefore x \in (-\infty, 0]$$

$$\boxed{x \in (-\infty, 0] \cup [3, \infty)}$$