Lecture 20

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Contents

1 Double Integrals

Example 1. Calculate $\iint\limits_D \sin(y^2)\,\mathrm{d}A$ for D given by the triangle enclosed by $(0,0)$, $(0,1)$, $(1,1)$.

Solution. The domain D is of the first and second kind.

$$\begin{aligned} D_I &: 0 \le x \le 1 \quad \& \quad x \le y \le 1 \\ D_{II} &: 0 \le y \le 1 \quad \& \quad 0 \le x \le y \end{aligned}$$

Therefore,

$$\iint\limits_{D_I} \sin(y^2)\,\mathrm{d}A = \int_0^1 \int_x^1 \sin(y^2)\,\mathrm{d}y\,\mathrm{d}x$$

This form is unsolvable.

$$\begin{aligned} \iint\limits_{D_{II}} \sin(y^2)\,\mathrm{d}A &= \int_0^1 \int_0^y \sin(y^2)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 \sin(y^2)x\Big|_{x=0}^{x=y}\,\mathrm{d}y \\ &= \int_0^1 y\sin(y^2)\,\mathrm{d}y \\ &= -\frac{1}{2}\cos(y^2)\Big|_0^1 \\ &= -\frac{1}{2}\cos 1 + \frac{1}{2} \end{aligned}$$

Theorem 1. *If D is not a domain of both kinds, but $D = D_I \cup D_{II}$, s.t. the area of $D_I \cap D_{II}$ is zero, then*

$$\iint\limits_D f(x,y)\,\mathrm{d}A = \iint\limits_{D_I} f(x,y)\,\mathrm{d}A + \iint\limits_{D_{II}} f(x,y)\,\mathrm{d}A$$

1.1 Applications

1. If $f_x(x, y)$ and $f_y(x, y)$ are continuous in D, then the area of the surface $\sigma : z = f(x, y)$ above D is equal to

$$S(\Sigma) = \iint\limits_D \sqrt{1 + \big(f_x(x, y)\big)^2 + \big(f_y(x, y)\big)^2}\,\mathrm{d}A$$

2. If $\rho(x, y)$ is the density function of a thin body,

$$m = \iint\limits_D \rho(x, y)\,\mathrm{d}A$$

$$(x_{\mathrm{COM}}, y_{\mathrm{COM}}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

where

$$\begin{aligned} M_x &= \iint\limits_D y\rho(x, y)\,\mathrm{d}A \\ M_y &= \iint\limits_D x\rho(x, y)\,\mathrm{d}A \end{aligned}$$

2 Triple Integrals

2.1 Triple Integrals on Parallelepiped Domains

Definition 1 (Triple integral on parallelepiped domain)**.** Consider a parallelepiped

$$B = \{(x, y, z) | a \leq x \leq b, c \leq y \leq d, e \leq z \leq g\}$$

Consider a partition T

$$\begin{aligned} &a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \\ &c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d \\ &e = z_0 < z_1 < \cdots < z_{n-1} < z_n = g \end{aligned}$$

Consider a parallelepiped

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y-j] \times [z_{k-1}, z_k]$$

Let

$$P_{ijk}{}^* = \left(x_{ijk}{}^*, y_{ijk}{}^*, z_{ijk}{}^*\right)$$

be a point in B_{ijk}. Let

$$\begin{aligned}
\Delta x_i &= x_i - x_{i-1} \\
\Delta y_j &= y_j - y_{j-1} \\
\Delta z_k &= z_k - z_{k-1}
\end{aligned}$$

Therefore,

$$\Delta V_{ijk} = \Delta x_i \cdot \Delta y_j \cdot \Delta z_k$$

Let

$$\Delta T = \max\{\Delta x_i, \Delta y_j, \Delta z_k\}$$

$\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} f(x_{ijk}{}^*, y_{ijk}{}^*, z_{ijk}{}^*)\Delta V_{ijk}$ is called a Riemann triple integral sum. The double integral of $f(x, y, z)$ over the domain of definition B is the limit, if it exists, of the Riemann double integral sum, as $\Delta T \to 0$.

$$\iiint\limits_B f(x, y, z)\,\mathrm{d}V = \lim_{\Delta T \to 0} \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} f(x_{ijk}{}^*, y_{ijk}{}^*, z_{ijk}{}^*)\Delta V_{ijk}$$

Theorem 2 (Fubini Theorem)**.** *If $f(x, y, z)$ is continuous on $B = [a, b] \times [c, d] \times [e, g]$, then the triple integral and six iterated integrals exist, and they are equal.*

$$\iiint\limits_B f(x, y, z)\,\mathrm{d}V = \int\limits_a^b \int\limits_c^d \int\limits_e^g f(x, y, z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \ldots$$

Example 2. Calculate

$$\iiint\limits_{[0,1]\times[-1,2]\times[1,3]} z^2 \sin(x + y)\,\mathrm{d}V$$

Solution.

$$\iiint\limits_{[0,1]\times[-1,2]\times[1,3]} z^2 \sin(x+y)\,\mathrm{d}V = \int\limits_0^1 \int\limits_{-1}^2 \int\limits_1^3 z^2 \sin(x+y)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int\limits_0^1 \int\limits_{-1}^2 \frac{z^3}{3}\sin(x+y)\Big|_{z=0}^{z=3}\,\mathrm{d}y\,\mathrm{d}x$$

$$= \frac{26}{3}\int\limits_0^1 \int\limits_{-1}^2 \sin(x+y)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \frac{26}{3}\int\limits_0^1 -\cos(x+y)\big|_{y=-1}^{y=2}\,\mathrm{d}x$$

$$= \frac{26}{3}\int\limits_0^1 \big(-\cos(x+2)+\cos(x-1)\big)\,\mathrm{d}x$$

$$= \frac{26}{3}\,\big(-\sin(x+2)+\sin(x-1)\big)\Big|_0^1$$

$$= \frac{26}{3}\Big(-\sin 3 + 0 - \big(-\sin 2 + \sin(-1)\big)\Big)$$

$$= \frac{26}{3}\,(-\sin 3 + \sin 2 + \sin 1)$$

2.2 Triple Integrals on Arbitrary Domains

Definition 2 (Triple integral on arbitrary domain)**.** Let E be an arbitrary solid which is bounded and closed in $\mathbb{R}^3$ and $f(x,y,z)$ be defined on E. Consider a parallelepiped B, s.t. $E \subset B$ and construct the function $F(x,y,z)$, s.t.

$$F(x,y,x) = \begin{cases} f(x,y,z) & ; \quad (x,y,z)\in E \\ f(x,y,z) & ; \quad 0 \notin E \end{cases}$$

If $\exists \iiint\limits_B F(x,y,z)\,\mathrm{d}V$, then $\exists \iiint\limits_E f(x,y,z)\,\mathrm{d}V$, and

$$\iiint\limits_E f(x,y,z)\,\mathrm{d}V = \iiint\limits_B F(x,y,z)\,\mathrm{d}V$$

Definition 3 (Solid of the first kind)**.** A solid E in $\mathbb{R}^3$ is called a solid of the first kind if there exist continuous functions $\varphi_1(x,y)$ and $\varphi_2(x,y)$, s.t.

$$E_I = \{(x,y,z)|(x,y) \in D, \varphi_1(x,y) \leq z \leq \varphi_2(x,y)\}$$

Theorem 3. *If $f(x,y,z)$ is continuous on E,*

$$\iiint\limits_{E_I} f(x,y,z)\,\mathrm{d}V = \iint\limits_{D} \left(\int\limits_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)\,\mathrm{d}x \right) \mathrm{d}A$$