Differential and Integral Methods - Exercise 8

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(1) Solve the following integrals

(a)

$$\int \frac{e^x}{1+e^x} \, \mathrm{d}x$$

Let

$$t = 1 + e^x$$
$$\therefore \mathrm{d}t = e^t$$

$$\therefore \int \frac{e^x}{1+e^x} \, \mathrm{d}x = \int \frac{\mathrm{d}t}{t}$$
$$= \ln t + c$$
$$= \ln(1 + e^x) + c$$

(b)

$$\int \frac{1}{1+e^x} \, \mathrm{d}x$$

$$\int \frac{1}{1+e^x} \, \mathrm{d}x = \int \frac{1+e^x}{1+e^x} \, \mathrm{d}x - \int \frac{e^x}{1+e^x} \, \mathrm{d}x$$
$$= \int \mathrm{d}x - \int \frac{e^x}{1+e^x} \, \mathrm{d}x$$

Let

$$t = 1 + e^x$$
$$\therefore \mathrm{d}t = e^x$$

$$\therefore \int \frac{1}{1+e^x} \, \mathrm{d}x = x - \int \frac{\mathrm{d}t}{t} + c$$
$$\therefore \int \frac{1}{1+e^x} \, \mathrm{d}x = x - \ln t + c$$
$$\therefore \int \frac{1}{1+e^x} \, \mathrm{d}x = x - \ln(1 + e^x) + c$$

(c)

$$\int \frac{5x+4}{(x-1)(x-2)}\,\mathrm{d}x$$

$$\begin{aligned}\int \frac{5x+4}{(x-1)(x-2)}\,\mathrm{d}x &= \int \frac{5x-5+9}{(x-1)(x-2)}\,\mathrm{d}x\\ &= \int \frac{5(x-1)+9}{(x-1)(x-2)}\,\mathrm{d}x\\ &= \int \left(\frac{3}{x-1}+\frac{2}{x+2}\right)\mathrm{d}x\\ &= 3\ln|x-1| + 2\ln|x+2| + c\end{aligned}$$

(d)

$$\int \frac{5x^2-7x+3}{x(x-1)^2}\,\mathrm{d}x$$

$$\begin{aligned}\int \frac{5x^2-7x+3}{x(x-1)^2}\,\mathrm{d}x &= \int \left(\frac{3}{x}+\frac{2}{x-1}+\frac{1}{(x-1)^2}\right)\mathrm{d}x\\ &= 3\ln|x| + 2\ln|x-1| - \frac{1}{x-1} + c\end{aligned}$$

(e)

$$\int \frac{2x^2-2x-2}{x^3-x}\,\mathrm{d}x$$

$$\begin{aligned}\int \frac{2x^2-2x-2}{x^3-x}\,\mathrm{d}x &= \int \frac{2x^2-2x-2}{x(x+1)(x-1)}\,\mathrm{d}x\\ &= \int \left(\frac{2}{x}-\frac{1}{x-1}+\frac{1}{x+1}\right)\mathrm{d}x\\ &= 2\ln|x| - \ln|x-1| + \ln|x+1| + c\\ &= \ln\left|\frac{x^2(x+1)}{x-1}\right| + c\end{aligned}$$

(2) Write as a sum of simple rational functions

$$f(x) = \frac{x-1}{(x^2+1)(x+1)}$$

Let

$$\begin{aligned}f(x) &= \frac{Ax+B}{x^2+1} + \frac{C}{x+1}\\ \therefore f(x) &= \frac{(Ax+B)(x+1)+C(x^2+1)}{(x^2+1)(x+1)}\\ &= \frac{Ax^2+(A+B)x+B+Cx^2+C}{(x^2+1)(x+1)}\\ &= \frac{x^2(A+C)+x(A+B)+C}{(x^2+1)(x+1)}\end{aligned}$$

Therefore,

$$\begin{aligned} A + B &= 1 \\ A + C &= 0 \\ C &= -1 \end{aligned}$$

$\therefore A = 1$; $B = 0$; $C = -1$

Therefore,

$$f(x) = \frac{x}{x^2+1} - \frac{1}{x+1}$$

(3) Solve the following integrals

(a)

$$\int (1 + \sqrt[3]{x^2})^2 \, \mathrm{d}x$$

Let

$$\begin{aligned} x &= t^3 \\ \therefore \mathrm{d}x &= 3t^2 \, \mathrm{d}t \end{aligned}$$

Therefore,

$$\begin{aligned} \int (1 + \sqrt[3]{x^2})^2 \, \mathrm{d}x &= \int (1 + t^2)^2 3t^2 \, \mathrm{d}t \\ &= \int (3t^6 + 6t^4 + 3t^2) \, \mathrm{d}t \\ &= \frac{3t^7}{7} + \frac{6t^5}{5} + t^3 + c \\ &= \frac{3}{7}(\sqrt[3]{x})^7 + \frac{6}{5}(\sqrt[3]{x})^5 + x + c \end{aligned}$$

(b)

$$\int \frac{x^3 - x}{x^4 - 2x^2 - 7} \, \mathrm{d}x$$

Let

$$\begin{aligned} t &= x^4 - 2x^2 - 7 \\ \therefore \mathrm{d}t &= (4x^3 - 4x) \, \mathrm{d}x \end{aligned}$$

Therefore,

$$\begin{aligned} \int \frac{x^3 - x}{x^4 - 2x^2 - 7} \, \mathrm{d}x &= \frac{1}{4} \int \frac{\mathrm{d}t}{t} \\ &= \frac{1}{4} \ln |x^4 - 2x^2 - 7| + c \end{aligned}$$

(c)

$$\int \frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x$$

$$\begin{aligned}
\int \frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x &= \frac{1}{3}\int \frac{1}{\sqrt{1-(x/3)^2}}\,\mathrm{d}x \\
&= \sin^{-1}\left(\frac{x}{3}\right) + c
\end{aligned}$$

(d)

$$\int \sin^2 x\,\mathrm{d}x$$

$$\begin{aligned}
\int \sin^2 x\,\mathrm{d}x &= \int \frac{1-\cos(2x)}{2}\,\mathrm{d}x \\
&= \int \frac{\mathrm{d}x}{2} - \int \frac{\cos(2x)}{2}\,\mathrm{d}x \\
&= \frac{x}{2} - \frac{1}{2}\cdot\frac{\sin(2x)}{2} + c \\
&= \frac{x}{2} - \frac{\sin(2x)}{4} + c
\end{aligned}$$

(e)

$$\int \tan x\,\mathrm{d}x$$

$$\int \tan x\,\mathrm{d}x = \int \frac{\sin x}{\cos x}\,\mathrm{d}x$$

Let

$$\begin{aligned}
t &= \cos x \\
\therefore \mathrm{d}t &= -\sin t
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int \frac{\sin x}{\cos x}\,\mathrm{d}x &= \int \frac{-\,\mathrm{d}t}{t} \\
&= -\ln|t| + c \\
&= -\ln|\cos x| + c
\end{aligned}$$

(f)

$$\int x(\ln x)^2\,\mathrm{d}x$$

Let

$$\begin{aligned}
x &= e^t \\
\therefore \mathrm{d}x &= e^t\,\mathrm{d}t
\end{aligned}$$

Therefore,

$$\begin{aligned}\int x(\ln x)^2\,\mathrm{d}x &= \int e^t t^2 e^t\,\mathrm{d}t\\ &= \int e^{2t}t^2\,\mathrm{d}t\\ &= \left(\frac{t^2}{2}-\frac{t}{2}+\frac{1}{4}\right)e^{2t}+c\\ &= \frac{(x\ln x)^2}{2}-\frac{x^2\ln x}{2}+\frac{x^2}{4}+c\end{aligned}$$

(g)

$$\int x^2e^x\,\mathrm{d}x$$

$$\begin{aligned}\int x^2e^x\,\mathrm{d}x &= x^2\int e^x\,\mathrm{d}x-\int 2x\int e^x\,\mathrm{d}x\,\mathrm{d}x\\ &= x^2e^x-\int 2xe^x\,\mathrm{d}x\\ &= x^2e^x-2x\int e^x\,\mathrm{d}x+\int 2\int e^x\,\mathrm{d}x\,\mathrm{d}x\\ &= x^2e^x-2xe^x+\int 2e^x\,\mathrm{d}x\\ &= x^2e^x-2xe^x+2e^x+c\end{aligned}$$

(h)

$$\int \frac{\ln x}{x^2}\,\mathrm{d}x$$

Let

$$\begin{aligned}t&=\ln x\\ \therefore\ \mathrm{d}t&=\frac{\mathrm{d}x}{x}\end{aligned}$$

Therefore,

$$\begin{aligned}\int \frac{\ln x}{x^2}\,\mathrm{d}x &= \int te^t\,\mathrm{d}t\\ &= -e^{-t}(t+1)+c\\ &= -\frac{1}{x}(\ln x+1)+c\end{aligned}$$

(i)

$$\int x\sinh x\,\mathrm{d}x$$

$$\begin{aligned}\int x\sinh x\,\mathrm{d}x &= \int x\frac{e^x-e^{-x}}{2}\,\mathrm{d}x\\ &= \frac{1}{2}\int x(e^x-e^{-x})\,\mathrm{d}x\\ &= \frac{1}{2}\left((x-1)e^x+(x+1)e^{-x}\right)+c\end{aligned}$$

(4) Solve by substitution

(a)

$$\int \frac{x}{\sqrt{x^2+4}}\,\mathrm{d}x$$

Let

$$\begin{aligned} t &= x^2+4 \\ \therefore \mathrm{d}t &= 2x \end{aligned}$$

Therefore,

$$\begin{aligned} \int \frac{x}{\sqrt{x^2+4}}\,\mathrm{d}x &= \int \frac{\mathrm{d}t}{2\sqrt{t}} \\ &= \sqrt{t}+c \\ &= \sqrt{x^2+4}+c \end{aligned}$$

(b)

$$\int \frac{\ln(2x)}{x\ln(4x)}\,\mathrm{d}x$$

Let

$$\begin{aligned} t &= \ln(4x) \\ \therefore t-\ln 2 &= \ln(2x) \\ \therefore \mathrm{d}t &= \frac{1}{x}\,\mathrm{d}x \end{aligned}$$

Therefore,

$$\begin{aligned} \int \frac{\ln(2x)}{x\ln(4x)}\,\mathrm{d}x &= \int \frac{t-\ln 2}{t}\,\mathrm{d}t \\ &= \int \mathrm{d}t - \int \frac{\ln 2}{t}\,\mathrm{d}t \\ &= t-(\ln 2)(\ln t)+c \\ &= \ln(4x)-(\ln 2)(\ln(\ln 4x))+c \end{aligned}$$

(c)

$$\int \frac{x^4}{\sqrt{x^{10}-2}}\,\mathrm{d}x$$

Let

$$\begin{aligned} y &= x^5 \\ \therefore \mathrm{d}y &= 5x^4\,\mathrm{d}x \end{aligned}$$

$$\begin{aligned} \therefore \int \frac{x^4}{\sqrt{x^{10}-2}}\,\mathrm{d}x &= \frac{1}{5}\int \frac{\mathrm{d}y}{\sqrt{y^2-2}} \\ &= \frac{1}{5\sqrt{2}}\int \frac{\mathrm{d}y}{\sqrt{\left({}^{y}\!/\!{}_{\sqrt{2}}\right)^2-1}} \end{aligned}$$

Let

$$y = \frac{\sqrt{2}}{\cos z}$$
$$\therefore \mathrm{d}y = \frac{\sqrt{2}\sin z}{\cos^2 z}$$

$$\begin{aligned}
\therefore \int \frac{x^4}{\sqrt{x^{10}-2}}\,\mathrm{d}x &= \frac{1}{5\sqrt{2}} \int \frac{1}{\sqrt{1/\cos^2 z - 1}} \cdot \sqrt{2} \cdot \frac{\sin z}{\cos^2 z}\,\mathrm{d}z \\
&= \frac{1}{5} \int \frac{1}{\tan z} \cdot \frac{\sin z}{\cos^2 z}\,\mathrm{d}x \\
&= \frac{1}{5} \int \frac{1}{\cos z}\,\mathrm{d}x \\
&= \ln|\sec z + \tan z| + c \\
&= \ln\left|\frac{x^5 + \sqrt{x^{10}-2}}{\sqrt{2}}\right| + c
\end{aligned}$$

(d)

$$\int \frac{1}{x\sqrt{2x+1}}\,\mathrm{d}x$$

Let

$$t = \sqrt{2x+1}$$
$$\therefore \mathrm{d}t = x\,\mathrm{d}x$$

$$\begin{aligned}
\int \frac{1}{x\sqrt{2x+1}}\,\mathrm{d}x &= \int \frac{t\,\mathrm{d}t}{\frac{t^2-1}{2} \cdot t} \\
&= \int \frac{2\,\mathrm{d}t}{(t+1)(t-1)} \\
&= \int \left(\frac{1}{t-1} - \frac{1}{t+1}\right)\mathrm{d}t \\
&= \ln\left|\frac{t-1}{t+1}\right| + c \\
&= \ln\left|\frac{\sqrt{2x+1}-1}{\sqrt{2x+1}+1}\right| + c
\end{aligned}$$

(5) Solve the following integrals

(a)

$$\int \cos(7x)\sin(7x)\,\mathrm{d}x$$

$$\begin{aligned}
\int \cos(7x)\sin(7x)\,\mathrm{d}x &= \frac{1}{2}\int \sin(14x)\,\mathrm{d}x \\
&= \frac{1}{2}\left(-\frac{\cos(14x)}{14}\right) + c \\
&= \frac{\cos(14x)}{28} + c
\end{aligned}$$

(b)

$$\int \cos(3x)\cos(2x)\,\mathrm{d}x$$

$$\begin{aligned}\int \cos(3x)\cos(2x)\,\mathrm{d}x &= \int (\cos(5x)+\cos(x))\,\mathrm{d}x\\ &= -\frac{\sin(5x)}{5}+\sin x + c\end{aligned}$$

(c)

$$\int \frac{1}{\cos x+\sin x+1}\,\mathrm{d}x$$

$$\begin{aligned}\int \frac{1}{\cos x+\sin x+1}\,\mathrm{d}x &= \int \frac{1}{2\cos^2(x/2)+2\sin(x/2)\cos(x/2)}\,\mathrm{d}x\\ &= \int \frac{1}{2\cos^2(x/2)(1+\tan(x/2))}\,\mathrm{d}x\\ &= \int \frac{\sec^2(x/2)}{1+\tan(x/2)}\,\mathrm{d}x\\ &= \ln\left|1+\tan(x/2)\right| + c\end{aligned}$$

(d)

$$\int \frac{1}{\sin x}\,\mathrm{d}x$$

$$\int \csc x\,\mathrm{d}x = \int \frac{\csc x(\csc x+\cot x)}{\csc x+\cot x}\,\mathrm{d}x$$

Let

$$t = \csc x + \cot x$$

$$\therefore \mathrm{d}t = -\csc x\cot x - \csc^2 x\,\mathrm{d}x$$

Therefore,

$$\begin{aligned}\int \frac{\csc x(\csc x+\cot x)}{\csc x+\cot x}\,\mathrm{d}x &= \int -\frac{\mathrm{d}t}{t}\\ &= -\ln|t| + c\\ &= -\ln|\csc x+\cot x| + c\end{aligned}$$

(e)

$$\int \frac{1}{1-\sin x}\,\mathrm{d}x$$

$$\begin{aligned}\int \frac{1}{1-\sin x}\,\mathrm{d}x &= \int \frac{1+\sin x}{1-\sin^2 x}\,\mathrm{d}x\\ &= \int \frac{1+\sin x}{\cos x}\,\mathrm{d}x\\ &= \int \frac{1}{\cos^2 x}\,\mathrm{d}x + \int \frac{\sin x}{\cos^2 x}\,\mathrm{d}x\\ &= \tan x + \sec x + c\end{aligned}$$

(f)

$$\int \sin^2 x \cos^4 x \, \mathrm{d}x$$

$$\begin{aligned}
\int \sin^2 x \cos^4 x \, \mathrm{d}x &= \int \frac{1-\cos(2x)}{2} \left(\frac{1+\cos(2x)}{2}\right)^2 \mathrm{d}x \\
&= \frac{1}{8} \int (1-\cos^2(2x))(1+\cos(2x)) \, \mathrm{d}x \\
&= \frac{1}{8} \int \left(\frac{1-\cos(4x)}{2}\right) \mathrm{d}x + \int \sin^2(2x)\cos(2x) \, \mathrm{d}x \\
&= \frac{1}{16} \left(x - \frac{\sin(4x)}{4} + \frac{\sin^3(2x)}{3}\right) + c
\end{aligned}$$

(g)

$$\int \frac{\sin(2x)}{\sqrt{3-\cos^4 x}} \, \mathrm{d}x$$

$$\int \frac{\sin(2x)}{\sqrt{3-\cos^4 x}} \, \mathrm{d}x = \int \frac{2\sin x \cos x}{\sqrt{3-\cos^4 x}} \, \mathrm{d}x$$

Let

$$\begin{aligned}
t^2 &= 3 - \cos^4 x \\
\therefore 2\sin x \cos x \, \mathrm{d}x &= \frac{t \, \mathrm{d}t}{\sqrt{3-t^2}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int \frac{2\sin x \cos x}{\sqrt{3-\cos^4 x}} \, \mathrm{d}x &= \int \frac{t \, \mathrm{d}t}{t\sqrt{3-t^2}} \\
&= \sin^{-1} \frac{t}{\sqrt{3}} + c \\
&= \sin^{-1} \sqrt{\frac{3-\cos^4 x}{3}} + c
\end{aligned}$$

(h)

$$\int e^{4x} \cos(2x) \, \mathrm{d}x$$

$$
\begin{aligned}
\int e^{4x}\cos(2x)\,\mathrm{d}x &= e^{4x}\int \cos(2x)\,\mathrm{d}x - \int 4e^{4x}\int \cos(2x)\,\mathrm{d}x\,\mathrm{d}x \\
&= \frac{e^{4x}\sin(2x)}{2} - \int 2e^{4x}\sin(2x)\,\mathrm{d}x \\
&= \frac{e^{4x}\sin(2x)}{2} - \left(2e^{4x}\int \sin(2x)\,\mathrm{d}x - \int 8e^{4x}\int \sin(2x)\,\mathrm{d}x\,\mathrm{d}x\right) \\
&= \frac{e^{4x}\sin(2x)}{2} - \left(-e^{4x}\cos(2x) + 4\int e^{4x}\cos(2x)\,\mathrm{d}x\right) \\
&= \frac{e^{4x}\sin(2x)}{2} + e^{4x}\cos(2x) - 4\int e^{4x}\cos(2x)\,\mathrm{d}x \\
\therefore 5\int e^{4x}\cos(2x)\,\mathrm{d}x &= \frac{e^{4x}\sin(2x)}{2} + e^{4x}\cos(2x) \\
\therefore \int e^{4x}\cos(2x)\,\mathrm{d}x &= \frac{1}{5}\left(\frac{e^{4x}\sin(2x)}{2} + e^{4x}\cos(2x)\right)
\end{aligned}
$$

(i)

$$\int \ln x\,\mathrm{d}x$$

$$
\begin{aligned}
\int \ln x\,\mathrm{d}x &= \int 1\cdot\ln x\,\mathrm{d}x \\
&= \ln x\int 1\cdot\mathrm{d}x - \int \frac{1}{x}\int 1\cdot\mathrm{d}x\,\mathrm{d}x \\
&= x\ln x - x + c
\end{aligned}
$$

(j)

$$\int \frac{1}{x(\ln x)^x}$$

Let

$$
\begin{aligned}
t &= \ln x \\
\therefore \mathrm{d}t &= \frac{1}{x}\,\mathrm{d}x
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int \frac{1}{x(\ln x)^x} &= \int \frac{\mathrm{d}t}{t^2} \\
&= -\frac{1}{t} + c \\
&= -\frac{1}{\ln x} + c
\end{aligned}
$$

(k)

$$\int \frac{1}{\sqrt{x}(1+\sqrt[3]{x})}\,\mathrm{d}x$$

Let

$$
\begin{aligned}
x &= t^6 \\
\therefore \mathrm{d}x &= 6t^5\,\mathrm{d}t
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int \frac{1}{\sqrt{x}(1+\sqrt[3]{x})}\,\mathrm{d}x &= \int \frac{6t^5}{t^3(1+t^2)}\,\mathrm{d}t \\
&= \int \frac{6t^2}{(1+t^2)}\,\mathrm{d}t \\
&= \int \frac{6(t^2+1)-6}{(t^2+1)}\,\mathrm{d}t \\
&= \int 6\,\mathrm{d}t - \int \frac{6}{t^2+1}\,\mathrm{d}t \\
&= 6t - 6\tan^{-1} t + c \\
&= 6\sqrt[6]{x} + 6\tan^{-1}\sqrt[6]{x} + c
\end{aligned}
$$

(l)

$$\int \frac{1}{x^2-1}\,\mathrm{d}x$$

$$
\begin{aligned}
\int \frac{1}{x^2-1} &= \frac{1}{2}\int \left(\frac{1}{x-1} - \frac{1}{x+1}\right)\mathrm{d}x \\
&= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + c
\end{aligned}
$$

(m)

$$\int \frac{x^3-1}{4x^3-x}\,\mathrm{d}x$$

$$
\begin{aligned}
\int \frac{x^3-1}{4x^3-x}\,\mathrm{d}x &= \frac{1}{4}\int \frac{4x^3-x+x-4}{4x^3-x}\,\mathrm{d}x \\
&= \frac{1}{4}\int \frac{4x^3-x}{4x^3-x}\,\mathrm{d}x - \int \frac{4-x}{4x^3-x}\,\mathrm{d}x \\
&= \frac{1}{4}\left(x + \int \frac{4}{x} - \frac{7}{2(2x-1)} - \frac{9}{2(2x+1)}\,\mathrm{d}x\right) \\
&= \frac{1}{4}\left(x + 4\ln|x| + \frac{7}{4}\ln|2x-1| - \frac{9}{4}\ln|2x+1|\right)
\end{aligned}
$$

(n)

$$\int \frac{1}{x(x+1)^2}\,\mathrm{d}x$$

$$
\begin{aligned}
\int \frac{1}{x(x+1)^2}\,\mathrm{d}x &= \int \left(\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}\right)\mathrm{d}x \\
&= \ln\left|\frac{x}{x+1}\right| + \frac{1}{x+1} + c
\end{aligned}
$$

(o)

$$\int \frac{x^2}{(x-1)^5}\,\mathrm{d}x$$

Let

$$\begin{aligned} t &= x-1 \\ \therefore x &= t+1 \\ \therefore \mathrm{d}t &= \mathrm{d}x \end{aligned}$$

$$\begin{aligned} \int \frac{x^2}{(x-1)^5}\,\mathrm{d}x &= \int \frac{(t+1)^2}{t^5}\,\mathrm{d}x \\ &= \int \frac{t^2+2t+1}{t^5}\,\mathrm{d}t \\ &= \int \left(\frac{1}{t^3}+\frac{2}{t^4}+\frac{1}{t^5}\right)\mathrm{d}t \\ &= -\frac{1}{2x^2}-\frac{2}{3x^3}-\frac{1}{4x^4}+c \end{aligned}$$