Differential and Integral Methods: Compendium

Aakash Jog

1 Functions

Definition 1 (Even function).

$$f(-x) = f(x)$$

Definition 2 (Odd function).

$$f(-x) = -f(x)$$

Definition 3 (Shifting with respect to y-axis). $f(x+a)$ is the graph of $f(x)$, shifted by a, in the direction of the x-axis, opposite to the sign of a.

Definition 4 (Shifting with respect to x-axis). $f(x)+a$ is the graph of $f(x)$, shifted by a, in the direction of the y-axis, according to the sign of a.

1.1 Hyperbolic Functions

Definition 5 (Hyperbolic functions).

$$\sinh x \doteq \frac{e^x - e^{-x}}{2} \qquad I(\sinh x) = \mathbb{R}$$

$$\cosh x \doteq \frac{e^x + e^{-x}}{2} \qquad I(\cosh x) = [1, \infty)$$

$$\tanh x \doteq \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad I(\tanh x) = (-1, 1)$$

1.1.1 Identities of Hyperbolic Functions

$$\begin{aligned}
\sinh(2x) &= 2\sinh x \cosh x \\
\cosh^2 x + \sinh^2 x &= \cosh(2x) \\
\cosh^2 x - \sinh^2 x &= 1 \\
\frac{\cosh(2x) - 1}{2} &= \sinh^2 x \\
\frac{\cosh(2x) + 1}{2} &= \cosh^2 x
\end{aligned}$$

1.2 Trigonometric Identities

$$1 - \cos x = 2\sin^2\left(\frac{x}{2}\right)$$

$$1 + \cos x = 2\cos^2\left(\frac{x}{2}\right)$$

2 Limits

Definition 6 (Cauchy's definition of a limit of a function)**.**

$$\forall\epsilon > 0 \exists\delta > 0 : 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$$

Definition 7 (Removable discontinuity point)**.**

$$\exists \lim_{x\to a} f(x), \text{ but either } \lim_{x\to a} f(x) \neq f(a) \text{ or } \nexists f(a)$$

Definition 8 (Discontinuity of first kind)**.**

$$\exists \lim_{x\to a^-} f(x), \exists \lim_{x\to a^+} f(x), \text{ but } \lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$$

Definition 9 (Discontinuity of second kind)**.** Atleast one of the two one-sided limits of f does not exist. (Limits are defined as finite numbers only.)

Theorem 1 (Sandwich Theorem)**.** *Let $f(x), g(x), h(x)$ be defined on an open interval about a, except possibly at a itself. Assume that $\forall x \neq a$ from the interval, it is satisfied that $f(x) \leq g(x) \leq h(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then,*

$$\lim_{x\to a} g(x) = L$$

.

Theorem 2. *If $\lim_{x\to a} f(x) = 0$ and $g(x)$ is bounded in an open interval about a, except possibly at a itself, then,*

$$\lim_{x\to a}(f(x)g(x)) = 0$$

2.1 Useful Limits

If $\lim_{x\to x_0} g(x) = 0$,

$$\lim_{x\to x_0} (1 + g(x))^{\frac{1}{g(x)}} = e$$

$$\lim_{x\to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{x\to -\infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

3 Derivatives

Definition 10 (Derivative of a function)**.**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x\to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = L$$

Theorem 3 (Derivative of inverse functions)**.**

$$(f^{-1})'(x) = \frac{1}{f'(x)}$$

Theorem 4 (Chain rule)**.**

$$f(g(x)) = \frac{\mathrm{d}f(g(x))}{\mathrm{d}g(x)} \cdot \frac{\mathrm{d}g(x)}{\mathrm{d}x}$$

Theorem 5 (Rolle's Theorem)**.** *Let $f(x)$ be defined on $[a, b]$, s.t.*

1. *f is continuous on $[a, b]$*
2. *f is differentiable on (a, b)*
3. *$f(a) = f(b)$*

Then, $\exists c \in (a, b)$, s.t. $f'(c) = 0$.

Theorem 6. *Let $f(x)$ be defined on $[a, b]$, s.t.*

1. *f is continuous on $[a, b]$*
2. *f is differentiable on (a, b)*

Then,

$$\exists c \in (a, b), \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a}$$