Lecture 5

Tuesday 11th November, 2014

Contents

1 Derivative of a Function

1.1 Definition

If there exists

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = L$$

then the limit is called the derivative of f at x_0.

1.1.1 Geometrical Interpretation

The slope of $\overleftrightarrow{AB}$ is

$$\tan \alpha = \frac{\Delta y}{\Delta x}$$

When $\Delta x \to 0$, $\overleftrightarrow{AB}$ tends to a straight line which is called the tangent to $y = f(x)$ at $(x_0, f(x_0))$.
The derivative is the slope of the tangent to $y = f(x)$ at $(x_0, f(x))$.

1.2 Notation

$$L = f'(x_0) = \frac{\mathrm{d}f(x_0)}{\mathrm{d}x} = \frac{\mathrm{d}y(x_0)}{\mathrm{d}x} = Df(x_0)$$

1.3 Derivative Function

If we calculate the derivative of $f(x)$ at any possible x , we get the derivative function.

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x} = Df$$

1.4 The Tangent Line

$$y - f(x_0) = f'(x_0)(x - x_0)$$

1.5 The Normal Line

$$\begin{aligned} y - f(x_0) &= -\frac{1}{f'(x_0)}(x - x_0) && ; f'(x_0) \neq 0 \\ x &= x_0 && ; f'(x_0) = 0 \end{aligned}$$

1.6 Proofs of Standard Derivatives

1.6.1 $y = f(x) = c$

$$
\begin{aligned}
f'(x) &= \lim_{x \to 0} \frac{f(x + \Delta x)}{\Delta x} \\
&= \lim_{x \to 0} \frac{c - c}{\Delta x} \\
&= 0
\end{aligned}
$$

1.6.2 $y = f(x) = x^n; n \in \mathbb{N}$

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} \\
a^n - b^n &= (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) \\
\therefore f'(x) &= \lim_{\Delta x \to 0} \frac{\Delta x((x + \Delta x)^{n-1} + (x + \Delta x)^{n-2}x + \cdots + (x + \Delta x)x^{n-2} + x^{n-1}}{\Delta x} \\
&= \lim_{\Delta x \to 0} (x + \Delta x)^{n-1} + (x + \Delta x)^{n-2}x + \cdots + (x + \Delta x)x^{n-2} + x^{n-1} \\
&= x^{n-1} + x \cdot x^{n-2} + \cdots + x^{n-2} \cdot x + x^{n-1} \\
&= nx^{n-1}
\end{aligned}
$$

1.6.3 $y = f(x) = x^{-n}; n \in \mathbb{N}, x \neq 0$

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^{-n} - x^{-n}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\dfrac{1}{(x + \Delta x)^n} - \dfrac{1}{x^n}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^n - (x + \Delta x)^n}{\Delta x(x^n(x + \Delta x)^n)} \\
&= \frac{-nx^{n-1}}{x^n x^n}
\end{aligned}
$$

1.6.4 $y = f(x) = \sin x$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2\sin\frac{\Delta x}{2}\cos(x+\frac{\Delta x}{2})}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cos\left(x+\frac{\Delta x}{2}\right) \\ &= \cos x \end{aligned}$$

1.7 Theorem: If $\exists f'(x)$, $\exists g'(x)$ and c is a constant, then, $(cf(x))' = cf'(x)$, $(f(x) \pm g(x))' = f'(x) \pm g'(x)$, $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$, $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

$$\begin{aligned} (f(x)g(x))' &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(f(x+\Delta x)g(x+\Delta x) - f(x)g(x+\Delta x)) + (f(x)g(x+\Delta x) - f(x)g(x))}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(f(x+\Delta x) - f(x))g(x+\Delta x)}{\Delta x} + \frac{f(x)((g(x+\Delta x) - g(x))}{\Delta x} \\ &= f'(x)g(x) + f(x)g'(x) \end{aligned}$$

1.8 Theorem: Let f be defined on an open interval about x_0. Then, $f(x)$ is differentiable at x_0, iff $\exists A \in \mathbb{R}$ and $\exists \alpha(\Delta x)$, with $\lim_{\Delta x \to 0} \alpha(\Delta x) = 0$, s.t. $\frac{\Delta y}{\Delta x} = A + \alpha(\Delta x); A = f'(x_0)$

1.9 If $y = f(x)$ is differentiable at x_0, then, $f(x)$ is continuous at x_0.

$$\begin{aligned} \exists f'(x_0) \Rightarrow \frac{\Delta y}{\Delta x} &= f'(x_0) + \alpha(\Delta x) \\ \therefore f(x_0+\Delta x) - f(x_0) = \Delta y &= \Delta x(f'(x_0) + \alpha(\Delta x) \\ \therefore f(x_0+\Delta x) &= f(x_0) + \Delta x(f'(x_0) + \alpha(\Delta x)) \\ \therefore \lim_{\Delta x \to 0} f(x_0+\Delta x) &= \lim_{\Delta x \to 0} f(x_0) + \Delta x(f'(x_0) + \alpha(\Delta x)) \\ &= x_0 \end{aligned}$$

The converse of this theorem is not true.