Recitation 12

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Contents

1 Functions of Two Variables

1.1 Differentiability

Example 1. Check the differentiability of $f(x, y)$ in $\mathbb{R}^2$.

$$f(x,y) = \begin{cases} \dfrac{x^3 + y^4}{x^2 + y^2} & ; \quad (x,y) \neq (0,0) \\ 0 & ; \quad (x,y) = (0,0) \end{cases}$$

Solution. If $(x, y) \neq (0, 0)$, $f_x(x, y)$ and $f_y(x, y)$ are continuous. Therefore, f is differentiable in $\mathbb{R}^2 - \{(0, 0)\}$.
f is differentiable at $(0, 0)$ if f_x and f_y are continuous at $(0, 0)$.

$$\begin{aligned} f_x(0,0) &= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h - 0} \\ &= \lim_{h \to 0} \frac{\frac{h^3}{h^2} - 0}{h} \\ &= 1 \end{aligned}$$

$$\begin{aligned} f_y(0,0) &= \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h - 0} \\ &= \lim_{h \to 0} \frac{\frac{h^4}{h^2} - 0}{h} \\ &= 0 \end{aligned}$$

$$\begin{aligned} &\lim_{\substack{x=0 \\ y \to 0}} f_x(x,y) = 0 \\ \therefore &\lim_{\substack{x=0 \\ y \to 0}} f_x(x,y) \neq f_x(0,0) \end{aligned}$$

Therefore, $f(x)$ is not continuous at $(0, 0)$.

Therefore, it needs to be checked by definition.

$$\begin{aligned}
\varepsilon(\Delta x, \Delta y) &= f(x,y) - f(0,0) - f_x(0,0)\Delta x - f_y(0,0)\Delta y \\
&= \frac{x^3 + y^4}{x^2 + y^2} - 1 \cdot \Delta x - 0 \cdot \Delta y \\
&= \frac{x^3 + y^4}{x^2 + y^2} - x \\
&= \frac{x^3 + y^4 - x(x^2 + y^2)}{x^2 + y^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{\substack{x \to 0 \\ y \to 0}} \frac{\varepsilon(x,y)}{\sqrt{x^2 + y^2}} &= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{\dfrac{x^3 + y^4 - x^3 + xy^2}{x^2 + y^2}}{\sqrt{x^2 + y^2}} \\
&= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{y^4 - xy^2}{\left(x^2 + y^2\right)^{3/2}}
\end{aligned}$$

Over the path $y = x$, $x \to 0$, the limit is $-\dfrac{1}{2^{3/2}}$, not 0. Therefore, f is not differentiable at $(0,0)$.

1.2 Tangent Plane

Example 2. Find the tangent plane to $F(x,y,z) = xyz - 8$ at $(-2, 2, -2)$.

Solution.

$$\begin{aligned}
F_x &= yz \\
F_y &= xz \\
F_z &= xy
\end{aligned}$$

Therefore,

$$\begin{aligned}
F_x(-2,2,-2) &= -4 \\
F_y(-2,2,-2) &= 4 \\
F_z(-2,2,-2) &= -4
\end{aligned}$$

Therefore, the tangent plane is

$$-4(x+2) + 4(y-2) - 4(z+2) = 0$$

Example 3. At which points will the tangent plane to $x^2 + y^2 = 4z$ be parallel to the plane $x - 2y - z = 5$?

Solution. Let

$$F(x, y, z) = x^2 + y^2 - 4z$$

Then the tangent plane to $F(x, y, z) = 0$ is given by

$$2x_0(x - x_0) + 2y_0(y - y_0) - 4(z - z_0) = 0$$

Therefore,

$$(2x_0, 2y_0, -4) = t(1, -2, 1)$$

Solving,

$$\begin{aligned} x_0 &= 2 \\ y_0 &= -4 \\ \therefore z_0 &= 5 \end{aligned}$$

Therefore, the planes are parallel at $(2, -4, 5)$.