Recitation 2

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Contents

1 One-to-one Functions (Injective Functions)

A function $f : A \rightarrow \mathbb{R}$ is one-to-one if $\forall x, yinA, f(x) = f(y) \Rightarrow x = y$.

2 Onto Functions

$f : A \rightarrow B$ is onto B if $\forall y \in B, \exists x \in A$, s.t. $f(x) = y$.

3 Strictly Monotonically Increasing Functions

A function is strictly monotonically increasing if $\forall x, y \in D(f)$, s.t., $x < y, f(x) < f(y)$.
A strictly monotonically increasing function is always ont-to-one.

4 Inverse Functions

If $f : A \rightarrow B$ is one-to-one and onto, we can define a function $f^{-1} : B \rightarrow A$, s.t. $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$.
f^{-1} is called the inverse function of f.

5 Check which of the following functions are one-to-one and find their inverses

5.1 $f(x) = e^{e^x}$

$f(x)$ is strictly monotonically increasing. Hence, it is one-to-one.

$I(f) = D(f^{-1}) = (0, \infty)$

$$\begin{aligned}
y &= e^{e^x} \\
\therefore \ln y &= e^x \\
\therefore \ln \ln y &= x
\end{aligned}$$

$$\begin{aligned}
\therefore f^{-1}(x) &= \ln \ln x \\
D(f^{-1}) &= (1, \infty)
\end{aligned}$$

5.2 $f(x) = 1 - x^3$

If $f(x) = f(y)$,

$$\begin{aligned}
1 - x^3 &= 1 - y^3 \\
\therefore x^3 &= y^3 \\
\therefore x &= y
\end{aligned}$$

Therefore, $f(x)$ is one-to-one over $\mathbb{R}$

$$y = 1 - x^3$$
$$\therefore x^3 = 1 - y$$
$$\therefore x = \sqrt[3]{1-y}$$

$$\therefore f^{-1}(x) = \sqrt[3]{1-x}$$
$$D(f^{-1}) = \mathbb{R}$$

5.3 $f(x) = \dfrac{x}{1+x}; x \neq -1$

$$y = \frac{x}{1+x}$$
$$\Leftrightarrow y(1+x) = x$$
$$\Leftrightarrow y + xy = x$$
$$\Leftrightarrow x = \frac{y}{1-y}$$

$$\therefore f^{-1}(x) = \frac{x}{1-x}$$
$$D(f^{-1}) = \mathbb{R} - \{1\}$$

6 Composition of Functions

If $f : A \to B, g : C \to D$ and $B \subseteq C$, then we can define the composition $d \circ f : A \to D$ as $(g \circ f)(x) = g(f(x))$.

7 Limits of Functions

Let f be defined in a punctured neighbourhood of x_0.
Then the limit of f at x_0 is l. It is denoted as $\lim\limits_{x \to x_0} f(x) = l$.
If $\forall \varepsilon > 0 \exists \delta > 0$, s.t. if$|x - x_0| > \delta$ then, $|f(x) - l| < \varepsilon$

7.1 Prove: $\lim\limits_{x \to 1}(2x + 5) = 7$

Let $\varepsilon > 0$. We have to find δ , s.t. if $|x - 1| < \delta$, then, $|2x + 5 - 7| < \varepsilon$.

$$|2x + 5 - 7| = |2x - 2|$$
$$= 2|x - 1|$$

Hence, if we take $d = \dfrac{\varepsilon}{2}$, we have the following.
If $|x-1| < \delta$, then $|f(x) - 7| = |2x + 5 - 7| = 2|x-1| < 2\delta = 2\dfrac{\varepsilon}{2} = \varepsilon$.
Therefore, $\lim_{x\to 1}(2x+5) = 7$

7.2 Prove: $\lim_{x\to 2} \dfrac{2x+6}{3x-1} = 2$

We want δ s.t.
if $|x-2| < \delta$ then $\left|\dfrac{2x+6}{3x-1} - 2\right| < \varepsilon$

$$\begin{aligned}\left|\frac{2x+6}{3x-1}\right| - 2 &= \left|\frac{2x+6-2(3x-1)}{3x-1}\right| \\ &= \left|\frac{-4x+8}{3x-1}\right| \\ &= \frac{4\,|x-2|}{|3x-1|}\end{aligned}$$

We can always take $\delta \leq 1$

$$\therefore \left|\frac{2x+6}{3x-1}\right| - 2 < \frac{1}{2} 4\,|x-2| < 2\delta = \varepsilon$$

Take $\delta = \min\left(\dfrac{\varepsilon}{2}, 1\right)$.
Then, if $|x-2| < \delta$, then

$$|f(x) - l| = \left|\frac{2x+6}{3x} - 2\right| = \frac{4\,|x-2|}{|3x-1|}$$