Recitation 11

Wednesday 7th January, 2015

Contents

1 Functions of Two Variables

1.1 Domain of Definition

Example 1. Find the domain of definition of

$$f(x, y) = \sqrt{(x^2 + y^2 - 16)(x^2 + y^2 - 9)}$$

Solution.

$$x^2 + y^2 - 16 \geq 0 \qquad \& \qquad x^2 + y^2 - 9 \geq 0$$

or

$$x^2 + y^2 - 16 \leq 0 \qquad \& \qquad x^2 + y^2 - 9 \leq 0$$

Therefore,

$$\sqrt{x^2 + y^2} \geq 4$$

and

$$\sqrt{x^2 + y^2} \leq 3$$

1.2 Disproving Existence of Limits

Example 2. Does $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ exist?

Solution.

$$\begin{aligned} \lim_{(x,y) \overset{y=kx}{\to} (0,0)} \frac{x^2 - y^2}{x^2 + y^2} &= \lim_{x\to 0} \frac{x^2 - k^2x^2}{x^2 + k^2x^2} \\ &= \frac{1 - k^2}{1 + k^2} \end{aligned}$$

Therefore, the limit is different for different values of k. Hence, the limit does not exist.

Example 3. Does $\lim_{(x,y)\to(0,0)} \frac{xy}{x^3 + y^{3/2}}$ exist?

Solution.

$$\lim_{(x,y)\overset{y=kx}{\to}(0,0)} \frac{xy}{x^3+y^{3/2}} = 0$$

However, this is not enough to prove the existence of the limit.

$$\lim_{(x,y)\overset{y=kx^2}{\to}(0,0)} \frac{xy}{x^3+y^{3/2}} = \frac{k}{1+k^{3/2}}$$

Therefore, the limit does not exist.

1.3 Proving Existence of Limits

Example 4. Check the continuity of

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & ; \quad (x,y) \neq (0,0) \\ 0 & ; \quad (x,y) = (0,0) \end{cases}$$

Solution.

$$0 \leq \left|\frac{x^2y}{x^2+y^2}\right| = \frac{x^2}{x^2+y^2}|y| \leq |y|$$

Therefore, by the Sandwich Theorem,

$$0 \leq \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y}{x^2+y^2}|y| \leq \lim_{y\to 0}|y|$$

Therefore,

$$\lim_{(x,y)\to(0,0)} f(x) = 0$$

Therefore, $f(x)$ is continuous.

Example 5. Check for which values of m the function

$$f(x,y) = \begin{cases} \dfrac{x^my}{x^2+4y^2} & ; \quad (x,y) \neq (0,0) \\ 0 & ; \quad (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

Solution. Let

$$\begin{aligned} x &= r\cos t \\ y &= r\sin t \end{aligned}$$

Therefore, when $(x, y) \to (0, 0)$, $r \to 0$.
Therefore,

$$\begin{aligned} \lim_{\substack{x\to 0\\ y\to 0}} f(x, y) &= \lim_{r\to 0} f(r\cos t, r\sin t) \\ &= \lim_{r\to 0} \frac{r^m \cos^m t \cdot r\sin t}{r^2\cos^2 t + 4r^2\sin^2 t} \\ &= \lim_{r\to 0} \frac{r^{m+1}\cos^m t \sin t}{r^2(\cos^2 t + 4\sin^2 t)} \\ &= \lim_{r\to 0} \frac{r^{m-1}\cos^m t\sin t}{1 + 3\sin^2 t} \end{aligned}$$

If the limit depends on t, the limit is path dependent. Hence, the limit does not exist.
If $m > 1$,

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x, y) &= \lim_{r\to 0} \frac{r^{m-1}\cos^m t\sin t}{1 + 3\sin^2 t} \\ &= 0 \end{aligned}$$

Therefore, the limit is 0, and $f(x, y)$ is continuous at $(0, 0)$.
If $m = 1$,

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{r\to 0} \frac{\cos t\sin t}{1 + 3\sin^2 t}$$

The limit depends on t, and hence does not exist.
If $m < 1$,

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x, y) &= \lim_{r\to 0} \frac{r^{m-1}\cos^m t\sin t}{1 + 3\sin^2 t} \\ &= \infty \end{aligned}$$

Therefore, the limit does not exist.