Differential and Integral Methods - Exercise 1

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5^{th} November, 2014

(1) Find the domain of the following functions

(a) $f(x) = 2x - 3\sin x$

There is no restriction on the values of x in $f(x)$.

$$\boxed{\therefore D(f) = \mathbb{R}}$$

(b) $f(x) = \dfrac{1}{\sqrt{|x| - x}}$

For the square root to exist, $|x| - x \geq 0$, and for the fraction to exist, $\sqrt{|x| - x} \neq 0$.

$\therefore |x| - x > 0$
$\therefore x \in (-\infty, 0)$

$$\boxed{\therefore D(f) = (-\infty, 0)}$$

(c) $f(x) = \sqrt{\dfrac{x+3}{|x^2 - 4|}}$

Conditions on x:

$$\frac{x+3}{|x^2-4|} \geq 0 \tag{1}$$

$$|x^2 - 4| \neq 0 \tag{2}$$

$$\begin{aligned}(1) &\Rightarrow x + 3 \geq 0\\ &\therefore x \geq -3\\ (2) &\Rightarrow x^2 \neq 4\\ &\therefore x \notin \{-2, 2\}\end{aligned}$$

$$\boxed{\therefore D(f) = [-3, \infty) - \{-2, 2\}}$$

(d) $f(x) = \ln(x + 2)$

Conditions on x:

$$x + 2 > 0 \tag{3}$$

$\therefore x > -2$

$$\boxed{\therefore D(f) = (-2, \infty)}$$

(e) $f(x) = \ln(|x+2|)$

Conditions on x:

$$|x+2| > 0 \tag{4}$$

$\therefore x \neq -2$

$\boxed{\therefore D(f) = \mathbb{R} - \{-2\}}$

(f) $f(x) = \log_2(\log_2 x)$

Conditions on x:

$$x > 0 \tag{5}$$
$$\log_2 x > 0 \tag{6}$$

$\therefore x > 1$

$\boxed{\therefore D(f) = (1, \infty)}$

(g) $\dfrac{1}{1-\cos x}$

Conditions on x:

$$1 - \cos x \neq 0 \tag{7}$$
$$\tag{8}$$

$$\therefore \cos x \neq 1$$
$$\therefore x \neq 2n\pi; n \in Z$$

$\boxed{\therefore D(f) = \mathbb{R} - \{2n\pi : n \in Z\}}$

(h) $f(x) = \dfrac{1+\sin x}{\sqrt{1 - sin^2 x}}$

$$\begin{aligned} f(x) &= \frac{1+\sin x}{\sqrt{1-\sin^2 x}} \\ &= \frac{1+\sin x}{\sqrt{\cos^2 x}} \\ &= \frac{1+\sin x}{\cos x} \end{aligned}$$

Therefore, conditions on x:

$$\cos x \neq 0 \tag{9}$$

$\therefore x \neq (2n+1)\dfrac{\pi}{2}$

$\boxed{\therefore x \in \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}}$

(i) $\quad f(x) = \sqrt{\dfrac{x-2}{x+2}} + \sqrt{\dfrac{1+x}{1-x}}$

Conditions on x:

$$\frac{x-2}{x+2} \geq 0 \tag{10}$$
$$x+2 \neq 0 \tag{11}$$
$$\frac{1+x}{1-x} \geq 0 \tag{12}$$
$$1-x \neq 0 \tag{13}$$

$\therefore (10) \Rightarrow x \in (\infty, -2] \cup [2, \infty)$
$\therefore (11) \Rightarrow x \neq -2$
$\therefore (10) \Rightarrow x \in [-1, 1]$
$\therefore (11) \Rightarrow x \neq 1$

$\boxed{D(f) = \{\}}$

(j) $\quad f(x) = \tan\left(x + \dfrac{\pi}{4}\right)$

Conditions on x:

$$x + \frac{\pi}{4} \neq (2n+1)\frac{\pi}{2}; n \in \mathbb{Z} \tag{14}$$

$\therefore x \neq (2n+1)\dfrac{\pi}{2} - \dfrac{\pi}{4}; n \in \mathbb{Z}$

$\boxed{\therefore D(f) = \mathbb{R} - \{(2n+1)\frac{\pi}{2} - \frac{\pi}{4} : n \in \mathbb{Z}\}}$

(2) Write which of the following functions are even, odd or neither odd nor even:

(a) $f(x) = x - x^3 + x^5$

$$\begin{aligned} f(-x) &= (-x) - (-x)^3 + (-x)^5 \\ &= (-x) - (-x^3) + (-x^5) \\ &= -x + x^3 - x^5 \\ &= -f(x) \end{aligned}$$

Therefore, $f(x)$ is odd.

(b) $f(x) = x^2 - x^3 + x^6$

$$\begin{aligned} f(-x) &= (-x)^2 - (-x)^3 + (-x)^6 \\ &= x^2 - (-x^3) + x^6 \\ &= x^2 + x^3 + x^6 \end{aligned}$$

Therefore, $f(x)$ is neither odd nor even.

(c) $f(x) = 5^x$

$$\begin{aligned} f(-x) &= 5^{-x} \\ &= \frac{1}{5^x} \end{aligned}$$

Therefore, $f(x)$ is neither odd nor even.

(d) $f(x) = \sin(\sin x)$

$$\begin{aligned} f(-x) &= \sin(\sin(-x)) \\ &= \sin(-\sin x) \\ &= -\sin(\sin x) \\ &= -f(x) \end{aligned}$$

Therefore, $f(x)$ is odd.

(e) $f(x) = \cos(\sin x)$

$$\begin{aligned} f(-x) &= \cos(\sin(-x)) \\ &= \cos(-\sin x) \\ &= \cos(\sin x) \\ &= f(x) \end{aligned}$$

Therefore, $f(x)$ is even.

(f) $f(x) = \sin(\cos x)$

$$\begin{aligned} f(-x) &= \sin(\cos(-x)) \\ &= \sin(\cos x) \\ &= f(x) \end{aligned}$$

Therefore, $f(x)$ is even.

(g) $f(x) = x \sin x$

$$\begin{aligned} f(-x) &= (-x)\sin(-x) \\ &= (-x)(-sinx) \\ &= x \sin x & = f(x) \end{aligned}$$

Therefore, $f(x)$ is even.

(h) $f(x) = \dfrac{1-x}{1+x^2}$

$$\begin{aligned} f(-x) &= \frac{1-(-x)}{1+(-x)^2} \\ &= \frac{1+x}{1+x^2} \end{aligned}$$

Therefore, $f(x)$ is neither odd nor even.

(3) Find the image of the following functions in the specified domains:

(a) $f(x) = \sin x + 3$ in the interval $\left[\pi, \frac{3\pi}{2}\right]$

The image of $\sin x$ in $\left[\pi, \frac{3\pi}{2}\right]$ is $[-1, 0]$.
$\therefore$ the image of $f(x)$ is $[(-1+3), (0+3)]$, i.e. $[2, 3]$.

(b) $f(x) = -x^2 + 8$ in the interval $(0, 7]$

The image of x^2 in $(0, 7]$ is $(0, 49]$.
$\therefore$ the image of $-x^2$ in $(0, 7]$ is $[-49, 0)$.
$\therefore$ the image of $f(x)$ in $(0, 7]$ is $[-41, 8)$.

(4) Let $f(x) = x^3$. Find the function obtained from performing the following actions on f:

(a) Translation downwards by 3

$$g(x) = f(x) - 3$$
$$\therefore g(x) = x^3 - 3$$

(b) Right translation by 4

$$g(x) = f(x-4)$$
$$\therefore g(x) = (x-4)^3$$

(c) Reflection around the x-axis.

$$g(x) = -f(x)$$
$$\therefore g(x) = -x^3$$

(d) Reflection around the y-axis.

$$g(x) = f(-x)$$
$$\therefore g(x) = (-x)^3 = -x^3$$

(e) Left translation by 1

$$g(x) = f(x+1)$$
$$\therefore g(x) = (x+1)^3$$
$$\therefore g(x) = x^3 + 3x^2 + 3x + 1$$

(5) Draw the graph of the function $f(x) = x - [x]$, where $[x]$ is the biggest integer smaller or equal to x.

(6) Solve the following inequalities:

(a) $x^2 - 5x + 6 \leq 0$

$$x^2 - 5x + 6 \leq 0$$
$$\therefore (x-2)(x-3) \leq 0$$

$$\therefore x \in [2, 3]$$

(b) $x^2 - 4x > 21$

$$x^2 - 4x > 21$$
$$\therefore x^2 - 4x - 21 > 0$$
$$\therefore (x-7)(x+3) > 0$$

$$\therefore x \in (-\infty, -3) \cup (7, \infty)$$

(c) $|-x + 3| < 7$

$$|-x+3| < 7$$
$$\therefore -7 < -x + 3 < 7$$
$$\therefore -10 < -x < 4$$
$$\therefore 10 > x > -4$$

$$\therefore x \in (-4, 10)$$

(d) $|2x - 3| \leq x + 3$

$$|2x-3| \leq x + 3$$
$$\therefore -x - 3 \leq 2x - 3 \leq x + 3$$
$$\therefore 0 \leq x \leq 6$$

(e) $|x+1| - |2-x| \geq 0$

$$\begin{aligned} |x+1| &\geq |2-x| \\ \therefore (x+1)^2 &\geq (2-x)^2 \\ \therefore x^2+2x+1 &\geq 4-4x+x^2 \\ \therefore 6x &\geq 3 \\ \therefore x &\geq \frac{1}{2} \end{aligned}$$

$$\therefore x \in \left[\frac{1}{2}, \infty\right)$$

(f) $|x-1| + |x-2| - |x-3| \geq x$

Case I: $x > 3$

$$\therefore (x-1)+(x-2)-(x-3)-x \geq 0$$

No contradiction

$$\therefore x \in (3, \infty)$$

Case II: $2 < x \leq 3$

$$\begin{aligned} (x-1)+(x-2)+(x-3)-x &\geq 0 \\ \therefore x &\geq 3 \end{aligned}$$

$$\therefore x = 3$$

Case III: $1 < x \leq 2$

$$\begin{aligned} (x-1)-(x-2)+(x-3)-x &\geq 0 \\ \therefore -2 &\geq 0 \end{aligned}$$

$\therefore$ no values of x exist.

Case IV: $x \leq 1$

$$\begin{aligned} -(x-1)-(x-2)+(x-3)-x &\geq 0 \\ \therefore x &\leq 0 \end{aligned}$$

$$\therefore x \in (-\infty, 0]$$

$$\boxed{x \in (\infty, 0] \cup [3, \infty)}$$