Differential and Integral Methods

Aakash Jog

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Part I

General Information

1 Contact Information

Dr. Yakov Yakubov
yakubov@post.tau.ac.il

2 Office Hours

Monday
16:15 - 17:15
Room 233, Schreiber Building
Tel: 03-6405357

Part II
Functions

3 Notation

$$\mathbb{N} = \text{Set of all natural numbers}$$
$$\mathbb{Z} = \text{Set of all integers}$$
$$\mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N} \right\}$$

4 Basic Definitions

Definition 1 (Domain, Range and Variables)**.** Let D and E be two sets of real numbers. A function f from D into E is a well defined law which, to each $x \in D$ corresponds to a unique number $y \in E$. The set D is called the domain of f and the set E is called the range of f.
Denote $f : D \to E$ or $y = f(x)$.
The variable x is called independent variable and the variable y is called dependent variable.
The variable x is also called the origin of y and y is also called the image of x.

Definition 2 (Image of a function)**.** Given $f : D \to E$. Then the image of f is a set of all $y \in E$ s.t. $\exists x \in D, y = f(x) : I(f) = \{y \in E : \exists x \in D, y = f(x)\}$

Definition 3 (Existence domain)**.** The biggest possible domain of a function f is called the existence domain of f.

Definition 4 (Graph of a function)**.** A set of point $\{(x, f(x)) : x \in D\}$ in the plane $\mathbb{R}^2$ is called a graph of a function $y = f(x)$.

Definition 5 (Even function)**.** If $f(-x) = f(x); (x, -x \in D)$ then, f is called an even function.
Each even function is symmeteric about the y-axis.

Definition 6 (Odd function)**.** If $f(-x) = -f(x); (x, -x \in D)$ then, f is called an odd function.
Each odd function is symmeteric about the origin.

Definition 7 (Periodical function)**.** A function $y = f(x)$ which is defined on D is called periodical if $\exists T \neq 0$ which is called a period of f s.t. $\forall x \in D \Rightarrow$

$x + T \in D$ and $f(x+T) = f(x)$.
The smallest such $T > 0$ (if it exists) is called the minimal period.

Definition 8 (Shifting with respect to y-axis)**.** $f(x+a)$ is the graph of $f(x)$, shifted by a, in the direction of the x-axis, opposite to the sign of a.

Definition 9 (Shifting with respect to x-axis)**.** $f(x)+a$ is the graph of $f(x)$, shifted by a, in the direction of the y-axis, according to the sign of a.

Definition 10 (Monotonic function)**.** A function $y = f(x)$ is called monotonic increasing (strongly increasing) in D if $\forall x_1, x_2 \in D, x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)(f(x_1) < f(x_2))$.
A function $y = f(x)$ is called monotonic decreasing (strongly increasing) in D if $\forall x_1, x_2 \in D, x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2)(f(x_1) > f(x_2))$.

Definition 11 (One-to-one function)**.** A function $f : D(f) \to E$ is called one-to-one if $\forall y \in I(f) \Rightarrow \exists! x \in D(f)$ s.t. $y = f(x)$.
Equivalently, $\forall x_1, x_2 \in D(f)$, if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

5 Inverse of a Function

Definition 12 (Inverse function)**.** If $f : D(f) \to I(f)$ is one-to-one and onto. Then, we can define $g : I(f) \to D(f)$, which is one-to-one and onto, by $g(y) = x$, where $y = f(x)$. Therefore, $g(f(x)) = x$. g is called the inverse function of f.
The inverse function is denoted as $g = f^{-1}$(Note: $f^{-1} \neq \frac{1}{f}$)

$$D(f) = I(f^{-1}) \tag{1}$$
$$I(f) = D(f^{-1}) \tag{2}$$

The graphs of a f and f^{-1} are symmeteric about $y = x$.

6 Elementary Operations between Functions

Definition 13 (Addition and subtraction of functions)**.**

$$h(x) = f(x) \pm g(x) \tag{3}$$
$$D(h) = D(f) \cap D(g) \tag{4}$$

Definition 14 (Multiplication of a function by a constant)**.**

$$h(x) = kf(x) \tag{5}$$
$$D(h) = D(f) \tag{6}$$

Definition 15 (Multiplication of functions).

$$h(x) = f(x)g(x) \tag{7}$$
$$D(h) = D(f) \cap D(g) \tag{8}$$

Definition 16 (Division of functions).

$$h(x) = \frac{f(x)}{g(x)} \tag{9}$$
$$D(h) = \{x \in D(f) \cap D(g) : g(x) \neq 0\} \tag{10}$$

7 Composite Functions

Definition 17 (Composition of Functions). Let $f : D(f) \to E$ and $g : D(g) \to F$ be two functions. A composition of f with g is a function $h : D(h) \to F$ where $h(x) = g(f(x))$. It is denoted as $g \circ f$

$$D(h) = \{x \in D(f) : f(x) \in D(g)\} \tag{11}$$

8 Elementary Functions

8.1 Polynomial

$$y = f(x) = a_0 + a_1x + \cdots + a_nx^n; a_0, \ldots, a_n \in \mathbb{R} \tag{12}$$
$$D(f) = \mathbb{R} \tag{13}$$

1. If $n = 0, y = f(x) = a_0$ represents a constant function.
2. If $n = 1, y = f(x) = a_0 + a_1x$ represents a straight line in the $X - Y$ plane.
3. If $n = 2, y = f(x) = a_0 + a_1x + a_2x^2$ represents a parabola in the $X - Y$ plane.

8.2 Power Function

$$y = f(x) = x^a; a \in R \tag{14}$$
$$D(f) \text{ depends on } a \tag{15}$$

8.3 Exponential Function

$$y = f(x) = a^x; a > 0, a \neq 1 \tag{16}$$
$$D(f) = \mathbb{R} \tag{17}$$
$$I(f) = (0, \infty) \tag{18}$$

8.4 Logarithmic Function

$$y = f(x) = \log_a x \tag{19}$$
$$D(f) = (0, \infty) \tag{20}$$
$$I(f) = \mathbb{R} \tag{21}$$

8.5 Trigonometric Functions

$$y = f(x) = \sin x \tag{22}$$
$$y = f(x) = \cos x \tag{23}$$
$$y = f(x) = \tan x = \frac{\sin x}{\cos x} \tag{24}$$
$$y = f(x) = \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x} \tag{25}$$
$$y = f(x) = \csc x = \frac{1}{\sin x} \tag{26}$$
$$y = f(x) = \sec x = \frac{1}{\cos x} \tag{27}$$

8.6 Inverse Trigonometeric Functions

$$y = f^{-1}(x) = \sin^{-1} x = \arcsin x \tag{28}$$
$$D(\arcsin x) = [-1, 1] \tag{29}$$
$$I(\arcsin x) = [-\frac{\pi}{2}, \frac{\pi}{2}] \tag{30}$$

$$y = f^{-1}(x) = \cos^{-1} x = \arccos x \tag{31}$$
$$D(\arccos x) = [-1, 1] \tag{32}$$
$$I(\arccos x) = [0, \pi] \tag{33}$$

$$y = f^{-1}(x) = \tan^{-1} x = \arctan x \tag{34}$$
$$D(\arctan x) = \mathbb{R} \tag{35}$$
$$I(\arctan x) = [-\frac{\pi}{2}, \frac{\pi}{2}] \tag{36}$$

8.7 Hyperbolic Functions

$$\sinh x \doteq \frac{e^x - e^{-x}}{2} \tag{37}$$
$$D(\sinh x) = \mathbb{R} \tag{38}$$
$$I(\sinh x) = \mathbb{R} \tag{39}$$

$$\cosh x \doteq \frac{e^x + e^{-x}}{2} \tag{40}$$
$$D(\cosh x) = \mathbb{R} \tag{41}$$
$$I(\cosh x) = [1, \infty) \tag{42}$$

$$\tanh x \doteq \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{43}$$
$$D(\tanh x) = \mathbb{R} \tag{44}$$
$$I(\tanh x) = (-1, 1) \tag{45}$$

8.7.1 Identities of Hyperbolic Functions

$$\sinh(2x) = 2 \sinh x \cosh x \tag{46}$$
$$\cosh^2 x + \sinh^2 x = \cosh(2x) \tag{47}$$
$$\cosh^2 x - \sinh^2 x = 1 \tag{48}$$
$$\frac{\cosh(2x) - 1}{2} = \sinh^2 x \tag{49}$$
$$\frac{\cosh(2x) + 1}{2} = \cosh^2 x \tag{50}$$

8.8 Absolute Value

$$y = f(x) = \begin{cases} x; x > 0 \\ 0; x = 0 \\ -x; x < 0 \end{cases} \tag{51}$$

8.9 Floor Function

$$y = f(x) = \lfloor x \rfloor = \text{the largest integer less than or equal to } x \tag{52}$$

Part III
Limits and Continuity

Definition 18 (Existence of a limit). If $x \to a^+$ then $f(x) \to L_2$, and if $x \to a^-$ then $f(x) \to L_1$.
We say that $\exists \lim_{x \to a} f(x)$ iff $L_1 = L_2$

Definition 19 (Continuity of a function). If $x \to a$ then $f(x) \to L$, we say that L is the limit of $f(x)$ at $x = a$.

$$\lim_{x \to a} f(x) = L$$

We say that $f(x)$ is continuous at $x = a$, iff

$$\lim_{x \to a} f(x) = L = f(a)$$

Definition 20 (Cauchy's definition of a limit of a function). Let $f(x)$ be defined on an open interval about a, except possibly at a itself.
A number L is called the limit of $f(x)$ at a if

$$\forall \epsilon > 0 \exists \delta > 0 : 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon \tag{53}$$

9 A Classification of Discontinuity Points

Let $f(x)$ be defined on an open interval about a, except possibly at a itself.

Definition 21 (Removable Discontinuity point). The point a is a removable discontinuity point of f if, $\lim_{x \to a} f(x)$ exists, but either $\lim_{x \to a} f(x) \neq f(a)$ or $f(a)$ does not exist.

Definition 22 (Discontinuity of First Kind). The point a is a discontinuity point of the first kind if both $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

Definition 23 (Discontinuity of Second Kind). The point a is a discontinuity point of the second kind if atleast one of the two one-sided limits of f does not exist.
Note that the limits are defined as finite numbers only.

Theorems

Theorem 1 (Sandwich Theorem). *Let $f(x), g(x), h(x)$ be defined on an open interval about a, except possibly at a itself. Assume that $\forall x \neq a$ from the interval, it is satified that $f(x) \leq g(x) \leq h(x)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$. Then, $\lim_{x \to a} g(x) = L$.*

Proof.

$$\forall \varepsilon > 0, \exists \delta > 0 : 0 < |x - a| < \delta$$
$$therefore \left|g(x) - L\right| < \varepsilon$$

i.e.,

$$L - \varepsilon < g(x) < L + \varepsilon$$

Given $\exists \delta_1 > 0 : 0 < |x - a| < \delta_1$

$$f(x) \leq g(x) \leq h(x)$$

For this $\varepsilon > 0$,

$$\exists \delta_2 > 0 : 0 < [x - a] < \delta_2$$
$$\therefore \left|f(x) - L\right| < \varepsilon$$

i.e.,

$$L - \varepsilon < f(x) < L + \varepsilon$$
$$\varepsilon > 0, \exists \delta_3 > 0 : 0 < [x - a] < \delta_3$$
$$\therefore \left|h(x) - L\right| < \varepsilon$$

i.e.,

$$L - \varepsilon < h(x) < L + \varepsilon$$

So, $\forall \varepsilon > 0$,

$$\exists \delta = \min\{\delta_1, \delta_2, \delta_3\} > 0 : 0 < |x - a| < \delta$$
$$\therefore L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon$$

□

Theorem 2 (Theorem 5)**.** *If* $\lim_{x \to a} f(x) = 0$ *and* $g(x)$ *is bounded in an open interval about* a*, except possibly at* a *itself, then,* $\lim_{x \to a} (f(x)g(x)) = 0$.

Proof. We have to prove that

$$\forall \varepsilon > 0, \exists \delta > 0 : 0 < |x - a| < \delta \Rightarrow \left|f(x)g(x) - 0\right| < \varepsilon$$

Given $\lim_{x \to a} f(x) = 0$,

$$\forall \varepsilon_1 > 0, \exists \delta_1 > 0 : 0 < |x - a| < \delta_1 \Rightarrow \left|f(x) - 0\right| < \varepsilon_1$$

As $g(x)$ is bounded, in an open interval about a, except possibly at a itself,

$$\exists\delta_2 > 0, \exists M > 0 : 0 < |x - a| < \delta_2 \Rightarrow \big|g(x)\big| \leq M$$

So, if we choose $\varepsilon = \dfrac{\varepsilon}{M}$,

$$\forall\varepsilon > 0, \exists\delta = \min\{\delta_1, \delta_2\} > 0 : 0 < |x - a|\,\delta \Rightarrow \big|f(x)g(x) - 0\big| = |f(x)||g(x)| < \varepsilon_1 M = \varepsilon$$

$\square$

10 Infinite Limits

$$\lim_{x\to a} f(x) = +\infty \Leftrightarrow \forall M > 0, \exists\delta > 0 : 0 < |x - a| < \delta \Rightarrow f(x) > M$$
$$\lim_{x\to a} f(x) = -\infty \Leftrightarrow \forall M < 0, \exists\delta > 0 : 0 < |x - a| < \delta \Rightarrow f(x) < M$$
$$\lim_{x\to+\infty} f(x) = L \Leftrightarrow \forall\varepsilon > 0, \exists M > 0 : x > M \Rightarrow |f(x) - L| < \varepsilon$$
$$\lim_{x\to-\infty} f(x) = L \Leftrightarrow \forall\varepsilon > 0, \exists M > 0 : x > M \Rightarrow |f(x) - L| < \varepsilon$$

11 Known Limits

$$\lim_{x\to+\infty} \left(1 + \frac{1}{x}\right)^x = e$$
$$\lim_{x\to-\infty} \left(1 + \frac{1}{x}\right)^x = e$$
$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

12 Examples

Example 1. Find

$$\lim_{x\to 0} \frac{\tan 2x}{x}$$

Solution.

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan 2x}{x} &= \lim_{x\to 0} \frac{\frac{\sin 2x}{\cos 2x}}{x} \\
&= \lim_{x\to 0} \frac{\sin 2x}{2x} \frac{2}{\cos x} \\
&= \lim_{x\to 0} \frac{\sin 2x}{2x} \lim_{x\to 0} \frac{2}{\cos x} \\
&= 1 \cdot 2 \qquad\qquad = 2
\end{aligned}$$

Part IV
Derivatives

13 Derivative of a Function

13.1 Algebraic Definition

Definition 24. If there exists

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = L$$

then the limit is called the derivative of f at x_0.

13.2 Geometrical Interpretation

The slope of $\overleftrightarrow{AB}$ is

$$\tan \alpha = \frac{\Delta y}{\Delta x}$$

When $\Delta x \to 0$, $\overleftrightarrow{AB}$ tends to a straight line which is called the tangent to $y = f(x)$ at $(x_0, f(x_0))$.
The derivative is the slope of the tangent to $y = f(x)$ at $(x_0, f(x))$.

13.3 Notation

$$L = f'(x_0) = \frac{\mathrm{d}f(x_0)}{\mathrm{d}x} = \frac{\mathrm{d}y(x_0)}{\mathrm{d}x} = Df(x_0)$$

14 Derivative Function

If we calculate the derivative of $f(x)$ at any possible x , we get the derivative function.

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x} = Df$$

15 The Tangent Line

$$y - f(x_0) = f'(x_0)(x - x_0)$$

16 The Normal Line

$$\begin{aligned} y - f(x_0) &= -\frac{1}{f'(x_0)}(x - x_0) && ; f'(x_0) \neq 0 \\ x &= x_0 && ; f'(x_0) = 0 \end{aligned}$$

16.1 Proofs of Standard Derivatives

16.1.1 $y = f(x) = c$

$$\begin{aligned} f'(x) &= \lim_{x\to 0} \frac{f(x+\Delta x)}{\Delta x} \\ &= \lim_{x\to 0} \frac{c-c}{\Delta x} \\ &= 0 \end{aligned}$$

16.1.2 $y = f(x) = x^n; n \in \mathbb{N}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x\to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \\ a^n - b^n &= (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) \\ \therefore f'(x) &= \lim_{\Delta x\to 0} \frac{\Delta x((x+\Delta x)^{n-1} + (x+\Delta x)^{n-2}x + \cdots + (x+\Delta x)x^{n-2} + x^{n-1}}{\Delta x} \\ &= \lim_{\Delta x\to 0} (x+\Delta x)^{n-1} + (x+\Delta x)^{n-2}x + \cdots + (x+\Delta x)x^{n-2} + x^{n-1} \\ &= x^{n-1} + x\cdot x^{n-2} + \cdots + x^{n-2}\cdot x + x^{n-1} \\ &= nx^{n-1} \end{aligned}$$

16.1.3 $y = f(x) = x^{-n}; n \in \mathbb{N}, x \neq 0$

$$\begin{aligned} f'(x) &= \lim_{\Delta x\to 0} \frac{(x+\Delta x)^{-n} - x^{-n}}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\dfrac{1}{(x+\Delta x)^n} - \dfrac{1}{x^n}}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{x^n - (x+\Delta x)^n}{\Delta x(x^n(x+\Delta x)^n)} \\ &= \frac{-nx^{n-1}}{x^n x^n} \end{aligned}$$

16.1.4 $y = f(x) = \sin x$

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2\sin\dfrac{\Delta x}{2}\cos(x+\dfrac{\Delta x}{2})}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sin\dfrac{\Delta x}{2}}{\dfrac{\Delta x}{2}} \cos\left(x + \frac{\Delta x}{2}\right) \\
&= \cos x
\end{aligned}
$$

Theorem 3. *Theorem: If $\exists f'(x)$, $\exists g'(x)$ and c is a constant, then,*

$$
\begin{aligned}
(cf(x))' &= cf'(x) \\
(f(x) \pm g(x))' &= f'(x) \pm g'(x) \\
(f(x)g(x))' &= f'(x)g(x) + f(x)g'(x) \\
\left(\frac{f(x)}{g(x)}\right)' &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}
\end{aligned}
$$

Proof.

$$
\begin{aligned}
(f(x)g(x))' &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(f(x+\Delta x)g(x+\Delta x) - f(x)g(x+\Delta x)) + (f(x)g(x+\Delta x) - f(x)g(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(f(x+\Delta x) - f(x))g(x+\Delta x)}{\Delta x} + \frac{f(x)((g(x+\Delta x) - g(x))}{\Delta x} \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}
$$

$\square$

Theorem 4. *Let f be defined on an open interval about x_0. Then, $f(x)$ is differentiable at x_0, iff $\exists A \in \mathbb{R}$ and $\exists \alpha(\Delta x)$, with $\lim_{\Delta x \to 0} \alpha(\Delta x) = 0$, s.t. $\dfrac{\Delta y}{\Delta x} = A + \alpha(\Delta x); A = f'(x_0)$*

Theorem 5. *If $y = f(x)$ is differentiable at x_0, then, $f(x)$ is continuous at x_0.*

Proof.

$$
\begin{aligned}
\exists f'(x_0) \Rightarrow \frac{\Delta y}{\Delta x} &= f'(x_0) + \alpha(\Delta x) \\
\therefore f(x_0 + \Delta x) - f(x_0) = \Delta y &= \Delta x(f'(x_0) + \alpha(\Delta x) \\
\therefore f(x_0 + \Delta x) &= f(x_0) + \Delta x(f'(x_0) + \alpha(\Delta x)) \\
\therefore \lim_{\Delta x \to 0} f(x_0 + \Delta x) &= \lim_{\Delta x \to 0} f(x_0) + \Delta x(f'(x_0) + \alpha(\Delta x)) \\
&= x_0
\end{aligned}
$$

Remark 1. The converse of this theorem is not true.

$\square$

Theorem 6 (Derivative of Inverse Functions)**.** *Let $f(x)$ be invertible and continuous in an open interval about x_0. If $\exists f'(x_0) \neq 0$, then, the inverse function $x = g(y)$ is differentiable at $y = f(x_0)$ and*

$$g'(y_0) = \frac{1}{f'(x_0)}$$

16.2 Examples

Example 2. Find the derivative of

$$y = f(x) = \tan x$$

Solution.

$$\begin{aligned} y = f(x) &= \tan x \\ \therefore (\tan^{-1})'y &= \frac{1}{\tan' x} \\ &= \frac{1}{\frac{1}{\cos^2 x}} \\ &= \frac{1}{1+\tan^2 x} \\ &= \frac{1}{1+y^2} \end{aligned}$$

Similarly,

$$(\cot^{-1})'x = -\frac{1}{1+x^2}$$

Theorem 7 (Chain Rule)**.** *Let $y = f(u)$ be differentiable at u_0, and $u = g(x)$ be differentiable at x_0, s.t. $u_o = g(x_0)$. Then, $y = f(g(x))$ is differentiable at x_0, and,*

$$y'(x_0) = f'(u_0) \cdot g'(x_0)$$

Proof.

$$g'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

Therefore, by Theorem 2,

$$\begin{aligned} \therefore \frac{\Delta u}{\Delta x} &= g'(x_0) + \alpha_1(\Delta x); \alpha_1(\Delta x) \to 0 \text{ if } \Delta x \to 0 \\ \therefore \frac{\Delta y}{\Delta u} &= f'(u_0) + \alpha_2(\Delta u); \alpha_1(\Delta u) \to 0 \text{ if } \Delta u \to 0 \end{aligned}$$

Therefore,

$$\begin{aligned}
\Delta u &= (g'(x_0) + \alpha_1)\Delta x \\
\Delta y &= (f'(u_0) + \alpha_2)\Delta u \\
\therefore \Delta y &= (f'(u_0) + \alpha_2)(g'(x_0) + \alpha_1)\Delta x \\
\therefore \frac{\Delta y}{\Delta x} &= (f'(u_0) + \alpha_2)(g'(x_0) + \alpha_1)
\end{aligned}$$

$$\begin{aligned}
\Delta x \to 0 &\Rightarrow \Delta u \to 0, \alpha_1 \to 0 \\
&\Rightarrow \alpha_2 \to 0
\end{aligned}$$

Substituting,

$$y'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \big(f'(u_0) + a_2\big)\big(g'(x_0) + \alpha_1\big) = f'(u_0) \cdot g'(x_0)$$

□

Theorem 8 (Fermat Theorem)**.** *Let $f(x)$ be defined on an open interval (a, b) and differentiable at $x_0 \in (a, b)$. If $f(x)$ has its extremum at x_0, then, $f'(x_0) = 0$*

Proof. Assume that $f(x_0)$ is the maximum value of $f(x)$ on (a, b). Then, $\forall \Delta x, f(x_0 + \Delta x) \le f(x_0)$.
Case 1. $\Delta x > 0$

$$\begin{aligned}
\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} &\le 0 \\
\therefore \text{RHD} = \lim_{\Delta x \to 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} &\le 0
\end{aligned}$$

Case 2. $\Delta x < 0$

$$\begin{aligned}
\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} &\ge 0 \\
\therefore \text{LHD} = \lim_{\Delta x \to 0^-} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} &\ge 0
\end{aligned}$$

$$\begin{aligned}
\exists f'(x_0) &\Rightarrow \text{LHD} = \text{RHD} \\
\therefore 0 &\le f'(x_0) \le 0 \\
\therefore f'(x_0) &= 0
\end{aligned}$$

□

Theorem 9 (Rolle's Theorem)**.** *Let $f(x)$ be defined on $[a, b]$, s.t.*

1. *f is continuous on $[a, b]$*

2. f is differentiable on (a,b)

3. $f(a) = f(b)$

Then, $\exists c \in (a,b)$, s.t. $f'(c) = 0$.

Proof. By Weirstrauss Theorem, as $f(x)$ is continuous on $[a,b]$, $f(x)$ has its maximum M and minimum m on $[a,b]$.
Case 3. $m = M$

$$\begin{aligned} f(x) &= \text{constant} \\ \therefore f'(x) &= 0 \text{ on } [a,b] \end{aligned}$$

Case 4. $m < M$
Atleast one of m and M must be in (a,b), otherwise $f(a) \neq f(b)$, which contradicts (3).
Let $M = c \in (a,b)$. Therefore, by Theorem 8, $f'(c) = 0$

□

17 Lagrange Theorem

Let $f(x)$ be defined on $[a,b]$, s.t.

1. f is continuous on $[a,b]$
2. f is differentiable on (a,b)

Then, $\exists c \in (a,b)$, s.t. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

18 Theorem

Let $f(x)$ be continuous on $(x_0 - \delta, x_0 + \delta)$ and differentiable on $(x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$.
If $\lim\limits_{x \to x_0^+} f'(x) = \lim\limits_{x \to x_0^-} f'(x) = L$, then, $\exists f'(x_0) = L$.