DIFFERENTIAL AND INTEGRAL METHODS - EXERCISE 10

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(1). Differentiate the following functions using the fundamental theorem of calculus:

(a). $\int\limits_{0}^{x} \sin(t^2) \,\mathrm{d}t$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int\limits_{0}^{x} \sin(t^2) \,\mathrm{d}t\right) = \sin(x^2)$$

(b). $\int\limits_{x^2}^{x^3} \sqrt{1+t^2} \,\mathrm{d}t$.

$$\begin{aligned}
\int\limits_{x^2}^{x^3} \sqrt{1+t^2} \,\mathrm{d}t &= \int\limits_{x^2}^{a} \sqrt{1+t^2} \,\mathrm{d}t + \int\limits_{a}^{x^3} \sqrt{1+t^2} \,\mathrm{d}t \\
&= -\int\limits_{a}^{x^2} \sqrt{1+t^2} \,\mathrm{d}t + \int\limits_{x^2}^{x^3} \sqrt{1+t^2} \,\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\int\limits_{x^2}^{x^3} \sqrt{1+t^2} \,\mathrm{d}t\right) &= -\sqrt{1+{x^2}^2} \cdot 2x + \sqrt{1+{x^2}^2} \cdot 3x^2 \\
&= \sqrt{1+x^4}(3x^2-2x)
\end{aligned}$$

(2). Calculate the volume of the body obtained by rotating the upper half of the circle $y = \sqrt{r^2-x^2}$ around the x-axis.

$$\begin{aligned}
V &= \pi \int\limits_{-r}^{r} (f(x))^2 \,\mathrm{d}x \\
&= \pi \int\limits_{-r}^{r} (r^2-x^2) \,\mathrm{d}x \\
&= \pi r^2 x - \pi \frac{x^3}{3}\Big|_{-r}^{r} \\
&= \pi r^3 - (-\pi r^3) - \left(\pi\frac{r^3}{3} - \pi\frac{-r^3}{3}\right) \\
&= 2\pi r^3 - \pi\frac{2r^3}{3} \\
&= \frac{4}{3}\pi r^3
\end{aligned}$$

(3). Calculate the volume of the body obtained by rotating the upper half of the circle $y = \sqrt{r^2 - x^2}$ around the y-axis.

$$
\begin{aligned}
y &= \sqrt{r^2 - x^2} \\
\therefore x &= \sqrt{r^2 - y^2} \\
\therefore V &= \pi \int_0^r (f(y))^2 \,\mathrm{d}y \\
&= \pi \int_0^r (r^2 - y^2) \,\mathrm{d}y \\
&= \pi r^2 x - \pi \frac{y^3}{3} \bigg|_0^r \\
&= \pi r^3 - \pi \frac{r^3}{3} \\
&= \pi r^3 - \pi \frac{r^3}{3} \\
&= \frac{2}{3} \pi r^3
\end{aligned}
$$

(4). Calculate the volume of the rotation body, obtained by rotating the area bounded by $f(x) = x^2$, $g(x) = \sqrt{x}$ around the x-axis.

The graphs of $y = f(x)$ and $f = g(x)$ intersect at $(0,0)$ and $(1,1)$.

$$
\begin{aligned}
V &= \left| \pi \int_0^1 (f(x))^2 \,\mathrm{d}x - \pi \int_0^1 (g(x))^2 \,\mathrm{d}x \right| \\
&= \pi \int_0^1 x \,\mathrm{d}x - \pi \int_0^1 x^4 \,\mathrm{d}x \\
&= \pi \frac{1^2}{2} - \pi \frac{1^5}{5} \\
&= \pi \frac{1}{2} - \pi \frac{1}{5} \\
&= \frac{3\pi}{10}
\end{aligned}
$$

(5). Calculate the improper integral $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2}$

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2} &= \int_{-\infty}^{0} \frac{\mathrm{d}x}{1+x^2} + \int_{0}^{\infty} \frac{\mathrm{d}x}{1+x^2} \\
&= \lim_{t\to-\infty} \int_{t}^{0} \frac{\mathrm{d}x}{1+x^2} + \lim_{u\to\infty} \int_{0}^{u} \frac{\mathrm{d}x}{1+x^2} \\
&= \lim_{t\to-\infty} \tan^{-1} x\Big|_t^0 + \lim_{u\to\infty} \tan^{-1} x\Big|_0^u \\
&= \lim_{t\to-\infty} -\tan^{-1} t + \lim_{u\to\infty} \tan^{-1} u \\
&= \frac{\pi}{2} + \frac{\pi}{2} \\
&= \pi
\end{aligned}$$

(6). Check convergence of the following integrals:

(a). $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2}\,\mathrm{d}x.$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2}\,\mathrm{d}x = \int_{-\infty}^{-1} \frac{\sin x}{x^2}\,\mathrm{d}x + \int_{-1}^{0} \frac{\sin x}{x^2}\,\mathrm{d}x + \int_{0}^{1} \frac{\sin x}{x^2}\,\mathrm{d}x + \int_{1}^{\infty} \frac{\sin x}{x^2}\,\mathrm{d}x$$

$$\frac{\sin x}{x^2}\,\mathrm{d}x \le \frac{1}{x^2}$$

Therefore, as $\frac{1}{x^2}$ converges in $(1,\infty)$ and $(-\infty,-1)$, $\frac{\sin x}{x^2}$ converges in $(1,\infty)$ and $(-\infty,-1)$.

However, the limit $\displaystyle\lim_{b\to0^+} \int_{b}^{1} \frac{\sin x}{x^2}\,\mathrm{d}x$ does not exist. Therefore the integral diverges.

(b). $\displaystyle\int_{0}^{\infty} \frac{\mathrm{d}x}{\sqrt{3x^4+x^2+x}}\,\mathrm{d}x.$

$$\int_{0}^{\infty} \frac{\mathrm{d}x}{\sqrt{3x^4+x^2+x}}\,\mathrm{d}x = \int_{0}^{1} \frac{\mathrm{d}x}{\sqrt{3x^4+x^2+x}}\,\mathrm{d}x + \int_{1}^{\infty} \frac{\mathrm{d}x}{\sqrt{3x^4+x^2+x}}\,\mathrm{d}x$$

$$\frac{\mathrm{d}x}{\sqrt{3x^4+x^2+x}} \le \frac{1}{\sqrt{x}}$$

$$\lim_{x\to0^+} \frac{\sqrt{x}}{\sqrt{3x^4+x^2+x}} = 1$$

Therefore, by the second comparison test, $\displaystyle\int_{0}^{1} \frac{\mathrm{d}x}{\sqrt{3x^4+x^2+x}}\,\mathrm{d}x$ converges.

$$\frac{\mathrm{d}x}{\sqrt{3x^4+x^2+x}} \le \frac{1}{x^2}$$

Therefore, by the first comparison test, as $\frac{1}{x^2}$ converges, $\int\limits_1^\infty \frac{\mathrm{d}x}{\sqrt{3x^4+x^2+x}}\,\mathrm{d}x$ converges.

Hence, $\int\limits_0^\infty \frac{\mathrm{d}x}{\sqrt{3x^4+x^2+x}}\,\mathrm{d}x$ converges.

(c). $\int\limits_1^\infty \frac{e^{-x}\sin 2x}{\sqrt{1+x^4}}\,\mathrm{d}x$.

$$\frac{e^{-x}\sin 2x}{\sqrt{1+x^4}}\,\mathrm{d}x \le \frac{1}{x^2}$$

Therefore, by the first comparison test, as $\frac{1}{x^2}$ converges, $\int\limits_1^\infty \frac{e^{-x}\sin 2x}{\sqrt{1+x^4}}\,\mathrm{d}x$ converges.

(d). $\int\limits_0^\infty \frac{\arctan x}{\sqrt{x+x^3}}\,\mathrm{d}x$.

$$\int\limits_0^\infty \frac{\arctan x}{\sqrt{x+x^3}}\,\mathrm{d}x = \int\limits_0^1 \frac{\arctan x}{\sqrt{x+x^3}}\,\mathrm{d}x + \int\limits_1^\infty \frac{\arctan x}{\sqrt{x+x^3}}\,\mathrm{d}x$$

$$\begin{aligned}
\frac{\arctan x}{\sqrt{x+x^3}} &= \frac{\arctan x}{\sqrt{x}\sqrt{1+x^2}} \\
\therefore \frac{\frac{\arctan x}{\sqrt{x+x^3}}}{\frac{\arctan x}{\sqrt{x}}} &= \frac{1}{\sqrt{1+x^2}} \\
\lim_{x\to 0^+} \frac{1}{\sqrt{1+x^2}} &= 1 \\
\frac{\arctan x}{\sqrt{x}} &\le \frac{1}{\sqrt{x}}
\end{aligned}$$

Therefore, as $\int\limits_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$ converges, $\int\limits_0^1 \frac{\arctan x}{\sqrt{x}}\,\mathrm{d}x$ converges. Hence, $\int\limits_0^1 \frac{\arctan x}{\sqrt{x+x^3}}\,\mathrm{d}x$ converges.

(e). $\int\limits_0^1 \frac{\arctan x}{x^2}\,\mathrm{d}x$.

$$\begin{aligned}
\frac{\arctan x}{x^2} &\le \frac{1}{x^2} \\
\lim_{x\to 0} \frac{\frac{\arctan x}{x^2}}{\frac{1}{x^2}} &= 1
\end{aligned}$$

Therefore, $\int\limits_0^1 \frac{\arctan x}{x^2}$ and $\int\limits_0^1 \frac{1}{x^2}$ converge or diverge simultaneously.

Therefore, as $\int\limits_0^1 \frac{1}{x^2}\,\mathrm{d}x$ diverges, $\int\limits_0^1 \frac{\arctan x}{x^2}\,\mathrm{d}x$ also diverges.

(7). $\Gamma(x) = \int\limits_0^\infty t^{x-1} e^{-t} \, \mathrm{d}t$

(a).

$$\begin{aligned}
\Gamma(1) &= \int\limits_0^\infty t^0 e^{-t} \, \mathrm{d}t \\
&= \lim_{a \to \infty} \int\limits_0^a e^{-t} \, \mathrm{d}t \\
&= \lim_{a \to \infty} -e^{-t}\Big|_0^a \\
&= \lim_{a \to \infty} -e^{-a} - (-e^0) \\
&= \lim_{a \to \infty} -e^{-a} + 1 \\
&= 1
\end{aligned}$$