Lecture 9

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Contents

1 Local Minimum and Maximum

Definition 1. Let $f(x)$ be defined on an open interval about x_0. We say that $f(x)$ has a local minimum (or local maximum) at x_0, if there exists an open interval about x_0, s.t. $f(x) \geq f(x_0)$ (or $f(x) \leq f(x_0)$), $\forall x$ in the interval. x_0 which is a local minimum or maximum is called a local extremum.

Definition 2. Let $f(x)$ be defined on an open interval about x_0. We say that x_0 is a critical point of f, if $f'(x_0) = 0$ or $\nexists f'(x_0)$.

Theorem 1 (Fermat Theorem - Necessary Condition for Extrema Existence)**.** *If $\exists f'(x_0)$ where x_0 is a local extremum, then $f'(x_0) = 0$.*

Remark 1. By Fermat Theorem - Necessary Condition for Extrema Existence, any local extremum point is a critical point, but the converse is not true.

Theorem 2 (Sufficient Condition for Extrema Existence)**.** *If $\exists f'(x)$ and $\exists f''(x)$ are continuous on an open interval about x_0 and $f'(x_0) = 0$, then,*

1. *x_0 is a local maximum if $f''(x_0) < 0$*
2. *x_0 is a local minimum if $f''(x_0) > 0$*
3. *there is no rule if $f''(x_0) = 0$*

Proof. If $f''(x_0) < 0$ and $f''(x)$ is continuous on an open interval about x_0, then, $f''(x) < 0$ on some interval about x_0.

By **??** with $n = 1$

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(c)}{2!}(x - x_0)^2$$

where c is between x and x_0.

$$\therefore f(x) = f(x_0) + \frac{f''(c)}{2}(x - x_0)^2$$
$$\therefore f(x) - f(x_0) = \frac{f''(c)}{2}(x - x_0)^2$$

$f'(c) < 0$ and $(x - x_0)^2 \geq 0$

$$\therefore f(x) - f(x_0) \leq 0$$
$$\therefore f(x) \leq f(x_0)$$

Similarly for the remaining cases. ☐

Theorem 3. *Let x_0 be a critical point of $f(x)$ and let $f(x)$ be continuous at x_0, and differentiable on an open interval about x_0 except possibly at x_0 itself. Then*

1. *If $f'(x)$ changes the sign from negative to positive at x_0, then x_0 is a local minimum.*
2. *If $f'(x)$ changes the sign from positive to negative at x_0, then x_0 is a local maximum.*
3. *If $f'(x)$ does not change the sign at x_0, then x_0 is not a local extremum.*

2 Absolute or Global Minimum and Maximum

Definition 3. Let $f(x)$ be defined on a domain D. $x_0 \in D$ is called an absolute minimum (or maximum) of $f(x)$ on D if $f(x) \geq f(x_0)$ (or $f(x) \leq f(x_0)$), $\forall x \in D$.

Theorem 4. *Let $f(x)$ be continuous on $[a, b]$. Then $f(x)$ has atleast one absolute maximum and atleast one absolute minimum in $[a, b]$. If x_0 is such a point, then x_0 must be a critical point of $f(x)$ or one of a or b.*

2.1 Algorithm for Finding Maxima and Minima of a Function

Step 1 Find all critical points of $f(x)$ on the domain, excluding the end points.

Step 2 Calculate the values of $f(x)$ at the critical points.

Step 3 Calculate the values of $f(x)$ at the end points of the domain.

Step 4 Select the maximum and minimum values from Step 2 and Step 3

2.2 Examples

Example 1. Find maximum and minimum values of

$$f(x) = x^{\frac{5}{3}} + 5x^{\frac{2}{3}}$$

on $[-1, 1]$

Solution.

$$\begin{aligned}
f'(x) &= \frac{5}{3}x^{\frac{2}{3}} + \frac{10}{3}x^{-\frac{1}{3}} \\
&= \frac{5}{3}\frac{x+2}{x^{\frac{1}{3}}} \\
f'(x) = 0 &\iff x = -2 \\
f'(0) &= \lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \Delta x^{\frac{2}{3}} + \frac{5}{\Delta x^{\frac{1}{3}}} \\
\therefore \nexists f'(0)
\end{aligned}$$

Therefore, $x = 0$ is a critical point

$$\begin{aligned} \therefore f(0) &= 0 \\ f(-1) &= 4 \\ f(1) &= 6 \\ \therefore \min_{[-1,1]} f(x) &= 0 \\ \therefore \max_{[-1,1]} f(x) &= 6 \end{aligned}$$

3 Convexity and Inflection Points

Definition 4. Let $f(x)$ be differentiable at x_0. $f(x)$ is said to be convex upwards (or downwards) at x_0, if there exists an open interval about x_0 in which the graph of $y = f(x)$ is below (or above) the line tangent to $y = f(x)$ at $(x_0, f(x_0))$, i.e. $\exists \delta > 0$, s.t.

$$f(x) \leq f(x_0) + f'(x_0)(x - x_0); \forall x \in (x_0 - \delta, x_0 + \delta)$$
$$\Big(\text{ or } f(x) \geq f(x_0) + f'(x_0)(x - x_0); \forall x \in (x_0 - \delta, x_0 + \delta)\Big)$$

Theorem 5. *Let $f(x)$ be twice differentiable on the interval (a, b).*

1. *If $f''(x) > 0, \forall x \in (a, b)$, then $f(x)$ is convex downwards on (a, b).*
2. *If $f''(x) < 0, \forall x \in (a, b)$, then $f(x)$ is convex upwards on (a, b).*

Definition 5. If $f(x)$ is continuous on an open interval about x_0 and differentiable at x_0 in a wide sense (i.e. the derivative may be infinite), we say that x_0 is an inflection point of $f(x)$ if there exists an interval $(x_0 - \delta, x_0 + \delta)$, s.t. the function changes its convexity passing through x_0.

Remark 2. At an inflection point x_0, there is no restriction of the value of $f'(x)$. It may be 0, a finite number or ∞.

Theorem 6. *If x_0 is an inflection point of $f(x)$ and $\exists f''(x)$ on an open interval about x_0 and $f''(x)$ is continuous at x_0, then $f''(x_0) = 0$.*

4 Asymptotes

Definition 6. Let $f(x)$ be defined on $(a - \delta)$ or $(a, a + \delta)$ or $(a - \delta, a + \delta) - \{a\}$ for $\delta > 0$. If atleast one of $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ is equal to $\pm\infty$, then the straight line $x = a$ is said to be a vertical asymptote of $f(x)$.

Definition 7. The straight line $y = ax + b$ is called an oblique asymptote of a function $y = f(x)$ at $+\infty$ (or $-\infty$), if

$$\lim_{x \to +\infty} (f(x) - (ax + b)) = 0$$
$$\Big(\text{ or } \lim_{x \to -\infty} (f(x) - (ax + b)) = 0\Big)$$

Theorem 7. *Let $f(x)$ be defined on $(c, +\infty)$. If there exist the limits $a_1 = \lim\limits_{x \to +\infty} \frac{f(x)}{x}$ and $b_1 = \lim\limits_{x \to +\infty} (f(x) - a_1 x)$, then the straight line $y = a_1 x + b_1$ is a unique oblique asymptote of $f(x)$ at $+\infty$.*

Let $f(x)$ be defined on $(-\infty, c)$. If there exist the limits $a_2 = \lim\limits_{x \to -\infty} \frac{f(x)}{x}$ and $b_2 = \lim\limits_{x \to -\infty} (f(x) - a_2 x)$, then the straight line $y = a_2 x + b_2$ is a unique oblique asymptote of $f(x)$ at $-\infty$.