Lecture 22

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1 The Fundamental Theorem of Line Integrals

Theorem 1 (The Fundamental Theorem of Line Integrals). *Let C be a smooth curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ given parametrically by $\overline{r}(t)$, $t : a \to b$. Let f be a continuous function of (x, y) or (x, y, z) respectively, on C and ∇f be a continuous vector function in a connected domain D which contains C. Then*

$$\begin{aligned} W &= \int\limits_C \nabla f \cdot \hat{T} \, \mathrm{d}s \\ &= f(r(b)) - f(r(a)) \\ &= f(B) - f(A) \end{aligned}$$

Remark 1. $\overline{F} = \nabla f$ is called a conservative vector field. The line integral of a vector field does not depend on the path, but only on the endpoints. The work done by it over a closed path is 0.

2 Application of Line Integrals

Example 1. If $\overline{r}(t), t : a \to b$ represents the position of a particle with mass m with respect to time t over a path C, find the work done between time a and b.

Solution.

$$
\begin{aligned}
W &= \int\limits_C \overline{F} \cdot \hat{T} \, \mathrm{d}s \\
&= \int\limits_a^b \overline{F}\left(\overline{r}(t)\right) \cdot \left(\overline{r}(t)\right)' \\
&= m \int\limits_a^b \left(\overline{r}(t)\right)'' \cdot \left(\overline{r}(t)\right)' \, \mathrm{d}t \\
&= \frac{m}{2} \int\limits_a^b \left(\left(\overline{r}(t)\right)' \cdot \left(\overline{r}(t)\right)'\right)' \mathrm{d}t \\
&= \frac{m}{2} \int\limits_a^b \left(\left|\left(\overline{r}(t)\right)'\right|^2\right)' \mathrm{d}t \\
&= \frac{m}{2} \left|\left(\overline{r}(t)\right)'\right|^2 \Bigg|_a^b \\
&= \frac{m\left|\overline{v}(b)\right|^2}{2} - \frac{m\left|\overline{v}(a)\right|^2}{2}
\end{aligned}
$$

3 Conservative Vector Field in a Plane

Definition 1 (Simple curve)**.** A curve C is called a simple curve if it does not intersect itself.

Definition 2 (Domain)**.** A domain $D \subset \mathbb{R}^2$ is called connected if for any two points from D, the is a path C which connects the points and remains in D.

Definition 3 (Simple connected domain)**.** A connected domain $D \subset \mathbb{R}^2$ is called simple connected if any simple closed curve from D contains inside itself only points in D.

Theorem 2. *If*

$$\overline{F}(x, y) = \big(P(x, y), Q(x, y)\big) = \nabla f(x, y)$$

is the conservative vector field in a connected domain D, where there exist first order partial derivatives of P and Q continuous in D, then

$$P_y(x, y) = Q_x(x, y) \qquad\qquad \forall (x, y) \in D$$

Proof. As $\overline{F} = \nabla f$,

$$(P, Q) = (f_x, f_y)$$

Therefore,

$$\begin{aligned} f_{xy} &= P_y \\ f_{yx} &= Q_x \\ \therefore P_y &= Q_x \end{aligned}$$

$\square$

Theorem 3. *Let*

$$\overline{F}(x, y) = \big(P(x, y), Q(x, y)\big)$$

be a vector field in an open, simple connected domain D. If there exist first order partial derivatives of P and Q which are continuous in D, and

$$P_y(x, y) = Q_x(x, y) \qquad \forall (x, y) \in D$$

Then, $\exists f(x, y)$ s.t.

$$\overline{F}(x, y) = \nabla f(x, y)$$

i.e. $\overline{F}$ is a conservative vector field.

Example 2. If

$$\overline{F}(x, y) = (3 + 2xy, x^2 - 3y^2)$$

a conservative vector field? If yes, find $f(x, y)$, s.t.

$$\overline{F}(x, y) = \nabla f(x, y)$$

and find the work done by the force $\overline{F}(x, y)$ over the curve

$$\overline{r}(t) = (e^t \sin t, e^t \cos t) \qquad t : 0 \to \pi$$

Solution.

$$\begin{aligned} P(x, y) &= 3 + 2xy \\ \therefore P_y &= 2x \\ Q(x, y) &= x^2 - 3y^2 \\ \therefore Q_x &= 2x \\ \therefore P_y &= Q_x \end{aligned}$$

Therefore, $\overline{F}(x, y)$ is a conservative vector field.

$$\begin{aligned} f_x &= P \\ &= 3 + 2xy \\ \therefore f &= 3x + x^2y + c(y) \\ \therefore f_y &= x^2 + c'(y) \end{aligned}$$

Comapring with $f_y = Q$,

$$\begin{aligned} c'(y) &= -3y^3 \\ \therefore c(y) &= -y^3 + c \\ \therefore f(x, y) &= 3x + x^2y - y^3 + c \end{aligned}$$

By the definition of work,

$$\begin{aligned} W &= \int\limits_C \overline{F} \cdot \hat{T} \, \mathrm{d}s \\ &= \int\limits_a^b \big(P(\overline{r}(t))x'(t) + Q(\overline{r}(t))y'(t)\big) \, \mathrm{d}t \end{aligned}$$

Alternatively, using The Fundamental Theorem of Line Integrals,

$$\begin{aligned} W &= \int\limits_C \overline{F} \cdot \hat{T} \, \mathrm{d}s \\ &= \int\limits_C \nabla d \cdot \hat{T} \, \mathrm{d}s \\ &= f(\overline{r}(\pi)) - f(\overline{r}(0)) \\ &= f(0, -e^{\pi}) - f(0, 1) \\ &= -(-e^{\pi})^3 - (-1)^3 \\ &= e^{3\pi} + 1 \end{aligned}$$

Definition 4 (Curve with positive orientation)**.** A simple closed curve C is called a curve with a positive orientation, or with anti-clockwise orientation if the domain D bounded by C always remains on the left when we circulate over C by $\overline{r}(t), t : a \to b$.