Lecture 19

Aakash Jog

Sunday 4^{th} January, 2015

Contents

1 Lagrange Multipliers

Assume $f(x,y,z)$ and $g(x,y,z)$ to be differentiable. In order to find maximum and minimum values of $f(x,y,z)$ subject to constraint $g(x,y,z)=k$, find the values of x, y, z and λ which satisfy

$$\begin{aligned}\nabla f(x,y,z) &= \lambda \nabla g(x,y,z)\\ g(x,y,z) &= k\end{aligned}$$

Calculate the values of f at the corresponding points from the above equations. The largest value is the maximum value of f and the smallest values is the minimum values of f subject to the constraints.

Example 1. A parallelepiped box without a top has to be built using a cardboard with area 12 m^2. Find the box with maximum volume.

Solution.

$$\begin{aligned}V &= xyz\\ S &= 2xz+2yz+xy\\ &= 12\end{aligned}$$

Let

$$\begin{aligned}f(x,y,z) &= xyz\\ g(x,y,z) &= 2xz+2yz+xy\end{aligned}$$

For finding maximum value,

$$\begin{aligned}\nabla f(x,y,z) &= \lambda \nabla g(x,y,z)\\ g(x,y,z) &= k\end{aligned}$$

Therefore,

$$\begin{aligned}f_x &= \lambda g_x\\ f_y &= \lambda g_y\\ f_z &= \lambda g_z\\ g &= k\end{aligned}$$

Therefore,

$$\begin{aligned}yz &= \lambda(2z+y)\\ xz &= \lambda(2z+x)\\ xy &= \lambda(2x+2y)\\ 2xz+2yz+xy &= 12\end{aligned}$$

Solving,

$$\begin{aligned}
x &= 2 \\
y &= 2 \\
z &= 1 \\
\therefore V &= 4\mathrm{m}^3
\end{aligned}$$

2 Double Integrals

2.1 Double Integrals on Rectangular Domains

Definition 1 (Double integral)**.** Consider $z = f(x, y)$ defined on a rectangle $R = [a, b] \times [c, d]$.
Let

$$a = x_0 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$$

$$c = y_0 < \cdots < y_{j-1} < y_j < \cdots < y_n = d$$

Let

$$\begin{aligned}
\Delta x_i &= x_{i-1} \\
\Delta y_i &= y_{j-1} \\
\Delta T &= \max\{\Delta x_i, \Delta y_j\}
\end{aligned}$$

Let

$$P_{ij}{}^* = (x_{ij}{}^*, y_{ij}{}^*) \in R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$

Let

$$\Delta A_{ij} = \Delta x_i \Delta y_j$$

be the area of R_{ij}.
Then, $\displaystyle\sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{ij}{}^*, y_{ij}{}^*) \Delta A_{ij}$ is called the Riemann double integral sum.
The double integral of $z = f(x, y)$ over the domain of definition R is the limit, if it exists, of the Riemann double integral sum, as $\Delta T \to 0$.

$$\iint\limits_R f(x, y)\,\mathrm{d}A = \lim_{\Delta T \to 0} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{ij}{}^*, y_{ij}{}^*) \Delta A_{ij}$$

$$\iint\limits_R f(x, y)\,\mathrm{d}A = \iint\limits_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

2.2 Iterated Integrals

Definition 2. Assume that $z = f(x, y)$ is integrable over $R = [a, b] \times [c, d]$.

Iterated Integrals	Analogy in Differential Calculus
$A(x) = \int\limits_c^d f(x, y)\,\mathrm{d}y$	$f_y(x, y)$
$I_1 = \int\limits_a^b A(x)\,\mathrm{d}x$	$f_{yx}(x, y)$
$B(y) = \int\limits_a^b f(x, y)\,\mathrm{d}x$	$f_x(x, y)$
$I_1 = \int\limits_c^d B(y)\,\mathrm{d}y$	$f_{xy}(x, y)$

Theorem 1 (Fubini Theorem)**.** *If $f(x, y)$ is continuous on R, then the double integral and two iterated integrals exist, and they are equal.*

$$\iint\limits_R f(x, y)\,\mathrm{d}A = \int\limits_a^b \int\limits_c^d f(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int\limits_c^d \int\limits_a^b f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

Remark 1. Existence of two iterated integrals does not necessarily guarantee existence of the double integral.

Example 2. Solve

$$\iint\limits_{[1,2]\times[0,3]} y\cos(x + y^2)\,\mathrm{d}A$$

Solution.

$$
\begin{aligned}
\iint\limits_{[1,2]\times[0,3]} y\cos(x+y^2)\,\mathrm{d}A &= \int\limits_1^2 \int\limits_0^3 y\cos(x+y^2)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int\limits_1^2 \left(\frac{1}{2}\sin(x+y^2)\Big|_0^3\right)\mathrm{d}x \\
&= \frac{1}{2}\int\limits_1^2 \big(\sin(x+9)-\sin x\big)\,\mathrm{d}x \\
&= \frac{1}{2}\big(-\cos(x+9)+\cos x\big)\Big|_1^2 \\
&= \frac{1}{2}(-\cos 11+\cos 2+\cos 10-\cos 1)
\end{aligned}
$$

2.3 Double Integrals on Arbitrary Domains

Definition 3 (Double integral on arbitrary domain)**.** Let D be a bounded and closed domain in $\mathbb{R}^2$ and the function $z = f(x,y)$ be defined on D. Consider a rectangle $R = [a,b]\times[c,d]$, s.t. $D \subset R$. Define a function $F(x,y)$, s.t.

$$
F(x,y) = \begin{cases} f(x,y) & ; \quad (x,y)\in D \\ 0 & ; \quad (x,y)\notin D \end{cases}
$$

If $F(x,y)$ is integrable over R, it is said to be integrable on D

$$
\iint\limits_D f(x,y)\,\mathrm{d}A = \iint\limits_R F(x,y)\,\mathrm{d}A
$$

Definition 4 (Domain of the first kind)**.** A domain D is said to be the domain of the first kind if there exist continuous functions $f_1(x)$ and $f_2(x)$, s.t.

$$
D_I = \{(x,y)|a\le x\le b, f_1(x)\le y\le f_2(x)\}
$$

Theorem 2. *If $f(x,y)$ is continuous in D_I, then*

$$
\iint\limits_R f(x,y)\,\mathrm{d}A = \int\limits_a^b \int\limits_{f_1(x)}^{f_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x
$$

Proof.

$$
\begin{aligned}
\iint\limits_{D_I} f(x,y)\,\mathrm{d}A &= \iint\limits_{R} F(x,y)\,\mathrm{d}A \\
&= \int\limits_a^b \int\limits_c^d F(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int\limits_a^b \left(\int\limits_c^{f_1(x)} F\,\mathrm{d}y + \int\limits_{f_1(x)}^{f_2(x)} F\,\mathrm{d}y + \int\limits_{f_2(x)}^{d} F\,\mathrm{d}y \right) \mathrm{d}x \\
&= \int\limits_a^b \left(\int\limits_c^{f_1(x)} 0\,\mathrm{d}y + \int\limits_{f_1(x)}^{f_2(x)} f\,\mathrm{d}y + \int\limits_{f_2(x)}^{d} 0\,\mathrm{d}y \right) \mathrm{d}x \\
&= \int\limits_a^b \int\limits_{f_1(x)}^{f_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x
\end{aligned}
$$

□

Definition 5 (Domain of the second kind)**.** A domain D is said to be the domain of the second kind if there exist continuous functions $g_1(y)$ and $g_2(y)$, s.t.

$$D_{II} = \{(x,y) | c \le y \le d, g_1(y) \le x \le g_2(y)\}$$

Theorem 3. *If $f(x,y)$ is continuous in D_{II}, then*

$$\iint\limits_{R} f(x,y)\,\mathrm{d}A = \int\limits_c^d \int\limits_{g_1(y)}^{g_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$