Differential and Integral Methods: Compendium

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1 Functions

Definition 1 (Even function)**.**

$$f(-x) = f(x)$$

Definition 2 (Odd function)**.**

$$f(-x) = -f(x)$$

1.1 Hyperbolic Functions

Definition 3 (Hyperbolic functions)**.**

$$\sinh x \doteq \frac{e^x - e^{-x}}{2} \qquad I(\sinh x) = \mathbb{R}$$

$$\cosh x \doteq \frac{e^x + e^{-x}}{2} \qquad I(\cosh x) = [1, \infty)$$

$$\tanh x \doteq \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad I(\tanh x) = (-1, 1)$$

1.1.1 Identities of Hyperbolic Functions

$$\sinh(2x) = 2\sinh x\cosh x$$
$$\cosh^2 x + \sinh^2 x = \cosh(2x)$$
$$\cosh^2 x - \sinh^2 x = 1$$
$$\frac{\cosh(2x)-1}{2} = \sinh^2 x$$
$$\frac{\cosh(2x)+1}{2} = \cosh^2 x$$

1.2 Trigonometric Identities

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$
$$\cot^2 x + 1 = \csc^2 x$$

$$\sin(x \pm y) = \sin x\cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x\cos y \mp \sin x \sin y$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$\sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$$
$$\cos x \cos y = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right)$$
$$\sin x \cos y = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right)$$
$$\cos x \sin y = \frac{1}{2}\left(\sin(x+y) - \sin(x-y)\right)$$

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$
$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$
$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

2 Limits

Definition 4 (Cauchy's definition of a limit of a function)**.**

$\forall \epsilon > 0 \exists \delta > 0 : 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$

Definition 5 (Removable discontinuity point)**.**

$\exists \lim_{x\to a} f(x)$, but either $\lim_{x\to a} f(x) \neq f(a)$ or $\nexists f(a)$

Definition 6 (Discontinuity of first kind)**.**

$\exists \lim_{x\to a^-} f(x), \exists \lim_{x\to a^+} f(x)$, but $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

Definition 7 (Discontinuity of second kind)**.** At least one of the two one-sided limits of f does not exist. (Limits are defined as finite numbers only.)

Theorem 1 (Sandwich Theorem)**.** *Let $f(x), g(x), h(x)$ be defined on an open interval about a, except possibly at a itself. Assume that $\forall x \neq a$ from the interval, it is satisfied that $f(x) \leq g(x) \leq h(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then,*

$$\lim_{x\to a} g(x) = L$$

Theorem 2. *If $\lim_{x\to a} f(x) = 0$ and $g(x)$ is bounded in an open interval about a, except possibly at a itself, then,*

$$\lim_{x\to a}(f(x)g(x)) = 0$$

2.1 Useful Limits

If $\lim_{x\to x_0} g(x) = 0$,

$$\lim_{x\to x_0}(1+g(x))^{\frac{1}{g(x)}} = e$$

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$$
$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$$
$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

3 Derivatives

Definition 8 (Derivative of a function)**.**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} = L$$

Theorem 3 (Derivative of inverse functions)**.**

$$(f^{-1})'(x) = \frac{1}{f'(x)}$$

Theorem 4 (Chain rule)**.**

$$\frac{\mathrm{d}f(g(x))}{\mathrm{d}x} = \frac{\mathrm{d}f(g(x))}{\mathrm{d}g(x)} \cdot \frac{\mathrm{d}g(x)}{\mathrm{d}x}$$

Theorem 5 (Rolle's Theorem)**.** *Let $f(x)$ be defined on $[a, b]$, s.t.*

1. *f is continuous on $[a,b]$*
2. *f is differentiable on (a,b)*
3. *$f(a) = f(b)$*

Then, $\exists c \in (a,b)$, s.t. $f'(c) = 0$.

Theorem 6 (Lagrange Theorem)**.** *Let $f(x)$ be defined on $[a,b]$, s.t.*

1. *f is continuous on $[a,b]$*
2. *f is differentiable on (a,b)*

Then,

$$\exists c \in (a,b),\ s.t.\ f'(c) = \frac{f(b)-f(a)}{b-a}$$

4 Taylor's Formula

Theorem 7 (Taylor's Formula)**.**

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i + \frac{f^{(n)}(c)}{(n+1)!}(x-a)^{n+1}$$

4.1 Common Derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x} x = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \tan x = \sec^2 c$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \sec x = \sec x \tan x$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \csc x = -\csc x \cot x$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \cot x = -\csc^2 c$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x = \frac{1}{1+x^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}x} a^x = a^x \ln a$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \log_a x = \frac{1}{x \ln a}$$

5 Full Investigation of Functions

1. Domain of definition of f
2. Points of intersection of $y = f(x)$ with x-axis and y-axis
3. Symmetry and periodicity
4. Extrema points
5. Monotonicity
6. Convexity
7. Inflection points
8. Asymptotes (vertical and oblique)
9. Graph

Definition 9 (Vertical asymptote)**.** Let $f(x)$ be defined on $(a-\delta)$ or $(a, a+\delta)$ or $(a-\delta, a+\delta) - \{a\}$ for $\delta > 0$. If atleast one of $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ is equal to $\pm\infty$, then the straight line $x = a$ is said to be a vertical asymptote of $f(x)$.

Definition 10 (Oblique asymptote)**.** The straight line $y = ax + b$ is called an oblique asymptote of a function $y = f(x)$ at $+\infty$ (or $-\infty$), if

$$\lim_{x\to+\infty} (f(x) - (ax+b)) = 0$$
$$\left(\text{or } \lim_{x\to-\infty} (f(x) - (ax+b)) = 0\right)$$

Example 1. Investigate

$$y = f(x) = \frac{(x-1)^3}{(x+1)^2}$$

Solution.

$$D(f) = \mathbb{R} - \{-1\}$$

$$\begin{aligned} y &= 0 && \implies x = 1 \\ x &= 0 && \implies y = -1 \end{aligned}$$

The function is not periodic.

$$\begin{aligned} f(-x) &\neq f(x) \\ &\neq -f(x) \end{aligned}$$

Therefore, the function is not symmetric.

$$f'(x) = \frac{(x-1)^2(x+5)}{(x+1)^3}$$

Therefore, $x = -5$ is a local maximum point.
The function is monotonically increasing in $(-\infty, -5) \cup (-1, +\infty)$ and is monotonically decreasing in $(-5, -1)$.

$$f''(x) = \frac{24(x-1)}{(x+1)^4}$$

Therefore, the function is convex upwards in $(-\infty, -1) \cup (-1, 1)$ and convex downwards in $(1, \infty)$.

$$\begin{aligned}\lim_{x \to -1^-} \frac{(x-1)^3}{(x+1)^2} &= \frac{-8}{+0} \\ &= -\infty \\ \lim_{x \to -1^+} \frac{(x-1)^3}{(x+1)^2} &= \frac{-8}{+0} \\ &= -\infty\end{aligned}$$

Therefore, $x = -1$ is a vertical asymptote of $f(x)$.

$$\begin{aligned}a_1 &= \lim_{x \to +\infty} \frac{f(x)}{x} &&= 1 \\ b_1 &= \lim_{x \to +\infty} \big(f(x) - a_1 x\big) &&= -5 \\ a_2 &= \lim_{x \to -\infty} \frac{f(x)}{x} &&= 1 \\ b_2 &= \lim_{x \to -\infty} \big(f(x) - a_1 x\big) &&= -5\end{aligned}$$

Therefore, $y = x - 5$ is an oblique asymptote of the function, at $+\infty$ and $-\infty$.

6 Integration

Definition 11 (Basic rational functions)**.** A simple rational function of one of the following forms is called a basic rational function.

$$\begin{aligned}&\frac{A}{x-\alpha} && ; A, \alpha \in \mathbb{R} \\ &\frac{A}{(x-\alpha)^n} && ; A, \alpha \in \mathbb{R}, n \in \mathbb{N} - \{1\} \\ &\frac{Ax+B}{x^2+px+q} && ; A, B, p, q \in \mathbb{R}, p^2 - 4q < 0 \\ &\frac{Ax+B}{(x^2+px+q)^n} && ; A, B, p, q \in \mathbb{R}, p^2 - 4q < 0, n \in \mathbb{N} - \{1\}\end{aligned}$$

Example 2. Solve $\displaystyle\int \frac{-x+2}{x(x-1)^2}\,\mathrm{d}x$.

Solution.

$$\int \frac{-x+2}{x(x-1)^2}\,\mathrm{d}x = \int \left(\frac{A_1}{x} + \frac{B_1}{x-1} + \frac{B_2}{(x-1)^2}\right)\mathrm{d}x$$

$$\begin{aligned}\frac{-x+2}{x(x-1)^2} &= \frac{A_1(x-1)^2 + B_1(x)(x-1) + B_2 x}{x(x-1)^2} \\ &= \frac{x^2(A_1+B_1) + x(-2A_1 - B_1 + B_2) + A_1}{x(x-1)^2}\end{aligned}$$

Therefore,

$$\begin{aligned}A_1 + B_1 &= 0 \\ -2A_1 - B_1 + B_2 &= -1 \\ A_1 &= 2\end{aligned}$$

Therefore,

$$\begin{aligned}A_1 &= 2 \\ B_1 &= -2 \\ B_2 &= 1\end{aligned}$$

Therefore,

$$\begin{aligned}\int \frac{-x+2}{x(x-1)^2}\,\mathrm{d}x &= \int \left(\frac{2}{x} + \frac{-2}{x-1} + \frac{1}{(x-1)^2}\right)\mathrm{d}x \\ &= 2\ln|x| - 2\ln|x-1| - \frac{1}{x-1} + c \\ &= 2\ln\left|\frac{x}{x-1}\right| - \frac{1}{x-1} + x\end{aligned}$$

Example 3. Solve $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\,\mathrm{d}x$.

Solution.

$$\begin{aligned}\int \frac{2x^2-x+4}{x^3+4x}\,\mathrm{d}x &= \int \frac{2x^2-x+4}{x(x^2+4)}\,\mathrm{d}x \\ &= \int \left(\frac{A}{x} + \frac{Bx+C}{x^2+4}\right)\mathrm{d}x\end{aligned}$$

$$\begin{aligned}\frac{2x^2-x+4}{x^3+4x} &= \frac{A(x^2+4) + (Bx+c)x}{x(x^2+4)} \\ &= \frac{x^2(A+B) + x(C) + 4A}{x(x^2+4)}\end{aligned}$$

Therefore,

$$\begin{aligned}A + B &= 2 \\ C &= -1 \\ 4A &= 4\end{aligned}$$

Therefore,

$$\begin{aligned}A &= 1 \\ B &= 1 \\ C &= -1\end{aligned}$$

Therefore,

$$\begin{aligned}\int \frac{2x^2-x+4}{x(x^2+4)}\,\mathrm{d}x &= \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right)\mathrm{d}x \\ &= \ln|x| \\ &\quad + \int \frac{x}{x^2+4}\,\mathrm{d}x - \int \frac{1}{x^2+4}\,\mathrm{d}x + d \\ &= \ln|x| \\ &\quad + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + d\end{aligned}$$

Theorem 8 (Integration by Parts)**.**

$$\int uv\,\mathrm{d}x = u\int v\,\mathrm{d}x - \int u'\left(\int v\,\mathrm{d}x\right)\mathrm{d}x$$

Theorem 9 (Fundamental Theorem of Calculus, Part 1)**.** *Let $f(x)$ be continuous on (a,b) and let $c \in (a,b)$. Then, the function $F(x) = \int_c^x f(t)\,\mathrm{d}t$ is an anti-derivative function of $f(x)$ on (a,b), i.e.*

$$F'(x) = f(x) \quad ; \quad \forall x \in (a,b)$$

Theorem 10 (Fundamental Theorem of Calculus, Part 2)**.** *Let $f(x)$ be continuous on $[a,b]$ and let $G(x)$ be an arbitrary anti-derivative function of $f(x)$. Then,*

$$\int_a^b f(x)\,\mathrm{d}x = G(b) - G(a)$$

6.1 Common Integrals

$$\int k\,\mathrm{d}x = kx + c$$

$$\int x^n\,\mathrm{d}x = \begin{cases} \frac{1}{n+1}x^{n+1} + c & ; \quad n \neq -1 \\ \ln|x| + c & ; \quad n = -1 \end{cases}$$

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + c$$

$$\int \ln x\,\mathrm{d}x = x\ln x - x + c$$

$$\int e^x\,\mathrm{d}x = e^x + c$$

$$\int \cos x\,\mathrm{d}x = \sin x + c$$

$$\int \sin x\,\mathrm{d}x = -\cos x + c$$

$$\int \sec^2 x\,\mathrm{d}x = \tan x + c$$

$$\int \csc^2 x\,\mathrm{d}x = -\cot x + c$$

$$\int \sec x \tan x\,\mathrm{d}x = \sec x + c$$

$$\int \csc x \cot x\,\mathrm{d}x = -\csc x + c$$

$$\int \tan x\,\mathrm{d}x = \ln|\sec x| + c$$

$$\int \sec x\,\mathrm{d}x = \ln|\sec x + \tan x| + c$$

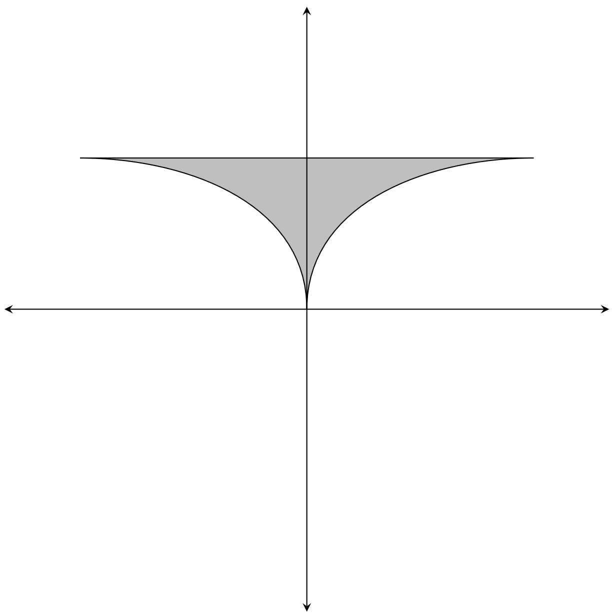
$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}\frac{x}{a} + c$$

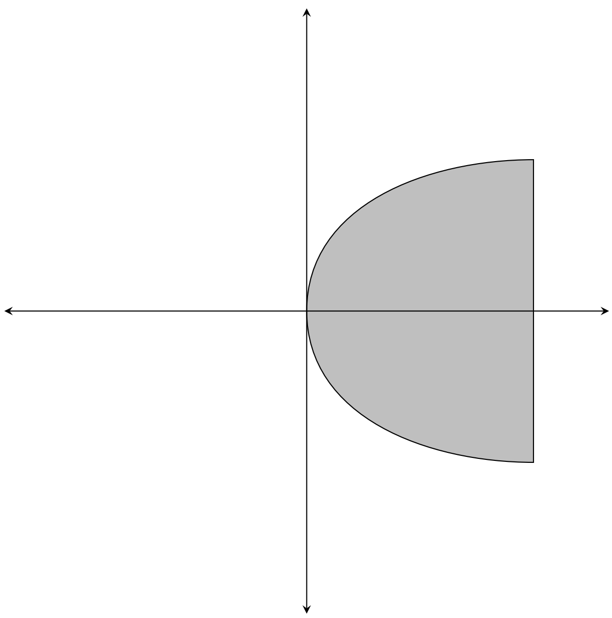
6.2 Length of a Curve

$$l = \int_a^b \sqrt{1+(f'(x))^2}\,\mathrm{d}x$$

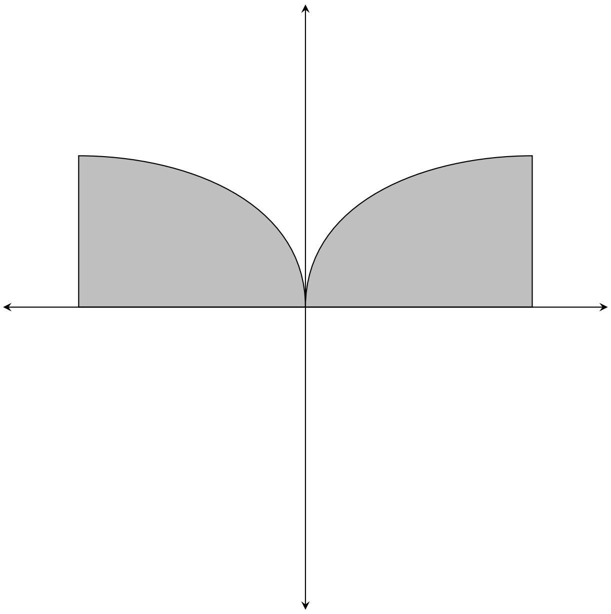
6.3 Volume of Solids of Rotation



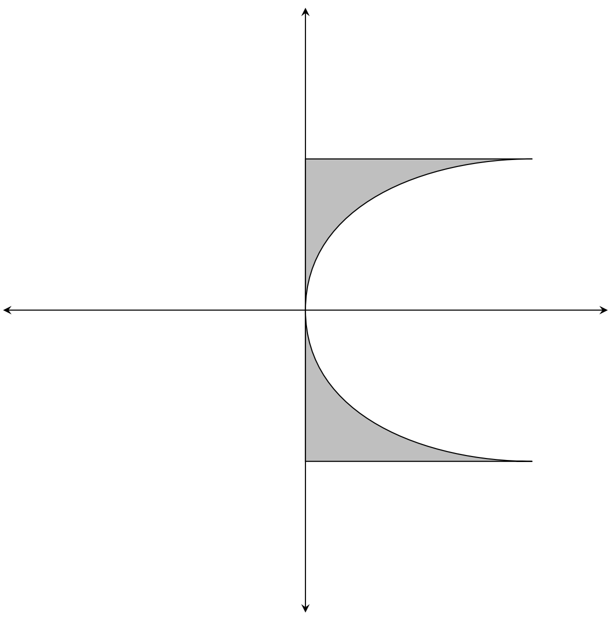
$$V = \pi\int_a^b \big(f(y)\big)^2\,\mathrm{d}y$$



$$V = \pi\int_a^b \big(f(x)\big)^2\,\mathrm{d}x$$



$$V = 2\pi \int_a^b x f(x)\,\mathrm{d}x$$



$$V = 2\pi \int_a^b y f(y)\,\mathrm{d}y$$

6.4 Improper Integrals

6.4.1 Direct Comparison Tests

Theorem 11 (First comparison test)**.** *Let $f(x)$ and $g(x)$ be two functions defined on $[a, +\infty)$ and Riemann integrable over $[a, t], \forall t \geq a$. Assume that $\exists b \geq a$, s.t. $f(x) \geq g(x) \geq 0, \forall x \geq b$. Then,*

1. *if $\int_a^{+\infty} f(x)\,\mathrm{d}x$ converges, then $\int_a^{+\infty} g(x)\,\mathrm{d}x$ converges.*
2. *if $\int_a^{+\infty} g(x)\,\mathrm{d}x$ diverges, then $\int_a^{+\infty} f(x)\,\mathrm{d}x$ diverges.*

Theorem 12 (Second comparison test)**.** *Assume $f(x) \geq g(x) \geq 0, \forall x \in (a, b)$. Assume that f, g are not bounded in a neighbourhood of b but integrable on intervals of the type $[a, t]$ for $a < t < b$. Assume that*

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = l > 0$$

Then,

$$\int_a^b f(x)\,\mathrm{d}x$$

and

$$\int_a^b g(x)\,\mathrm{d}x$$

converge or diverge simultaneously.

7 Multi-variable Functions

Theorem 13 (Existence of limits)**.** *Let $\exists g(r, \theta)$, s.t. $f(x, y) = g(r, \theta)$. Then, if it exists,*

$$\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{r \to 0} g(r, \theta)$$

Definition 12 (Critical point)**.** If both of $f_x(a, b)$ and $f_y(a, b)$ are zero, or if at least one of them does not exist, then (a, b) is said to be a critical point.

Theorem 14 (A necessary condition for local extrema existence)**.** *If the function $z = f(x, y)$ has a local extrema at the point (a, b) and $\exists f_x(a, b)$ and $\exists f_y(a, b)$ then $f_x(a, b) = f_y(a, b) = 0$*

Theorem 15 (A sufficient condition for local extrema point)**.** *Assume that there exist second order partial derivates of $z = f(x, y)$, they are continuous on some open neighbourhood of (a, b) and $f_x(a, b) = f_y(a, b) = 0$. Denote*

$$D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - \big(f_{xy}(a, b)\big)^2$$

1. *If $D(a, b) > 0$ and $f_{xx} < 0$ then (a, b) is a local maximum point.*
2. *If $D(a, b) > 0$ and $f_{xx} > 0$ then (a, b) is a local minimum point.*
3. *If $D(a, b) < 0$ then (a, b) is called a saddle point.*

Example 4. Find all critical points of

$$z = f(x, y) = x^4 + y^4 - 4xy + 1$$

and classify them.

Solution.

$$f_x(x,y) = 4x^3 - 4y$$
$$f_y(x,y) = 4y^3 - 4x$$

For critical points,

$$f_x(x,y) = 0$$
$$f_y(x,y) = 0$$

Solving, $(0,0)$, $(1,1)$, $(-1,-1)$ are critical points.

$$\begin{aligned} f_{xx}(x,y) &= 12x^2 \\ f_{xy}(x,y) &= -4 \\ f_{yy}(x,y) &= 12y^2 \\ \therefore D(x,y) &= 144x^2y^2 - 16 \end{aligned}$$

For $(0,0)$,

$$D = -16$$

Therefore, $(0,0)$ is a saddle point.
For $(1,1)$,

$$D = 144 - 16$$

Therefore, $(1,1)$ is a local minimum point.
For $(-1,-1)$,

$$D = 144 - 16$$

Therefore, $(-1,-1)$ is a local minimum point.

Definition 13 (Gradient)**.**

$$\nabla f(x,y,z) = (f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)) \neq 0$$

Theorem 16. *If $F(x,y,z)$ is differentiable at some point $P_0(x_0,y_0,z_0)$ on the surface, then the tangent plane α to the surface at the point can be calculated by the formula*

$$F_x(P_0)(x-x_0) + F_y(P_0)(y-y_0) + F_z(P_0)(z-z_0) = 0$$

7.1 Lagrange Multipliers

To find the extrema of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$, solve

$$\begin{aligned} \nabla f(x,y,z) &= \lambda \nabla g(x,y,z) \\ g(x,y,z) &= k \end{aligned}$$

8 Double Integrals

Theorem 17 (Area of a surface)**.** *If $f_x(x,y)$ and $f_y(x,y)$ are continuous in D, then the area of the surface $\sigma : z = f(x,y)$ above D is equal to*

$$S(\sigma) = \iint\limits_D \sqrt{1 + \left(f_x(x,y)\right)^2 + \left(f_y(x,y)\right)^2}\,\mathrm{d}A$$

Definition 14 (Centre of mass)**.** If $\rho(x,y)$ is the density function of a thin body,

$$m = \iint\limits_D \rho(x,y)\,\mathrm{d}A$$

$$(x_{\mathrm{COM}}, y_{\mathrm{COM}}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

where

$$\begin{aligned} M_x &= \iint\limits_D y\rho(x,y)\,\mathrm{d}A \\ M_y &= \iint\limits_D x\rho(x,y)\,\mathrm{d}A \end{aligned}$$

Definition 15 (Domain of the first kind)**.** A domain D is said to be the domain of the first kind if there exist continuous functions $f_1(x)$ and $f_2(x)$, s.t.

$$D_{\mathrm{I}} = \{(x,y) | a \le x \le b, f_1(x) \le y \le f_2(x)\}$$

Theorem 18. *If $f(x,y)$ is continuous in D_{I}, then*

$$\iint\limits_R f(x,y)\,\mathrm{d}A = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

Definition 16 (Domain of the second kind)**.** A domain D is said to be the domain of the second kind if there exist continuous functions $g_1(y)$ and $g_2(y)$, s.t.

$$D_{\mathrm{II}} = \{(x,y) | c \le y \le d, g_1(y) \le x \le g_2(y)\}$$

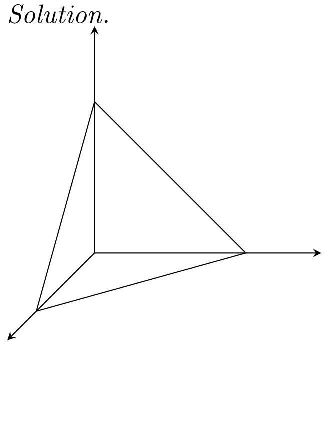
Theorem 19. *If $f(x,y)$ is continuous in D_{II}, then*

$$\iint\limits_R f(x,y)\,\mathrm{d}A = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

9 Triple Integrals

Example 5. Find $\iiint\limits_E x^2 + y^2 + z^2\,\mathrm{d}V$ where E is bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a, a > 0$.

Solution.



Therefore,

$$\iiint_E x^2+y^2+z^2\,\mathrm{d}V = \int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2+y^2+z^2\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

Example 6. Calculate $\iiint_E xe^z\,\mathrm{d}V$ where E is bounded by $z=x^2+y^2$ and $z=8-x^2-y^2$.

Solution. The two boundaries intersect at $x^2+y^2=4$. Therefore the projection of the volume is the circle. Therefore,

$$\iiint_E xe^z\,\mathrm{d}V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-x^2-y^2} xe^z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

10 Line Integrals of Scalar Functions

Definition 17 (Smooth curve)**.** Let C be given parametrically as

$$\overline{r}(t) = \big(x(t), y(t)\big) \quad t : a \to b$$

The curve is said to be smooth if

$$\overline{r}'(t) = \big(x'(t), y'(t)\big)$$

is a continuous function on $[a,b]$, $\overline{r}'(t) \neq \overline{0}$ on (a,b) and $\overline{r}'(t)$ is also continuous on (a,b).

Theorem 20. *If $f(x,y)$ is continuous and C is smooth, then*

$$\int_C f(x,y)\,\mathrm{d}s = \int_a^b f(x(t),y(t))\sqrt{\big(x'(t)\big)^2+\big(y'(t)\big)^2}\,\mathrm{d}t$$

Example 7. Calculate $\int_C x^2+y^2\,\mathrm{d}s$ where C is a circle of radius 2.

Solution.

$$\begin{aligned}\int_C x^2+y^2\,\mathrm{d}s &= \int_0^{2\pi} \Big((2\cos t)^2+(2\sin t)^2\Big)\cdot 2\,\mathrm{d}t \\ &= 16\pi\end{aligned}$$

11 Line Integrals of Vector Functions

Definition 18 (Line integral of vector function)**.**

$$\begin{aligned}W &= \int_C \overline{F}\cdot\hat{T}\,\mathrm{d}s \\ &= \int_C \overline{F}\cdot\mathrm{d}\overline{z} \\ &= \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z\end{aligned}$$

Example 8. Find the work W done by the force $\overline{F}(x,y) = (x, xy)$ over the curve $C : \overline{r}(t) = (2\cos t, 2\sin t), t : \pi \to 2\pi$.

Solution.

$$\begin{aligned}W &= \int_C \overline{F}\cdot\hat{T}\,\mathrm{d}s \\ &= \int_\pi^{2\pi} \big(2\cos t(-2\sin t) + 2\cos t\cdot 2\sin t\cdot 2\cos t\big)\,\mathrm{d}t \\ &= \int_\pi^{2\pi} \Big(-2\sin(2t) + 8\cos^2 t\sin t\Big)\,\mathrm{d}t \\ &= \cos(2t) - \frac{8}{3}\cos^3 t\Big|_\pi^{2\pi} \\ &= \left(1-\frac{8}{3}\right) - \left(1+\frac{8}{3}\right) \\ &= -\frac{16}{3}\end{aligned}$$

Example 9. Calculate $\int_C \frac{x}{y}\,\mathrm{d}x + \frac{y-x}{x}\,\mathrm{d}y$ where C is the path over the parabola $y=x^2$ from $(2,4)$ to $(1,1)$.

Solution.

$$\begin{aligned}\int_C \left(\frac{x}{y}, \frac{y-x}{x}\right)\mathrm{d}r &= \int_2^1 \left(\frac{t}{t^2} + \frac{t^2-t}{t}\right)\cdot(1,2t)\,\mathrm{d}t \\ &= \int_2^1 \left(\frac{1}{t} + (t-1)\cdot 2t\right)\mathrm{d}t \\ &= \ln t + \frac{2t^3}{3} - t^2\Big|_2^1 \\ &= \ln\frac{1}{2} + \frac{2}{3} - \frac{16}{3} - 1 + 4 \\ &= 3 - \frac{14}{3} - \ln 2 \\ &= \frac{5}{3} - \ln 2\end{aligned}$$

Theorem 21 (The Fundamental Theorem of Line Integrals)**.** *Let C be a smooth curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ given parametrically by $\overline{r}(t)$, $t : a \to b$. Let f be a continuous function of (x, y) or (x, y, z) respectively, on C and ∇f be a continuous vector function in a connected domain D which contains C. Then*

$$\begin{aligned} W &= \int\limits_C \nabla f \cdot \hat{T} \, \mathrm{d}s \\ &= f(r(b)) - f(r(a)) \\ &= f(B) - f(A) \end{aligned}$$

Definition 19 (Simple curve)**.** A curve C is called a simple curve if it does not intersect itself.

Definition 20 (Domain)**.** A domain $D \subset \mathbb{R}^2$ is called connected if for any two points from D, the is a path C which connects the points and remains in D.

Definition 21 (Simple connected domain)**.** A connected domain $D \subset \mathbb{R}^2$ is called simple connected if any simple closed curve from D contains inside itself only points in D.

Theorem 22. *If*

$$\overline{F}(x, y) = \big(P(x, y), Q(x, y)\big) = \nabla f(x, y)$$

is the conservative vector field in a connected domain D, where there exist first order partial derivatives of P and Q continuous in D, then

$$P_y(x, y) = Q_x(x, y) \qquad \forall (x, y) \in D$$

Theorem 23. *Let*

$$\overline{F}(x, y) = \big(P(x, y), Q(x, y)\big)$$

be a vector field in an open, simple connected domain D. If there exist first order partial derivatives of P and Q which are continuous in D, and

$$P_y(x, y) = Q_x(x, y) \qquad \forall (x, y) \in D$$

Then, $\exists f(x, y)$ s.t.

$$\overline{F}(x, y) = \nabla f(x, y)$$

i.e. $\overline{F}$ is a conservative vector field.

Example 10. If

$$\overline{F}(x, y) = (3 + 2xy, x^2 - 3y^2)$$

a conservative vector field? If yes, find $f(x, y)$, s.t.

$$\overline{F}(x, y) = \nabla f(x, y)$$

and find the work done by the force $\overline{F}(x, y)$ over the curve

$$\overline{r}(t) = (e^t \sin t, e^t \cos t) \qquad t : 0 \to \pi$$

Solution.

$$\begin{aligned} P(x, y) &= 3 + 2xy \\ \therefore P_y &= 2x \\ Q(x, y) &= x^2 - 3y^2 \\ \therefore Q_x &= 2x \\ \therefore P_y &= Q_x \end{aligned}$$

Therefore, $\overline{F}(x, y)$ is a conservative vector field.

$$\begin{aligned} f_x &= P \\ &= 3 + 2xy \\ \therefore f &= 3x + x^2 y + c(y) \\ \therefore f_y &= x^2 + c'(y) \end{aligned}$$

Comapring with $f_y = Q$,

$$\begin{aligned} c'(y) &= -3y^3 \\ \therefore c(y) &= -y^3 + c \\ \therefore f(x, y) &= 3x + x^2 y - y^3 + c \end{aligned}$$

By the definition of work,

$$\begin{aligned} W &= \int\limits_C \overline{F} \cdot \hat{T} \, \mathrm{d}s \\ &= \int\limits_a^b \big(P(\overline{r}(t)) x'(t) + Q(\overline{r}(t)) y'(t)\big) \, \mathrm{d}t \end{aligned}$$

Alternatively, using The Fundamental Theorem of Line Integrals,

$$\begin{aligned} W &= \int\limits_C \overline{F} \cdot \hat{T} \, \mathrm{d}s \\ &= \int\limits_C \nabla d \cdot \hat{T} \, \mathrm{d}s \\ &= f(\overline{r}(\pi)) - f(\overline{r}(0)) \\ &= f(0, -e^\pi) - f(0, 1) \\ &= -(-e^\pi)^3 - (-1)^3 \\ &= e^{3\pi} + 1 \end{aligned}$$

Theorem 24 (Green's Theorem)**.** *Let C be a piecewise smooth, simple, and closed curve in $\mathbb{R}^2$ with positive orientation. Let D be a domain bounded by C. If there exist continuous first order partial derivatives of $P(x, y)$ and $Q(x, y)$ in an open domain which contains D, then*

$$W = \int\limits_C \overline{F} \cdot \hat{T} \, \mathrm{d}s = \int\limits_C P \, \mathrm{d}x + Q \, \mathrm{d}y = \iint\limits_D (Q_x - P_y) \, \mathrm{d}A$$

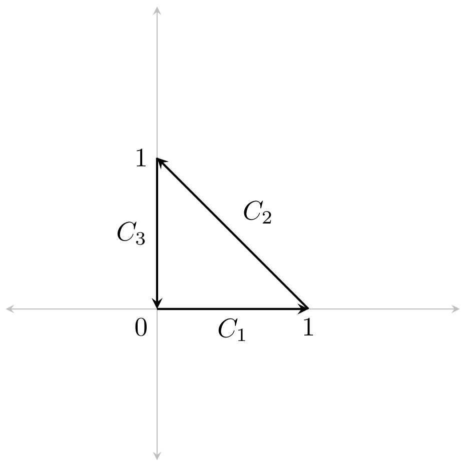
Remark 1. Green's Theorem is also true for domains with holes.

Example 11. Find the work done by the force

$\overline{F}(x,y) = (x^4, xy)$

over the path

$C = C_1 \cup C_2 \cup C_3$



Solution. By Green's Theorem,

$$
\begin{aligned}
W &= \int\limits_C P\,\mathrm{d}x + Q\,\mathrm{d}y \\
&= \iint\limits_D (Q_x - P_y)\,\mathrm{d}A \\
&= \iint\limits_D (y-0)\,\mathrm{d}A \\
&= \int\limits_0^1 \int\limits_0^{1-x} y\,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{1}{6}
\end{aligned}
$$

Example 12. Calculate $\int\limits_C \overline{F}\cdot\hat{T}\,\mathrm{d}s$ when

$$\overline{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

and C is a simple, closed, piecewise smooth curve with positive orientation which does not pass through $(0,0)$.

Solution.

$$
\begin{aligned}
P &= \frac{y}{x^2+y^2} \\
Q &= \frac{x}{x^2+y^2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
P_y &= -\frac{(x^2+y^2) - y\cdot 2y}{(x^2+y^2)^2} \\
&= \frac{y^2-x^2}{(x^2+y^2)^2} \\
Q_x &= \frac{(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2}
\end{aligned}
$$

If $(0,0) \notin D$, Green's Theorem is applicable. Therefore,

$$
\begin{aligned}
\int\limits_C \overline{F}\cdot\hat{T}\,\mathrm{d}s &= \iint\limits_D (Q_x - P_y)\,\mathrm{d}A \\
&= 0
\end{aligned}
$$

If $(0,0) \in D$, Green's Theorem is not applicable as P_y and Q_x are not continuous in D.
Let C_1 be a circle of radius a, with the same orientation as C. Let $\widetilde{C} = C \cup (-C_1)$. Green's Theorem can be applied on the domain $D \setminus D_1$ which is enclosed by $\widetilde{C}$.

$$
\begin{aligned}
\int\limits_{C\cup(-C_1)} P\,\mathrm{d}x + Q\,\mathrm{d}y &= \iint\limits_{D\setminus D_1} (Q_x - P_y)\,\mathrm{d}A \\
&= 0
\end{aligned}
$$

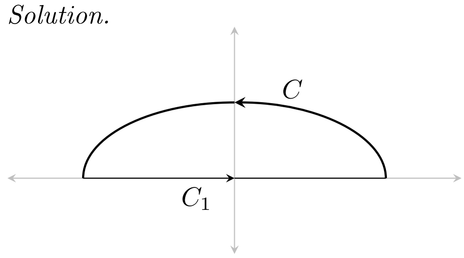
$$\int\limits_C P\,\mathrm{d}x + Q\,\mathrm{d}y + \int\limits_{-C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y = 0$$

Therefore,

$$
\begin{aligned}
\int\limits_C P\,\mathrm{d}x + Q\,\mathrm{d}y &= \int\limits_{C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y \\
&= \int\limits_0^{2\pi} P\big(x(t),y(t)\big)\,x'(t)\,\mathrm{d}t \\
&\quad + \int\limits_0^{2} Q\big(x(t),y(t)\big)\,\mathrm{d}t \\
&= \int\limits_0^{2\pi} (\sin^2 t + \cos^2 t)\,\mathrm{d}t \\
&= 2\pi
\end{aligned}
$$

Example 13. Calculate $\int\limits_C -2e^{2x-y}\cos y\,\mathrm{d}x + \left(e^{2x-y}(\sin y + \cos y) + 2xy\right)\mathrm{d}y$ when C is the half ellipse $\left\{\frac{x^2}{4} + y^2 = 1, y \geq 0\right\}$ oriented from the point $(2,0)$ to the point $(-2,0)$.

Solution.



Let C_1 be the line segment as shown.

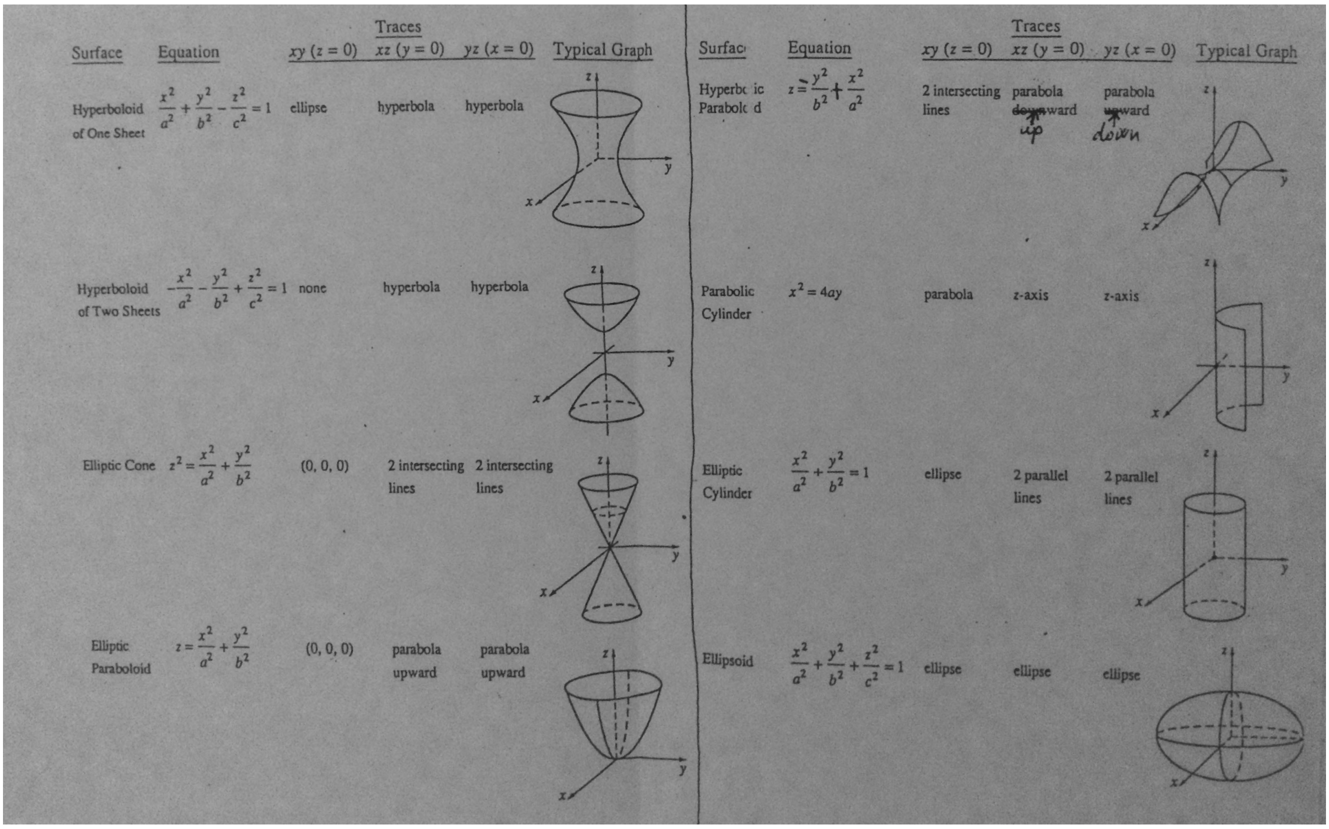
$$
\begin{aligned}
P &= -2e^{2x-y}\cos y \\
Q &= e^{2x-y}(\sin y + \cos y) + 2xy
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
P_y &= 2e^{2x-y}\cos y + 2e^{2x-y}\sin y \\
&= 2e^{2x-y}(\cos y + \sin y) \\
Q_x &= 2e^{2x-y}(\sin x + \cos y) + 2y
\end{aligned}
$$

The domain is of the first kind.

$$
\begin{aligned}
\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y &= \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y + \int_{C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y \\
&\quad - \int_{C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y \\
&= \int_{C\cup C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y - \int_{C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y \\
&= \iint_D (Q_x - P_y)\,\mathrm{d}A - \int_{C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y
\end{aligned}
$$



Surface	Equation	Traces: xy ($z=0$)	Traces: xz ($y=0$)	Traces: yz ($x=0$)	Typical Graph
Hyperboloid of One Sheet	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	ellipse	hyperbola	hyperbola	
Hyperboloid of Two Sheets	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	none	hyperbola	hyperbola	
Elliptic Cone	$z^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}$	(0, 0, 0)	2 intersecting lines	2 intersecting lines	
Elliptic Paraboloid	$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$	(0, 0, 0)	parabola upward	parabola upward	
Hyperbolic Paraboloid	$z=-\frac{y^2}{b^2}+\frac{x^2}{a^2}$	2 intersecting lines	parabola ~~downward~~ up	parabola ~~upward~~ down	
Parabolic Cylinder	$x^2=4ay$	parabola	z-axis	z-axis	
Elliptic Cylinder	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	ellipse	2 parallel lines	2 parallel lines	
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	ellipse	ellipse	ellipse	