Lecture 25

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Contents

1 Vector Form of Green's Theorem

Theorem 1 (Green's Theorem)**.**

$$\int_C \overline{F} \cdot \hat{n} \, \mathrm{d}s = \iint_D \operatorname{div} \overline{F} \, \mathrm{d}A$$

2 Gauss Theorem

Theorem 2 (Gauss Theorem)**.** *Let E be a solid bounded by a surface S with positive orientation. Let $\overline{F}(x,y,z) = \big(P(x,y,z), Q(x,y,z), R(x,y,z)\big)$ be a vector field in $\mathbb{R}^3$, s.t. there exist continuous first order partial derivatives of P, Q, R in some open domain which contain E. Then*

$$\iint_S \overline{F} \cdot \hat{n} \, \mathrm{d}S = \iiint_E \operatorname{div} \overline{F} \, \mathrm{d}V$$

Remark 1. Gauss Theorem is an analogy of Green's Theorem

Example 1. Calculate the flux of $\overline{F} = (xy, xe^z, 2+z)$ through the surface S which is a boundary of a solid E bounded by two paraboloids

$$z = 12 - 2x^2 - 2y^2$$
$$z = x^2 + y^2$$

Solution. Let

$$g_1(x,y) = 12 - 2x^2 - 2y^2$$
$$g_2(x,y) = x^2 + y^2$$

The intersection of $g_1(x,y)$ and $g_2(x,y)$ is $x^2+y^2=4$.

$$\begin{aligned}
\iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S &= \iiint_E \operatorname{div}\overline{F}\,\mathrm{d}V \\
&= \iiint_E (y+0+1)\,\mathrm{d}V \\
&= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{12-2x^2-2y^2} (y+1)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (y+1)(12-3x^2-3y^2)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \big(y(12-3x^2-3y^2)+(12-3x^2-3y^2)\big)\,\mathrm{d}y\,\mathrm{d}x
\end{aligned}$$

$y(12-3x^2-3y^2)$ is odd and $(12-3x^2-3y^2)$ is even.

$$\begin{aligned}
\therefore \iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S &= \int_{-2}^{2} 2\int_{0}^{\sqrt{4-x^2}} (12-3x^2-3y^2)\,\mathrm{d}y\,\mathrm{d}x \\
&= 6\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} (4-x^2-y^2)\,\mathrm{d}y\,\mathrm{d}x \\
&= 6\int_{-2}^{2} \left((4-x^2)y-\frac{y^3}{3}\right)\Bigg|_{y=0}^{y=\sqrt{4-x^2}}\,\mathrm{d}x \\
&= 6\int_{-2}^{2} (4-x^2)\left(\sqrt{4-x^2}-\frac{1}{3}(4-x^2)\sqrt{4-x^2}\right)\mathrm{d}x
\end{aligned}$$

Let

$$\begin{aligned}
x &= 2\sin\theta \\
\therefore \mathrm{d}x &= 2\cos\theta\,\mathrm{d}\theta
\end{aligned}$$

Therefore, solving,

$$\iint_S \overline{F} \cdot \hat{n} \, \mathrm{d}S = 24\pi$$