DIGITAL LOGIC SYSTEMS : ASSIGNMENT 4

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Exercise 1.
Let p, q, r denote boolean formulas. Prove that if p is logically equivalent to q, and q is logically equivalent to r, then p is logically equivalent to r.

Solution 1.

p	q	r	$p \leftrightarrow q$	$q \leftrightarrow r$	$(p \leftrightarrow q) \land (q \leftrightarrow r)$	$p \leftrightarrow r$
0	0	0	1	1	1	1
0	0	1	1	0	0	0
0	1	0	0	0	1	1
0	1	1	0	1	0	0
1	0	0	0	1	0	0
1	0	1	0	0	1	1
1	1	0	1	0	0	0
1	1	1	1	1	1	1

Therefore, as the columns of $((p \leftrightarrow q) \land (q \leftrightarrow r))$ and $(p \leftrightarrow r)$ are identical, they are equivalent. Hence, if p is logically equivalent to q, and q is logically equivalent to r, then p is logically equivalent to r.

Exercise 2.
Let $L(\varphi)$ denote the number of vertices in the parse tree of φ that are not labelled by negation. Prove that $L\left(\mathrm{DM}(\varphi)\right) = L(\varphi)$, for every boolean formula φ that uses the logical connectives $\{$ NOT , OR , AND $\}$.

Solution 2.
In the De Morgan dual of φ, all vertices labeled by OR are changed to vertices labeled by AND , and vice-versa. Depending on φ, vertices labelled by NOT are be added or removed.
Therefore, the number of vertices labelled by AND in φ is equal to the number of vertices labelled by OR in $\mathrm{DM}(\varphi)$. Similarly for OR in φ and AND in $\mathrm{DM}(\varphi)$.

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The number of vertices labelled by variables is unchanged in φ and $\text{DM}(\varphi)$.
Therefore, the total number of vertices not labelled by negation, i.e. labelled only be variables, AND , and OR is the same in the parse trees of φ and $\text{DM}(\varphi)$. Therefore, $L\big(\text{DM}(\varphi)\big) = L(\varphi)$. $\square$

Exercise 4.
Add the following two reduction rules to Algorithm $\text{DM}(\varphi)$ so that you can also deal with the XOR and NXOR connectives:

(1) If $\varphi = (\varphi_1 \text{ XOR } \varphi_2)$, then return $(\text{DM}(\varphi_1) \text{ NXOR } \text{DM}(\varphi_2))$.
(2) If $\varphi = (\varphi_1 \text{ NXOR } \varphi_2)$, then return $(\text{DM}(\varphi_1) \text{ XOR } \text{DM}(\varphi_2))$.

Prove that, even after this modification, $\text{DM}(\varphi) \leftrightarrow \neg\varphi$ is a tautology.

Solution 4.

$$\begin{aligned} x \text{ XOR } y &= (x \vee y) \wedge (x \wedge (\neg y)) \\ x \text{ NXOR } y &= \neg((x \vee y) \wedge (x \wedge (\neg y))) \end{aligned}$$

Therefore, if

$$\begin{aligned} \varphi &\equiv \varphi_1 \text{ XOR } \varphi_2 \\ \therefore \varphi &\equiv (\varphi_1 \vee \varphi_2) \wedge (\varphi_1 \wedge \neg\varphi_2) \\ \therefore \text{DM}(\varphi) &\equiv \text{DM}((\varphi_1 \vee \varphi_2) \wedge (\varphi_1 \wedge \neg\varphi_2)) \\ &\equiv (\neg\varphi_1 \wedge \neg\varphi_2) \vee (\neg\varphi_1 \vee \varphi_2) \\ &\equiv \neg(\varphi_1 \vee \varphi_2) \vee \neg(\varphi_1 \wedge \neg\varphi_2) \\ &\equiv \neg((\varphi_1 \vee \varphi_2) \wedge (\varphi_1 \wedge \neg\varphi_2)) \\ &\equiv \neg(\varphi_1 \text{ XOR } \varphi_2) \\ &\equiv \neg(\varphi) \end{aligned}$$

Therefore, as $\text{DM}(\varphi) \equiv \neg\varphi$, $\text{DM}(\varphi) \leftrightarrow \neg\varphi$ is a tautology.

Similarly, if

$$\begin{aligned} \varphi &\equiv \varphi_1 \text{ NXOR } \varphi_2 \\ \therefore \varphi &\equiv \neg((\varphi_1 \vee \varphi_2) \wedge (\varphi_1 \wedge \neg\varphi_2)) \\ \therefore \text{DM}(\varphi) &\equiv \text{DM}(\neg((\varphi_1 \vee \varphi_2) \wedge (\varphi_1 \wedge \neg\varphi_2))) \\ &\equiv \text{DM}(\neg(\varphi_1 \vee \varphi_2) \vee \neg(\varphi_1 \wedge \neg\varphi_2)) \\ &\equiv \text{DM}((\neg\varphi_1 \wedge \neg\varphi_2) \vee (\neg\varphi_1 \vee \varphi_2)) \\ &\equiv (\varphi_1 \vee \varphi_2) \wedge (\varphi_1 \wedge \neg\varphi_2) \\ &\equiv \varphi_1 \text{ XOR } \varphi_2 \\ &\equiv \neg(\varphi_1 \text{ NXOR } \varphi_2) \\ &\equiv \neg(\varphi) \end{aligned}$$

Therefore, as $\text{DM}(\varphi) \equiv \neg\varphi$, $\text{DM}(\varphi) \leftrightarrow \neg\varphi$ is a tautology. $\square$

Exercise 5.
Let φ_k denote the boolean formula in which the variable X is negated k times. Run algorithm NNF(φ_k). What is the outcome? Prove your result. Hint: distinguish between an even k and an odd k.

Solution 5.

$$\begin{aligned}
\varphi_k &= \underbrace{\neg(\neg(\ldots(\neg X)))}_{k \text{ times}} \\
\therefore \mathrm{NNF}(\varphi_k) &= \mathrm{DM}(\mathrm{NNF}(\underbrace{\neg(\neg(\ldots(\neg X))))}_{k-1 \text{ times}})) \\
&= \underbrace{\mathrm{DM}(\mathrm{DM}(\ldots(\mathrm{DM}}_{k-1 \text{ times}}(\mathrm{NNF}(X))))) \\
&= \underbrace{\mathrm{DM}(\mathrm{DM}(\ldots(\mathrm{DM}}_{k-1 \text{ times}}(\neg X))))
\end{aligned}$$

Case 1 (k is odd). If k is odd, $k-1$ must be even.
Therefore, as DM(DM(φ)) is logically equivalent to φ, the $\dfrac{k-1}{2}$ pairs of DM will be cancelled out. Hence, NNF(φ_k) is logically equivalent to $\neg X$.
Therefore,

$$\mathrm{NNF}(\varphi_k) \equiv \neg X$$

Case 2 (k is even). If k is odd, $k-1$ must be odd.
Therefore, as DM(DM(φ)) is logically equivalent to φ, the $\dfrac{k-2}{2}$ pairs of DM will be cancelled out. Hence, NNF(φ_k) is logically equivalent to DM($\neg X$).
Therefore,

$$\mathrm{NNF}(\varphi_k) \equiv X$$