DIGITAL LOGIC SYSTEMS : ASSIGNMENT 5

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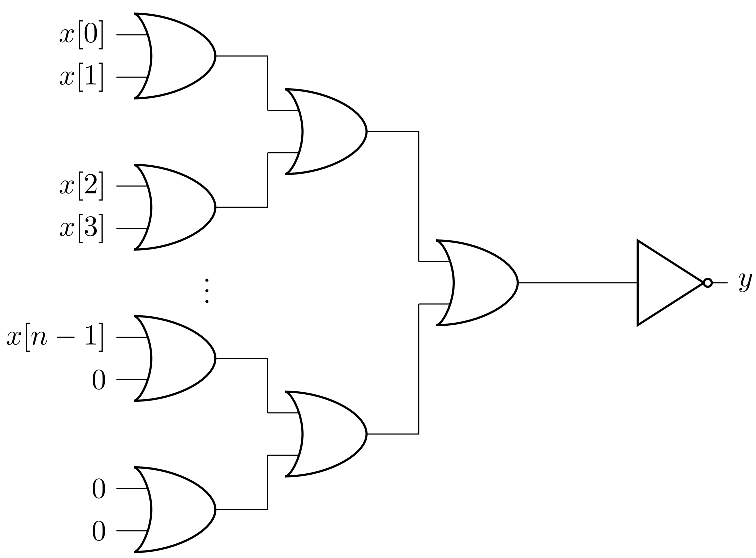
Exercise 1.
Design a zero-tester, defined as follows.

Input	$x[n-1:0]$
Output	$y \in \{0, 1\}$
Functionality	$y = 1 \iff x[n-1:0] = 0^n$

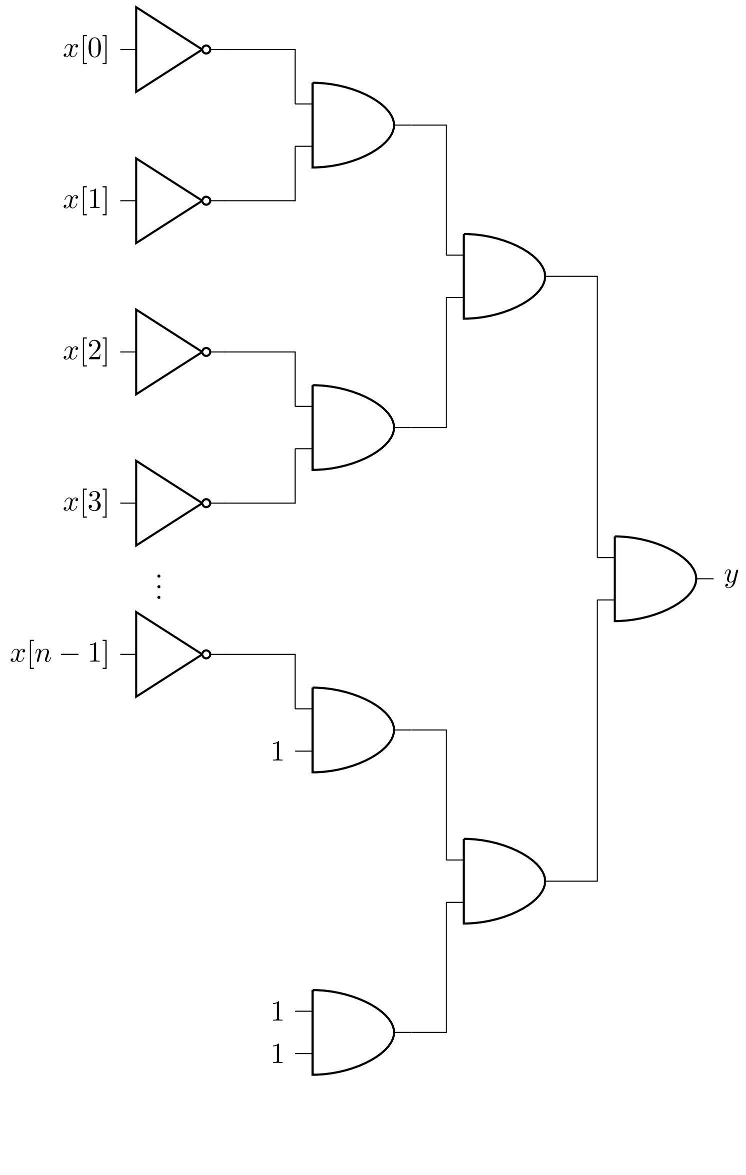
(1) Suggest a design based on an OR -tree.
(2) Suggest a design based on an AND -tree.
(3) What do you think about a design based on a tree of nor gates?

Solution 1.
(1) The output must be 1 if and only if all n bits of the input are 0.
As OR is monotonic, at least one NOT operator is necessary for implementation of the function.
Therefore, as 0 OR $X \equiv X$, any tree can be made perfect by adding 0s as inputs for completing the number of input bits to the smallest power of 2 greater than or equal to n.

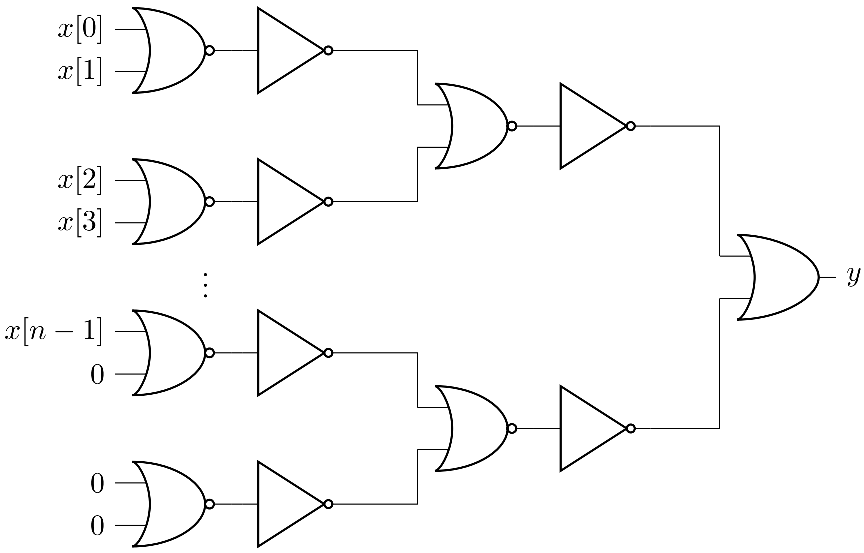


Date: Tuesday 12th May, 2015.

(2) The output must be 1 if and only if all n bits of the input are 0.
As AND is monotonic, at least one NOT operator is necessary for implementation of the function.
Therefore, as 1 AND $X \equiv X$, any tree can be made perfect by adding 1s as inputs for completing the number of input bits to the smallest power of 2 greater than or equal to n.



(3) As OR $\equiv$ NOT (NOR), a tree using NOR gates can be constructed by replacing all OR gates in the above tree by a combination of NOR and NOT gates.



Exercise 2.
Prove that each of the following functions $f : \{0,1\}^n \to \{0,1\}$ is associative:
(1) constant 0
(2) constant 1
(3) x_1
(4) x_n
(5) AND $_n$
(6) OR $_n$
(7) XOR $_n$
(8) NXOR $_n$

Solution 2.
If $f\{0,1\}^2 \to \{0,1\}$ is an associative boolean function, then $f_n(x_1, x_2, \ldots, x_n) = f\left(f_{nk}(x_1, \ldots, x_{nk}), f_k(x_{nk+1}, \ldots, x_n)\right)$ for every $n \geq 2$ and $k \in [1, n1]$. Therefore, it is enough to show that the functions are associative for 2 bits.

(1)

$f(x_1, x_2, x_3) = 0$

x_1	x_2	x_3	$f((x_1, x_2), x_3)$	$f(x_1, f(x_2, x_3))$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	0	0
1	0	0	0	0
1	0	1	0	0
1	1	0	0	0
1	1	1	0	0

Therefore, as the columns of $f((x_1, x_2), x_3)$ and $f(x_1, f(x_2, x_3))$ are identical, the function is associative.

(2)

$f(x_1, x_2, x_3) = 1$

x_1	x_2	x_3	$f((x_1, x_2), x_3)$	$f(x_1, f(x_2, x_3))$
0	0	0	1	1
0	0	1	1	1
0	1	0	1	1
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

Therefore, as the columns of $f((x_1, x_2), x_3)$ and $f(x_1, f(x_2, x_3))$ are identical, the function is associative.

(3)

$f(x_1, x_2, x_3) = x_1$

x_1	x_2	x_3	$f((x_1, x_2), x_3)$	$f(x_1, f(x_2, x_3))$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	0	0
1	0	0	1	0
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

Therefore, as the columns of $f((x_1, x_2), x_3)$ and $f(x_1, f(x_2, x_3))$ are identical, the function is associative.

(4)

$f(x_1, x_2, x_3) = x_3$

x_1	x_2	x_3	$f((x_1, x_2), x_3)$	$f(x_1, f(x_2, x_3))$
0	0	0	0	0
0	0	1	1	1
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1
1	1	0	0	0
1	1	1	1	1

Therefore, as the columns of $f((x_1, x_2), x_3)$ and $f(x_1, f(x_2, x_3))$ are identical, the function is associative.

(5)

$f(x_1, x_2, x_3) = x_1 \wedge x_2 \wedge x_3$

x_1	x_2	x_3	$f((x_1,x_2),x_3)$	$f(x_1,f(x_2,x_3))$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	0	0
1	0	0	0	0
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

Therefore, as the columns of $f((x_1,x_2),x_3)$ and $f(x_1,f(x_2,x_3))$ are identical, the function is associative.

(6)

$$f(x_1,x_2,x_3) = x_1 \vee x_2 \vee x_3$$

x_1	x_2	x_3	$f((x_1,x_2),x_3)$	$f(x_1,f(x_2,x_3))$
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

Therefore, as the columns of $f((x_1,x_2),x_3)$ and $f(x_1,f(x_2,x_3))$ are identical, the function is associative.

(7)

$$f(x_1,x_2,x_3) = x_1 \oplus x_2 \oplus x_3$$

x_1	x_2	x_3	$f((x_1,x_2),x_3)$	$f(x_1,f(x_2,x_3))$
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	0	0
1	0	0	1	1
1	0	1	0	0
1	1	0	0	0
1	1	1	0	0

Therefore, as the columns of $f((x_1,x_2),x_3)$ and $f(x_1,f(x_2,x_3))$ are identical, the function is associative.

(8)

$$f(x_1,x_2,x_3) = x_1 \text{ NXOR } x_2 \text{ NXOR } x_3$$

x_1	x_2	x_3	$f((x_1, x_2), x_3)$	$f(x_1, f(x_2, x_3))$
0	0	0	1	1
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

Therefore, as the columns of $f((x_1, x_2), x_3)$ and $f(x_1, f(x_2, x_3))$ are identical, the function is associative.

Exercise 3.

Prove that there is only one minimum depth binary tree with n leaves if and only if n is a power of 2.

Solution 3.

If n is a power of 2, then the minimum depth tree is a perfect tree. Therefore, it is symmetrical for any arrangement of the leaves.
Therefore, the tree is unique.

Conversely, let the minimum depth binary tree be unique.
If possible, let n not be a power of 2.
Therefore, the minimum depth tree is not perfect.
Therefore, it is not symmetric with respect to the leaves. Hence, there can exist multiple trees with minimum depth.
This contradicts the assumption the tree is unique.
Therefore, n must be a power of 2.