Digital Logic Systems

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2014-15

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1 Lecturer Information

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2 Required Reading

Guy Even and Moti Medina: *Digital Logic Design*

Part I

Introduction to Discrete Math

1 Sets and Functions

Definition 1 (Universal set)**.** The universal set is a set that contains all the possible objects.

Definition 2 (Set)**.** A set is a collection of objects from a universal set.

Definition 3 (Subset)**.** A is a subset of B if every element in A is also an element in B. It is denoted as $A \subseteq B$

Definition 4 (Equal sets)**.** Two sets A and B are said to be equal if $A \subseteq B$ and $B \subseteq A$.

Definition 5 (Strict containment)**.** $A \subsetneq B \iff A \subseteq B$ and $A \neq B$.

Definition 6 (Empty set)**.** The empty set is the set that does not contain any element. It is usually denoted by $\emptyset$.

Definition 7 (Power set)**.** The power set of a set A is the set of all the subsets of A. The power set of A is denoted by $P(A)$ or 2^A.

Definition 8 (Ordered pair)**.** Two objects (possibly equal) with an order (i.e., the first object and the second object) are called an ordered pair.

Definition 9 (Cartesian product)**.** The Cartesian product of the sets A and B is the set

$$A \times B \triangleq \{(a, b) | a \in A \text{ and } b \in B\}$$

Theorem 1 (De Morgan's Laws)**.**

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Theorem 2 (Binary relation)**.** *A subset $R \subseteq A \times B$ is called a binary relation.*

Definition 10 (Function)**.** A binary relation $R \subseteq A \times B$ is a function if for every $a \in A$ there exists a unique element $b \in B$ such that $(a, b) \in R$.

Definition 11 (Extension)**.** Let f and g denote two functions. g is an extension of f if $f \subseteq g$, i.e., if every ordered pair in f is also an ordered pair in g.

Definition 12 (Boolean function)**.** A function B : 0, 1n 0, 1k is called a Boolean function.

1.1 Important Boolean Functions

Definition 13 (NOT)**.**

$$\text{NOT}(x) = 1 - x$$

Definition 14 (AND)**.**

$$\text{AND}(x) = x \cdot y$$

Definition 15 (OR)**.**

$$\text{OR}(x) = x + y - (x \cdot y)$$

Definition 16 (XOR)**.**

$$\text{XOR}(x) = (x + y) \bmod 2$$

2 Mathematical Induction

Theorem 3. *For every $n \geq 2$, and for sets $A_1, \ldots, A_n$,*

$$\overline{A_1 \cup \cdots \cup A_n} = \overline{A_1} \cap \cdots \cap \overline{A_n}$$

Proof. If $n = 2$, the statement is true, by De Morgan's Laws.
Let

$$B = A_1 \cup \cdots \cup A_n$$

If possible, let

$$\overline{B} = \overline{A_1} \cap \cdots \cap \overline{A_n}$$

Therefore,

$$\begin{aligned}\overline{A_1 \cup \cdots \cup A_n \cup A_{n+1}} &= \overline{B \cup A_{n+1}} \\ &= \overline{B} \cap \overline{A_{n+1}} \\ &= (\overline{A_1} \cap \cdots \cap \overline{A_n}) \cap \overline{A_{n+1}}\end{aligned}$$

□

Theorem 4 (Pigeon-hole Principle)**.** *If there are n holes and $m > n$ pigeons then there exists at least one hole with more than one pigeon in it. Let $f : A \to \{1, ..., n\}$, and $|A| > n$, then f is not one-to-one, i.e., there are distinct $a_1, a_2 \in A; a_1 \neq a_2$, such that $f(a_1) = f(a_2)$.*

3 Sequences and Series

3.1 Sequences

3.1.1 Arithmetic Sequences

Definition 17 (Arithmetic Sequence)**.** An arithmetic sequence $\{a_n\}_{n=0}^{\infty}$ is defined as

$$a_n \doteq a_0 + n \cdot d$$

3.1.2 Geometric Sequences

Definition 18 (Geometric Sequence)**.** A geometric sequence $\{b_n\}_{n=0}^{\infty}$ is defined as

$$b_n \doteq b_0 \cdot q^n$$

3.2 Series

Definition 19 (Series)**.** The sum of the elements of a sequence is called a series.

3.2.1 Arithmetic Series

Theorem 5.

$$S_n = a_0 \cdot (n+1) + d \cdot \frac{n(n+1)}{2}$$

Proof.

$$\begin{aligned} S_n &= \sum_{i=0}^{n} a_i \\ &= \sum_{i=0}^{n} a_n + id \\ &= \sum_{i=0}^{n} a_0 + \sum_{i=0}^{n} id \\ &= a_0(n+1) + d \sum_{i=0}^{n} i \\ &= a_0(n+1) + d \cdot \frac{n(n+1)}{2} \end{aligned}$$

□

For estimation,

$$\begin{aligned} S_n &= \sum_{i=0}^{n} a_i \\ &\approx \int\limits_0^n f(x)\,\mathrm{d}x \\ &= \int\limits_0^n (a_0 + d \cdot x)\,\mathrm{d}x \\ \therefore S_n &\approx a_0 n + \frac{d}{2} n^2 \end{aligned}$$

3.2.2 Geometric Series

Theorem 6.

$$S_n = b_0 \cdot \frac{q^{n+1} - 1}{q - 1}$$

For estimation,

$$\begin{aligned} S_n &= \sum_{i=0}^{n} b_i \\ &\approx \int\limits_0^n g(x)\,\mathrm{d}x \\ &= \int\limits_0^n b_0 q^x\,\mathrm{d}x \\ \therefore S_n &\propto e^n \end{aligned}$$

3.2.3 Harmonic Series

Theorem 7. *Let*

$$H_n \doteq \sum_{i=1}^{n} \frac{1}{i}$$

Then, $\forall k \in \mathbb{N}$,

$$1 + \frac{k}{2} \leq H_{2^k} \leq k + 1$$

Proof. If $k = 0$,

$$\begin{aligned} H_{2^0} &= H_1 \\ &= 1 \\ \therefore 1 &\leq k + 1 \end{aligned}$$

If possible, let $H_{2^k} \leq k + 1$.
Therefore,

$$H_{2^{k+1}} = H_{2^k} + \sum_{i=2^k+1}^{2 \cdot 2^k} \frac{1}{i}$$

Therefore, by the induction hypothesis,

$$H_{2^{k+1}} \leq k + 1 + \sum_{i=2^k+1}^{2 \cdot 2^k} \frac{1}{i}$$

As the harmonic sequence monotonically decreases, the largest term between $2^k + 1$ and $2 \cdot 2^k$ is $\frac{1}{2^k}$. Therefore, $\sum_{i=2^k+1}^{2 \cdot 2^k} \frac{1}{i} \leq \frac{1}{2^k} \cdot 2^k$. Therefore,

$$\begin{aligned} H_{2^{k+1}} &\leq k + 1 + \frac{1}{2^k} \cdot 2^k \\ \therefore H_{2^{k+1}} &\leq k + 2 \end{aligned}$$

Therefore, by induction,

$$H_{2^k} \leq k + 1$$

If $k = 0$,

$$\begin{aligned} H_{2^0} &= 1 \\ &\geq 1 + \frac{k}{2} \end{aligned}$$

If possible, let $H_{2^k} \geq 1 + \frac{k}{2}$.

Therefore,

$$\begin{aligned}
H_{2^{k+1}} &= H_{2^k} + \sum_{i=2^k+1}^{2\cdot 2^k} \frac{1}{i} \\
\therefore H_{2^{k+1}} &\geq 1 + \frac{k}{2} + \frac{1}{2\cdot 2^k} \cdot 2^k \\
\therefore H_{2^{k+1}} &\geq 1 + \frac{k}{2} + \frac{1}{2} \\
\therefore H_{2^{k+1}} &\geq 1 + \frac{k+1}{2}
\end{aligned}$$

Therefore, by induction,

$$H_{2^k} \geq 1 + \frac{k+1}{2}$$

Therefore,

$$1 + \frac{k}{2} \leq H_{2^k} \leq k + 1$$

□

For estimation,

$$\begin{aligned}
H_n &\approx \int_1^n \frac{1}{x}\,\mathrm{d}x \\
&= \ln x
\end{aligned}$$

4 Directed Graphs

Definition 20 (Directed graph)**.** Let V denote a finite set and $E \subset V \times V$. The pair $G = (V, E)$ is called a directed graph. An element $v \in V$ is called a vertex or a node. An element $(u, v) \in E$ is called an arc or a directed edge.

Definition 21 (Path or walk)**.** A path or a walk of length l in a directed graph $G = (V, E)$ is a sequence $(v_0, e_0, v_1, e_1, \ldots, v_{l-1}, e_{l-1}, v_l)$, such that

1. $v_i \in V \forall 0 \leq i \leq l$
2. $e_i \in E \forall 0 \leq i < l$

3. $e_i = (v_i, v_{i+1}) \forall 0 \leq i < l$

Definition 22 (Closed path or cycle)**.** A path is said to be closed if the first and last vertices are equal.

Definition 23 (Open path)**.** A path is said to be open if the first and last vertices are distinct.

Definition 24 (Simple path)**.** An open path is said to be simple if every vertex in the path appears only once in the path.

Definition 25 (Closed path)**.** A closed path is said to be simple if every interior vertex appears only once in the path.

Definition 26 (Directed acyclic graph)**.** A directed acyclic graph (DAG) is directed graph that does not contain any cycles.

Definition 27 (In-degree)**.** The in-degree of a vertex v is the number of edges that enter v. It is denoted as $\deg_{\text{in}}(v)$.

Definition 28 (Out-degree)**.** The out-degree of a vertex v is the number of edges that enter v. It is denoted as $\deg_{\text{out}}(v)$.

Definition 29 (Source and sink)**.** A vertex is said to be a source if $\deg_{\text{out}}(v) = 0$. A vertex is said to be a sink if $\deg_{\text{in}}(v) = 0$.

Theorem 8. *Each DAG has at least one sink.*

Theorem 9. *Each DAG has at least one source.*

Definition 30 (Topological ordering)**.** Let $G = (V, E)$ denote a DAG with $|V| = n$. A bijection $\pi : V \to \{0, \ldots, n-1\}$ is a topological ordering if $(u, v) \in E$ implies that $\pi(u) < \pi(v)$.

Algorithm 1 TS(V, E) - An algorithm for sorting the vertices of a DAG $G = (V, E)$ in topological ordering.

1: Base Case: If $|V| = 1$, then let $v \in V$ and return $(\pi(v) = 0)$.
2: Reduction Rule
   1: Let $v \in V$ denote a sink and let E_v denote any path which begins from or ends in v.
   2: return $(TS(V \setminus v, E \setminus E_v)$ extended by $(\pi(v) = |V| - 1))$.

Theorem 10. TS(V, E) *computes a topological ordering of a DAG* $G = (V, E)$.

Definition 31 (Longest path ending in a node)**.** A path Γ that ends in vertex v is a longest path ending in v if $|\Gamma'| \leq |\Gamma|$, for every path Γ' that ends in v.

Definition 32 (Longest path)**.** A path Γ is a longest path if $|\Gamma'| \leq |\Gamma|$, for every path Γ'.

Theorem 11. *If* $G = (V, E)$ *is a DAG, then there exists a longest path that ends in* v*, for every* v*. In addition, there exists a longest path in* G*.*

Algorithm 2 longest-path-lengths(V, E) - An algorithm for computing the lengths of longest paths in a DAG. Returns a delay function $d(v)$.

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topological sort: $(v_0, \ldots, v_{n1}) \leftarrow$ TS(V, E).
For j=0 to $(n1)$ do
    If v_j is a source then $d(v_j) \leftarrow 0$
    Else $d(v_j) \leftarrow 1 + \max\{d(v_i) i < j and (v_i, v_j) \in E\}$

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5 Binary Representation

Definition 33 (Binary string)**.** A binary string is a finite sequence of bits.

Definition 34 (Least significant and most significant bit)**.** The least significant bit of the string $A[i : j]$ is the bit $A[k]$, where $k = \min\{i, j\}$. The most significant bit of the string A[i : j] is the bit A[l], where $l = \max\{i, j\}$.

Definition 35 (Binary number)**.** The natural number a is represented in binary representation by the binary string $A[n - 1 : 0]$ as

$$a \doteq \sum_{i=0}^{n-1} A[i] \cdot 2^i$$

The term 2^i is called the weight of the bit $A[i]$.

Algorithm 3 BR(x, k) - An algorithm for computing a binary representation of a natural number a using k bits

if $x \geq 2^k$ **then**
  return fail
else if $k = 1$ **then**
  return x
else if $x \geq 2^{k-1}$ **then**
  return $\left(1 \circ \mathrm{BR}\left(x - 2^{k-1}, k-1\right)\right)$
else if $x \leq 2^{k-1}$ **then**
  return $\left(0 \circ \mathrm{BR}\left(x, k-1\right)\right)$
end if

6 Propositional Logic

Definition 36 (Constant or bit)**.** A constant or a bit is either 0 or 1. As in the case of bits, 1 is interpreted as "true and a 0 as a false.The term bit is used in Boolean functions and in circuits while the term constants is used in Boolean formulas.

Definition 37 (Variable)**.** A variable is an element in a set of variables. The set of variables is denoted by U. The set U does not contain constants. Variables are usually denoted by upper case letters.

Definition 38 (Connectives)**.** Connectives are operators used to build longer formulas from shorter ones. The set of connectives is denoted by $\mathcal{C}$.

Definition 39 (Arity)**.** The arity of a connective is the number of operands or inputs that it accepts.

Definition 40 (Parse tree)**.** A parse tree is a pair (G, π) where $G = (V, E)$ is a rooted tree and $\pi : V \to \{0, 1\} \cup U \cup \mathcal{C}$ is a labelling function that satisfies

1. A leaf is labelled by a constant or a variable, i.e. if $v \in V$ is a leaf, then $\pi(v) \in \{0, 1\} \cup U$.

2. An interior vertex v is labelled by a connective whose arity is equal to the in-degree of v, i.e. if $v \in V$ is an interior vertex, then $\pi(v) \in \mathcal{C}$ is a connective with arity $\deg_{\text{in}}(v)$.