HARMONIC ANALYSIS : ASSIGNMENT 7

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Exercise 1.
Consider $C[-1,2]$, the space of all complex continuous functions on $[-1,2]$. Which of the following expressions define inner product on $C[-1,2]$? Explain.

(1) $\langle f, g\rangle = \int\limits_{-1}^{2} \left|f(t)+g(t)\right| \mathrm{d}t$
(2) $\langle f, g\rangle = \int\limits_{-1}^{2} f(t)\overline{g(t)} \,\mathrm{d}t + f\left(-\frac{1}{2}\right)\overline{g\left(-\frac{1}{2}\right)}$
(3) $\langle f, g\rangle = f(0)\overline{g(0)} + f(1)\overline{g(1)}$

Solution 1.
(1)

$$\langle f, g\rangle = \int\limits_{-1}^{2} \left|f(t)+g(t)\right| \mathrm{d}t$$

Therefore,

$$\begin{aligned}\overline{\langle g, f\rangle} &= \int\limits_{-1}^{2} \overline{\left|f(t)+g(t)\right|} \,\mathrm{d}t \\ &= \int\limits_{-1}^{2} \left|f(t)+g(t)\right| \\ &= \langle f, g\rangle\end{aligned}$$

Therefore,

$$\begin{aligned}\langle f+h, g\rangle &= \int\limits_{-1}^{2} \left|f(t)+h(t)+g(t)\right| \mathrm{d}t \\ &\neq \int\limits_{-1}^{2} \left|f(t)+g(t)\right| + \left|h(t)+g(t)\right| \mathrm{d}t\end{aligned}$$

$\therefore \langle f+h, g\rangle \neq \langle f, g\rangle + \langle h,$

Therefore, it is not an inner product.

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(2)

$$\langle f, g \rangle = \int_{-1}^{2} f(t)\overline{g(t)}\,\mathrm{d}t + f\left(-\frac{1}{2}\right)\overline{g\left(-\frac{1}{2}\right)}$$

Therefore,

$$\begin{aligned}\overline{\langle g, f \rangle} &= \int_{-1}^{2} \overline{g(t)\overline{f(t)}}\,\mathrm{d}t + \overline{g\left(-\frac{1}{2}\right)\overline{f\left(-\frac{1}{2}\right)}} \\ &= \int_{-1}^{2} f(t)\overline{g(t)}\,\mathrm{d}t + f\left(-\frac{1}{2}\right)\overline{g\left(-\frac{1}{2}\right)} \\ &= \langle f, g \rangle\end{aligned}$$

Therefore,

$$\begin{aligned}\langle f + h, g \rangle &= \int_{-1}^{2} \big(f(t) + h(t)\big)\,\overline{g(t)}\,\mathrm{d}t + \left(f\left(-\frac{1}{2}\right) + h\left(-\frac{1}{2}\right)\right)\overline{g\left(-\frac{1}{2}\right)} \\ &= \int_{-1}^{2} f(t)\overline{g(t)}\,\mathrm{d}t + f\left(-\frac{1}{2}\right)\overline{g\left(-\frac{1}{2}\right)} + \int_{-1}^{2} h(t)\overline{g(t)}\,\mathrm{d}t + h\left(-\frac{1}{2}\right)\overline{g\left(-\frac{1}{2}\right)} \\ &= \langle f, g \rangle + \langle h, g \rangle\end{aligned}$$

Therefore,

$$\begin{aligned}\langle \alpha f, g \rangle &= \int_{-1}^{2} \alpha f(t)\overline{g(t)}\,\mathrm{d}t + \alpha f\left(-\frac{1}{2}\right)\overline{g\left(-\frac{1}{2}\right)} \\ &= \alpha \langle f, g \rangle\end{aligned}$$

Therefore,

$$\begin{aligned}\langle f, f \rangle &= \int_{-1}^{2} f(t)\overline{f(t)}\,\mathrm{d}t + f\left(-\frac{1}{2}\right)\overline{f\left(-\frac{1}{2}\right)} \\ &= \int_{-1}^{2} \left|f(t)\right|^2\,\mathrm{d}t + \left|f\left(-\frac{1}{2}\right)\right|^2 \\ &\geq 0\end{aligned}$$

Therefore, it is an inner product.

(3)

$$\langle f, g \rangle = f(0)\overline{g(0)} + f(1)\overline{g(1)}$$

Therefore,

$$\begin{aligned}\overline{\langle g, f\rangle} &= \overline{g(0)\overline{f(0)} + g(1)\overline{f(1)}}\\ &= \overline{g(0)\overline{f(0)}} + \overline{g(1)\overline{f(1)}}\\ &= f(0)\overline{g(0)} + f(1)\overline{g(1)}\\ &= \langle f, g\rangle\end{aligned}$$

Therefore,

$$\begin{aligned}\langle f + h, g\rangle &= \big(f(0) + h(0)\big)\,\overline{g(0)} + \big(f(1) + h(1)\big)\,\overline{g(1)}\\ &= f(0)\overline{g(0)} + f(1)\overline{g(1)} + h(0)\overline{g(0)} + h(1)\overline{g(1)}\\ &= \langle f, g\rangle + \langle h, g\rangle\end{aligned}$$

Therefore,

$$\begin{aligned}\langle \alpha f, g\rangle &= \alpha f(0)\overline{g(0)} + \alpha f(1)\overline{g(1)}\\ &= \alpha\langle f, g\rangle\end{aligned}$$

Therefore,

$$\begin{aligned}\langle f, f\rangle &= f(0)\overline{f(0)} + f(1)\overline{f(1)}\\ &= \big|f(0)\big|^2 + \big|f(1)\big|^2\\ &\geq 0\end{aligned}$$

Therefore, it is an inner product.

Exercise 2.
Let V be the space of all real, twice continuously differentiable functions of $[-\pi, \pi]$. Is

$$\langle f, g\rangle = f(-\pi)g(-\pi) + \int\limits_{-\pi}^{\pi} f''(x)g''(x)\,\mathrm{d}x$$

an inner product on V?

Solution 2.

$$\langle f, g\rangle = f(-\pi)g(-\pi) + \int\limits_{-\pi}^{\pi} f''(x)g''(x)\,\mathrm{d}x$$

Therefore,

$$\begin{aligned}\overline{\langle g, f\rangle} &= \overline{g(-\pi)f(-\pi)} + \int\limits_{-\pi}^{\pi} \overline{f''(x)g''(x)}\,\mathrm{d}x\\ &= g(-\pi)f(-\pi) + \int\limits_{-\pi}^{\pi} f''(x)g''(x)\,\mathrm{d}x\\ &= \langle f, g\rangle\end{aligned}$$

Therefore,

$$\begin{aligned}\langle f+h, g\rangle &= \big(f(-\pi)+h(-\pi)\big)\, g(-\pi) + \int\limits_{-\pi}^{\pi} \big(f''(x)+h''(x)\big)\, g''(x)\,\mathrm{d}x \\ &= f(-\pi)g(-\pi) + \int\limits_{-\pi}^{\pi} \big(f''(x)g''(x)\big)\,\mathrm{d}x + h(-\pi)g(-\pi) + \int\limits_{-\pi}^{\pi} \big(h''(x)g''(x)\big)\,\mathrm{d}x \\ &= \langle f, g\rangle + \langle h, g\rangle\end{aligned}$$

Therefore,

$$\begin{aligned}\langle \alpha f, g\rangle &= \alpha f(-\pi)g(-\pi) + \int\limits_{-\pi}^{\pi} \big(\alpha f''(x)g''(x)\big)\,\mathrm{d}x \\ &= \alpha\langle f, g\rangle\end{aligned}$$

Therefore,

$$\begin{aligned}\langle f, f\rangle &= f(-\pi)f(-\pi) + \int\limits_{-\pi}^{\pi} f''(x)f''(x)\,\mathrm{d}x \\ &= \big(f(-\pi)\big)^2 + \int\limits_{-\pi}^{\pi} \big(f''(x)\big)^2\,\mathrm{d}x \\ &\geq 0\end{aligned}$$

Therefore, it is an inner product.

Exercise 3.
Consider $C^1[0,1]$, the space of all complex continuously differentiable functions on $[0,1]$. Which of the following expressions define inner product on $C^1[0,1]$? Explain.

(1) $\langle f, g\rangle = f(0)\overline{g(0)} + \int\limits_0^1 f'(t)\overline{g'(t)}\,\mathrm{d}t$
(2) $\langle f, g\rangle = f(0)\overline{g(0)} + f'(1)\overline{g'(1)}$

Solution 3.
(1)

$$\langle f, g\rangle = f(0)\overline{g(0)} + \int\limits_0^1 f'(t)\overline{g'(t)}\,\mathrm{d}t$$

Therefore,

$$\begin{aligned}\overline{\langle g, f\rangle} &= \overline{g(0)f(0)} + \int_0^1 \overline{g'(t)\overline{f'(t)}}\,\mathrm{d}t \\ &= f(0)\overline{g(0)} + \int_0^1 f'(t)\overline{g'(t)}\,\mathrm{d}t \\ &= \langle f, g\rangle\end{aligned}$$

Therefore,

$$\begin{aligned}\langle f+g, h\rangle &= \big(f(0)+h(0)\big)\,\overline{g(0)} + \int_0^1 \big(f'(t)+h'(t)\big)\,\overline{g'(t)}\,\mathrm{d}t \\ &= f(0)\overline{g(0)} + h(0) + \overline{g(0)} + \int_0^1 f'(t)\overline{g'(t)}\,\mathrm{d}t + \int_0^1 h'(t)\overline{g'(t)}\,\mathrm{d}t \\ &= \langle f, g\rangle + \langle h, g\rangle\end{aligned}$$

Therefore,

$$\begin{aligned}\langle \alpha f, g\rangle &= \alpha f(0)\overline{g(0)} + \int_0^1 \alpha f'(t)\overline{g'(t)}\,\mathrm{d}t \\ &= \alpha\langle f, g\rangle\end{aligned}$$

Therefore,

$$\begin{aligned}\langle f, f\rangle &= f(0)\overline{f(0)} + \int_0^1 f'(t)\overline{f'(t)}\,\mathrm{d}t \\ &= \big|f(0)\big|^2 + \int_0^1 \big|f'(t)\big|^2\,\mathrm{d}t \\ &\geq 0\end{aligned}$$

Therefore, it is an inner product.

(2)

$$\langle f, g\rangle = f(0)\overline{g(0)} + f'(1)\overline{g'(1)}$$

Therefore,

$$\begin{aligned}\overline{\langle g, f\rangle} &= \overline{g(0)\overline{f(0)} + g'(1)\overline{f'(1)}} \\ &= \overline{g(0)\overline{f(0)}} + \overline{g'(1)\overline{f'(1)}} \\ &= f(0)\overline{g(0)} + f'(1)\overline{g'(1)} \\ &= \langle f, g\rangle\end{aligned}$$

Therefore,

$$\begin{aligned}\langle f+h, g\rangle &= \big(f(0)+h(0)\big)\,\overline{g(0)} + \big(f'(1)+h'(1)\big)\,\overline{g'(1)} \\ &= f(0)\overline{g(0)} + h(0)\overline{g(0)} + f'(1)\overline{g'(1)} + h'(1)\overline{g'(1)} \\ &= \langle f, g\rangle + \langle h, g\rangle\end{aligned}$$

Therefore,

$$\begin{aligned}\langle \alpha f, g\rangle &= \alpha f(0)\overline{g(0)} + \alpha f'(1)\overline{g'(1)} \\ &= \alpha\langle f, g\rangle\end{aligned}$$

Therefore,

$$\begin{aligned}\langle f, f\rangle &= f(0)\overline{f(0)} + f'(1)\overline{f'(1)} \\ &= \big|f(0)\big|^2 + \Big|f'(1)^2\Big| \\ &\geq 0\end{aligned}$$

Therefore, it is an inner product.

Exercise 4.
Let V be an inner product space. Prove that for all $u, v \in V$,

$$\langle u, v\rangle = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2$$

Solution 4.

$$\begin{aligned}\frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2 &= \frac{1}{4}\Big(\|u+v\|^2 - \|u-v\|^2\Big) \\ &= \frac{1}{4}\left(\langle u+v, u+v\rangle - \langle u-v, u-v\rangle\right) \\ &= \frac{1}{4}\Big(\big(\langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle\big) - \big(\langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v, \\ &= \frac{1}{4}\Big(2\left(\langle u,v\rangle + \langle v,u\rangle\right)\Big) \\ &= \frac{1}{2}\Big(\langle u,v\rangle + \overline{\langle u,v\rangle}\Big) \\ &= \Re\left(\langle u,v\rangle\right)\end{aligned}$$

Therefore, if V is a subset of $\mathbb{R}$, then the equality holds.