

Harmonic Analysis

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1 Lecturer Information

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2 Required Reading

1. Folland, G.B.: Fourier Analysis and its applications, Wadsworth & Brooks/Cole mathematics series, 1992

3 Additional Reading

1. Katznelson, Yitzhak. An introduction to Harmonic analysis. Cambridge University Press, 2004.

Part I

Basic Definitions and Theorems

1 Sequences and Series

Definition 1 (Convergent series). The series $\sum_{n=0}^{\infty} a_n$ is said to converge if the sequence of partial sums $S_N = \sum_{n=0}^N a_n$ converges to a finite limit.

Definition 2 (Pointwise convergence of sequence of functions). Let $D \subseteq \mathbb{R}$, and $\{f_n(x) : D \rightarrow \mathbb{R}\}$ be a sequence of functions. $f_n(x)$ is said to converge pointwise, to a limit function $f(x)$ on D , if $\forall \varepsilon > 0, \forall x \in D, \exists N \in \mathbb{N}$, such that $\forall n > N, |f_n(x) - f(x)| < \varepsilon$.

Definition 3 (Uniform convergence of sequence of functions). Let $D \subseteq \mathbb{R}$, and $\{f_n(x) : D \rightarrow \mathbb{R}\}$ be a sequence of functions. $f_n(x)$ is said to converge uniformly to $f(x)$ on D if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$, such that, $\forall n > N, \forall x \in D, |f_n(x) - f(x)| < \varepsilon$.

Theorem 1. If $\{f_n(x)\}_{n=1}^{\infty}$ are continuous functions, and $f_n(x) \xrightarrow{U} f(x)$, then $f(x)$ is also continuous.

Theorem 2. If a sequence of functions converges pointwise as well as uniformly, then the limit function must be the same.

Theorem 3 (Weierstrass M-test). If $|u_k(x)| \leq c_k$ on D for $k \in \{1, 2, 3, \dots\}$ and the numerical series $\sum_{k=1}^{\infty} c_k$ converges, then the series of functions $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on D .

2 Periodic Functions

Definition 4 (Periodic functions). A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be periodic if $\exists 0 < L \in \mathbb{R}$, such that $\forall x \in \mathbb{R}$,

$$f(x) = f(x + L)$$

For a function $f(x) = k$, as any positive number is a period, there is no minimum L . Hence, $\nexists L^*$.

If there exists a minimum L , it is called L^* , the fundamental period.

3 Odd and Even Functions

Definition 5 (Odd functions). A function is said to be odd if $f(-x) = -f(x)$.

Odd functions are symmetric about the origin.

Definition 6 (Even functions). A function is said to be even if $f(-x) = f(x)$.

Even functions are symmetric about the y -axis.

Theorem 4. *If $h(x)$ is odd,*

$$\int_{-L}^L h(x) \, dx = 0$$

Part II

Introduction to Fourier Series

1 Real Fourier Series

Definition 7. Let $f : [-L, L] \rightarrow \mathbb{R}$, where $L > 0$. If $\forall x \in [-L, L]$, then

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right)$$

Theorem 5. Let $L > 0$, $m \in \mathbb{W}$, $n \in \mathbb{W}$.

Then

$$\int_{-L}^L \cos\left(\frac{m\pi}{L}x\right) \cos\left(\frac{n\pi}{L}x\right) dx = \begin{cases} 0 & ; \quad m \neq n \\ L & ; \quad m = n \neq 0 \\ 2L & ; \quad m = n = 0 \end{cases}$$

Proof.

$$\begin{aligned} E &= \int_{-L}^L \cos\left(\frac{m\pi}{L}x\right) \cos\left(\frac{n\pi}{L}x\right) dx \\ &= \int_{-L}^L \frac{1}{2} \left(\cos\left((m+n)\frac{\pi}{L}x\right) + \cos\left((m-n)\frac{\pi}{L}x\right) \right) dx \end{aligned}$$

$$\because \cos \alpha \cos \beta = \frac{\cos(\alpha+\beta)}{2} + \frac{\cos(\alpha-\beta)}{2}$$

If $m \neq n$,

$$\begin{aligned} E &= \frac{1}{2} \left(\frac{\sin\left((m+n)\frac{\pi}{L}x\right)}{(m+n)\frac{\pi}{L}} + \frac{\sin\left((m-n)\frac{\pi}{L}x\right)}{(m-n)\frac{\pi}{L}} \right) \Bigg|_{-L}^L \\ &= 0 \end{aligned}$$

If $m = n \neq 0$,

$$\begin{aligned} E &= \int_{-L}^L \frac{1}{2} \left(\cos\left(2m\frac{\pi}{L}x\right) + 1 \right) dx \\ &= \frac{1}{2} \int_{-L}^L \cos\left(2m\frac{\pi}{L}x\right) dx + \frac{1}{2}x \Bigg|_{-L}^L \\ &= L \end{aligned}$$

If $m = n = 0$,

$$\begin{aligned} E &= \int_{-L}^L \cos(0) \cos(0) \, dx \\ &= x \Big|_{-L}^L \\ &= 2L \end{aligned}$$

□

Theorem 6. Let $L > 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}$.

Then

$$\int_{-L}^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) \, dx = \begin{cases} 0 & ; \quad m \neq n \\ L & ; \quad m = n \end{cases}$$

Theorem 7. Let $L > 0$, $m \in \mathbb{W}$, $n \in \mathbb{W}$.

Then

$$\int_{-L}^L \sin\left(\frac{m\pi}{L}x\right) \cos\left(\frac{n\pi}{L}x\right) \, dx = 0$$

Assuming $f(x)$ is known, and assuming that it can be integrated term by term,

$$\begin{aligned} \int_{-L}^L f(x) \, dx &= \int_{-L}^L \frac{1}{2} a_0 \, dx + \sum_{n=1}^{\infty} a_n \int_{-L}^L \cos\left(n \frac{\pi}{L} x\right) \, dx + b_n \int_{-L}^L \sin\left(n \frac{\pi}{L} x\right) \, dx \\ \therefore \int_{-L}^L f(x) \, dx &= \frac{1}{2} \int_{-L}^L a_0 \, dx \\ &= \frac{1}{2} a_0 \cdot 2L \\ \therefore a_0 &= \frac{1}{L} \int_{-L}^L f(x) \, dx \end{aligned}$$

Similarly, multiplying the series with $\cos\left(m \frac{\pi}{L} x\right)$ for $m \neq 0$ and integrating,

$$a_m = \frac{1}{L} \int_{-L}^L f(x) \cos\left(m \frac{\pi}{L} x\right) \, dx$$

for $m \in \mathbb{N}$.

Similarly, for $m \in \mathbb{N} \setminus \{0\}$,

$$b_m = \frac{1}{L} \int_{-L}^L f(x) \sin\left(m \frac{\pi}{L} x\right) dx$$

Definition 8. The expansion

$$f(x) \approx \frac{1}{2}a_0 + \sum_{i=1}^{\infty} \left(a_n \cos\left(n \frac{\pi}{L} x\right) + b_n \sin\left(n \frac{\pi}{L} x\right) \right)$$

where, for $m \in \mathbb{N}$,

$$a_m = \frac{1}{L} \int_{-L}^L f(x) \cos\left(m \frac{\pi}{L} x\right) dx$$

and, for $m \in \mathbb{N} \setminus \{0\}$,

$$b_m = \frac{1}{L} \int_{-L}^L f(x) \sin\left(m \frac{\pi}{L} x\right) dx$$

is called the Fourier Series of $f(x)$.

2 Complex Fourier Series

By Euler's formula,

$$\begin{aligned} \cos \theta &= \frac{1}{2} (e^{i\theta} + e^{-i\theta}) \\ \sin \theta &= \frac{1}{2i} (e^{i\theta} - e^{-i\theta}) \end{aligned}$$

Therefore,

$$\frac{1}{2i} = -\frac{i}{2}$$

Therefore, substituting in the Fourier series,

$$\begin{aligned}
f(x) &\approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \frac{1}{2} \left(e^{\frac{in\pi}{L}x} + e^{-\frac{in\pi}{L}x} \right) - b_n \frac{i}{2} \left(e^{\frac{in\pi}{L}x} - e^{-\frac{in\pi}{L}x} \right) \right) \\
&\approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(e^{\frac{in\pi}{L}x} \left(\frac{1}{2}a_n - \frac{i}{2}b_n \right) + e^{-\frac{in\pi}{L}x} \left(\frac{1}{2}a_n + \frac{i}{2}b_n \right) \right) \\
&\approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(e^{\frac{in\pi}{L}x} \left(\frac{1}{2}a_n - \frac{i}{2}b_n \right) \right) + \sum_{n=-\infty}^{-1} \left(e^{\frac{in\pi}{L}x} \left(\frac{1}{2}a_n + \frac{i}{2}b_n \right) \right) \\
&\approx \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{L}x}
\end{aligned}$$

3 Bessel's Inequality

Definition 9 (Piecewise continuous functions). $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be piecewise continuous if, for every finite interval $[a, b]$ there is a finite number of discontinuity points, and the one-sided limits at each of these points are also finite.

Definition 10 (Piecewise continuously differentiable functions). $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be piecewise continuously differentiable if it is piecewise continuous, and

$$\lim_{h \rightarrow 0^+} \frac{f(x+h) - f(x^+)}{h} < \infty$$

and

$$\lim_{h \rightarrow 0^-} \frac{f(x+h) - f(x^-)}{h} < \infty$$

Theorem 8 (Bessel's Inequality). *Let $f(x)$ be a piecewise continuous function defined on $[-L, L]$. Then*

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \leq \frac{1}{L} \int_{-L}^L f(x)^2 dx$$

4 Riemann-Lebesgue's Lemma

Theorem 9 (Riemann-Lebesgue's Lemma). *If $f(x)$ is piecewise continuous on $[-L, L]$, then*

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 0$$

Proof. By Bessel's Inequality,

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \leq \int_{-L}^L f(x)^2 dx$$

$$\therefore \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 < \infty$$

As the function is piecewise continuous in $[-L, L]$, its integral from $-L$ to L is finite.

Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n^2 &\leq \lim_{n \rightarrow \infty} a_n^2 + b_n^2 \\ \therefore \lim_{n \rightarrow \infty} &\leq 0 \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} a_n = 0$$

Similarly,

$$\lim_{n \rightarrow \infty} b_n = 0$$

□

Exercise 1.

If $f(x)$ is piecewise continuous on $[-\pi, \pi]$, show that

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} f(x) \sin \left(\left(n + \frac{1}{2} \right) x \right) dx = 0$$

Solution 1.

$$\sin \left(\left(n + \frac{1}{2} \right) x \right) = \sin(nx) \cos \left(\frac{1}{2}x \right) + \cos(nx) \sin \left(\frac{1}{2}x \right)$$

Therefore, the limit is

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} f(x) \cos \left(\frac{1}{2}x \right) \sin(nx) dx \\ &\quad + \lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} f(x) \sin \left(\frac{1}{2}x \right) \cos(nx) dx \end{aligned}$$

Let

$$g_1 = f(x) \cos\left(\frac{1}{2}x\right)$$

$$g_2 = f(x) \sin\left(\frac{1}{2}x\right)$$

Therefore,

$$\lim_{n \rightarrow \infty} (\pi b_n(g_1) + \pi a_n(g_2)) = 0$$

Therefore,

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} f(x) \sin\left(\left(n + \frac{1}{2}\right)x\right) dx = 0$$

5 Dirichlet's Kernel

Definition 11 (Dirichlet kernel).

$$D_m(t) = \frac{1}{2} + \sum_{n=1}^m \cos(nt)$$

is called the Dirichlet kernel of order m .

Theorem 10 (Second representation of Dirichlet's kernel). *Let $m \in \mathbb{N}$. Then, for $t \neq 2\pi k$, where $k \in \mathbb{Z}$,*

$$D_m(t) = \frac{1}{2} + \cos(t) + \cos(2t) + \cdots + \cos(mt)$$

$$= \frac{\sin\left(\left(m + \frac{1}{2}\right)t\right)}{2 \sin\left(\frac{1}{2}t\right)}$$

Theorem 11. *Let*

$$S_m(f, x) = \frac{1}{2}a_0 + \sum_{n=1}^m a_n \cos(nx) + b_n \sin(nx)$$

Then,

$$S_m(f, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \left(\frac{1}{2} \sum_{n=1}^m \cos(nt) \right) dt$$

Proof.

$$\begin{aligned}
S_m(f, x) &= \frac{1}{2}a_0 + \sum_{n=1}^m a_n \cos(nx) + b_n \sin(nx) \\
&= \frac{1}{2} \underbrace{\left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \, ds \right)}_{a_0} \\
&\quad + \sum_{n=1}^m \underbrace{\left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \cos(ns) \, ds \right)}_{a_n} \cos(nx) \\
&\quad + \sum_{n=1}^m \underbrace{\left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \sin(ns) \, ds \right)}_{b_n} \sin(nx) \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \left(\frac{1}{2} + \sum_{n=1}^m \cos(ns) \cos(nx) + \sin(ns) \sin(nx) \right) \, ds \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \left(\frac{1}{2} + \sum_{n=1}^m \cos(n(s-x)) \right) \, ds
\end{aligned}$$

Let

$$\begin{aligned}
t &= s - x \\
\therefore dt &= ds
\end{aligned}$$

Therefore,

$$\begin{aligned}
S_m(f, x) &= \frac{1}{\pi} \int_{-\pi-x}^{\pi-x} f(t+x) \left(\frac{1}{2} + \sum_{n=1}^m \cos(nt) \right) \, dt \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t+x) D_m(t) \, dt \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t+x) D_m(-t) \, dt \\
&= \frac{1}{\pi} (f(t) * D_m(t))
\end{aligned}$$

As the function is 2π -periodic, the limits can be changed from $-\pi-x$ and $\pi-x$ to $-\pi$ and π .

□

Theorem 12 (Dirichlet Theorem). *Let $f : [-\pi, \pi] \rightarrow \mathbb{R}$ be a piecewise continuously differentiable function.* *This theorem is also valid for $[-L, L]$.*

Then, $\forall x \in (-\pi, \pi)$,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = \frac{f(x^-) + f(x^+)}{2}$$

and for $x = \pi$ or $x = -\pi$,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = \frac{f(\pi^-) + f(-\pi^+)}{2}$$