Harmonic Analysis : Recitations

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Contents

1 Instructor Information

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Part I
Fourier Series

1 Fourier Series

Definition 1 (Real Fourier series)**.** Let $f : [-L, L] \in \mathbb{C}$ be a piecewise continuous function.
The series

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

is called the Fourier series of $f(x)$, where

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, \mathrm{d}x$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(nx) \, \mathrm{d}x$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(nx) \, \mathrm{d}x$$

Theorem 1. *If $f(x)$ is an even function, then the appropriate Fourier series is*

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

If $f(x)$ is an odd function, then the appropriate Fourier series is

$$f(x) \approx \sum_{n=1}^{\infty} a_n \sin(nx)$$

If $f(x)$ is odd, its graph always passes through the origin. Therefore, it can be represented by a summation of sine functions, which also pass through the origin, and there is no need for a term, i.e. $\frac{a_0}{2}$, to change its position at the origin.

Definition 2 (Complex Fourier series)**.** Let $f : [-L, L] \in \mathbb{C}$ be a piecewise continuous function.
The series

$$f(x) \approx \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

is called the complex Fourier series of $f(x)$, where

$$c_n = \frac{1}{2L} \int\limits_{-L}^{L} f(x) e^{-inx} \, \mathrm{d}x$$

Recitation 1 – Exercise 1.
Calculate the real Fourier series of

$$f(x) = 2x - 2\pi$$

Recitation 1 – Solution 1.
As x is an odd function, the real Fourier series of x, in the interval $[-\pi, \pi]$ is

$$x \approx \sum_{n=1}^{\infty} b_n \sin(nx)$$

where

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int\limits_{-\pi}^{\pi} x \sin(nx) \, \mathrm{d}x \\
&= \frac{1}{\pi} \left(x \int \sin(nx) \, \mathrm{d}x - \int 1 \left(\int \sin(nx) \, \mathrm{d}x \right) \mathrm{d}x \right) \Bigg|_{-\pi}^{\pi} \\
&= \frac{1}{\pi} \left(- \frac{x \cos(nx)}{n} \right) \Bigg|_{-\pi}^{\pi} + \frac{1}{\pi} \int\limits_{-\pi}^{\pi} \frac{\cos(nx)}{n} \, \mathrm{d}x \\
&= \frac{1}{\pi} \left(- \frac{\pi \cos(n\pi) + \pi \cos(-n\pi)}{n} \right) + \frac{1}{\pi} \cancelto{0}{\frac{\sin(nx)}{n^2} \Bigg|_{-\pi}^{\pi}} \\
&= - \frac{\cos(n\pi) + \cos(n\pi)}{n} \\
&= -2 \frac{\cos(n\pi)}{n} \\
&= -2 \frac{(-1)^n}{n}
\end{aligned}$$

Therefore,

$$x \approx 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

Therefore,

$$2x - 2\pi \approx \left(4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) \right) - 2\pi$$

2 Bessel's Inequality

Definition 3 (Piecewise continuous functions). $f : \mathbb{R} \to \mathbb{R}$ is said to be piecewise continuous if, for every finite interval $[a, b]$ there is a finite number of discontinuity points, and the one-sided limits at each of these points are also finite.

Definition 4 (Piecewise continuously differentiable functions). $f : \mathbb{R} \to \mathbb{R}$ is said to be piecewise continuously differentiable if it is piecewise continuous, and

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x^+)}{h} < \infty$$

and

$$\lim_{h \to 0^-} \frac{f(x+h) - f(x^-)}{h} < \infty$$

Theorem 2 (Bessel's Inequality). *Let $f(x)$ be a piecewise continuous function defined on $[-L, L]$. Then*

$$\frac{1}{2} {a_0}^2 + \sum_{n=1}^{\infty} {a_n}^2 + {b_n}^2 \leq \frac{1}{L} \int\limits_{-L}^{L} f(x)^2 \, \mathrm{d}x$$

3 Riemann-Lebesgue's Lemma

Theorem 3 (Riemann-Lebesgue's Lemma). *If $f(x)$ is piecewise continuous on $[-L, L]$, then*

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = 0$$

4 Dirichlet's Kernel

Definition 5 (Dirichlet kernel)**.**

$$\begin{aligned} D_m(t) &= \frac{1}{2}\sum_{n=-m}^{m} e^{-int} \\ &= \frac{1}{2} + \sum_{n=1}^{m} \cos(nt) \\ &= \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{2\sin\frac{t}{2}} \end{aligned}$$

is called the Dirichlet kernel of order m.

Theorem 4 (Second representation of Dirichlet's kernel)**.** *Let $m \in \mathbb{N}$. Then, for $t \neq 2\pi k$, where $k \in \mathbb{Z}$,*

$$\begin{aligned} D_m(t) &= \frac{1}{2} + \cos(t) + \cos(2t) + \cdots + \cos(mt) \\ &= \frac{\sin\left(\left(m+\frac{1}{2}\right)t\right)}{2\sin\left(\frac{1}{2}t\right)} \end{aligned}$$

Theorem 5. *Let*

$$S_m(f,x) = \frac{1}{2}a_0 + \sum_{n=1}^{m} a_n\cos(nx) + b_n\sin(nx)$$

Then,

$$S_m(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\left(\frac{1}{2}\sum_{n=1}^{m}\cos(nt)\right)\mathrm{d}t$$

Theorem 6 (Dirichlet Theorem)**.** *Let $f : [-\pi,\pi] \to \mathbb{R}$ be a piecewise continuously differentiable function.*
Then, $\forall x \in (-\pi,\pi)$,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx) = \frac{f(x^-)+f(x^+)}{2}$$

and for $x=\pi$ or $x=-\pi$,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx) = \frac{f(\pi^-)+f(-\pi^+)}{2}$$

Recitation 2 – Exercise 1.
The Fourier series of x^2 of given to be

$$x^2 \approx \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx)$$

Calculate

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Recitation 2 – Solution 1.
As x^2 is continuous, with a continuous derivative, Dirichlet Theorem is applicable.
Therefore, let

$$x = \pi$$

Therefore, by Dirichlet Theorem,

$$\begin{aligned}
\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx) &= \frac{\left(\pi^-\right)^2 + \left((-\pi)^+\right)^2}{2} \\
\therefore \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(-1)^n &= \pi^2 \\
\therefore \frac{\pi^2}{4} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} &= \pi^2 \\
\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{1}{4}\left(\pi^2 - \frac{\pi^2}{3}\right) \\
&= \frac{\pi^2}{6}
\end{aligned}$$

Recitation 2 – Exercise 2.
The Fourier series of

$$f(x) = \begin{cases} x & ; \quad 0 \leq x \leq \pi \\ 0 & ; \quad -\pi \leq x \leq 0 \end{cases}$$

is given to be

$$f(x) \approx \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}}{n}\sin(nx) - \frac{2}{\pi(2n-1)^2}\cos\left((2n-1)x\right)\right)$$

Let this Fourier series be denoted by $S(x)$.
Calculate

1. $1+\frac{1}{3^2}+\frac{1}{5^2}+\dots$
2. $S\left(\frac{\pi}{2}\right)$

Recitation 2 – Solution 2.
1.

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\dots=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

Therefore, for $x=0$, by Dirichlet Theorem,

$$\begin{aligned}\frac{\pi}{4}+\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n}\sin(0)-\frac{2}{\pi(2n-1)^2}\cos(0)\right)&=\frac{f(0^-)+f(0^+)}{2}\\ \therefore\frac{\pi}{4}-\sum_{n=1}^{\infty}\left(\frac{2}{\pi(2n-1)^2}\right)&=0\\ \therefore\frac{\pi}{4}-\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{(2n-1)^2}\right)&=0\\ \therefore\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}&=\frac{\pi^2}{8}\end{aligned}$$

2. By Dirichlet Theorem,

$$\begin{aligned}S\left(\frac{\pi}{2}\right)&=\frac{f\left(\frac{\pi}{2}^-\right)+f\left(\frac{\pi}{2}^+\right)}{2}\\ &=\frac{\pi}{2}\end{aligned}$$

Theorem 7. *If f is a piecewise continuous and periodic function with period of 2π, then*

$$\begin{aligned}S_m(x)&=\frac{a_0}{2}+\sum_{n=1}^{m}\left(a_n\cos(nx)+b_n\sin(nx)\right)\\ &=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)D_m(t)\,\mathrm{d}t\end{aligned}$$

Recitation 3 – Exercise 1.
Calculate the limit

$$L = \lim_{n\to\infty} \int_{-n}^{n} \sin\left(\frac{2n+1}{2}t\right) \frac{\cos^2\left(\frac{\pi}{4}+t\right)+\pi^2}{\sin\left(\frac{t}{2}\right)}\,\mathrm{d}t$$

Recitation 3 – Solution 1.

$$\begin{aligned} L &= \lim_{n\to\infty} \int_{-n}^{n} \sin\left(\frac{2n+1}{2}t\right) \frac{\cos^2\left(\frac{\pi}{4}+t\right)+\pi^2}{\sin\left(\frac{t}{2}\right)}\,\mathrm{d}t \\ &= 2\lim_{n\to\infty} \int_{-\pi}^{\pi} \left(\cos^2\left(\frac{\pi}{4}+t\right)+\pi^2\right) \frac{\sin\left(n+\frac{1}{2}t\right)}{2\sin\left(\frac{t}{2}\right)}\,\mathrm{d}t \\ &= 2\lim_{n\to\infty} \int_{-\pi}^{\pi} \left(\cos^2\left(\frac{\pi}{4}+t\right)+\pi^2\right) D_n(t)\,\mathrm{d}t \end{aligned}$$

Let

$$f(x) = \cos^2 x + \pi^2$$

Let S_n be the partial sum of the Fourier series.
Therefore,

$$S_n = \frac{1}{\pi}\int_{-\pi}^{\pi} D_n(t)\left(\cos^2(x+t)+\pi^2\right)\mathrm{d}t$$

Therefore,

$$\begin{aligned} L &= 2\pi \lim_{n\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} \left(\cos^2\left(\frac{\pi}{4}+t\right)+\pi^2\right) D_n(t)\,\mathrm{d}t \\ &= 2\pi \lim_{n\to\infty} S_n\left(\frac{\pi}{4}\right) \\ &= 2\pi \frac{f\left(\frac{\pi}{4}^+\right)+f\left(\frac{\pi}{4}^-\right)}{2} \\ &= 2\pi f\left(\frac{\pi}{4}\right) \\ &= 2\pi\left(\cos^2\left(\frac{\pi}{4}\right)+\pi^2\right) \\ &= \pi + 2\pi^3 \end{aligned}$$

5 Fourier Series in a General Interval

Definition 6. Let f be a piecewise continuous function defined on $[a,b]$. The Fourier series over $[a,b]$ is defined as

$$\begin{aligned} f(x) &\approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n x}{b-a}\right) + b_n \sin\left(\frac{2\pi n x}{b-a}\right) \right) \\ &\approx \sum_{-\infty}^{\infty} c_n e^{\frac{2\pi i n x}{b-a}} \end{aligned}$$

where

$$\begin{aligned} a_0 &= \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x \\ a_n &= \frac{2}{b-a} \int_a^b f(x) \cos\frac{2\pi n x}{b-a}\,\mathrm{d}x \\ b_n &= \frac{2}{b-a} \int_a^b f(x) \sin\frac{2\pi n x}{b-a}\,\mathrm{d}x \\ c_n &= \frac{1}{b-a} \int_a^b f(x) e^{\frac{2\pi i n x}{b-a}}\,\mathrm{d}x \end{aligned}$$

Recitation 3 – Exercise 2.
Develop the Fourier series for $\operatorname{sign}(x)$ over $[0,\pi]$.

Recitation 3 – Solution 2.

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^\pi \operatorname{sign}(x)\,\mathrm{d}x \\ &= 2 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^\pi \operatorname{sign}(x) \cos\left(\frac{2\pi n x}{\pi}\right) \mathrm{d}x \\ &= \frac{2}{\pi} \int_0^\pi \cos(2nx)\,\mathrm{d}x \\ &= 0 \end{aligned}$$

$$
\begin{aligned}
b_n &= \frac{2}{\pi} \int\limits_0^{\pi} \operatorname{sign}(x) \sin\left(\frac{2\pi n x}{\pi}\right) \mathrm{d}x \\
&= \frac{2}{\pi} \int\limits_0^{\pi} \sin(2nx)\, \mathrm{d}x \\
&= 0
\end{aligned}
$$

Therefore, over $[0, \pi]$,

$$
\begin{aligned}
\operatorname{sign}(x) &= \frac{2}{2} + \sum_{n=1}^{\infty} 0 \\
&= 1
\end{aligned}
$$

Theorem 8. *Let f be continuous in $[-\pi, \pi]$, with piecewise continuous derivative, and $f(-\pi) = f(\pi)$. Then, the Fourier series converges uniformly on $[-\pi, \pi]$.*

Theorem 9 (Percival Equality)**.** *Let f be a piecewise continuous function in $[-\pi, \pi]$. Then,*

$$
\begin{aligned}
\frac{1}{\pi} \int\limits_{-\pi}^{\pi} |f(x)|^2 \, \mathrm{d}x &= \frac{{a_0}^2}{2} + \sum_{n=1}^{\infty} \left({a_n}^2 + {b_n}^2\right) \\
&= 2 \sum_{n=-\infty}^{\infty} |c_n|^2
\end{aligned}
$$

Recitation 4 – Exercise 1.
Use the Fourier series

$$
x^2 \approx \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos(nx)
$$

to calculate $\sum\limits_{n=1}^{\infty} \frac{1}{n^4}$.

Recitation 4 – Solution 1.
As x^2 is continuous, by Percival Equality,

$$\begin{aligned}\frac{1}{\pi}\int\limits_{-\pi}^{\pi}\left|x^2\right|^2 \mathrm{d}x &= \frac{{a_0}^2}{2} + \sum_{n=1}^{\infty} {a_n}^2 + {b_n}^2 \\ &= \frac{2\pi^4}{9} + \sum_{n=1}^{\infty}\frac{16}{n^2}\end{aligned}$$

Therefore,

$$\begin{aligned}16\sum_{n=1}^{\infty}\frac{1}{n^4} &= \frac{1}{\pi}\int\limits_{\pi}^{\pi} x^4 \,\mathrm{d}x - \frac{2\pi^4}{9} \\ &= \frac{1}{\pi}\left.\frac{x^5}{5}\right|_{-\pi}^{\pi} - \frac{2\pi^4}{9} \\ &= \frac{2\pi^4}{5} - \frac{2\pi^4}{9} \\ &= \frac{8}{45}\pi^4 \\ \therefore \sum_{n=1}^{\infty}\frac{1}{n^4} &= \frac{\pi^4}{90}\end{aligned}$$

Recitation 4 – Exercise 2.
Use the Fourier series

$$e^x \approx \sum_{n=\infty}^{\infty}(-1)^n\frac{e^{\pi}-e^{-\pi}}{2\pi(1-in)}e^{inx}$$

to calculate $\sum\limits_{n-\infty}^{\infty}\frac{1}{n^2+1}$.

Recitation 4 – Solution 2.
As x^2 is continuous, by Percival Equality,

$$\begin{aligned}\frac{1}{\pi}\int\limits_{-\pi}^{\pi}|e^x|^2 \,\mathrm{d}x &= 2\sum_{n=-\infty}^{\infty}|c_n|^2 \\ &= 2\sum_{n=-\infty}^{\infty}\frac{\left(e^{\pi}-e^{-\pi}\right)^2}{4\pi^2|1-in|^2} \\ &= \frac{2\left(e^{\pi}-e^{-\pi}\right)^2}{4\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{n^2+1}\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1} &= \frac{4\pi}{2\left(e^{\pi}-e^{-\pi}\right)^2} \int\limits_{-\pi}^{\pi} \left|e^{x}\right|^2 \mathrm{d}x \\
&= \frac{4\pi}{2\left(e^{\pi}-e^{-\pi}\right)^2} \left.\frac{e^{2x}}{2}\right|_{-\pi}^{\pi} \left|e^{x}\right|^2 \mathrm{d}x \\
&= \frac{4\pi}{2\left(e^{\pi}-e^{-\pi}\right)^2} \frac{e^{2\pi}-e^{-2\pi}}{2} \\
&= \frac{e^{2\pi}-e^{-2\pi}}{\left(e^{\pi}-e^{-\pi}\right)^2} \\
&= \frac{\Big(e^{\pi}+e^{-\pi}\Big)\Big(e^{\pi}-e^{-\pi}\Big)}{\left(e^{\pi}-e^{-\pi}\right)^2} \\
&= \frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}
\end{aligned}$$