HARMONIC ANALYSIS : ASSIGNMENT 8

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Exercise 1.
Find the minimum point of

$$F(\alpha, \beta, \gamma) = \int\limits_{-\pi}^{\pi} \left| f(x) - \big(\alpha + \beta \cos(x) - \gamma \cos(10x)\big) \right|^2 \mathrm{d}x$$

for the following functions.

(1) $\cos^2 x$
(2) x^3
(3) $\sin x$
(4) $1 - \cos(2x)$
(5) $|x|$
(6) $|\sin x|$

Find the projection of the given function on the space spanned by the basis elements and use the best approximation theorem.

Solution 1.

(1)

$$\mathrm{proj}_W(f) = \frac{\langle f, 1 \rangle}{\|1\|^2} 1 + \frac{\big\langle f, \cos(x) \big\rangle}{\big\|\cos(x)\big\|} \cos(x) + \frac{\big\langle f, \cos(10x) \big\rangle}{\big\|\cos(10x)\big\|} \cos(10x)$$

Therefore,

$$\begin{aligned}
\alpha &= \frac{\langle f, 1 \rangle}{\|1\|^2} \\
&= \frac{\big\langle \cos^2(x), 1 \big\rangle}{\|1\|^2} \\
&= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\beta &= \frac{\big\langle f, \cos(x) \big\rangle}{\big\|\cos(x)\big\|^2} \\
&= \frac{\big\langle \cos^2(x), \cos(x) \big\rangle}{\big\|\cos(x)\big\|^2} \\
&= \frac{0}{1} \\
&= 0
\end{aligned}$$

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$$\begin{aligned}\gamma &= \frac{\big\langle f, \cos(10x)\big\rangle}{\big\|\cos(10x)\big\|^2} \\ &= \frac{\big\langle \cos^2(x), \cos(10x)\big\rangle}{\big\|\cos(10x)\big\|^2} \\ &= \frac{0}{1} \\ &= 0\end{aligned}$$

Therefore,

$$\begin{aligned}F(x) &= \alpha + \beta\cos(x) + \gamma\cos(10x) \\ &= \frac{1}{2}\end{aligned}$$

(2)

$$\text{proj}_W(f) = \frac{\langle f, 1\rangle}{\|1\|^2}1 + \frac{\big\langle f, \cos(x)\big\rangle}{\big\|\cos(x)\big\|}\cos(x) + \frac{\big\langle f, \cos(10x)\big\rangle}{\big\|\cos(10x)\big\|}\cos(10x)$$

Therefore,

$$\begin{aligned}\alpha &= \frac{\langle f, 1\rangle}{\|1\|^2} \\ &= \frac{\big\langle x^3, 1\big\rangle}{\|1\|^2} \\ &= \frac{0}{2} \\ &= 0\end{aligned}$$

$$\begin{aligned}\beta &= \frac{\big\langle f, \cos(x)\big\rangle}{\big\|\cos(x)\big\|^2} \\ &= \frac{\big\langle x^3, \cos(x)\big\rangle}{\big\|\cos(x)\big\|^2} \\ &= \frac{0}{1} \\ &= 0\end{aligned}$$

$$\begin{aligned}\gamma &= \frac{\big\langle f, \cos(10x)\big\rangle}{\big\|\cos(10x)\big\|^2} \\ &= \frac{\big\langle x^3, \cos(10x)\big\rangle}{\big\|\cos(10x)\big\|^2} \\ &= \frac{0}{1} \\ &= 1\end{aligned}$$

Therefore,

$$\begin{aligned} F(x) &= \alpha + \beta\cos(x) + \gamma\cos(10x) \\ &= 0 \end{aligned}$$

(3)

$$\mathrm{proj}_W(f) = \frac{\langle f, 1\rangle}{\|1\|^2}1 + \frac{\big\langle f, \cos(x)\big\rangle}{\big\|\cos(x)\big\|}\cos(x) + \frac{\big\langle f, \cos(10x)\big\rangle}{\big\|\cos(10x)\big\|}\cos(10x)$$

Therefore,

$$\begin{aligned} \alpha &= \frac{\langle f, 1\rangle}{\|1\|^2} \\ &= \frac{\big\langle \sin(x), 1\big\rangle}{\|1\|^2} \\ &= \frac{0}{2} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \beta &= \frac{\big\langle f, \cos(x)\big\rangle}{\big\|\cos(x)\big\|^2} \\ &= \frac{\big\langle \sin(x), \cos(x)\big\rangle}{\big\|\cos(x)\big\|^2} \\ &= \frac{0}{1} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \gamma &= \frac{\big\langle f, \cos(10x)\big\rangle}{\big\|\cos(10x)\big\|^2} \\ &= \frac{\big\langle \sin(x), \cos(10x)\big\rangle}{\big\|\cos(10x)\big\|^2} \\ &= \frac{0}{1} \\ &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} F(x) &= \alpha + \beta\cos(x) + \gamma\cos(10x) \\ &= 0 \end{aligned}$$

(4)

$$\mathrm{proj}_W(f) = \frac{\langle f, 1\rangle}{\|1\|^2}1 + \frac{\big\langle f, \cos(x)\big\rangle}{\big\|\cos(x)\big\|}\cos(x) + \frac{\big\langle f, \cos(10x)\big\rangle}{\big\|\cos(10x)\big\|}\cos(10x)$$

Therefore,

$$\begin{aligned}\alpha &= \frac{\langle f, 1\rangle}{\|1\|^2} \\ &= \frac{\big\langle 1-\cos(2x), 1\big\rangle}{\|1\|^2} \\ &= \frac{2}{2} \\ &= 1\end{aligned}$$

$$\begin{aligned}\beta &= \frac{\big\langle f, \cos(x)\big\rangle}{\big\|\cos(x)\big\|^2} \\ &= \frac{\big\langle 1-\cos(2x), \cos(x)\big\rangle}{\big\|\cos(x)\big\|^2} \\ &= \frac{0}{1} \\ &= 0\end{aligned}$$

$$\begin{aligned}\gamma &= \frac{\big\langle f, \cos(10x)\big\rangle}{\big\|\cos(10x)\big\|^2} \\ &= \frac{\big\langle 1-\cos(2x), \cos(10x)\big\rangle}{\big\|\cos(10x)\big\|^2} \\ &= \frac{0}{1} \\ &= 0\end{aligned}$$

Therefore,

$$\begin{aligned}F(x) &= \alpha + \beta\cos(x) + \gamma\cos(10x) \\ &= 1\end{aligned}$$

(5)

$$\mathrm{proj}_W(f) = \frac{\langle f, 1\rangle}{\|1\|^2}1 + \frac{\big\langle f, \cos(x)\big\rangle}{\big\|\cos(x)\big\|}\cos(x) + \frac{\big\langle f, \cos(10x)\big\rangle}{\big\|\cos(10x)\big\|}\cos(10x)$$

Therefore,

$$\begin{aligned}\alpha &= \frac{\langle f, 1\rangle}{\|1\|^2} \\ &= \frac{\big\langle |x|, 1\big\rangle}{\|1\|^2} \\ &= \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}
\beta &= \frac{\big\langle f, \cos(x) \big\rangle}{\big\|\cos(x)\big\|^2} \\
&= \frac{\big\langle |x|, \cos(x) \big\rangle}{\big\|\cos(x)\big\|^2} \\
&= \frac{-\frac{4}{\pi}}{1} \\
&= -\frac{4}{\pi}
\end{aligned}$$

$$\begin{aligned}
\gamma &= \frac{\big\langle f, \cos(10x) \big\rangle}{\big\|\cos(10x)\big\|^2} \\
&= \frac{\big\langle |x|, \cos(10x) \big\rangle}{\big\|\cos(10x)\big\|^2} \\
&= \frac{0}{1} \\
&= 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
F(x) &= \alpha + \beta\cos(x) + \gamma\cos(10x) \\
&= \frac{\pi}{2} - \frac{4}{\pi}\cos(x)
\end{aligned}$$

(6)

$$\mathrm{proj}_W(f) = \frac{\langle f, 1 \rangle}{\|1\|^2} 1 + \frac{\big\langle f, \cos(x) \big\rangle}{\big\|\cos(x)\big\|} \cos(x) + \frac{\big\langle f, \cos(10x) \big\rangle}{\big\|\cos(10x)\big\|} \cos(10x)$$

Therefore,

$$\begin{aligned}
\alpha &= \frac{\langle f, 1 \rangle}{\|1\|^2} \\
&= \frac{\big\langle \big|\sin(x)\big|, 1 \big\rangle}{\|1\|^2} \\
&= \frac{\frac{4}{\pi}}{2} \\
&= \frac{2}{\pi}
\end{aligned}$$

$$\begin{aligned}
\beta &= \frac{\big\langle f, \cos(x) \big\rangle}{\big\|\cos(x)\big\|^2} \\
&= \frac{\big\langle \big|\sin(x)\big|, \cos(x) \big\rangle}{\big\|\cos(x)\big\|^2} \\
&= \frac{0}{1} \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\gamma &= \frac{\left\langle f, \cos(10x) \right\rangle}{\left\|\cos(10x)\right\|^2} \\
&= \frac{\left\langle \left|\sin(x)\right|, \cos(10x) \right\rangle}{\left\|\cos(10x)\right\|^2} \\
&= -\frac{4}{99}1 \\
&= -\frac{4}{99}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
F(x) &= \alpha + \beta\cos(x) + \gamma\cos(10x) \\
&= \frac{2}{\pi} - \frac{4}{99}\cos(10x)
\end{aligned}
$$

Exercise 2.
In the space $C[-1,1]$, let the inner product be defined as

$$
\langle f, g \rangle = \int\limits_{-1}^{1} f(x)\overline{g(x)}\,\mathrm{d}x
$$

Let the subspace W be defined as

$$
W = \operatorname{span}\left\{1, e^{i\pi x}, e^{-i\pi x}\right\}
$$

and let

$$
f(x) = \left|\sin\left(\frac{\pi x}{2}\right)\right|
$$

Find a function $g \in W$ such that $\|f-g\|$ is minimal, and calculate the norm.

Solution 2.

$$
\begin{aligned}
\alpha &= \frac{\langle f, 1 \rangle}{\|1\|^2} \\
&= \frac{\left\langle \left|\sin\left(\frac{\pi x}{2}\right)\right|, 1 \right\rangle}{\|1\|^2} \\
&= \frac{\int\limits_{-1}^{1} \left|\sin\left(\frac{\pi x}{2}\right) \cdot 1\,\mathrm{d}x\right|}{2} \\
&= \frac{2\int\limits_{0}^{1} \sin\left(\frac{\pi x}{2}\right) \cdot 1\,\mathrm{d}x}{2} \\
&= \frac{\frac{4}{\pi}}{2} \\
&= \frac{2}{\pi}
\end{aligned}
$$

$$\begin{aligned}
\beta &= \frac{\left\langle f, e^{i\pi x} \right\rangle}{\left\| e^{i\pi x} \right\|^2} \\
&= \frac{\left\langle \left| \sin\left(\frac{\pi x}{2}\right) \right|, e^{i\pi x} \right\rangle}{\left\| e^{i\pi x} \right\|^2} \\
&= \frac{\int\limits_{-1}^{1} \left| \sin\left(\frac{\pi x}{2}\right) \right| \overline{e^{inx}} \, \mathrm{d}x}{2} \\
&= \frac{2 \int\limits_{0}^{1} \sin\left(\frac{\pi x}{2}\right) e^{-inx} \, \mathrm{d}x}{2} \\
&= \frac{-\frac{4}{3\pi}}{2} \\
&= -\frac{2}{3\pi}
\end{aligned}$$

$$\begin{aligned}
\gamma &= \frac{\left\langle f, e^{-i\pi x} \right\rangle}{\left\| e^{-i\pi x} \right\|^2} \\
&= \frac{\left\langle \left| \sin\left(\frac{\pi x}{2}\right) \right|, e^{-i\pi x} \right\rangle}{\left\| e^{-i\pi x} \right\|^2} \\
&= \frac{\int\limits_{-1}^{1} \left| \sin\left(\frac{\pi x}{2}\right) \right| \overline{e^{-inx}} \, \mathrm{d}x}{2} \\
&= \frac{2 \int\limits_{0}^{1} \sin\left(\frac{\pi x}{2}\right) e^{inx} \, \mathrm{d}x}{2} \\
&= \frac{-\frac{4}{3\pi}}{2} \\
&= -\frac{2}{3\pi}
\end{aligned}$$

Therefore,

$$\begin{aligned}
g &= \alpha + \beta e^{inx} + \gamma e^{-inx} \\
&= \frac{2}{\pi} - \frac{2}{3\pi} e^{inx} - \frac{2}{3\pi} e^{-inx}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle f - g \rangle = \Bigg\langle & \left| \sin\left(\frac{\pi x}{2}\right) \right| - \frac{2}{\pi} - \frac{2}{3\pi} e^{inx} - \frac{2}{3\pi} e^{-inx}, \\
& \left| \sin\left(\frac{\pi x}{2}\right) \right| - \frac{2}{\pi} - \frac{2}{3\pi} e^{inx} - \frac{2}{3\pi} e^{-inx} \Bigg\rangle
\end{aligned}$$

Exercise 3.
Let

$$\lambda_n = \min_{\alpha \in \mathbb{C}} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sqrt{|\cos x|} - \alpha \cos(nx) \right|^2 \mathrm{d}x \right)$$

Calculate $\lim\limits_{n\to\infty} \lambda_n$.

Exercise 4.
Consider the function

$$f(x) = |\cos x|$$

Use the Fourier series of f,

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(2nx)}{4n^2 - 1}$$

to prove that for all $a, b \in \mathbb{C}$, the following condition holds.

$$\int_{-\pi}^{\pi} \big| |\cos x| - a \sin x - b \cos(2x) \big|^2 \mathrm{d}x \geq \pi - \frac{88}{9\pi}$$

Solution 4.
Let

$$W_n = \operatorname{span}\left\{\cos(nx)\right\}$$

Therefore,

$$\operatorname{proj}_{W_n}(f) = \frac{\langle f, \cos(nx) \rangle}{\left\|\cos(nx)\right\|^2} \cos(nx)$$

Therfore, by the best approximation theorem,

$$\lambda_n = \left\| f - \operatorname{proj}_{W_n}(f) \right\|$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} \lambda_n &= \lim_{n\to\infty} \left\| f - \operatorname{proj}_{W_n}(f) \right\| \\ &= \lim_{n\to\infty} \left\| f - \frac{\langle f, \cos(nx) \rangle}{\left\|\cos(nx)\right\|^2} \cos(nx) \right\| \end{aligned}$$

Solution 4.
The standard Fourier coefficients are

$$\begin{aligned} a_2 &= -\frac{4}{\pi} \frac{(-1)^1}{4 \cdot 1^2 - 1} \\ &= \frac{4}{3\pi} \\ b_1 &= 0 \end{aligned}$$

Let

$$I = \int_{-\pi}^{\pi} \big| |\cos x| - a\sin x - b\cos(2x) \big|^2 \,\mathrm{d}x$$

Therefore, by the best approximation theorem, as the Fourier coefficients give the best approximation,

$$\begin{aligned}
I &\geq \int_{-\pi}^{\pi} \big| |\cos x| - b_1 \sin x - a_2\cos(2x) \big|^2 \,\mathrm{d}x \\
&\geq \int_{-\pi}^{\pi} \left| |\cos x| - \frac{4}{3\pi}\cos(2x) \right|^2 \mathrm{d}x \\
&\geq 2\int_{0}^{\pi} \left(\cos(x) - \frac{4}{3\pi}\cos(2x) \right)^2 \mathrm{d}x \\
&\geq 2\int_{0}^{\pi} \cos^2 x - \frac{8}{3\pi}|\cos x|\cos(2x) + \frac{16}{9\pi^2}\cos^2(2x)\,\mathrm{d}x \\
&\geq 2\int_{0}^{\pi} \cos^2 x - \frac{16}{3\pi}\int_{0}^{\pi} |\cos x|\cos(2x)\,\mathrm{d}x + \frac{32}{9\pi^2}\int_{0}^{\pi}\cos^2(2x)\,\mathrm{d}x \\
&\geq 2\frac{\pi}{2} - \frac{16}{3\pi}\frac{2}{3} + \frac{32}{9\pi^2}\frac{\pi}{2} \\
&\geq \pi - \frac{32}{9\pi} + \frac{16}{9\pi^2} \\
&\geq \pi - \frac{88}{9\pi}
\end{aligned}$$

$\square$