Harmonic Analysis : Recitations

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1 Instructor Information

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1 Fourier Series

Definition 1 (Real Fourier series)**.** Let $f : [-L, L] \in \mathbb{C}$ be a piecewise continuous function.
The series

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

is called the Fourier series of $f(x)$, where

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, \mathrm{d}x$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(nx)\, \mathrm{d}x$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(nx)\, \mathrm{d}x$$

Theorem 1. *If $f(x)$ is an even function, then the appropriate Fourier series is*

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

If $f(x)$ is an odd function, then the appropriate Fourier series is

$$f(x) \approx \sum_{n=1}^{\infty} a_n \sin(nx)$$

If $f(x)$ is odd, its graph always passes through the origin. Therefore, it can be represented by a summation of sine functions, which also pass through the origin, and there is no need for a term, i.e. $\frac{a_0}{2}$, to change its position at the origin.

Definition 2 (Complex Fourier series)**.** Let $f : [-L, L] \in \mathbb{C}$ be a piecewise continuous function.
The series

$$f(x) \approx \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

is called the complex Fourier series of $f(x)$, where

$$c_n = \frac{1}{2L} \int\limits_{-L}^{L} f(x) e^{-inx} \,\mathrm{d}x$$

Recitation 1 – Exercise 1.
Calculate the real Fourier series of

$$f(x) = 2x - 2\pi$$

Recitation 1 – Solution 1.
As x is an odd function, the real Fourier series of x, in the interval $[-\pi, \pi]$ is

$$x \approx \sum_{n=1}^{\infty} b_n \sin(nx)$$

where

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int\limits_{-\pi}^{\pi} x \sin(nx) \,\mathrm{d}x \\
&= \frac{1}{\pi} \left(x \int \sin(nx) \,\mathrm{d}x - \int 1 \left(\int \sin(nx) \,\mathrm{d}x \right) \mathrm{d}x \right) \Bigg|_{-\pi}^{\pi} \\
&= \frac{1}{\pi} \left(-\frac{x \cos(nx)}{n} \right) \Bigg|_{-\pi}^{\pi} + \frac{1}{\pi} \int\limits_{-\pi}^{\pi} \frac{\cos(nx)}{n} \,\mathrm{d}x \\
&= \frac{1}{\pi} \left(-\frac{\pi \cos(n\pi) + \pi \cos(-n\pi)}{n} \right) + \frac{1}{\pi} \cancelto{0}{\frac{\sin(nx)}{n^2} \Bigg|_{-\pi}^{\pi}} \\
&= -\frac{\cos(n\pi) + \cos(n\pi)}{n} \\
&= -2 \frac{\cos(n\pi)}{n} \\
&= -2 \frac{(-1)^n}{n}
\end{aligned}$$

Therefore,

$$x \approx 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

Therefore,

$$2x - 2\pi \approx \left(4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) \right) - 2\pi$$