Harmonic Analysis : Review Session

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Exercise 1.
Is $\{1, \cos(nx)\}$ orthogonal and complete in $L^2[0, \pi]$?

Solution 1.
$L^2[0, \pi]$ is the inner product space of all functions defined on the interval $[0, \pi]$ with inner product.
For all $f(x)$, let

$$F(x) = \begin{cases} f(x) & ; \quad 0 \leq x \leq \pi \\ f(-x) & ; \quad -\pi \leq x \leq 0 \end{cases}$$

Therefore, $F(x)$ is even. Therefore, the Fourier series of $F(x)$ will be of the form

$$F(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

Therefore, as $\left\{\frac{1}{\sqrt{2}}, \sin(nx), \cos(nx)\right\}$ is complete and orthogonal in $L^2[-\pi, \pi]$. Therefore, for any $f \in L^2[0, \pi]$, and its equivalent $F(x)$, the function can be described with $\left\{\frac{1}{\sqrt{2}}, \cos(nx)\right\}$. Therefore, the set $\left\{\frac{1}{\sqrt{2}}, \cos(nx)\right\}$ is orthonormal and complete.
Hence, the set $\{1, \cos(nx)\}$ is orthogonal and complete in $L^2[0, \pi]$.