Harmonic Analysis

Aakash Jog

2015-16

Contents

1 Lecturer Information

Barak Sober

Office: Shenkar Physics 201
E-mail: barakino@gmail.com
Telephone: +972 3-640-6024

2 Required Reading

1. Folland, G.B.: Fourier Analysis and its applications, Wadsworth & Brooks/Cole mathematics series, 1992

3 Additional Reading

1. Katznelson, Yitzhak. An introduction to Harmonic analysis. Cambridge University Press, 2004.

Part I
Basic Definitions and Theorems

1 Sequences and Series

Definition 1 (Convergent series)**.** The series $\sum_{n=0}^{\infty} a_n$ is said to converge if the sequence of partial sums $S_N = \sum_{n=0}^{N} a_n$ converges to a finite limit.

Definition 2 (Pointwise convergence of sequence of functions)**.** Let $D \subseteq \mathbb{R}$, and $\{f_n(x) : D \to \mathbb{R}\}$ be a sequence of functions. $f_n(x)$ is said to converge pointwise, to a limit function $f(x)$ on D, if $\forall \varepsilon > 0$, $\forall x \in D$, $\exists N \in \mathbb{N}$, such that $\forall n > N$, $|f_n(x) - f(x)| < \varepsilon$.

Definition 3 (Uniform convergence of sequence of functions)**.** Let $D \subseteq \mathbb{R}$, and $\{f_n(x) : D \to \mathbb{R}\}$ be a sequence of functions. $f_n(x)$ is said to converge uniformly to $f(x)$ on D¡ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, such that, $\forall n > N$, $\forall x \in D$, $|f_n(x) - f(x)| < \varepsilon$.

Theorem 1. *If $\{f_n(x)\}_{n=1}^{\infty}$ are continuous functions, and $f_n(x) \xrightarrow{U} f(x)$, then $f(x)$ is also continuous.*

Theorem 2. *If a sequence of functions converges pointwise as well as uniformly, then the limit function must be the same.*

Theorem 3 (Weierstrass M-test)**.** *If $|u_k(x)| \leq c_k$ on D for $k \in \{1, 2, 3, \dots\}$ and the numerical series $\sum_{k=1}^{\infty} c_k$ converges, then the series of functions $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on D.*

2 Periodic Functions

Definition 4 (Periodic functions)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is said to be periodic if $\exists 0 < L \in \mathbb{R}$, such that $\forall x \in \mathbb{R}$,

$$f(x) = f(x + L)$$

If there exists a minimum L, it is called L^*, the fundamental period.

For a function $f(x) = k$, as any positive number is a period, there is no minimum L. Hence, $\nexists L^*$.

3 Odd and Even Functions

Definition 5 (Odd functions)**.** A function is said to be odd if $f(-x) = -f(x)$.

Odd functions are symmeteric about the origin.

Definition 6 (Even functions)**.** A function is said to be even if $f(-x) = f(x)$.

Odd functions are symmeteric about the y-axis.

Theorem 4. *If $h(x)$ is odd,*

$$\int_{-L}^{L} h(x)\,\mathrm{d}x = 0$$

Part II
Introduction to Fourier Series

1 Real Fourier Series

Definition 7. Let $f : [-L, L] \to \mathbb{R}$, where $L > 0$. If $\forall x \in [-L, L]$, then

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right)$$

Theorem 5. *Let $L > 0$, $m \in \mathbb{W}$, $n \in \mathbb{W}$.*
Then

$$\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right) \cos\left(\frac{n\pi}{L}x\right) \mathrm{d}x = \begin{cases} 0 & ; \quad m \neq n \\ L & ; \quad m = n \neq 0 \\ 2L & ; \quad m = n = 0 \end{cases}$$

Proof.

$$E = \int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right) \cos\left(\frac{n\pi}{L}x\right) \mathrm{d}x$$

$$= \int_{-L}^{L} \frac{1}{2} \left(\cos\left((m+n)\frac{\pi}{L}x\right) + \cos\left((m-n)\frac{\pi}{L}x\right) \right) \mathrm{d}x$$

$\because \left(\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)\right)$

If $m \neq n$,

$$E = \frac{1}{2} \left(\frac{\sin\left((m+n)\frac{\pi}{L}x\right)}{(m+n)\frac{\pi}{L}} + \frac{\sin\left((m-n)\frac{\pi}{L}x\right)}{(m-n)\frac{\pi}{L}} \right) \Bigg|_{-L}^{L}$$

$$= 0$$

If $m = n \neq 0$,

$$E = \int_{-L}^{L} \frac{1}{2} \left(\cos\left(2m\frac{\pi}{L}x\right) + 1 \right) \mathrm{d}x$$

$$= \frac{1}{2} \int_{-L}^{L} \cos\left(2m\frac{\pi}{L}x\right) \mathrm{d}x + \frac{1}{2}x \Big|_{-L}^{L}$$

$$= L$$

If $m = n = 0$,

$$\begin{aligned} E &= \int\limits_{-L}^{L} \cos(0)\cos(0)\,\mathrm{d}x \\ &= x|_{-L}^{L} \\ &= 2L \end{aligned}$$

□

Theorem 6. *Let $L > 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}$.*
Then

$$\int\limits_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x = \begin{cases} 0 & ; \quad m \neq n \\ L & ; \quad m = n \end{cases}$$

Theorem 7. *Let $L > 0$, $m \in \mathbb{W}$, $n \in \mathbb{W}$.*
Then

$$\int\limits_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x = 0$$

Assuming $f(x)$ is known, and assuming that it can be integrated term by term,

$$\begin{aligned} \int\limits_{-L}^{L} f(x)\,\mathrm{d}x &= \int\limits_{-L}^{L} \frac{1}{2}a_0\,\mathrm{d}x + \sum_{n=1}^{\infty} a_n \int\limits_{-L}^{L} \cos\left(n\frac{\pi}{L}x\right)\mathrm{d}x + b_n \int\limits_{-L}^{L} \sin\left(n\frac{\pi}{L}x\right)\mathrm{d}x \\ \therefore \int\limits_{-L}^{L} f(x)\,\mathrm{d}x &= \frac{1}{2}\int\limits_{-L}^{L} a_0\,\mathrm{d}x \\ &= \frac{1}{2}a_0 \cdot 2L \\ \therefore a_0 &= \frac{1}{L}\int\limits_{-L}^{L} f(x)\,\mathrm{d}x \end{aligned}$$

Similarly, multiplying the series with $\cos\left(m\frac{\pi}{L}x\right)$ for $m \neq 0$ and integrating,

$$a_m = \frac{1}{L}\int\limits_{-L}^{L} f(x)\cos\left(m\frac{\pi}{L}x\right)\mathrm{d}x$$

for $m \in \mathbb{N}$.
Similarly, for $m \in \mathbb{N} \setminus \{0\}$,

$$b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(m \frac{\pi}{L} x\right) \mathrm{d}x$$

Definition 8. The expansion

$$f(x) \approx \frac{1}{2} a_0 + \sum_{i=1}^{\infty} \left(a_n \cos\left(n \frac{\pi}{L} x\right) + b_n \sin\left(n \frac{\pi}{L} x\right)\right)$$

where, for $m \in \mathbb{N}$,

$$a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(m \frac{\pi}{L} x\right) \mathrm{d}x$$

and, for $m \in \mathbb{N} \setminus \{0\}$,

$$b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(m \frac{\pi}{L} x\right) \mathrm{d}x$$

is called the Fourier Series of $f(x)$.

2 Complex Fourier Series

By Euler's formula,

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$
$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

Therefore,

$$\frac{1}{2i} = -\frac{i}{2}$$

Therefore, substituting in the Fourier series,

$$\begin{aligned}
f(x) &\approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\frac{1}{2}\left(e^{\frac{in\pi}{L}x}+e^{-\frac{in\pi}{L}x}\right) - b_n\frac{i}{2}\left(e^{\frac{in\pi}{L}x}-e^{-\frac{in\pi}{L}x}\right)\right)\\
&\approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(e^{\frac{in\pi}{L}x}\left(\frac{1}{2}a_n-\frac{i}{2}b_n\right)+e^{-\frac{in\pi}{L}x}\left(\frac{1}{2}a_n+\frac{i}{2}b_n\right)\right)\\
&\approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(e^{\frac{in\pi}{L}x}\left(\frac{1}{2}a_n-\frac{i}{2}b_n\right)\right)+\sum_{n=-\infty}^{1}\left(e^{\frac{in\pi}{L}x}\left(\frac{1}{2}a_n+\frac{i}{2}b_n\right)\right)\\
&\approx \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{L}x}
\end{aligned}$$

2.1 Bessel's Inequality

Definition 9 (Piecewise continuous functions)**.** $f:\mathbb{R}\to\mathbb{R}$ is said to be piecewise continuous if, for every finite interval $[a,b]$ there is a finite number of discontinuity points, and the one-sided limits at each of these points are also finite.

Definition 10 (Piecewise continuously differentiable functions)**.** $f:\mathbb{R}\to\mathbb{R}$ is said to be piecewise continuously differentiable if it is piecewise continuous, and

$$\lim_{h\to 0^+}\frac{f(x+h)-f(x^+)}{h}<\infty$$

and

$$\lim_{h\to 0^-}\frac{f(x+h)-f(x^-)}{h}<\infty$$

Theorem 8 (Bessel's Inequality)**.** *Let $f(x)$ be a piecewise continuous function defined on $[-L,L]$. Then*

$$\frac{1}{2}{a_0}^2+\sum_{n=1}^{\infty}{a_n}^2+{b_n}^2\leq\frac{1}{L}\int\limits_{-L}^{L}f(x)^2\,\mathrm{d}x$$