HARMONIC ANALYSIS : ASSIGNMENT 3

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Exercise 1.
Let $m \in \mathbb{Z}$ and $D_N(t)$ be the Dirichlet kernel. Calculate

(1) $\int\limits_{-\pi}^{\pi} D_N(t)\sin(100t)\,\mathrm{d}t$

(2) $\int\limits_{-\pi}^{\pi} D_N(t)\cos(100t)\,\mathrm{d}t$

(3) $\int\limits_{-\pi}^{\pi} \big(D_N(t)\big)^2\,\mathrm{d}t$ for $N = 100$.

Solution 1.
(1)

$$\int\limits_{-\pi}^{\pi} D_N(t)\sin(100t)\,\mathrm{d}t = \int\limits_{-\pi}^{\pi} \left(1 + 2\sum_{n=1}^{N}\cos(nt)\right)\sin(100t)\,\mathrm{d}t$$

Therefore, as $D_N(t)$ is even, and $\sin(100t)$ is odd, the function is odd. Hence,

$$\int\limits_{-\pi}^{\pi} D_N(t)\sin(100t)\,\mathrm{d}t = 0$$

Date: Tuesday 17^{th} November, 2015.

(2)

$$
\begin{aligned}
\int_{-\pi}^{\pi} D_N(t)\cos(100t)\,\mathrm{d}t &= \int_{-\pi}^{\pi}\left(1+2\sum_{n=1}^{N}\cos(nt)\right)\cos(100t)\,\mathrm{d}t \\
&= \int_{-\pi}^{\pi}\cos(100t)+2\cos(100t)\sum_{n=1}^{m}\cos(nt)\,\mathrm{d}t \\
&= \quad \int_{-\pi}^{\pi}\cos(100t)\,\mathrm{d}t \\
&\quad + \int_{-\pi}^{\pi} 2\cos(100t)\sum_{n=1}^{99}\cos(nt)\,\mathrm{d}t \\
&\quad + \int_{-\pi}^{\pi} 2\cos(100t)\cos(100t)\,\mathrm{d}t \\
&\quad + \int_{-\pi}^{\pi} 2\cos(100t)\sum_{n=101}^{N}\cos(nt)\,\mathrm{d}t \\
&= 0+0+2\pi+0 \\
&= 2\pi
\end{aligned}
$$

(3)

$$
\begin{aligned}
\int_{-\pi}^{\pi}\left(D_{100}(t)\right)^2\mathrm{d}t &= \int_{-\pi}^{\pi}\left(1+2\sum_{n=1}^{100}\cos(nt)\right)^2\mathrm{d}t \\
&= \int_{-\pi}^{\pi} 1+4\sum_{n=1}^{100}\cos(nt)+4\left(\sum_{n=1}^{100}\cos(nt)\right)^2\mathrm{d}x \\
&= 402\pi
\end{aligned}
$$

Exercise 2.
Decide whether the series of function $f_n(x) = f(nx)$, where

$$
f(x) = \begin{cases} 1-x^2 & ; \quad |x| \le 1 \\ 0 & ; \quad |x| > 1 \end{cases}
$$

converges to the zero function point wise, mean squarely, or uniformly. Explain.

Solution 2.

$$\begin{aligned} f_n(0) &= f(0n) \\ &= f(0) \\ &= 1 \\ &\neq 0 \end{aligned}$$

Therefore, for $x = 0$, the series of functions does not converge point wise to the zero function. Hence, it does not converge uniformly either.

Exercise 3.
Let

$$f(x) = \begin{cases} Ax + B & ; \quad -\pi \leq x < 0 \\ \cos x & ; \quad 0 \leq x \leq \pi \end{cases}$$

For which values of A and B does the Fourier series of f uniformly converge in $[-\pi, \pi]$?

Solution 3.
For f to be uniformly convergent, f must be continuous.
Therefore,

$$\begin{aligned} f(0^-) &= f(0^+) \\ \therefore B &= \cos(0) \end{aligned}$$

Therefore,

$$B = 1$$

Also, for f to be uniformly convergent, $f(-\pi) = f(\pi)$.
Therefore,

$$-\pi A + B = \cos \pi$$

Therefore,

$$A = \frac{2}{\pi}$$

Exercise 4.

Calculate the integral $\displaystyle\int_{-\pi}^{\pi} \left| \sum_{n=1}^{\infty} \frac{1}{2^n} e^{inx} \right|^2 \mathrm{d}x$.

Solution 4.
Let

$$\begin{aligned}
\int_{-\pi}^{\pi} \left| \sum_{n=1}^{\infty} \frac{1}{2^n} e^{inx} \right|^2 \mathrm{d}x &= \int_{-\pi}^{\pi} \left| \sum_{n=1}^{\infty} \frac{\cos(nx) + i\sin(nx)}{2^n} \right|^2 \mathrm{d}x \\
&= \int_{-\pi}^{\pi} \left(\sqrt{\left(\sum_{n=1}^{\infty} \frac{\cos(nx)}{2^n} \right)^2 + \left(\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2} \right)^2} \right)^2 \mathrm{d}x \\
&= \int_{-\pi}^{\pi} \left(\frac{1}{4} + \frac{1}{4^2} + \dots \right) \mathrm{d}x \\
&= \int_{-\pi}^{\pi} \frac{\frac{1}{4}}{1 - \frac{1}{4}} \mathrm{d}x \\
&= \frac{2\pi}{3}
\end{aligned}$$

Exercise 5.
Let f be a periodic piecewise continuous function with period 2π, and Fourier series

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx) \big)$$

on $[-\pi, \pi]$. Express $\displaystyle\frac{1}{\pi} \int_{-\pi}^{\pi} \big| f(x+\pi) - f(x) \big|^2 \mathrm{d}x$ using a_n, b_n.

Solution 5.

$$\begin{aligned}
f(x) &\approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \\
\therefore f(x+\pi) &\approx \frac{a_0}{2} + \sum_{n=1}^{\infty} (-1)^n \big(a_n \cos(nx) + b_n \sin(nx) \big)
\end{aligned}$$

Therefore, by Percival's identity,

$$\begin{aligned}
\frac{1}{\pi} \int_{-\pi}^{\pi} \big| f(x+\pi) - f(x) \big|^2 \mathrm{d}x &= 0 + \sum_{n=1}^{\infty} \big((-1)^n - 1 \big)^2 \left({a_n}^2 + {b_n}^2 \right) \\
&= \sum_{n=1}^{\infty} 2 \left(1 - (-1)^n \right) \left({a_n}^2 + {b_n}^2 \right) \\
&= 4 \sum_{n=1}^{\infty} \left({a_{2n-1}}^2 + {b_{2n-1}}^2 \right)
\end{aligned}$$

Exercise 6.
For all natural n, we define

$$f_n(x) = 1 + \sum_{k=1}^{n} \big(\cos(kx) - \sin(kx)\big)$$

Calculate the integral $\displaystyle\int_{-\pi}^{\pi} \big|f_n(x)\big|^2 \,\mathrm{d}x$

Solution 6.

$$f_n(x) = 1 + \sum_{k=1}^{n} \big(\cos(kx) - \sin(kx)\big)$$

Therefore, by Percival's identity,

$$\begin{aligned}\int_{-\pi}^{\pi} \big|f_n(x)\big|^2 \,\mathrm{d}x &= \pi\left(\frac{2^2}{2} + \sum_{n=1}^{\infty} 1^2 + (-1)^2\right)\\ &= 2\pi(n+1)\end{aligned}$$

Exercise 7.

(1) Develop the Fourier series of the function $f(x) = x^2$ and $[0, 2\pi]$.
(2) Compare the above series and the series of f on $[-\pi, \pi]$ and explain the difference.

Solution 7.
(1)

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Therefore,

$$\begin{aligned}a_0 &= \frac{2}{2\pi}\int_0^{2\pi} x^2 \,\mathrm{d}x\\ &= \frac{8\pi^2}{3}\end{aligned}$$

$$\begin{aligned}a_n &= \frac{2}{2\pi}\int_0^{2\pi} x^2 \cos\left(\frac{2\pi n x}{2\pi}\right) \mathrm{d}x\\ &= \frac{4}{n^2}\end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{2\pi}\int\limits_0^{2\pi} x^2 \sin\left(\frac{2\pi n x}{2\pi}\right) \mathrm{d}x \\ &= -\frac{4\pi}{n} \end{aligned}$$

Therefore, on $[0, 2\pi]$,

$$x^2 \approx \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} - \frac{\pi \sin(nx)}{n}$$

(2) There is a difference between the Fourier seies on $[-\pi, \pi]$ and $[0, 2\pi]$, as the extensions are different for the two intervals.

Exercise 8.
Develop the Fourier series of

$$f(x) = \begin{cases} \frac{x}{2} & ; \quad 0 < x < 2 \\ 1 & ; \quad 2 < x < 3 \end{cases}$$

on $[0, 3]$.

Solution 8.

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Therefore,

$$\begin{aligned} a_0 &= \frac{2}{3}\int\limits_0^3 f(x)\,\mathrm{d}x \\ &= \frac{2}{3}\int\limits_0^2 \frac{x}{2}\,\mathrm{d}x + \frac{2}{3}\int\limits_2^3 \mathrm{d}x \\ &= \frac{4}{3} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{3}\int\limits_0^3 f(x)\cos\left(\frac{2\pi n x}{3}\right)\mathrm{d}x \\ &= \frac{2}{3}\int\limits_0^2 \frac{x}{2}\cos\left(\frac{2\pi n x}{3}\right)\mathrm{d}x + \frac{2}{3}\int\limits_2^3 \cos\left(\frac{2\pi n x}{3}\right)\mathrm{d}x \\ &= \frac{3}{2\pi^2 n^2}\sin^2\left(\frac{2\pi n}{3}\right) \end{aligned}$$

$$\begin{aligned}
b_n &= \frac{2}{3}\int_0^3 f(x)\sin\left(\frac{2\pi nx}{3}\right)\mathrm{d}x \\
&= \frac{2}{3}\int_0^2 \frac{x}{2}\sin\left(\frac{2\pi nx}{3}\right)\mathrm{d}x + \frac{2}{3}\int_2^3 \sin\left(\frac{2\pi nx}{3}\right)\mathrm{d}x \\
&= \frac{3}{4\pi^2 n^2}\sin\left(\frac{4\pi n}{3}\right) - \frac{1}{n\pi}
\end{aligned}$$

Therefore, on $[0,3]$,

$$\begin{aligned}
f(x) \approx \quad & \frac{2}{3} + \sum_{n=1}^{\infty}\frac{3}{2\pi^2 n^2}\sin^2\left(\frac{2\pi n}{3}\right)\cos\left(\frac{2\pi nx}{3}\right) \\
& - \sum_{n=1}^{\infty}\left(\frac{3}{4\pi^2 n^2}\sin\left(\frac{4\pi n}{3} - \frac{1}{n\pi}\right)\sin\left(\frac{2\pi nx}{3}\right)\right)
\end{aligned}$$

Exercise 9.
Develop the Fourier series of e^x on $[0,1]$.

Solution 9.

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

Therefore,

$$\begin{aligned}
a_0 &= \frac{2}{1}\int_0^1 e^x\,\mathrm{d}x \\
&= 2(e-1)
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{1}\int_0^1 e^x\cos(2\pi nx)\,\mathrm{d}x \\
&= \frac{2(e-1)}{4\pi^2 n^2 + 1}
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{2}{1}\int_0^1 e^x\sin(2\pi nx)\,\mathrm{d}x \\
&= -\frac{4\pi n(e-1)}{4\pi^2 n^2 + 1}
\end{aligned}$$

Therefore, on $[0,1]$,

$$f(x) \approx (e-1) + 2(e-1)\sum_{n=1}^{\infty}\frac{\cos(2\pi nx) - 2\pi n\sin(2\pi nx)}{4\pi^2 n^2 + 1}$$