HARMONIC ANALYSIS

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Part 1. Fourier Series

1. Fourier Series

Definition 1 (Real Fourier series). Let $f : [-L, L] \in \mathbb{C}$ be a piecewise continuous function. $\boxed{f(x) \approx \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)}$

$\boxed{f(x) \approx \sum\limits_{n=-\infty}^{\infty} c_n e^{inx}}$ $\boxed{a_0 = \frac{1}{L} \int\limits_{-L}^{L} f(x) \mathrm{d}x}$ $\boxed{a_n = \frac{1}{L} \int\limits_{-L}^{L} f(x) \cos(nx) \mathrm{d}x}$

$\boxed{b_n = \frac{1}{L} \int\limits_{-L}^{L} f(x) \sin(nx) \mathrm{d}x}$ $\boxed{c_n = \frac{1}{2L} \int\limits_{-L}^{L} f(x) e^{-inx} \mathrm{d}x}$

Definition 2 (Piecewise continuous functions). $f : \mathbb{R} \to \mathbb{R}$ is said to be piecewise continuous if, for every finite interval $[a,b]$ there is a finite number of discontinuity points, and the one-sided limits at each of these points are also finite.

Definition 3 (Piecewise continuously differentiable functions). $f : \mathbb{R} \to \mathbb{R}$ is said to be piecewise continuously differentiable if it is piecewise continuous, and $\boxed{\lim\limits_{h \to 0^+} \frac{f(x+h) - f(x^+)}{h} < \infty}$ $\boxed{\lim\limits_{h \to 0^-} \frac{f(x-h) - f(x^-)}{h} < \infty}$

Theorem 1 (Bessel's Inequality). *Let $f(x)$ be a piecewise continuous function defined on $[-L,L]$. Then* $\boxed{\frac{1}{2} {a_0}^2 + \sum\limits_{n=1}^{\infty} {a_n}^2 + {b_n}^2 \le \frac{1}{L} \int\limits_{-L}^{L} f(x)^2 \mathrm{d}x}$

Theorem 2 (Riemann-Lebesgue's Lemma). *If $f(x)$ is piecewise continuous on $[-L,L]$, then* $\boxed{\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} b_n = 0}$

Definition 4 (Dirichlet kernel).

$$D_m(t) = \frac{1}{2} \sum_{n=-m}^{m} e^{-int} = \frac{1}{2} + \sum_{n=1}^{m} \cos(nt) = \frac{\sin\left(\left(n + \frac{1}{2}\right)t\right)}{2 \sin \frac{t}{2}}$$

is called the Dirichlet kernel of order m.

Theorem 3 (Second representation of Dirichlet's kernel). *Let $m \in \mathbb{N}$. Then, for $t \ne 2\pi k$, where $k \in \mathbb{Z}$,*

$$D_m(t) = \frac{1}{2} + \cos(t) + \cos(2t) + \dots + \cos(mt) = \frac{\sin\left(\left(m + \frac{1}{2}\right)t\right)}{2 \sin\left(\frac{1}{2}t\right)}$$

Theorem 4. *Let* $\boxed{S_m(f,x) = \frac{1}{2} a_0 + \sum\limits_{n=1}^{m} a_n \cos(nx) + b_n \sin(nx)}$ *Then,* $\boxed{S_m(f,x) = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(x+t) \left(\frac{1}{2} \sum\limits_{n=1}^{m} \cos(nt)\right) \mathrm{d}t}$

Theorem 5 (Dirichlet Theorem). *Let $f : [-\pi, \pi] \to \mathbb{R}$ be a piecewise continuously differentiable function.*

Then, $\forall x \in (-\pi,\pi)$, $\boxed{\frac{1}{2} a_0 + \sum\limits_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = \frac{f(x^-) + f(x^+)}{2}}$ *and for $x = \pi$ or $x = -\pi$,* $\boxed{\frac{1}{2} a_0 + \sum\limits_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = \frac{f(\pi^-) + f(-\pi^+)}{2}}$

Theorem 6. *If f is a piecewise continuous and periodic function with period of 2π, then* $\boxed{S_m(x) = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(x+t) D_m(t) \mathrm{d}t}$

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Theorem 7 (Cauchy-Schwartz Inequality for Generalized Fourier Series). *Let $u,v \in V$, where V is an inner product space over $\mathbb{F}$. Then,* $\left|\langle u,v \rangle\right| \le \|u\| \|v\|$

Theorem 8 (Best Approximation Theorem). *Let $\{u_i\}_{i=1}^{n}$ be an orthonormal system in an normed inner product space V. Let $v \in V$. Then, $\forall \{c_i\}_{i=1}^{n} \subset \mathbb{F}$,*

(1) $\left\| v - \sum\limits_{k=1}^{n} \langle v, u_k \rangle u_k \right\|^2 \le \left\| v - \sum\limits_{k=1}^{n} c_k u_k \right\|^2$

(2) $\left\| v - \sum\limits_{k=1}^{n} \langle v, u_k \rangle u_k \right\|^2 = \|v\|^2 - \sum\limits_{k=1}^{n} \left|\langle v, u_k \rangle\right|^2$

2. Fourier Series in a General Interval

Definition 5. Let f be a piecewise continuous function defined on $[a,\ b]$. The Fourier series over $[a,\ b]$ is defined as $\boxed{f(x) \approx \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n x}{b-a}\right) + b_n \sin\left(\frac{2\pi n x}{b-a}\right)\right)}$ $\boxed{f(x) \approx \sum\limits_{-\infty}^{\infty} c_n e^{\frac{2\pi i n x}{b-a}}}$

where $\boxed{a_0 = \frac{1}{b-a} \int\limits_{a}^{b} f(x) \mathrm{d}x}$ $\boxed{a_n = \frac{2}{b-a} \int\limits_{a}^{b} f(x) \cos \frac{2\pi n x}{b-a} \mathrm{d}x}$ $\boxed{b_n = \frac{2}{b-a} \int\limits_{a}^{b} f(x) \sin \frac{2\pi n x}{b-a} \mathrm{d}x}$

$\boxed{c_n = \frac{1}{b-a} \int\limits_{a}^{b} f(x) e^{\frac{2\pi i n x}{b-a}} \mathrm{d}x}$

Theorem 9. *Let f be continuous in $[-\pi, \pi]$, with piecewise continuous derivative, and $f(-\pi) = f(\pi)$. Then, the Fourier series converges uniformly on $[-\pi,\pi]$.*

Theorem 10 (Perceval Equality). *Let f be a piecewise continuous function in $[-\pi,\ \pi]$. Then,* $\boxed{\frac{1}{\pi} \int\limits_{-\pi}^{\pi} \left|f(x)\right|^2 \mathrm{d}x = \frac{{a_0}^2}{2} + \sum\limits_{n=1}^{\infty} \left({a_n}^2 + {b_n}^2\right)}$

$\boxed{\frac{1}{\pi} \int\limits_{-\pi}^{\pi} \left|f(x)\right|^2 \mathrm{d}x = 2 \sum\limits_{n=-\infty}^{\infty} |c_n|^2}$

Theorem 11. *If f is piecewise continuous with Fourier series* $\boxed{f(x) \approx \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)}$ *the, for all $x \in [-\pi,\ \pi]$,* $\boxed{\int\limits_{0}^{x} f(t) \mathrm{d}t = \frac{a_0}{2} x + \sum\limits_{n=1}^{\infty} \frac{a_n}{n} \sin(nx) - \frac{b_n}{n} \left(\cos(nx) - 1\right)}$ *This is not a Fourier series due to the x in $\frac{a_0}{2} x$.*

Therefore, substituting the Fourier series of x,

$$\int\limits_{0}^{x} f(t) \mathrm{d}t = \sum_{n=1}^{\infty} \frac{b_n}{n} + \sum_{n=1}^{\infty} \left(\frac{a_n + (-1)^n a_0}{n} \sin(nx) - \frac{b_n}{n} \cos(nx)\right)$$

Theorem 12 (Bessel's Inequality for a General Interval). *Let $f : [a,b] \to \mathbb{R}$ be a piecewise continuous function. Then,*

$$\frac{1}{2} (a_0)^2 + \sum_{n=1}^{\infty} (a_n)^2 + (b_n)^2 \le \frac{2}{b-a} \int\limits_{a}^{b} \left|f(x)\right|^2 \mathrm{d}x$$

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \le \frac{1}{b-a} \int\limits_{a}^{b} \left|f(x)\right|^2 \mathrm{d}x$$

Theorem 13 (Dirichlet's Theorem for a General Interval). *Let $f:[a,b]\to\mathbb{R}$ be a piecewise continuously differentiable function. Then,*

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\left(\frac{2\pi}{b-a}nx\right)+b_n\sin\left(\frac{2\pi}{b-a}nx\right)$$
$$=\sum_{n=-\infty}^{\infty}c_n e^{i\frac{2\pi}{b-a}nx}$$
$$=\frac{f\left(x^+\right)+f\left(x^-\right)}{2}$$

Theorem 14 (Term-by-term Differentiation for a General Interval). *Let $f:[a,b]\to\mathbb{R}$ be a piecewise continuous function, such that $f(a)=f(b)$, and $f'(x)$ is piecewise continuous. Then, if*

$$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{i\frac{2\pi}{b-a}nx}$$

then

$$f(x)\approx\sum_{n=-\infty}^{\infty}i\frac{2\pi}{b-a}nc_n e^{i\frac{2\pi}{b-a}nx}$$

Theorem 15. *If f is 2π periodic and k times differentiable, such that the k derivatives are continuous and $f^{(k+1)}(x)$ is piecewise continuous, then,* $\boxed{\lim\limits_{n\to\infty}\left|n^k a_n\right|=\lim\limits_{n\to\infty}\left|n^k b_n\right|=\lim\limits_{n\to\infty}\left|n^k c_n\right|=0}$

Theorem 16. *If the Fourier coefficients of a 2π periodic function satisfy* $\boxed{|c_n|\le\frac{c}{n^{k+1+\varepsilon}}}$ *where $\varepsilon>0$, and c is constant, then f is k times differentiable.*

Theorem 17. *Let k be the largest integer such that* $\boxed{\lim\limits_{n\to\infty}n^k f(x)=0}$ *Then, $f(x)$ is continuously differentiable at most k times. Also, if* $\boxed{f(x)\le\frac{1}{n^{k+1+\varepsilon}}}$ *then, k is differentiable k times.*

Theorem 18 (Term-by-term differentiation). *Let the Fourier coefficients of $f(x)$ be a_0, a_n, b_n, c_n. Then, the Fourier coefficients of $f'(x)$ are*
$\alpha_0=0$
$\alpha_n=nb_n$
$\beta_n=-na_n$
$\gamma_n=inc_n$

Theorem 19 (Term-by-term integration). *Let*

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx)+b_n\sin(nx)$$

Then, the integral of $f(x)$ is

$$F(x)\approx\frac{a_0}{2}x+\sum_{n=1}^{\infty}\frac{a_n\sin(nx)+b_n\left(1-\cos(nx)\right)}{n}$$

3. Inner Product Spaces

Theorem 20 (Cauchy-Schwartz Inequality).

$$\left|\left\langle\overrightarrow{x},\overrightarrow{y}\right\rangle\right|\le\left\|\overrightarrow{x}\right\|\left\|\overrightarrow{y}\right\|$$

Definition 6 (Norm). Let V be a vector space. A function $\|\cdot\|:V\to\mathbb{R}^+$, such that
(1) $\forall v\in V$, $\boxed{\|v\|\ge0}$ and $\|v\|=0$ if an only if $v=\overrightarrow{0}$.
(2) $\forall v\in V$ and $\alpha\in\mathbb{F}$, $\boxed{\|\alpha v\|=|\alpha|\|v\|}$
(3) $\forall u,v\in V$, $\boxed{\|u+v\|\le\|u\|+\|v\|}$
is called a norm.
It is usually defined as $\boxed{\|v\|=\sqrt{\langle v,v\rangle}}$

Theorem 21 (Pythagoras Theorem). *If $u,v\in V$ are orthogonal vectors in an inner product space, then* $\boxed{\|u+v\|^2=\|u\|^2+\|v\|^2}$

Definition 7. A set is said to be orthonormal if $\forall u,v\in V$, $\boxed{\langle u,v\rangle=0}$ and $\boxed{\|v\|=1}$

Theorem 22. *In the space $C^0[-\pi,\pi]$ with* $\boxed{\langle f,g\rangle=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}f(x)g(x)\mathrm{d}x}$ *the set $\left\{\frac{1}{\sqrt{2}}\right\}\cup\left\{\cos(nx)\right\}_{n=1}^{\infty}\cup\left\{\sin(nx)\right\}_{n=1}^{\infty}$ is orthonormal.*

Definition 8 (Orthonormal set). A set $\{c_1,\dots,c_n\}$ is said to be orthonormal if $\boxed{\langle c_i,c_j\rangle=\delta_{ij}}$

Definition 9 (Projection). Let W be a subspace of a vector space V, such that $\boxed{W=\operatorname{span}\{c_1,\dots,c_n\}}$ The projection of a vector v with respect to W is defined as $\boxed{\operatorname{proj}_W(v)=v_W=\sum\limits_{k=1}^{n}\frac{\langle v,c_k\rangle}{\|c_k\|^2}c_k}$

Theorem 23. *Let W be a subspace of a vector space V. Let $v\in V$. Then, v_W is the best approximation for v in W, i.e.* $\boxed{\left\|v-\operatorname{proj}_W(v)\right\|=\min\limits_{w\in W}\|v-w\|}$

Theorem 24. *Let W be a subspace of a vector space V. Let $v\in V$. Then, $v-v_W\in W^\perp$, and $v_W\in W$.*

Theorem 25. *Let W be a subspace of a vector space V. Let $v\in V$.* $\boxed{\|v\|^2=\|v_W\|^2+\|v-v_W\|^2}$

Definition 10 (Complete set). An orthonormal set $\{u_k\}_{k=1}^{\infty}$ is said to be complete if the only vector $v\in V$, such that $\forall k$, $\boxed{\langle v,u_k\rangle=0}$ is the zero vector.

Definition 11 (Closed system). Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal system in V. The system is said to be a closed system if for each $u\in V$,

$$\lim_{N\to\infty}\left\|u-\sum_{n=1}^{N}\langle u,e_n\rangle e_n\right\|=0$$

Alternatively,

$$u\overset{\|\cdot\|}{=}\sum_{n=1}^{\infty}\langle u,e_n\rangle e_n$$

Definition 12 (Continuity of functions). Let $f:V\to\mathbb{F}$. f is said to be continuous at v_0 if for each $\varepsilon>0$, $\exists\delta>0$ such that $\forall v$ such that $\|v-v_0\|<\delta$, $\left|f(v)-f(v_0)\right|<\varepsilon$.

Theorem 26. *The function $\langle\cdot,\cdot\rangle:V\times V\to\mathbb{F}$ is continuous with respect to both variables.*

Theorem 27. *A closed orthonormal set is complete.*

Theorem 28. *The sequence $\{S_k\}_{k=1}^{\infty}$ where*

$$S_k=\sum_{n=1}^{k}\langle f,\varphi_n\rangle\varphi_n$$

is a Cauchy sequence.

Theorem 29. *For a Hilbert space, a sequence is Cauchy if and only if it is converging.*

Definition 13 (Hilbert space). A inner product, normed, complete vector space is said to be a Hilbert space.

Theorem 30 (Central Theorem about Complete Orthonormal Systems). *Let V be a Hilbert space. Then, the following conditions are equivalent for an orthonormal set $\{u_k\}_{k=1}^{\infty}$,*

(1) *For $v\in V$, if $\forall k$,* $\boxed{\langle v,u_k\rangle=0}$ *then* $\boxed{v=\overrightarrow{0}}$ *That is, if any vector is orthogonal to the entire orthonormal set, then the vector must be the zero vector.*
(2) *For $v\in V$,* $\boxed{\lim\limits_{n\to\infty}\left\|\sum\limits_{k=1}^{n}\langle v,u_k\rangle u_k-v\right\|=0}$
(3) *For $v\in V$,* $\boxed{\sum\limits_{k=1}^{\infty}\left|\langle v,u_k\rangle\right|^2=\|v\|^2}$ *That is, Perceval's Equality holds.*

Exercise 1.
Let

$$\lambda_n=\min_{\alpha\in\mathbb{R}}\frac{1}{\pi}\int_{-\pi}^{\pi}\left|\sqrt{\cos x}-\alpha\cos(nx)\right|^2\mathrm{d}x$$

Calculate $\lim\limits_{n\to\infty}\lambda_n$.

Solution 1.

$$\langle f,g\rangle=\frac{1}{\pi}\int_{-\pi}^{\pi}f\overline{g}\,\mathrm{d}x$$

Therefore,

$$\langle f,f\rangle=\frac{1}{\pi}\int_{-\pi}^{\pi}f\overline{f}\,\mathrm{d}x$$
$$=\frac{1}{\pi}\int_{-\pi}^{\pi}|f|^2\mathrm{d}x$$

Therefore, if
$f(x)=\sqrt{\cos x}$

then,

$$\lambda_n=\min_{\alpha\in\mathbb{R}}\left|\sqrt{\cos x}-\alpha\cos(nx)\right|^2$$
$$=\min_{\alpha\in\mathbb{R}}\left|f(x)-\alpha\cos(nx)\right|^2$$

Therefore, by the best approximation theorem, α will be the coefficient corresponding to the best approximation, i.e.,

$$\alpha\cos(nx)=\mathrm{proj}_W\left(\sqrt{\cos x}\right)$$

Therefore,

$$\alpha=\frac{\left\langle\sqrt{\cos x},\cos(nx)\right\rangle}{\left\|\cos(nx)\right\|^2}$$

Therefore,

$$\lambda_n=\left\|\sqrt{\cos x}-\alpha\cos(nx)\right\|^2$$
$$=\left\|\sqrt{\cos x}-\frac{\left\langle\sqrt{\cos x},\cos(nx)\right\rangle}{\left\|\cos(nx)\right\|^2}\cos(nx)\right\|^2$$
$$=\left\|\sqrt{\cos x}\right\|^2-\frac{\left\langle\sqrt{\cos x},\cos(nx)\right\rangle^2}{\left\|\cos(nx)\right\|^2}\left\|\cos(nx)\right\|^2$$
$$=\left\|\sqrt{\cos x}\right\|^2-\left\langle\sqrt{\cos x},\cos(nx)\right\rangle$$

Therefore,

$$\left\|\sqrt{\cos x}\right\|^2=\frac{1}{\pi}\int_{-\pi}^{\pi}|\cos x|\mathrm{d}x$$
$$=\frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\mathrm{d}x$$
$$=\frac{2}{\pi}\sin x|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$=\frac{4}{\pi}$$

Therefore,

$$\lambda_n=\frac{4}{\pi}-{a_n}^2$$

Therefore,

$$\lim_{n\to\infty}\lambda_n=\lim_{n\to\infty}\left(\frac{4}{\pi}-{a_n}^2\right)$$
$$=\frac{4}{\pi}-\lim_{n\to\infty}{a_n}^2$$

By Riemann-Lebesgue's Lemma,

$$\lim_{n\to\infty}a_n=0$$

Therefore,

$$\lim_{n\to\infty}\lambda_n=\frac{4}{\pi}$$

Theorem 31 (Bessel's Inequality)**.** *Let $\{u_k\}_{k=1}^{\infty}$ be an orthonormal system in V. Then, $\forall v\in V$,*

$$\sum_{k=1}^{\infty}\left|\langle v,u_k\rangle\right|^2\leq\|v\|^2$$

Theorem 32 (Perceval's Equality)**.** *The system $\{u_k\}_{k=1}^{\infty}$ is a closed orthonormal system if and only if*

$$\sum_{k=1}^{\infty}\left|\langle v,u_k\rangle\right|^2=\|v\|^2$$

Part 2. Integral Identities

$$\int u\sin u\mathrm{d}u=\sin u-u\cos u+c$$
$$\int u\cos u\mathrm{d}u=\cos u+u\sin u+c$$
$$\int u^2\sin u\mathrm{d}u=2u\sin u+\left(2-u^2\right)\cos u+c$$
$$\int u^2\cos u\mathrm{d}u=2u\cos u+\left(u^2-2\right)\sin u+c$$
$$\int e^{ax}\sin(px)\mathrm{d}x=\frac{e^{ax}\left(a\sin(px)-p\cos(px)\right)}{a^2+p^2}$$
$$\int e^{ax}\cos(px)\mathrm{d}x=\frac{e^{ax}\left(a\cos(px)+p\sin(px)\right)}{a^2+p^2}$$
$$\int\cos(mu)\cos(nu)\mathrm{d}u=\frac{\sin\left((m+n)u\right)}{2(m+n)}+\frac{\sin\left((m-n)u\right)}{2(m-n)}+c$$
$$\int\sin(mu)\cos(nu)\mathrm{d}u=-\frac{\cos\left((m+n)u\right)}{2(m+n)}-\frac{\cos\left((m-n)u\right)}{2(m-n)}+c$$
$$\int\sin(mu)\sin(nu)\mathrm{d}u=-\frac{\sin\left((m+n)u\right)}{2(m+n)}+\frac{\sin\left((m-n)u\right)}{2(m-n)}+c$$
$$\int ue^{au}\mathrm{d}u=\frac{e^{au}}{a^2}(au-1)+c$$
$$\int u^me^{au}\mathrm{d}u=\frac{u^me^{au}}{a}-\frac{m}{a}\int u^{m-1}e^{au}\mathrm{d}u+c$$
$$\int x^a\mathrm{d}x=\frac{x^{a+1}}{a+1}+c$$
$$\int\frac{\mathrm{d}x}{x}=\ln|x|+c$$
$$\int e^x\mathrm{d}x=e^x+c$$
$$\int a^x\mathrm{d}x=\frac{a^x}{\ln a}+c$$
$$\int\sin x\mathrm{d}x=-\cos x+c$$
$$\int\cos x\mathrm{d}x=\sin x+c$$
$$\int\sec x\mathrm{d}x=\tan x+c$$
$$\int\csc x\mathrm{d}x=-\cot x+c$$
$$\int\tan x\mathrm{d}x=-\ln|\cos x|+c$$
$$\int\cot x\mathrm{d}x=\ln|\sin x|+c$$
$$\int\frac{1}{1+x^2}\mathrm{d}x=\tan^{-1}x+c$$
$$\int\frac{1}{\sqrt{1-x^2}}\mathrm{d}x=\sin^{-1}x+c$$
$$\int\csc x\mathrm{d}x=\ln\left|\tan\frac{x}{2}\right|+c$$
$$\int\sec x\mathrm{d}x=\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|+c$$
$$\int\frac{1}{a^2+x^2}\mathrm{d}x=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$$
$$\int\frac{1}{x^2-a^2}\mathrm{d}x=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+c$$
$$\int\frac{1}{a^2-x^2}\mathrm{d}x=\frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|+c$$
$$\int\frac{1}{\sqrt{x^2\pm a^2}}\mathrm{d}x=\ln\left|x+\sqrt{x^2+a^2}\right|+c$$
$$\int\sqrt{a^2-x^2}\mathrm{d}x=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$$
$$\int\sqrt{x^2\pm a^2}\mathrm{d}x=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2\pm a^2}\right|+c$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x = \begin{cases} 0 & ; \quad m \neq n \\ L & ; \quad m = n \neq 0 \\ 2L & ; \quad m = n = 0 \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x = 0$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right) = \begin{cases} 0 & ; \quad m \neq n \\ L & ; \quad m = n \end{cases}$$

Part 3. Trigonometric Identities

$$\begin{aligned}
\cos(2a) &= 1 - 2\sin^2 a \\
&= 2\cos^2 a - 1 \\
\sin(a \pm b) &= \sin a \cos b \pm \cos a \sin b \\
\cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b \\
\sin^2 a + \cos^2 a &= 1 \\
\sec^2 a &= 1 + \tan^2 a \\
\csc^2 a &= 1 + \cot^2 a \\
\sin a \sin b &= \frac{\cos(a-b) - \cos(a+b)}{2} \\
\sin a \cos b &= \frac{\sin(a+b) + \sin(a-b)}{2} \\
\cos a \cos b &= \frac{\cos(a+b) + \cos(a-b)}{2}
\end{aligned}$$