HARMONIC ANALYSIS : ASSIGNMENT 6

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Exercise 1.
Calculate as exactly as you can how many times the series is continuously differentiable.

(1) $f(x) = \sum\limits_{n=1}^{\infty} \frac{\sin(nx)}{n^4}$
(2) $f(x) = \sum\limits_{n=1}^{\infty} \frac{\sin(nx)}{n^{4.01}}$
(3) $f(x) = \sum\limits_{n=1}^{\infty} \frac{1}{n^4} e^{inx}$
(4) $f(x) = \sum\limits_{n=1}^{\infty} \frac{1}{n^{5.01}} e^{inx}$
(5) $f(x) = \sum\limits_{n=0}^{\infty} \frac{1}{2^n} e^{i2^n x}$
(6) $f(x) = \sum\limits_{n=2}^{\infty} \frac{e^{inx}}{n \log^2 n}$
(7) $f(x) = \sum\limits_{n=2}^{\infty} \frac{e^{inx}}{n \log n}$

Solution 1.
(1)

$$\lim_{n\to\infty} n \frac{\sin(nx)}{n^4} = 0$$
$$\lim_{n\to\infty} n^2 \frac{\sin(nx)}{n^4} = 0$$
$$\lim_{n\to\infty} n^3 \frac{\sin(nx)}{n^4} = 0$$
$$\lim_{n\to\infty} n^4 \frac{\sin(nx)}{n^4} \neq 0$$

Therefore, the function is continuously differentiable at most 3 times.

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(2)

$$\lim_{n\to\infty} n\frac{\sin(nx)}{n^{4.01}} = 0$$
$$\lim_{n\to\infty} n^2\frac{\sin(nx)}{n^{4.01}} = 0$$
$$\lim_{n\to\infty} n^3\frac{\sin(nx)}{n^{4.01}} = 0$$
$$\lim_{n\to\infty} n^4\frac{\sin(nx)}{n^{4.01}} = 0$$
$$\lim_{n\to\infty} n^5\frac{\sin(nx)}{n^{4.01}} \neq 0$$

Therefore, the function is continuously differentiable at most 4 times.

(3)

$$\lim_{n\to\infty} n\frac{e^{inx}}{n^4} = 0$$
$$\lim_{n\to\infty} n^2\frac{e^{inx}}{n^4} = 0$$
$$\lim_{n\to\infty} n^3\frac{e^{inx}}{n^4} = 0$$
$$\lim_{n\to\infty} n^4\frac{e^{inx}}{n^4} \neq 0$$

Therefore, the function is continuously differentiable at most 3 times.

$$\frac{1}{n^4} \leq \frac{1}{n^{2+1+\varepsilon}}$$

Therefore, the function is differentiable 2 times.

(4)

$$\lim_{n\to\infty} n\frac{e^{inx}}{n^{5.01}} = 0$$
$$\lim_{n\to\infty} n^2\frac{e^{inx}}{n^{5.01}} = 0$$
$$\lim_{n\to\infty} n^3\frac{e^{inx}}{n^{5.01}} = 0$$
$$\lim_{n\to\infty} n^4\frac{e^{inx}}{n^{5.01}} = 0$$
$$\lim_{n\to\infty} n^5\frac{e^{inx}}{n^{5.01}} = 0$$
$$\lim_{n\to\infty} n^6\frac{e^{inx}}{n^{5.01}} \neq 0$$

Therefore, the function is continuously differentiable at most 5 times.

$$\frac{1}{n^{5.01}} \leq \frac{1}{n^{4+1+\varepsilon}}$$

Therefore, the function is differentiable 4 times.

(5) Let

$$k = 2^n$$

Therefore,

$$f(x) = \sum_{k=1,2,4,\dots}^{\infty} \frac{1}{k} e^{ikx}$$

Therefore, $f(x)$ is differentiable 0 times.

(6)

$$\lim_{n\to\infty} n \frac{e^{inx}}{n \log^2 n} = 0$$

$$\lim_{n\to\infty} n^2 \frac{e^{inx}}{n \log^2 n} \neq 0$$

Therefore, the function is continuously differentiable at most 2 times.

(7)

$$\lim_{n\to\infty} n \frac{e^{inx}}{n \log n} = 0$$

$$\lim_{n\to\infty} n^2 \frac{e^{inx}}{n \log n} \neq 0$$

Therefore, the function is continuously differentiable at most 2 times.

Exercise 2.
Let

$$f(x) = 3x^5 - 10\pi^2 x^3 + 7\pi^4 x$$

Prove

$$\lim_{n\to\infty} n^3 c_n = 0$$

where c_n are the Fourier coefficients of $f(x)$.

Solution 2.
The function is 5 times differentiable.
Therefore,

$$\lim_{n\to\infty} \left| n^5 c_n \right| = 0$$

$$\lim_{n\to\infty} \left| n^3 c_n \right| = 0$$

Exercise 3.
For the function

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{\sin nx}{3n^4 + 5n^6} + \frac{\cos nx}{n^9 + 3n^7} \right)$$

Find the maximal k, such that $f^{(k)}(x)$ is continuous.

Solution 3.

$$\lim_{n\to\infty} nf(x) = 0$$
$$\lim_{n\to\infty} n^2 f(x) = 0$$
$$\lim_{n\to\infty} n^3 f(x) = 0$$
$$\lim_{n\to\infty} n^4 f(x) = 0$$
$$\lim_{n\to\infty} n^5 f(x) \neq 0$$

Therefore, f is differentiable 4 times.
Therefore, $f^{(4)}(x)$ exists.
Therefore, $f^{(3)}(x)$ must be differentiable, and hence continuous.
Therefore, the maximal k is 3.