Harmonic Analysis : Recitations

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1 Fourier Series

Definition 1 (Real Fourier series)**.** Let $f : [-L, L] \in \mathbb{C}$ be a piecewise continuous function.
The series

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

is called the Fourier series of $f(x)$, where

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, \mathrm{d}x$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(nx)\, \mathrm{d}x$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(nx)\, \mathrm{d}x$$

Theorem 1. *If $f(x)$ is an even function, then the appropriate Fourier series is*

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

If $f(x)$ is an odd function, then the appropriate Fourier series is

$$f(x) \approx \sum_{n=1}^{\infty} a_n \sin(nx)$$

If $f(x)$ is odd, its graph always passes through the origin. Therefore, it can be represented by a summation of sine functions, which also pass through the origin, and there is no need for a term, i.e. $\frac{a_0}{2}$, to change its position at the origin.

Definition 2 (Complex Fourier series)**.** Let $f : [-L, L] \in \mathbb{C}$ be a piecewise continuous function.
The series

$$f(x) \approx \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

is called the complex Fourier series of $f(x)$, where

$$c_n = \frac{1}{2L}\int\limits_{-L}^{L} f(x)e^{-inx}\,\mathrm{d}x$$

Recitation 1 – Exercise 1.
Calculate the real Fourier series of

$$f(x) = 2x - 2\pi$$

Recitation 1 – Solution 1.
As x is an odd function, the real Fourier series of x, in the interval $[-\pi, \pi]$ is

$$x \approx \sum_{n=1}^{\infty} b_n \sin(nx)$$

where

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int\limits_{-\pi}^{\pi} x\sin(nx)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left(x\int \sin(nx)\,\mathrm{d}x - \int 1\left(\int \sin(nx)\,\mathrm{d}x\right)\mathrm{d}x\right)\Bigg|_{-\pi}^{\pi} \\
&= \frac{1}{\pi}\left(-\frac{x\cos(nx)}{n}\right)\Bigg|_{-\pi}^{\pi} + \frac{1}{\pi}\int\limits_{-\pi}^{\pi}\frac{\cos(nx)}{n}\,\mathrm{d}x \\
&= \frac{1}{\pi}\left(-\frac{\pi\cos(n\pi) + \pi\cos(-n\pi)}{n}\right) + \frac{1}{\pi}\cancelto{0}{\frac{\sin(nx)}{n^2}\Bigg|_{-\pi}^{\pi}} \\
&= -\frac{\cos(n\pi) + \cos(n\pi)}{n} \\
&= -2\frac{\cos(n\pi)}{n} \\
&= -2\frac{(-1)^n}{n}
\end{aligned}$$

Therefore,

$$x \approx 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$$

Therefore,

$$2x - 2\pi \approx \left(4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) \right) - 2\pi$$

2 Bessel's Inequality

Definition 3 (Piecewise continuous functions)**.** $f : \mathbb{R} \to \mathbb{R}$ is said to be piecewise continuous if, for every finite interval $[a, b]$ there is a finite number of discontinuity points, and the one-sided limits at each of these points are also finite.

Definition 4 (Piecewise continuously differentiable functions)**.** $f : \mathbb{R} \to \mathbb{R}$ is said to be piecewise continuously differentiable if it is piecewise continuous, and

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x^+)}{h} < \infty$$

and

$$\lim_{h \to 0^-} \frac{f(x+h) - f(x^-)}{h} < \infty$$

Theorem 2 (Bessel's Inequality)**.** *Let $f(x)$ be a piecewise continuous function defined on $[-L, L]$. Then*

$$\frac{1}{2} {a_0}^2 + \sum_{n=1}^{\infty} {a_n}^2 + {b_n}^2 \leq \frac{1}{L} \int_{-L}^{L} f(x)^2 \, \mathrm{d}x$$

3 Riemann-Lebesgue's Lemma

Theorem 3 (Riemann-Lebesgue's Lemma)**.** *If $f(x)$ is piecewise continuous on $[-L, L]$, then*

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = 0$$

4 Dirichlet's Kernel

Definition 5 (Dirichlet kernel)**.**

$$D_m(t) = \frac{1}{2}\sum_{n=1}^{m} \cos(nt)$$

is called the Dirichlet kernel of order m.

Theorem 4 (Second representation of Dirichlet's kernel)**.** *Let $m \in \mathbb{N}$. Then, for $t \neq 2\pi k$, where $k \in \mathbb{Z}$,*

$$\begin{aligned} D_m(t) &= \frac{1}{2} + \cos(t) + \cos(2t) + \cdots + \cos(mt) \\ &= \frac{\sin\left(\left(m + \frac{1}{2}\right) t\right)}{2\sin\left(\frac{1}{2}t\right)} \end{aligned}$$

Theorem 5. *Let*

$$S_m(f,x) = \frac{1}{2}a_0 + \sum_{n=1}^{m} a_n \cos(nx) + b_n \sin(nx)$$

Then,

$$S_m(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\left(\frac{1}{2}\sum_{n=1}^{m}\cos(nt)\right) \mathrm{d}t$$

Theorem 6 (Dirichlet Theorem)**.** *Let $f : [-\pi, \pi] \to \mathbb{R}$ be a piecewise continuously differentiable function.*
Then, $\forall x \in (-\pi, \pi)$,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = \frac{f(x^-) + f(x^+)}{2}$$

and for $x = \pi$ or $x = -\pi$,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = \frac{f(\pi^-) + f(-\pi^+)}{2}$$

Recitation 2 – Exercise 1.
The Fourier series of x^2 of given to be

$$x^2 \approx \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

Calculate

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Recitation 2 – Solution 1.
As x^2 is continuous, with a continuous derivative, Dirichlet Theorem is applicable.
Therefore, let

$$x = \pi$$

Therefore, by Dirichlet Theorem,

$$\begin{aligned} \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) &= \frac{\left(\pi^-\right)^2 + \left((-\pi)^+\right)^2}{2} \\ \therefore \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n &= \pi^2 \\ \therefore \frac{\pi^2}{4} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} &= \pi^2 \\ \therefore \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{1}{4}\left(\pi^2 - \frac{\pi^2}{3}\right) \\ &= \frac{\pi^2}{6} \end{aligned}$$

Recitation 2 – Exercise 2.
The Fourier series of

$$f(x) = \begin{cases} x & ; \quad 0 \le x \le \pi \\ 0 & ; \quad -\pi \le x \le 0 \end{cases}$$

is given to be

$$f(x) \approx \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}}{n} \sin(nx) - \frac{2}{\pi(2n-1)^2} \cos\left((2n-1)x\right) \right)$$

Let this Fourier series be denoted by $S(x)$.
Calculate

1. $1+\frac{1}{3^2}+\frac{1}{5^2}+\dots$

2. $S\left(\frac{\pi}{2}\right)$

Recitation 2 – Solution 2.

1.

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\dots=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

Therefore, for $x=0$, by Dirichlet Theorem,

$$\begin{aligned}\frac{\pi}{4}+\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n}\sin(0)-\frac{2}{\pi(2n-1)^2}\cos(0)\right)&=\frac{f(0^-)+f(0^+)}{2}\\ \therefore\frac{\pi}{4}-\sum_{n=1}^{\infty}\left(\frac{2}{\pi(2n-1)^2}\right)&=0\\ \therefore\frac{\pi}{4}-\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{(2n-1)^2}\right)&=0\\ \therefore\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}&=\frac{\pi^2}{8}\end{aligned}$$

2. By Dirichlet Theorem,

$$\begin{aligned}S\left(\frac{\pi}{2}\right)&=\frac{f\left(\frac{\pi}{2}^-\right)+f\left(\frac{\pi}{2}^+\right)}{2}\\&=\frac{\pi}{2}\end{aligned}$$