

Harmonic Analysis : Recitations

Aakash Jog

2015-16

Contents

1	Instructor Information	2
I	Fourier Series	3
1	Fourier Series	3
2	Bessel's Inequality	5
3	Riemann-Lebesgue's Lemma	5
4	Dirichlet's Kernel	6
5	Fourier Series in a General Interval	10



This work is licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License. To view a copy of this license, visit <http://creativecommons.org/licenses/by-nc-sa/4.0/>.

1 Instructor Information

Yaron Yeger

Office: Shenkar Physics 201

E-mail: yaronyeg@mail.tau.ac.il

Part I

Fourier Series

1 Fourier Series

Definition 1 (Real Fourier series). Let $f : [-L, L] \in \mathbb{C}$ be a piecewise continuous function.

The series

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

is called the Fourier series of $f(x)$, where

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos(nx) dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin(nx) dx$$

Theorem 1. If $f(x)$ is an even function, then the appropriate Fourier series is

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

If $f(x)$ is an odd function, then the appropriate Fourier series is

$$f(x) \approx \sum_{n=1}^{\infty} a_n \sin(nx)$$

Definition 2 (Complex Fourier series). Let $f : [-L, L] \in \mathbb{C}$ be a piecewise continuous function.

The series

$$f(x) \approx \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

If $f(x)$ is odd, its graph always passes through the origin. Therefore, it can be represented by a summation of sine functions, which also pass through the origin, and there is no need for a term, i.e. $\frac{a_0}{2}$, to change its position at the origin.

is called the complex Fourier series of $f(x)$, where

$$c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-inx} dx$$

Recitation 1 – Exercise 1.

Calculate the real Fourier series of

$$f(x) = 2x - 2\pi$$

Recitation 1 – Solution 1.

As x is an odd function, the real Fourier series of x , in the interval $[-\pi, \pi]$ is

$$x \approx \sum_{n=1}^{\infty} b_n \sin(nx)$$

where

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx \\ &= \frac{1}{\pi} \left(x \int \sin(nx) dx - \int 1 \left(\int \sin(nx) dx \right) dx \right) \Big|_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left(-\frac{x \cos(nx)}{n} \right) \Big|_{-\pi}^{\pi} + \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\cos(nx)}{n} dx \\ &= \frac{1}{\pi} \left(-\frac{\pi \cos(n\pi) + \pi \cos(-n\pi)}{n} \right) + \frac{1}{\pi} \frac{\sin(nx)}{n^2} \Big|_{-\pi}^{\pi} \xrightarrow{0} \\ &= -\frac{\cos(n\pi) + \cos(n\pi)}{n} \\ &= -2 \frac{\cos(n\pi)}{n} \\ &= -2 \frac{(-1)^n}{n} \end{aligned}$$

Therefore,

$$x \approx 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

Therefore,

$$2x - 2\pi \approx \left(4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) \right) - 2\pi$$

2 Bessel's Inequality

Definition 3 (Piecewise continuous functions). $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be piecewise continuous if, for every finite interval $[a, b]$ there is a finite number of discontinuity points, and the one-sided limits at each of these points are also finite.

Definition 4 (Piecewise continuously differentiable functions). $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be piecewise continuously differentiable if it is piecewise continuous, and

$$\lim_{h \rightarrow 0^+} \frac{f(x+h) - f(x^+)}{h} < \infty$$

and

$$\lim_{h \rightarrow 0^-} \frac{f(x+h) - f(x^-)}{h} < \infty$$

Theorem 2 (Bessel's Inequality). *Let $f(x)$ be a piecewise continuous function defined on $[-L, L]$. Then*

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \leq \frac{1}{L} \int_{-L}^L f(x)^2 dx$$

3 Riemann-Lebesgue's Lemma

Theorem 3 (Riemann-Lebesgue's Lemma). *If $f(x)$ is piecewise continuous on $[-L, L]$, then*

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 0$$

4 Dirichlet's Kernel

Definition 5 (Dirichlet kernel).

$$\begin{aligned} D_m(t) &= \frac{1}{2} \sum_{n=-m}^m e^{-int} \\ &= \frac{1}{2} + \sum_{n=1}^m \cos(nt) \\ &= \frac{\sin\left(\left(n + \frac{1}{2}\right)t\right)}{2 \sin \frac{t}{2}} \end{aligned}$$

is called the Dirichlet kernel of order m .

Theorem 4 (Second representation of Dirichlet's kernel). *Let $m \in \mathbb{N}$. Then, for $t \neq 2\pi k$, where $k \in \mathbb{Z}$,*

$$\begin{aligned} D_m(t) &= \frac{1}{2} + \cos(t) + \cos(2t) + \cdots + \cos(mt) \\ &= \frac{\sin\left(\left(m + \frac{1}{2}\right)t\right)}{2 \sin\left(\frac{1}{2}t\right)} \end{aligned}$$

Theorem 5. *Let*

$$S_m(f, x) = \frac{1}{2}a_0 + \sum_{n=1}^m a_n \cos(nx) + b_n \sin(nx)$$

Then,

$$S_m(f, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \left(\frac{1}{2} \sum_{n=1}^m \cos(nt) \right) dt$$

Theorem 6 (Dirichlet Theorem). *Let $f : [-\pi, \pi] \rightarrow \mathbb{R}$ be a piecewise continuously differentiable function.*

Then, $\forall x \in (-\pi, \pi)$,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = \frac{f(x^-) + f(x^+)}{2}$$

and for $x = \pi$ or $x = -\pi$,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = \frac{f(\pi^-) + f(-\pi^+)}{2}$$

Recitation 2 – Exercise 1.

The Fourier series of x^2 is given to be

$$x^2 \approx \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

Calculate

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Recitation 2 – Solution 1.

As x^2 is continuous, with a continuous derivative, Dirichlet Theorem is applicable.

Therefore, let

$$x = \pi$$

Therefore, by Dirichlet Theorem,

$$\begin{aligned} \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) &= \frac{(\pi^-)^2 + ((-\pi)^+)^2}{2} \\ \therefore \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n &= \pi^2 \\ \therefore \frac{\pi^2}{4} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} &= \pi^2 \\ \therefore \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{1}{4} \left(\pi^2 - \frac{\pi^2}{3} \right) \\ &= \frac{\pi^2}{6} \end{aligned}$$

Recitation 2 – Exercise 2.

The Fourier series of

$$f(x) = \begin{cases} x & ; \quad 0 \leq x \leq \pi \\ 0 & ; \quad -\pi \leq x \leq 0 \end{cases}$$

is given to be

$$f(x) \approx \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}}{n} \sin(nx) - \frac{2}{\pi(2n-1)^2} \cos((2n-1)x) \right)$$

Let this Fourier series be denoted by $S(x)$.

Calculate

1. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$
2. $S\left(\frac{\pi}{2}\right)$

Recitation 2 – Solution 2.

1.

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Therefore, for $x = 0$, by Dirichlet Theorem,

$$\begin{aligned} \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}}{n} \sin(0) - \frac{2}{\pi(2n-1)^2} \cos(0) \right) &= \frac{f(0^-) + f(0^+)}{2} \\ \therefore \frac{\pi}{4} - \sum_{n=1}^{\infty} \left(\frac{2}{\pi(2n-1)^2} \right) &= 0 \\ \therefore \frac{\pi}{4} - \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)^2} \right) &= 0 \\ \therefore \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} &= \frac{\pi^2}{8} \end{aligned}$$

2. By Dirichlet Theorem,

$$\begin{aligned} S\left(\frac{\pi}{2}\right) &= \frac{f\left(\frac{\pi}{2}^-\right) + f\left(\frac{\pi}{2}^+\right)}{2} \\ &= \frac{\pi}{2} \end{aligned}$$

Theorem 7. *If f is a piecewise continuous and periodic function with period of 2π , then*

$$\begin{aligned} S_m(x) &= \frac{a_0}{2} + \sum_{n=1}^m (a_n \cos(nx) + b_n \sin(nx)) \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) D_m(t) dt \end{aligned}$$

Recitation 3 – Exercise 1.

Calculate the limit

$$L = \lim_{n \rightarrow \infty} \int_{-n}^n \sin\left(\frac{2n+1}{2}t\right) \frac{\cos^2\left(\frac{\pi}{4} + t\right) + \pi^2}{\sin\left(\frac{t}{2}\right)} dt$$

Recitation 3 – Solution 1.

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \int_{-n}^n \sin\left(\frac{2n+1}{2}t\right) \frac{\cos^2\left(\frac{\pi}{4} + t\right) + \pi^2}{\sin\left(\frac{t}{2}\right)} dt \\ &= 2 \lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} \left(\cos^2\left(\frac{\pi}{4} + t\right) + \pi^2\right) \frac{\sin\left(n + \frac{1}{2}t\right)}{2 \sin\left(\frac{t}{2}\right)} dt \\ &= 2 \lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} \left(\cos^2\left(\frac{\pi}{4} + t\right) + \pi^2\right) D_n(t) dt \end{aligned}$$

Let

$$f(x) = \cos^2 x + \pi^2$$

Let S_n be the partial sum of the Fourier series.

Therefore,

$$S_n = \frac{1}{\pi} \int_{-\pi}^{\pi} D_n(t) \left(\cos^2(x+t) + \pi^2\right) dt$$

Therefore,

$$\begin{aligned} L &= 2\pi \lim_{n \rightarrow \infty} \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\cos^2\left(\frac{\pi}{4} + t\right) + \pi^2\right) D_n(t) dt \\ &= 2\pi \lim_{n \rightarrow \infty} S_n\left(\frac{\pi}{4}\right) \\ &= 2\pi \frac{f\left(\frac{\pi}{4}^+\right) + f\left(\frac{\pi}{4}^-\right)}{2} \\ &= 2\pi f\left(\frac{\pi}{4}\right) \\ &= 2\pi \left(\cos^2\left(\frac{\pi}{4}\right) + \pi^2\right) \\ &= \pi + 2\pi^3 \end{aligned}$$

5 Fourier Series in a General Interval

Definition 6. Let f be a piecewise continuous function defined on $[a, b]$. The Fourier series over $[a, b]$ is defined as

$$\begin{aligned} f(x) &\approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \left(\frac{2\pi nx}{b-a} \right) + b_n \sin \left(\frac{2\pi nx}{b-a} \right) \right) \\ &\approx \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi i n x}{b-a}} \end{aligned}$$

where

$$\begin{aligned} a_0 &= \frac{1}{b-a} \int_a^b f(x) \, dx \\ a_n &= \frac{2}{b-a} \int_a^b f(x) \cos \frac{2\pi nx}{b-a} \, dx \\ b_n &= \frac{2}{b-a} \int_a^b f(x) \sin \frac{2\pi nx}{b-a} \, dx \\ c_n &= \frac{1}{b-a} \int_a^b f(x) e^{\frac{2\pi i n x}{b-a}} \, dx \end{aligned}$$

Recitation 3 – Exercise 2.

Develop the Fourier series for $\text{sign}(x)$ over $[0, \pi]$.

Recitation 3 – Solution 2.

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} \text{sign}(x) \, dx \\ &= 2 \\ a_n &= \frac{2}{\pi} \int_0^{\pi} \text{sign}(x) \cos \left(\frac{2\pi nx}{\pi} \right) \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} \cos(2nx) \, dx \\ &= 0 \end{aligned}$$

$$\begin{aligned}
b_n &= \frac{2}{\pi} \int_0^{\pi} \text{sign}(x) \sin\left(\frac{2\pi nx}{\pi}\right) dx \\
&= \frac{2}{\pi} \int_0^{\pi} \sin(2nx) dx \\
&= 0
\end{aligned}$$

Therefore, over $[0, \pi]$,

$$\begin{aligned}
\text{sign}(x) &= \frac{2}{\pi} + \sum_{n=1}^{\infty} 0 \\
&= 1
\end{aligned}$$

Theorem 8. *Let f be continuous in $[-\pi, \pi]$, with piecewise continuous derivative, and $f(-\pi) = f(\pi)$. Then, the Fourier series converges uniformly on $[-\pi, \pi]$.*

Theorem 9 (Parseval Equality). *Let f be a piecewise continuous function in $[-\pi, \pi]$. Then,*

$$\begin{aligned}
\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx &= \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \\
&= 2 \sum_{n=-\infty}^{\infty} |c_n|^2
\end{aligned}$$

Recitation 4 – Exercise 1.

Use the Fourier series

$$x^2 \approx \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos(nx)$$

to calculate $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Recitation 4 – Solution 1.

As x^2 is continuous, by Parseval Equality,

$$\begin{aligned}\frac{1}{\pi} \int_{-\pi}^{\pi} |x^2|^2 dx &= \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \\ &= \frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^2}\end{aligned}$$

Therefore,

$$\begin{aligned}16 \sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 dx - \frac{2\pi^4}{9} \\ &= \frac{1}{\pi} \left. \frac{x^5}{5} \right|_{-\pi}^{\pi} - \frac{2\pi^4}{9} \\ &= \frac{2\pi^4}{5} - \frac{2\pi^4}{9} \\ &= \frac{8}{45} \pi^4 \\ \therefore \sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{\pi^4}{90}\end{aligned}$$

Recitation 4 – Exercise 2.

Use the Fourier series

$$e^x \approx \sum_{n=-\infty}^{\infty} (-1)^n \frac{e^{\pi} - e^{-\pi}}{2\pi(1 - in)} e^{inx}$$

to calculate $\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1}$.

Recitation 4 – Solution 2.

As x^2 is continuous, by Parseval Equality,

$$\begin{aligned}\frac{1}{\pi} \int_{-\pi}^{\pi} |e^x|^2 dx &= 2 \sum_{n=-\infty}^{\infty} |c_n|^2 \\ &= 2 \sum_{n=-\infty}^{\infty} \frac{(e^{\pi} - e^{-\pi})^2}{4\pi^2 |1 - in|^2} \\ &= \frac{2(e^{\pi} - e^{-\pi})^2}{4\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + 1}\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + 1} &= \frac{4\pi}{2(e^\pi - e^{-\pi})^2} \int_{-\pi}^{\pi} |e^x|^2 dx \\
&= \frac{4\pi}{2(e^\pi - e^{-\pi})^2} \frac{e^{2x}}{2} \Big|_{-\pi}^{\pi} |e^x|^2 dx \\
&= \frac{4\pi}{2(e^\pi - e^{-\pi})^2} \frac{e^{2\pi} - e^{-2\pi}}{2} \\
&= \frac{e^{2\pi} - e^{-2\pi}}{(e^\pi - e^{-\pi})^2} \\
&= \frac{(e^\pi + e^{-\pi})(e^\pi - e^{-\pi})}{(e^\pi - e^{-\pi})^2} \\
&= \frac{e^\pi + e^{-\pi}}{e^\pi - e^{-\pi}}
\end{aligned}$$

Recitation 5 – Exercise 1.

Use

$$|x| \approx \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \cos((2n-1)x)$$

to compute the Fourier series of

$$\text{sign}(x) = \begin{cases} 1 & ; \quad x \geq 0 \\ -1 & ; \quad x < 0 \end{cases}$$

Recitation 5 – Solution 1.

As $|x|$ is continuous, piecewise differentiable, and $|- \pi| = |\pi|$, its Fourier series converges uniformly. Hence, the Fourier series can be differentiated term by term.

Therefore,

$$|x|' = \text{sign}(x)$$

Therefore,

$$\begin{aligned}
\text{sign}(x) &\approx \left(\frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \cos((2n-1)x) \right)' \\
&\approx \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \sin((2n-1)x)
\end{aligned}$$

Recitation 5 – Exercise 2.

f is given to be continuous, piecewise differentiable, and periodic with period 2π . Also

$$f(x) \approx \frac{a_0}{2} \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Determine whether the following are true or false.

$$\lim_{n \rightarrow \infty} na_n = \lim_{n \rightarrow \infty} nb_n = 0$$

True or false.

Recitation 5 – Solution 2.

na_n and nb_n are the Fourier coefficients for $f'(x)$. Therefore, as f is piecewise differentiable, and by Riemann-Lebesgue's Lemma,

$$\lim_{n \rightarrow \infty} na_n = \lim_{n \rightarrow \infty} nb_n = 0$$

Hence, the statement is true.

Theorem 10. *If f is piecewise continuous with Fourier series*

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

the, for all $x \in [-\pi, \pi]$,

$$\int_0^x f(t) dt = \frac{a_0}{2}x + \sum_{n=1}^{\infty} \frac{a_n}{n} \sin(nx) - \frac{b_n}{n} (\cos(nx) - 1)$$

This is not a Fourier series due to the x in $\frac{a_0}{2}x$.

Therefore, substituting the Fourier series of x ,

$$\int_0^x f(t) dt = \sum_{n=1}^{\infty} \frac{b_n}{n} + \sum_{n=1}^{\infty} \left(\frac{a_n + (-1)^n a_0}{n} \sin(nx) - \frac{b_n}{n} \cos(nx) \right)$$

Recitation 6 – Exercise 1.

Let f be piecewise continuous with Fourier series

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Prove

$$\sum_{n=1}^{\infty} \frac{b_n}{n} \leq \infty$$

Recitation 6 – Solution 1.

Let

$$F(x) = \int_0^x f(t) dt$$

Therefore, as $f(x)$ is piecewise continuous, $F(x)$ is also piecewise continuous. Therefore,

$$F(x) = \frac{a_0}{2}x + \sum_{n=1}^{\infty} \frac{a_n}{n} \sin(nx) - \frac{b_n}{n} (\cos(nx) - 1)$$

Therefore, the Fourier series of $F(x) - \frac{a_0}{2}x$ is

$$\begin{aligned} F(x) - \frac{a_0}{2}x &\approx \sum_{n=1}^{\infty} \frac{a_n}{n} \sin(nx) - \frac{b_n}{n} (\cos(nx) - 1) \\ &\approx \sum_{n=1}^{\infty} \frac{b_n}{n} + \sum_{n=1}^{\infty} \frac{a_n}{n} \sin(nx) - \frac{b_n}{n} \cos(nx) \end{aligned}$$

Therefore, as $F(x) - \frac{a_0}{2}x$ is piecewise continuous and finite, $\sum_{n=1}^{\infty} \frac{b_n}{n}$ is also finite.