HARMONIC ANALYSIS : ASSIGNMENT 2

AAKASH JOG
ID : 989323563

Exercise 1.
Is there a piecewise continuous function f on $[-\pi, \pi]$ such that $b_n = \frac{1}{n}$, $a_n = (-1)^n$?

Solution 1.
By Riemann-Lebesgue's Lemma, if f is piecewise continuous, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} = 0$$

Therefore, as $\lim\limits_{n\to\infty} a_n$ does not exist, f cannot be piecewise continuous.

Exercise 2.
Calculate the limit $L = \lim\limits_{n\to\infty} \int\limits_{-\pi}^{\pi} \left(\sin^{\frac{1}{3}} x + \sin nx\right) \sin nx \, \mathrm{d}x$.

Solution 2.
As $\sin^{\frac{1}{3}} x$ is piecewise continuous, by Riemann-Lebesgue's Lemma,

$$\begin{aligned}
L &= \lim_{n\to\infty} \int\limits_{-\pi}^{\pi} \left(\sin^{\frac{1}{3}} x + \sin nx\right) \sin nx \, \mathrm{d}x \\
&= \lim_{n\to\infty} \left(\int\limits_{-\pi}^{\pi} \sin^{\frac{1}{3}} x \sin nx \, \mathrm{d}x + \int\limits_{-\pi}^{\pi} \sin^2 nx \, \mathrm{d}x \right) \\
&= \pi \lim_{n\to\infty} \frac{1}{\pi} \int\limits_{-\pi}^{\pi} \sin^{\frac{1}{3}} x \sin nx \, \mathrm{d}x + \lim_{n\to\infty} \int\limits_{-\pi}^{\pi} \sin^2 nx \, \mathrm{d}x \\
&= 0 + \lim_{n\to\infty} \int\limits_{-\pi}^{\pi} \sin^2 nx \\
&= \frac{1}{2} \lim_{n\to\infty} \int\limits_{-\pi}^{\pi} 1 - \cos 2nx \, \mathrm{d}x \\
&= \pi
\end{aligned}$$

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Exercise 3.
Let f be a piecewise periodic function with period β. Show that the integral $\int\limits_a^{a+\beta} f(x)\,\mathrm{d}x$ is independent of a. Hint: Define $g(a) = \int\limits_a^{a+\beta} f(x)\,\mathrm{d}x$ and show that g is constant.

Solution 3.
Let

$$\begin{aligned} g(a) &= \int\limits_a^{a+\beta} f(x)\,\mathrm{d}x \\ &= \int\limits_a^{0} f(x)\,\mathrm{d}x + \int\limits_0^{a+\beta} f(x)\,\mathrm{d}x \\ &= \int\limits_0^{a+\beta} f(x)\,\mathrm{d}x - \int\limits_0^{a} f(x)\,\mathrm{d}x \end{aligned}$$

Therefore,

$$g'(a) = f(a+\beta) - f(a)$$

As f is periodic, $\forall a$,

$$\begin{aligned} f(a+\beta) &= f(a) \\ \therefore g'(a) &= 0 \end{aligned}$$

Therefore, $g(a)$ is independent of a. $\square$

Exercise 4.
Prove that for all $x \in (-\pi, \pi)$, the following equality holds.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx = \frac{x}{2}$$

Hint: Develop the Fourier series of $f(x) = x$ and use Dirichlet theorem to show point wise continuity.

Solution 4.
Let

$$f(x) = x$$

Therefore,

$$
\begin{aligned}
a_0 &= \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi} \int\limits_{-\pi}^{\pi} x\,\mathrm{d}x \\
&= \frac{1}{\pi} \left.\frac{x^2}{2}\right|_{-\pi}^{\pi} \\
&= \frac{1}{\pi} \left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x \\
&= \frac{1}{\pi} \int\limits_{-\pi}^{\pi} x\cos(nx)\,\mathrm{d}x \\
&= \frac{1}{\pi} \left.\frac{\cos(nx)}{n^2} + \frac{x\sin(nx)}{n}\right|_{-\pi}^{\pi} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \\
&= \frac{1}{\pi} \int\limits_{-\pi}^{\pi} x\sin(nx)\,\mathrm{d}x \\
&= \frac{1}{\pi} \left.\frac{\sin(nx)}{n^2} - \frac{x\cos(nx)}{n}\right|_{-\pi}^{\pi} \\
&= 2\frac{(-1)^{n+1}}{n}
\end{aligned}
$$

Therefore,

$$
x \approx 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(nx)}{n}
$$

Therefore, by Dirichlet theorem,

$$
\begin{aligned}
2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx &= \frac{f(x^-)+f(x^+)}{2} \\
&= x \\
\therefore \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx &= \frac{x}{2}
\end{aligned}
$$

$\square$

Exercise 5.

Using the function $g(x) = x\left(\pi - |x|\right)$ calculate the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}$.

Solution 5.

$$g(x) = x\left(\pi - |x|\right)$$

Therefore, as the function is odd, its Fourier series is

$$g(x) \approx \sum_{n=1}^{\infty} b_n \sin(nx)\,\mathrm{d}x$$

Therefore,

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int\limits_{-\pi}^{\pi} x\left(\pi - |x|\right)\sin(nx)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int\limits_{-\pi}^{0} x(\pi+x)\sin(nx)\,\mathrm{d}x + \frac{1}{\pi}\int\limits_{0}^{\pi} x(\pi-x)\sin(nx)\,\mathrm{d}x \\
&= \frac{1}{\pi}\frac{2-2\cos(n\pi)-n\pi\sin(n\pi)}{n^3} + \frac{1}{\pi}\frac{2-2\cos(n\pi)-n\pi\sin(n\pi)}{n^3} \\
&= \frac{4}{\pi n^3}\left(1-(-1)^n\right)
\end{aligned}
$$

Therefore,

$$
b_n = \begin{cases} 0 & ; \quad n \text{ is even} \\ \frac{8}{\pi n^3} & ; \quad n \text{ is odd} \end{cases}
$$

Therefore,

$$\frac{g\left(\frac{\pi}{2}^{+}\right) + g\left(\frac{\pi}{2}^{-}\right)}{2} = \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\sin\left((2k-1)x\right)$$

Therefore,

$$\sum_{k=1}^{\infty}\frac{(-1)^{n+1}}{(2k-1)^3} = \frac{\pi^3}{32}$$

Exercise 6.
Find the Fourier series of $f(x) = 1 - x^2$. For which values does it converge when $x = 5\pi$ and $x = 6\pi$.

Solution 6.
As x^2 is even, its Fourier series is

$$x^2 \approx \frac{a_0}{2} \sum_{n=1}^{\infty} a_n \cos(nx)$$

Therefore,

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, \mathrm{d}x \\ &= \frac{1}{\pi} \left. \frac{x^3}{3} \right|_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \frac{2\pi^3}{3} \\ &= \frac{2\pi^2}{3} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) \, \mathrm{d}x \\ &= \frac{1}{\pi} \left. \frac{2x\cos(nx)}{n^2} + \frac{\left(n^2x^2 - 2\right)\sin(nx)}{n^3} \right|_{-\pi}^{\pi} \\ &= \frac{4}{n^2}(-1)^n \end{aligned}$$

Therefore,

$$\begin{aligned} x^2 &\approx \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos(nx) \\ \therefore 1 - x^2 &\approx 1 - \frac{\pi^2}{3} - \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos(nx) \end{aligned}$$

Exercise 7.
Out of the Fourier series of $\cos \alpha x$ for non-integral α, find $\cot(\alpha\pi)$.

Solution 7.
As $\cos(\alpha x)$ is even, its Fourier series is

$$\cos(\alpha x) \approx \frac{a_0}{2} \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(\alpha x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left.\frac{\sin(\alpha x)}{\alpha}\right|_{-\pi}^{\pi} \\
&= \frac{2\sin(\alpha\pi)}{\alpha\pi}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(\alpha x)\cos(nx)\,\mathrm{d}x \\
&= \frac{(-1)^n}{\pi}\sin(\alpha\pi)\frac{2\alpha}{\alpha^2 - n^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\cos(\alpha x) &\approx \frac{\sin(\alpha\pi)}{\alpha\pi} + \sum_{n=1}^{\infty}\frac{(-1)^n}{\pi}\sin(\alpha\pi)\frac{2\alpha}{\alpha^2 - n^2}\cos(nx) \\
&= \frac{\sin(\alpha\pi)}{\alpha\pi} + \frac{2\alpha\sin(\alpha\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha^2 - n^2}\cos(nx)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\cos(\alpha\pi) &= \frac{\sin(\alpha\pi)}{\alpha\pi} + \frac{2\alpha\sin(\alpha\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha^2 - n^2}(-1)^n \\
&= \frac{\sin(\alpha\pi)}{\alpha\pi} + \frac{2\alpha\sin(\alpha\pi)}{\pi}\sum_{n=1}^{\infty}\frac{1}{\alpha^2 - n^2}
\end{aligned}$$

$$\therefore \cot(\alpha\pi) = \frac{1}{\alpha\pi} + \frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\frac{1}{\alpha^2 - n^2}$$