HARMONIC ANALYSIS : ASSIGNMENT 4

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Exercise 1.
Let

$$g(x) = \begin{cases} \cos x & ; \quad -\pi \le x \le 0 \\ \sin x & ; \quad 0 < x \le \pi \end{cases}$$

(1) Calculate Fourier series of g.
(2) Let us define

$$h(x) = \int\limits_{-\pi}^{x} g(t) \,\mathrm{d}t + a \sin \frac{x}{2}$$

For which values of a, will the Fourier series of h uniformly converge?

Solution 1.
(1)

$$g(x) = \begin{cases} \cos x & ; \quad -\pi \le x \le 0 \\ \sin x & ; \quad 0 < x \le \pi \end{cases}$$

Therefore,

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int\limits_{-\pi}^{\pi} g(x) \,\mathrm{d}x \\ &= \frac{1}{\pi} \int\limits_{-\pi}^{0} \cos x \,\mathrm{d}x + \frac{1}{\pi} \int\limits_{0}^{\pi} \sin x \,\mathrm{d}x \\ &= \frac{2}{\pi} \end{aligned}$$

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$$
\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(x)\cos(nx)\,\mathrm{d}x \\
&= \frac{1}{\pi} \int_{-\pi}^{0} \cos x \cos(nx)\,\mathrm{d}x + \frac{1}{\pi} \int_{0}^{\pi} \sin x \cos(nx)\,\mathrm{d}x \\
&= \begin{cases} \frac{1}{2} & ; \quad n = 1 \\ \frac{1+(-1)^n}{\pi\left(n^2-1\right)} & ; \quad n \neq 1 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(x)\sin(nx)\,\mathrm{d}x \\
&= \frac{1}{\pi} \int_{-\pi}^{0} \cos x \sin(nx)\,\mathrm{d}x + \frac{1}{\pi} \int_{0}^{\pi} \sin x \sin(nx)\,\mathrm{d}x \\
&= \begin{cases} \frac{1}{2} & ; \quad n = 1 \\ -\frac{n\left(1+(-1)^n\right)}{n^2-1} & ; \quad n \neq 1 \end{cases}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
a_{2n} &= \frac{2}{\pi\left(1-4n^2\right)} \\
b_{2n} &= \frac{4n}{\pi\left(1-4n^2\right)}
\end{aligned}
$$

Therefore,

$$
g(x) \approx \frac{1}{\pi} + \frac{\sin x + \cos x}{2} + \sum_{n=1}^{\infty} \left(\frac{2\cos(2nx)}{\pi\left(1-4n^2\right)} + \frac{4n\sin(2nx)}{\pi\left(1-4n^2\right)} \right)
$$

(2)

$$
h(x) = \int_{-\pi}^{x} g(t)\,\mathrm{d}t + a \sin\frac{x}{2}
$$

Therefore,

$$
\begin{aligned}
h(x) &= \begin{cases} \int_{-\pi}^{x} \cos t\,\mathrm{d}t + a\sin\frac{x}{2} & ; \quad -\pi \leq x \leq 0 \\ \int_{-\pi}^{0} \cos t\,\mathrm{d}t + \int_{0}^{x} \sin t\,\mathrm{d}t + a\sin\frac{x}{2} & ; \quad 0 < x \leq \pi \end{cases} \\
&= \begin{cases} \sin x + a\sin\frac{x}{2} & ; \quad -\pi \leq x \leq 0 \\ 1 - \cos x + a\sin\frac{x}{2} & ; \quad 0 < x \leq \pi \end{cases}
\end{aligned}
$$

Therefore, for the Fourier series to converge uniformly, $h(-\pi) = h(\pi)$.
Therefore,

$$\sin(-\pi) + a \sin \frac{-\pi}{2} = 1 - \cos \pi + a \sin \frac{\pi}{2}$$

Therefore,

$$a = 0$$

Exercise 2.
Let $\hat{f}(n)$ represent coefficients. Let $\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$ be the Fourier series of $f(x) = |x|$. Prove that the series $\sum_{n=-\infty}^{\infty} n\hat{f}(n)e^{inx}$ converges for all x on $[-\pi, \pi]$.

Solution 2.
$f(x)$ is continuous, $f'(x)$ is piecewise continuous, and $f(-\pi) = f(\pi)$.
Therefore, differentiating term by term,

$$f'(x) \approx i \sum_{n=-\infty}^{\infty} n\hat{f}(n)e^{inx}$$

Therefore, the series $\sum_{n=-\infty}^{\infty} n\hat{f}(n)e^{inx}$ converges for all $x \in [-\pi, \pi]$.

Exercise 3.
Let

$$f(x) = \begin{cases} \sin 2x & ; \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & ; \quad \text{otherwise} \end{cases}$$

(1) Find the Fourier series of f and f'.
(2) Where does the Fourier series of f' converge at $x = \pm\frac{\pi}{2}$?

Solution 3.
(1)

$$f(x) = \begin{cases} \sin 2x & ; \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Therefore, as $f(x)$ is odd, its Fourier series is

$$f(x) \approx \sum_{n=1}^{\infty} b_n \sin(nx)$$

Therefore,

$$b_n = \frac{1}{\pi} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

Therefore,

$$b_2 = \frac{1}{2}$$

$$b_{2n-1} = \frac{(-1)^n}{2\pi\left(n^2 - n - 1\right)}$$

Therefore,

$$f(x) \approx \frac{\sin(2x)}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2\pi\left(n^2 - n - 1\right)}$$

$f(x)$ is continuous, $f'(x)$ is piecewise continuous, and $f(-\pi) = f(\pi)$. Therefore, differentiating term by term,

$$f'(x) = \cos(2x) + \sum_{n=1}^{\infty} \frac{(-1)^n n \cos(nx)}{2\pi\left(n^2 - n - 1\right)}$$

Therefore,

$$f'\left(-\frac{\pi}{2}\right) = -1$$

$$f'\left(\frac{\pi}{2}\right) = -1$$

Exercise 4.
Using the Fourier series of $f(x) = x^2$, calculate the Fourier series of

$$g(x) = x^3 - \pi^2 x$$

Hint: Use integration by terms.

Solution 4.

$$f(x) = x^2$$

$f(x)$ is continuous, $f'(x)$ is piecewise continuous, and $f(-\pi) = f(\pi)$. Therefore, integrating term by term,

$$x^2 \approx \pi^2 x + 12 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(nx)$$

$$x \approx 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(nx)}{n}$$

Therefore,

$$g(x) = x^3 - \pi^2 x$$

$$= 12 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(nx)$$

Exercise 5.
Let f be a piecewise continuous function with period 2π. The Fourier series of f is

$$f \approx \sum_{-\infty, n\neq 0}^{\infty} \hat{f}(n)e^{inx}$$

Let

$$g(x) = \int_{-\pi}^{\pi} \big(f(t) + f(\pi - t)\big)\,\mathrm{d}t$$

Find the Fourier coefficients $\hat{g}(n)$ using $\hat{f}(n)$.

Solution 5.

$$f(t) \approx \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$$

Therefore,

$$\begin{aligned} f(\pi - t) &\approx \sum_{n=-\infty}^{\infty} \hat{f}(n)(-1)^n e^{-inx} \\ &\approx \sum_{n=-\infty}^{\infty} \hat{f}(-n)(-1)^{-n} e^{inx} \end{aligned}$$

Therefore,

$$\begin{aligned} g(x) &= \int_{-\pi}^{x} \big(f(t) + f(\pi - t)\big)\,\mathrm{d}t \\ &\approx \int_{-\pi}^{x} \left(\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{int} + \sum_{n=-\infty}^{\infty} \hat{f}(-n)(-1)^n e^{int} \right) \mathrm{d}t \\ &\approx \sum_{n=\infty}^{\infty} \int_{-\pi}^{x} e^{int} \Big(\hat{f}(n) + (-1)^n \hat{f}(-n)\Big)\,\mathrm{d}t \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{\hat{f}(n) + (-1)^n \hat{f}(-n)}{in} \right) \Bigg|_{-\pi}^{x} \end{aligned}$$

Therefore,

$$\hat{g}(n) = \frac{\hat{f}(n) + (-1)^n \hat{f}(-n)}{in}$$

Exercise 6.
Let f be the 2π periodic function such that

$$f(x) = e^x$$

for $x \in [-\pi, \pi]$, and $\sum\limits_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$ its Fourier series. So for $|x| < \pi$, we have

$$e^x = \sum_{-\infty}^{\infty} \hat{f}(n)e^{inx}$$

We will formally differentiate both side and get

$$e^x = \sum_{-\infty}^{\infty} in\hat{f}(n)e^{inx}$$

So we have

$$\hat{f}(n) = in\hat{f}(n)$$
$$\therefore \hat{f}(n) = 0$$

Where was our mistake?

Solution 6.
As $f(-\pi) \neq f(\pi)$, term by term differentiation is not possible.