INTRODUCTION TO LINEAR SYSTEMS : ASSIGNMENT 2

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Exercise 1.
The transfer function of a system is given by

$$G(s) = \frac{s+2}{s^3 + 8s^2 + 19s + 12}$$

The initial conditions of the system are unknowns denoted by

$$\begin{aligned} y(0^-) &= a \\ y'(0^-) &= b \\ y''(0^-) &= c \end{aligned}$$

(1) Find the ODE that represents the system.
(2) Find the initial conditions of the system, i.e. find a, b, c, and the missing λ_1, λ_3, if it is known that the homogeneous solution of the system is

$$y_h(t) = \left(e^{\lambda_1 t} + 3e^{-3t} + 4e^{\lambda_3 t}\right)\delta_{-1}(t)$$

such that $|\lambda_1| < |\lambda_3|$.
(3) Find the full response of the system to a unit step input.

Solution 1.
(1)

$$\begin{aligned} G(s) &= \frac{s+2}{s^3 + 8s^2 + 19s + 12} \\ \therefore \frac{Y(s)}{U(s)} &= \frac{s+2}{s^3 + 8s^2 + 19s + 12} \\ \therefore Y(s)\left(s^3 + 8s^2 + 19s + 12\right) &= U(s)(s+2) \\ \therefore y^{(3)}(t) + 8y^{(2)}(t) + 19y^{(1)}(t) + 12y(t) &= u^{(1)}(t) + 2u(t) \end{aligned}$$

(2) The corresponding homogeneous ODE is

$$y^{(3)}(t) + 8y^{(2)}(t) + 19y^{(1)}(t) + 12y(t) = 0$$

Therefore, the characteristic equation is

$$\begin{aligned} \lambda^3 + 8\lambda^2 + 19\lambda + 12 &= 0 \\ \therefore (\lambda+3)(\lambda^2 + 5\lambda + 4) &= 0 \\ \therefore (\lambda+3)(\lambda+1)(\lambda+4) &= 0 \end{aligned}$$

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Therefore,

$$\lambda_1 = -1$$
$$\lambda_2 = -3$$
$$\lambda_3 = -4$$

Therefore,

$$\begin{aligned} y_h(t) &= e^{-t} + 3e^{-3t} + 4e^{-4t} \\ \therefore {y_h}^{(1)}(t) &= -e^{-t} - 9e^{-3t} - 16e^{-4t} \\ \therefore {y_h}^{(2)}(t) &= e^{-t} + 27e^{-3t} + 64e^{-4t} \end{aligned}$$

Therefore,

$$\begin{aligned} y_h(0) &= 1 + 3 + 4 \\ \therefore a &= 8 \\ {y_h}^{(1)}(0) &= -1 - 9 - 16 \\ \therefore b &= -26 \\ {y_h}^{(2)}(0) &= 1 + 27 + 64 \\ \therefore c &= 92 \end{aligned}$$

(3)

$$\begin{aligned} Y_p(s) &= G(s)U(s) \\ &= \frac{s+2}{(s+1)(s+3)(s+4)}U(s) \end{aligned}$$

Therefore, as $u(t) = \delta_{-1}(t)$,

$$\begin{aligned} Y_p(s) &= \frac{s+1}{(s+1)(s+3)(s+4)}\frac{1}{s} \\ &= \frac{s+1}{s(s+1)(s+3)(s+4)} \end{aligned}$$

Therefore, let

$$Y(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3} + \frac{D}{s+4}$$

Therefore,

$$\begin{aligned}
A &= \left.\frac{s+2}{s(s+1)(s+3)(s+4)}\right|_{s=0} \\
&= \frac{1}{6} \\
B &= \left.\frac{s+2}{s(s+3)(s+4)}\right|_{s=1} \\
&= -\frac{1}{6} \\
C &= \left.\frac{s+2}{s(s+1)(s+4)}\right|_{s=3} \\
&= -\frac{1}{6} \\
D &= \left.\frac{s+2}{s(s+1)(s+3)}\right|_{s=4} \\
&= \frac{1}{6}
\end{aligned}$$

Therefore,

$$\begin{aligned}
Y_p(s) &= \frac{1}{6s} - \frac{1}{6(s+1)} - \frac{1}{6(s+3)} + \frac{1}{6(s+4)} \\
y_p(t) &= \left(\frac{1}{6} - \frac{1}{6}e^{-t} - \frac{1}{6}e^{-3t} + \frac{1}{6}e^{-4t}\right)\delta_{-1}(t)
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(t) &= y_h(t) + y_p(t) \\
&= \left(e^{-t} + 3e^{-3t} + 4e^{-4t}\right)\delta_{-1}(t) + \left(\frac{1}{6} - \frac{1}{6}e^{-t} - \frac{1}{6}e^{-3t} + \frac{1}{6}e^{-4t}\right)\delta_{-1}(t) \\
&= \left(\frac{1}{6} + \frac{5}{6}e^{-t} + \frac{17}{6}e^{-3t} + \frac{25}{6}e^{-4t}\right)\delta_{-1}(t)
\end{aligned}$$

Exercise 2.
The following ODE is given.

$$\begin{aligned}
y'''(t) + 6y''(t) + 11y'(t) + 6y(t) &= au'(t) + bu(t) \\
y(0^-) &= -2 \\
y'(0^-) &= 1 \\
y''(0^-) &= 0
\end{aligned}$$

(1) Find the system's response to the initial conditions. Hint: One of the characteristic polynomial zeros is at $\lambda = -2$.
(2) It is given that the system's full response to the input $u(t) = 2e^{-4t}\delta_{-1}(t)$, and the initial conditions is

$$y(t) = \left(-\frac{5}{2}e^{-t} + 3e^{-2t} - \frac{11}{2}e^{-3t} + 3e^{-4t}\right)\delta_{-1}(t)$$

Find

   (a) the values of a and b
   (b) the system's transfer function
(3) Find the total response, including initial conditions, to the input $f(t) = 3e^{-2t}\delta_{-1}(t)$.

Solution 2.
(1)

$$y'''(t) + 6y''(t) + 11y'(t) + 6y(t) = au'(t) + bu(t)$$

Therefore, the corresponding homogeneous ODE is

$$y'''(t) + 6y''(t) + 11y'(t) + 6y(t) = 0$$

Therefore, the characteristic equation is

$$\begin{aligned} \lambda^3 + 6\lambda^2 + 11\lambda + 6 &= 0 \\ \therefore (\lambda+2)(\lambda^2 + 4\lambda + 3) &= 0 \\ \therefore (\lambda+2)(\lambda+1)(\lambda+3) &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= -2 \\ \lambda_3 &= -3 \end{aligned}$$

$$\begin{aligned} y_h(t) &= Ae^{-t} + Be^{-2t} + Ce^{-3t} \\ \therefore {y_h}'(t) &= -Ae^{-t} + -2Be^{-2t} + -3Ce^{-3t} \\ \therefore {y_h}''(t) &= Ae^{-t} + 4Be^{-2t} + 9Ce^{-3t} \end{aligned}$$

Therefore, substituting in initial conditions,

$$\begin{aligned} -2 &= A + B + C \\ 1 &= -A - 2B - 3C \\ 0 &= A + 4B + 9C \end{aligned}$$

Therefore, solving,

$$\begin{aligned} A &= -\frac{7}{2} \\ B &= 2 \\ C &= -\frac{1}{2} \end{aligned}$$

Therefore,

$$y_h(t) = \left(-\frac{7}{2}e^{-t} + 2e^{-2t} - \frac{1}{2}e^{-3t}\right)\delta_{-1}(t)$$

(2) (a) Taking the Laplace transform of the ODE,

$$\begin{aligned} Y_p(s) &= \frac{as+b}{s^3+6s^2+11s+6}U(s) \\ &= \frac{2(as+b)}{(s+1)(s+2)(s+3)(s+4)} \end{aligned}$$

Let

$$Y_p(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} + \frac{D}{s+4}$$

Therefore,

$$\begin{aligned} A &= \left.\frac{2(as+b)}{(s+2)(s+3)(s+4)}\right|_{s=-1} \\ &= \frac{b-a}{3} \\ B &= \left.\frac{2(as+b)}{(s+1)(s+3)(s+4)}\right|_{s=-2} \\ &= 2a-b \\ C &= \left.\frac{2(as+b)}{(s+1)(s+2)(s+4)}\right|_{s=-3} \\ &= b-3a \\ D &= \left.\frac{2(as+b)}{(s+1)(s+2)(s+3)}\right|_{s=-4} \\ &= \frac{4a-b}{3} \end{aligned}$$

Comparing to the given response,

$$\begin{aligned} 1 &= A \\ &= \frac{b-a}{3} \\ 1 &= B \\ &= 2a-b \\ -5 &= C \\ &= b-3a \\ 3 &= D \\ &= \frac{4a-b}{3} \end{aligned}$$

Therefore, solving,

$$\begin{aligned} a &= 4 \\ b &= 7 \end{aligned}$$

(b) If the RHS is $\delta(t)$, the ODE is

$$
\begin{aligned}
{y_\delta}'''(t) + 6{y_\delta}''(t) + 11{y_\delta}'(t) + 6y_\delta(t) &= 0 \\
y_\delta(0^+) &= 0 \\
{y_\delta}'(0^+) &= 0 \\
{y_\delta}''(0^+) &= 1
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
y_\delta(t) &= Ae^{-t} + Be^{-2t} + Ce^{-3t} \\
\therefore {y_\delta}'(t) &= -Ae^{-t} - 2Be^{-2t} - 3Ce^{-3t} \\
\therefore {y_\delta}''(t) &= Ae^{-t} + 4Be^{-2t} + 9Ce^{-3t}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
0 &= A + B + C \\
0 &= -A - 2B - 3C \\
1 &= A + 4B + 9C
\end{aligned}
$$

Therefore, solving,

$$
\begin{aligned}
A &= \frac{1}{2} \\
B &= -1 \\
C &= \frac{1}{2}
\end{aligned}
$$

Therefore,

$$
y_\delta(t) = \left(\frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}\right)\delta_{-1}(t)
$$

Therefore, the impulse response is

$$
\begin{aligned}
g(t) &= 4{y_\delta}'(t) + 7y_\delta(t) \\
&= \left(\frac{3}{2}e^{-t} + e^{-2t} - \frac{5}{2}e^{-3t}\right)\delta_{-1}(t)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
G(s) &= \frac{3}{2}\frac{1}{s+1} + \frac{1}{s+2} - \frac{5}{2}\frac{1}{s+3} \\
&= \frac{4s+7}{s^3 + 6s^2 + 11s + 6}
\end{aligned}
$$

(3)

$$
u(t) = 3e^{-2t}\delta_{-1}(t)
$$

Therefore,

$$
U(s) = \frac{3}{s+2}
$$

Therefore,

$$\begin{aligned}
Y(s) &= \frac{-2s^2 - 11s - 16}{(s+1)(s+2)(s+3)} + 3\frac{4s+7}{(s+1)(s+2)^2(s+3)} \\
&= \frac{-2s^3 - 15s^2 - 26s - 11}{(s+1)(s+2)^2(s+3)}
\end{aligned}$$

Let

$$Y(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{s+3}$$

Therefore, solving,

$$\begin{aligned}
A &= 1 \\
B &= -10 \\
C &= 3 \\
D &= 7
\end{aligned}$$

Therefore,

$$Y(s) = \frac{1}{s+1} - \frac{10}{s+2} + \frac{3}{(s+2)^2} + \frac{7}{s+3}$$

Therefore,

$$y(t) = \left(e^{-t} - 10e^{-2t} + 3te^{-2t} + 7e^{-3t}\right)\delta_{-1}(t)$$

Exercise 3.
The system's input response (particular solution) to a unit step response is

$$y_{-1}(t) = \left(\frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}\right)\delta_{-1}(t)$$

(1) Find the system's transfer function. Find the ODE that represents the system.

(2) Find the final value of the system's response to the input

$$f(t) = \left(3e^{-t}\cos(t) + \frac{1}{2}\right)\delta_{-1}(t)$$

(3) A system is described by the following ODE.

$$y'(t) + ay(t) + bu(t)$$

It is given that for that the total response for the input

$$f(t) = 2e^{-4t}\delta_{-1}(t)$$

is

$$y(t) = \left(3e^{2t} - e^{-4t}\right)\delta_{-1}(t)$$

Find the system's transfer function. Find the initial condition of the system, i.e. $y(0)$.

Solution 3.
(1)

$$y_{-1}(t) = \left(\frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}\right)\delta_{-1}(t)$$

Therefore,

$$\begin{aligned}
g(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(y_{-1}(t)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}\right)\delta_{-1}(t)\right) \\
&= \left(-\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t}\right)\delta_{-1}(t) + \left(\frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}\right)\delta(t) \\
&= \left(-\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t}\right)\delta_{-1}(t)
\end{aligned}$$

Therefore,

$$\begin{aligned}
G(s) &= -\frac{1}{2}\frac{1}{s+1} + \frac{3}{2}\frac{1}{s+3} \\
&= \frac{s}{(s+1)(s+3)}
\end{aligned}$$

Therefore,

$$Y(s)\left(s^2 + 4s + 3\right) = sU(s)$$

Therefore, the ODE is

$$y''(t) + 4y'(t) + 3y(t) = u'(t)$$

(2)

$$\begin{aligned}
u(t) &= \left(3e^{-t}\cos(t) + \frac{1}{2}\right)\delta_{-1}(t) \\
\therefore U(s) &= \frac{3(s+1)}{(s+1)^2 + 1} + \frac{1}{2s} \\
&= \frac{7s^2 + 8s + 2}{2s\left(s^2 + 2s + 2\right)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
Y(s) &= G(s)U(s) \\
&= \frac{s}{s^2 + 4s + 3}\frac{7s^2 + 8s + 2}{2s\left(s^2 + 2s + 2\right)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{t\to\infty} y(t) &= \lim_{s\to 0} sY(s) \\
&= \lim_{s\to 0}\frac{s}{s^2 + 4s + 3}\frac{7s^2 + 8s + 2}{2s\left(s^2 + 2s + 2\right)} \\
&= 0
\end{aligned}$$

(3)

$$y'(t) + ay(t) = bu(t)$$

Therefore,

$$\begin{aligned} sY(s) - y(0^-) + aY(s) &= bU(s) \\ \therefore Y(s)(s+a) &= bU(s) + y(0^-) \\ \therefore Y(s) &= \frac{b}{s+a}U(s) + \frac{y(0^-)}{s+a} \end{aligned}$$

According to the given data,

$$\begin{aligned} y(t) &= \left(3e^{2t} - e^{-4t}\right)\delta_{-1}(t) \\ \therefore Y(s) &= \frac{3}{s-2} - \frac{1}{s+4} \\ u(t) &= \left(2e^{-4t}\right)\delta_{-1}(t) \\ \therefore U(s) &= \frac{2}{s+4} \end{aligned}$$

Therefore, comparing and solving,

$$\begin{aligned} a &= -2 \\ b &= 3 \\ y(0^-) &= 2 \end{aligned}$$

Therefore,

$$\begin{aligned} G(s) &= \left.\left(\frac{Y(s)}{U(s)}\right)\right|_{y(0^-)=0} \\ &= \frac{b}{s+a} \\ &= \frac{3}{s-2} \end{aligned}$$

Exercise 4.
A system's input response (particular solution) is given

$$y_{-1}(t) = \left(1 + e^{-3t}\sin(t) - e^{-3t}\cos(t)\right)\delta_{-1}(t)$$

(1) Find the system's transfer function.
(2) Find the ODE that represents the system.
(3) Find the system's input response (particular solution) to the input

$$f(t) = e^{-4t}\delta_{-1}(t)$$

assuming zero initial conditions.

Exercise 5.

The transfer function of a system is given by

$$G(s) = \frac{s+3}{(s+1)(s+2)^2(s-4)}$$

Find the system's impulse response $g(t)$ using the Residue Theorem.

Solution 5.

$$G(s) = \frac{s+3}{(s+1)(s+2)^2(s-4)}$$

Therefore,

$$g(t) = \sum_{k=1}^{3} \text{Residue}\left(\frac{s+3}{(s+1)(s+2)^2(s-4)}e^{st}, \lambda_k\right)$$

Therefore,

$$X_1(s) = \frac{s+3}{(s+2)^2(s-4)}$$

$$\therefore X_1(-1) = -\frac{2}{5}e^{-t}$$

$$X_3(s) = \frac{s+3}{(s+1)(s+2)^2}$$

The residue at $\lambda_2 = 2$ is

$$\lim_{s\to-2} \frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{(s+3)e^{st}}{(s+1)(s-4)}\right) = \frac{1}{6}te^{-2t} + \frac{7}{36}e^{-2t}$$

Therefore,

$$g(t) = -\frac{2}{5}e^{-t} + \frac{1}{6}te^{-2t} + \frac{7}{36}e^{-2t} + \frac{7}{180}e^{4t}$$