INTRODUCTION TO LINEAR SYSTEMS : ASSIGNMENT 1

AAKASH JOG
ID : 989323563

Exercise 1.
Consider the following system:

$$y^{(2)}(t) + 3y^{(1)}(t) + 2y(t) = u^{(1)}(t) + 3u(t)$$
$$y(0^-) = 1$$
$$y^{(1)}(0^-) = 1$$

(1) Find the homogeneous solution.
(2) Find the system's impulse response, $g(t)$.
(3) Find the system's total response for the input $f(t) = \sin(2t)\delta_{-1}(t)$.
Hint: $\int\limits_0^t e^{a\tau}\sin(b\tau)\,\mathrm{d}\tau = \frac{e^{at}\left(a\sin(bt) - b\cos(bt)\right)}{a^2+b^2} + \frac{b}{a^2+b^2}$.
(4) Find the particular solution for the input $f(t) = \left(4\sin(2t) + \frac{1}{2}\cos(2t)\right)\delta_{-1}(t)$.

Solution 1.
(1) The corresponding homogeneous ODE is

$$y''(t) + 3y'(t) + 2y(t) = 0$$

Therefore, the characteristic equation is

$$\lambda^2 + 3\lambda + 2 = 0$$

Therefore,

$$\lambda = \frac{-3 \pm \sqrt{9-8}}{2}$$

Therefore,

$$\lambda_1 = -1$$
$$\lambda_2 = -2$$

Therefore,

$$y_1 = e^{-t}$$
$$y_2 = e^{-2t}$$

Therefore, the homogeneous solution is

$$y_h(t) = Ae^{-t} + Be^{-2t}$$

Therefore,

$${y_h}'(t) = -Ae^{-t} - 2Be^{-2t}$$

Date: Thursday 22[nd] October, 2015.

Substituting the initial conditions,

$$1 = A + B$$
$$1 = -A - 2B$$

Therefore, solving,

$$A = 3$$
$$B = -2$$

Therefore,

$$y_h(t) = \left(3e^{-t} - 2e^{-2t}\right) \delta_{-1}(t)$$

(2) The corresponding homogeneous ODE is

$$y''(t) + 3y'(t) + 2y(t) = 0$$

Therefore, the characteristic equation is

$$\lambda^2 + 3\lambda + 2 = 0$$

Therefore,

$$\lambda = \frac{-3 \pm \sqrt{9 - 8}}{2}$$

Therefore,

$$\lambda_1 = -1$$
$$\lambda_2 = -2$$

Therefore,

$$y_1 = e^{-t}$$
$$y_2 = e^{-2t}$$

Therefore, the homogeneous solution is

$$y_\delta = Ae^{-t} + Be^{-2t}$$

Therefore,

$$y_\delta{}' = -Ae^{-t} - 2Be^{-2t}$$

As the input is a delta function, the initial conditions are

$$y_\delta(0^+) = 0$$
$$y_\delta{}^{(1)}(0^+) = 1$$

Therefore, substituting,

$$0 = A + B$$
$$1 = -A - 2B$$

Therefore, solving,

$$A = 1$$
$$B = -1$$

Therefore,

$$y_\delta(t) = e^{-t} - e^{-2t}$$
$$\therefore {y_\delta}'(t) = -e^{-t} + 2e^{-2t}$$

Therefore,

$$\begin{aligned} g(t) &= {y_\delta}'(t) + 3y_\delta(t) \\ &= \left(2e^{-t} - e^{-2t}\right)\delta_{-1}(t) \end{aligned}$$

(3)

$$\begin{aligned} y_p(t) &= g(t) * f(t) \\ &= \int\limits_{-\infty}^{\infty} g(t-\tau) f(\tau)\,\mathrm{d}\tau \\ &= \int\limits_{0}^{t} \left(2e^{-(t-\tau)} - e^{-2(t-\tau)}\right)\sin(2\tau)\,\mathrm{d}\tau \\ &= 2e^{-t}\int\limits_{0}^{t} e^{\tau}\sin(2\tau)\,\mathrm{d}\tau - e^{-2t}\int\limits_{0}^{t} e^{2\tau}\sin(2\tau)\,\mathrm{d}\tau \\ &= \quad 2e^{-t}\left(\frac{e^{t}\left(\sin(2t) - 2\cos(2t)\right)}{1^2+2^2} + \frac{2}{1^2+2^2}\right) \\ &\quad - e^{-2t}\left(\frac{e^{2t}\left(2\sin(2t) - 2\cos(2t)\right)}{2^2+2^2} + \frac{2}{2^2+2^2}\right) \\ &= \frac{2}{5}\left(\sin(2t) - 2\cos(2t) + \frac{4}{5}e^{-t}\right) - \left(\frac{1}{4}\left(\sin(2t) - \cos(2t)\right) + \frac{1}{4}e^{-2t}\right) \\ &= \left(\frac{3}{20}\sin(2t) - \frac{11}{20}\cos(2t) + \frac{4}{5}e^{-t} - \frac{1}{4}e^{-2t}\right) \end{aligned}$$

Therefore, the particular solution is

$$y_p(t) = \left(\frac{3}{20}\sin(2t) - \frac{11}{20}\cos(2t) + \frac{4}{5}e^{-t} - \frac{1}{4}e^{-2t}\right)\delta_{-1}(t)$$

Therefore, the total solution is

$$\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ &= \left(3e^{-t} - 2e^{-2t}\right)\delta_{-1}(t) + \left(\frac{3}{20}\sin(2t) - \frac{11}{50}\cos(2t) + \frac{4}{5}e^{-t} - \frac{1}{4}e^{-2t}\right)\delta_{-1}(t) \\ &= \left(\frac{3}{20}\sin(2t) - \frac{11}{20}\cos(2t) + \frac{19}{5}e^{-t} - \frac{9}{4}e^{-2t}\right)\delta_{-1}(t) \end{aligned}$$

(4)

$$\begin{aligned} f(t) &= 4\sin(2t) + \frac{1}{2}\cos(2t) \\ &= 4\sin(2t) + \frac{1}{4}\sin(2t)' \end{aligned}$$

Let y_{p_1} be the particular solution for the input $\sin(2t)$.
Therefore,

$$y_{p_1}(t) = \left(\frac{3}{20}\sin(2t) - \frac{11}{20}\cos(2t) + \frac{4}{5}e^{-t} - \frac{1}{4}e^{2t}\right)\delta_{-1}(t)$$

$$\therefore {y_{p_1}}'(t) = \left(\frac{3}{10}\cos(2t) + \frac{11}{20}\sin(2t) - \frac{4}{5}e^{-t} + \frac{1}{2}e^{-2t}\right)$$

Let y_{p_2} be the particular solution for the input $f(t) = 4\sin(2t) + \frac{1}{2}\cos(2t)$.
Therefore,

$$\begin{aligned}
y_{p_2}(t) &= 4y_{p_1}(t) + \frac{1}{4}{y_{p_1}}'(t) \\
&= \quad 4\left(\frac{3}{20}\sin(2t) - \frac{11}{20}\cos(2t) + \frac{4}{5}e^{-t} - \frac{1}{4}e^{-2t}\right) \\
&\quad + \frac{1}{4}\left(\frac{3}{10}\cos(2t) + \frac{11}{20}\sin(2t) - \frac{4}{5}e^{-t} + \frac{1}{2}e^{-2t}\right) \\
&= \frac{7}{8}\sin(2t) - \frac{17}{8}\cos(2t) + 3e^{-t} - \frac{7}{8}e^{-2t}
\end{aligned}$$

Therefore, the particular solution is

$$y_{p_2}(t) = \frac{7}{8}\sin(2t) - \frac{17}{8}\cos(2t) + 3e^{-t} - \frac{7}{8}e^{-2t}$$

Therefore, the total solution is

$$\begin{aligned}
y(t) &= y_h + y_{p_2}(t) \\
&= \left(3e^{-t} - 2e^{-2t}\right)\delta_{-1}(t) + \left(\frac{7}{8}\sin(2t) - \frac{17}{8}\cos(2t) + 3e^{-t} - \frac{7}{8}e^{-2t}\right)\delta_{-1}(t) \\
&= \left(\frac{7}{8}\sin(2t) - \frac{17}{8}\cos(2t) + 6e^{-t} - \frac{23}{8}e^{-2t}\right)\delta_{-1}(t)
\end{aligned}$$

Exercise 2.
A linear system is given by the following ODE

$$y^{(2)}(t) + ay^{(1)}(t) + by(t) = 3u^{(1)}(t) + 5u(t)$$

It is known that the system's response to the initial conditions' response is

$$y_h(t) = \left(2e^{-t} + 4e^{-5t}\right)\delta_{-1}(t)$$

(1) Find the values of a and b.
(2) What are the system's initial conditions at $t = 0^-$?
(3) Irrespective of what you found so far, assume from here and on that $a = 6$, $b = 5$, and that the initial conditions are

$$y(0^-) = 0$$
$$y^{(1)}(0^-) = 1$$

Find the system's total response to an impulse input.
(4) Find the system's total response for a unit step input.

Solution 2.

(1) As the homogeneous solution is

$$y_h(t) = \left(2e^{-t} + 4e^{-5t}\right)\delta_{-1}(t)$$

the roots of the characteristic equation are

$$\lambda_1 = -1$$
$$\lambda_2 = -5$$

Therefore, the characteristic equation is

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$
$$\therefore (\lambda + 1)(\lambda + 5) = 0$$
$$\therefore \lambda^2 + 6\lambda + 5 = 0$$

Therefore, the homogeneous ODE is

$$y^{(2)}(t) + 6y^{(1)}(t) + 5y(t) = 0$$

Therefore, comparing,

$$a = 6$$
$$b = 5$$

(2) The system's inital conditions' response is

$$y_h(t) = \left(2e^{-t} + 4e^{-5t}\right)\delta_{-1}(t)$$
$$\therefore {y_h}'(t) = \left(-2e^{-t} - 20e^{-5t}\right)\delta_{-1}(t)$$

Therefore, substituting the initial conditions,

$$\begin{aligned} y_h(0^+) &= y_h(0^-) \\ &= 2 + 4 \\ &= 6 \\ {y_h}^{(1)}(0^+) &= {y_h}^{(1)}(0^-) \\ &= -2 - 20 \\ &= -22 \end{aligned}$$

(3) The homogeneous ODE is

$$y^{(2)}(t) + 6y^{(1)}(t) + 5y(t) = 0$$

Therefore,

$$\lambda_1 = -1$$
$$\lambda_2 = -5$$

Therefore,

$$\begin{aligned} y_\delta(t) &= Ae^{-t} + Be^{-5t} \\ \therefore {y_\delta}^{(1)}(t) &= -Ae^{-t} - 5Be^{-5t} \end{aligned}$$

As the input is a delta function

$$
\begin{aligned}
y_\delta(0) &= 0 \\
{y_\delta}^{(1)}(0) &= 1
\end{aligned}
$$

Therefore, substituting

$$
\begin{aligned}
0 &= A + B \\
1 &= -A - 5B
\end{aligned}
$$

Therefore, solving

$$
\begin{aligned}
A &= \frac{1}{4} \\
B &= -\frac{1}{4}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
y_\delta(t) &= \frac{1}{4}e^{-t} - \frac{1}{4}e^{-5t} \\
\therefore {y_\delta}^{(1)}(t) &= -\frac{1}{4}e^{-t} + \frac{5}{4}e^{-5t}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
g(t) &= 3{y_\delta}^{(1)}(t) + 5y_\delta(t) \\
&= -\frac{3}{4}e^{-t} + \frac{15}{4}e^{-5t} + \frac{5}{4}e^{-t} - \frac{5}{4}e^{-5t} \\
&= \frac{1}{2}e^{-t} + \frac{5}{2}e^{-5t}
\end{aligned}
$$

(4)

$$
\begin{aligned}
y_h(t) &= Ae^{-t} + Be^{-5t} \\
\therefore {y_h}^{(1)}(t) &= -Ae^{-t} - 5Be^{-5t}
\end{aligned}
$$

Therefore, substituting the initial values,

$$
\begin{aligned}
0 &= A + B \\
1 &= -A - 5B
\end{aligned}
$$

Therefore, solving,

$$
\begin{aligned}
A &= \frac{1}{4} \\
B &= -\frac{1}{4}
\end{aligned}
$$

Therefore, the homogeneous solution is

$$
y_h(t) = \left(\frac{1}{4}e^{-t} - \frac{1}{4}e^{-5t}\right)\delta_{-1}(t)
$$

The system's impulse response is

$$
g(t) = \left(\frac{1}{2}e^{-t} + \frac{5}{2}e^{-5t}\right)\delta_{-1}(t)
$$

Therefore,

$$\begin{aligned}
y_p(t) &= g(t) * u(t) \\
&= \int_0^t \left(\frac{1}{2}e^{-\tau} + \frac{5}{2}e^{-5\tau}\right) \mathrm{d}\tau \\
&= -\frac{1}{2}e^{-t}\Big|_0^t - \frac{1}{2}e^{-5t}\Big|_0^t \\
&= -\frac{1}{2}\left(e^{-t} - 1\right) - \frac{1}{2}\left(e^{-5t} - 1\right) \\
&= -\frac{1}{2}e^{-t} - \frac{1}{2}e^{-5t} + 1
\end{aligned}$$

Therefore, the particular solution is

$$y_p(t) = \left(-\frac{1}{2}e^{-t} - \frac{1}{2}e^{-5t} + 1\right)\delta_{-1}(t)$$

Therefore, the total solution is

$$\begin{aligned}
y(t) &= y_h(t) + y_p(t) \\
&= \left(\frac{1}{4}e^{-t} - \frac{1}{4}e^{-5t}\right)\delta_{-1}(t) + \left(-\frac{1}{2}e^{-t} - \frac{1}{2}e^{-5t} + 1\right)\delta_{-1}(t) \\
&= \left(-\frac{1}{4}e^{-t} - \frac{3}{4}e^{-5t} + 1\right)\delta_{-1}(t)
\end{aligned}$$

Exercise 3.

Consider the system given by

$$\begin{aligned}
y^{(2)}(t) + 6y^{(1)}(t) + 5y(t) &= u^{(1)}(t) + 4u(t) \\
y(0^-) &= 1 \\
y^{(1)}(0^-) &= -1
\end{aligned}$$

(1) Find the homogeneous solution.
(2) Find the system's impulse response.
(3) Find the system's total response for the input $f(t) = \left(e^{-t} + 5e^{-2t}\right)\delta_{-1}(t)$.
(4) It is given that the system's total response for the input $f(t) = \left(Ae^{-t} + Be^{-4t}\right)\delta_{-1}(t)$ is $y(t) = e^{-t}\delta_{-1}(t)$. Find the coefficients A and B.

Solution 3.

(1)

$$y^{(2)}(t) + 6y^{(1)}(t) + 5y(t) = u^{(1)}(t) + 4u(t)$$

Therefore, the corresponding homogeneous ODE is

$$y^{(2)}(t) + 6y^{(1)}(t) + 5y(t) = 0$$

Therefore, the characteristic equation is

$$\lambda^2 + 6\lambda + 5 = 0$$

Therefore,

$$\lambda_1 = -1$$
$$\lambda_2 = -5$$

Therefore,

$$y_1 = e^{-t}$$
$$y_2 = e^{-5t}$$

Therefore, the homogeneous solution is

$$y_h(t) = Ae^{-t} + Be^{-5t}$$
$$\therefore y_h{}'(t) = -Ae^{-t} - 5Be^{-5t}$$

Substituting the initial conditions,

$$1 = A + B$$
$$-1 = -A - 5B$$

Therefore, solving,

$$A = 1$$
$$B = 0$$

Therefore,

$$y_h(t) = e^{-t}\delta_{-1}(t)$$

(2)

$$y^{(2)}(t) + 6y^{(1)}(t) + 5y(t) = u^{(1)}(t) + 4u(t)$$

Therefore, the corresponding homogeneous ODE is

$$y^{(2)}(t) + 6y^{(1)}(t) + 5y(t) = 0$$

Therefore, the characteristic equation is

$$\lambda^2 + 6\lambda + 5 = 0$$

Therefore,

$$\lambda_1 = -1$$
$$\lambda_2 = -5$$

Therefore,

$$y_1 = e^{-t}$$
$$y_2 = e^{-5t}$$

Therefore, the homogeneous solution is

$$y_h(t) = Ae^{-t} + Be^{-5t}$$
$$\therefore y_h{}'(t) = -Ae^{-t} - 5Be^{-5t}$$

As the input is a delta function,

$$y_\delta(0^+) = 0$$
$$y_\delta{}'(0^+) = 1$$

Therefore, substituting,

$$0 = A + B$$
$$1 = -A - 5B$$

Therefore, solving,

$$A = \frac{1}{4}$$
$$B = -\frac{1}{4}$$

Therefore,

$$y_\delta(t) = \left(\frac{1}{4}e^{-t} - \frac{1}{4}e^{-5t}\right)\delta_{-1}(t)$$
$$\therefore y_\delta(t) = \left(-\frac{1}{4}e^{-t} + \frac{5}{4}e^{-5t}\right)\delta_{-1}(t)$$

Therefore,

$$\begin{aligned} g(t) &= {y_\delta}'(t) + 4y_\delta(t) \\ &= \left(-\frac{1}{4}e^{-t} + \frac{5}{4}e^{-5t}\right) + 4\left(\frac{1}{4}e^{-t} - \frac{1}{4}e^{-5t}\right) \\ &= \frac{3}{4}e^{-t} + \frac{1}{4}e^{-5t} \end{aligned}$$

Therefore, the system's total impulse response is

$$g(t) = \left(\frac{3}{4}e^{-t} + \frac{1}{4}e^{-5t}\right)\delta_{-1}(t)$$

(3)

$$\begin{aligned} y_p(t) &= g(t) * u(t) \\ &= \int_0^t \left(\frac{3}{4}e^{-(t-\tau)} + \frac{1}{4}e^{-5(t-\tau)}\right)\left(e^{-\tau} + 5e^{-2\tau}\right)\mathrm{d}\tau \\ &= \frac{3}{4}e^{-t}\int_0^t \left(1 + 5e^{-t}\right)\mathrm{d}\tau + \frac{1}{4}e^{-5t}\int_0^t \left(e^{4\tau} + 5e^{3\tau}\right)\mathrm{d}\tau \\ &= \frac{3}{4}e^{-t}\left(\tau - 5e^{-\tau}\right)\Big|_0^t + \frac{1}{4}e^{-5t}\left(\frac{1}{4}e^{4\tau} + \frac{5}{3}e^{3\tau}\right)\Big|_0^t \\ &= \frac{3}{4}e^{-t}\left(t - 5e^{-t} + 5\right) + \frac{1}{4}e^{-5t}\left(\frac{1}{4}e^{4t} + \frac{5}{3}e^{3t} - \left(\frac{1}{4} + \frac{5}{3}\right)\right) \\ &= \frac{61}{16}e^{-t} + \frac{3}{4}te^{-t} - \frac{10}{3}e^{-2t} - \frac{23}{48}e^{-5t} \end{aligned}$$

Therefore, the particular solution is

$$y_p(t) = \left(\frac{61}{16}e^{-t} + \frac{3}{4}te^{-t} - \frac{10}{3}e^{-2t} - \frac{23}{48}e^{-5t}\right)\delta_{-1}(t)$$

Therefore, the total solution is

$$\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ &= \left(\frac{77}{16} e^{-t} + \frac{3}{4} t e^{-t} - \frac{10}{3} e^{-2t} - \frac{23}{48} e^{-5t} \right) \delta_{-1}(t) \end{aligned}$$

(4)

$$\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ \therefore e^{-t} \delta_{-1}(t) &= e^{-t} \delta_{-1}(t) + y_p(t) \end{aligned}$$

Therefore,

$$\begin{aligned} y_p(t) &= g(t) * u(t) \\ \therefore 0 &= \int\limits_0^t \left(\frac{3}{4} e^{-t} + \frac{1}{4} e^{-5t} \right) \left(A e^{-(t-\tau)} + B e^{-4(t-\tau)} \right) \mathrm{d}\tau \\ &= A e^{-t} \int\limits_0^t \left(\frac{3}{4} + \frac{1}{4} e^{-4\tau} \right) + B e^{-4t} \int\limits_0^t \left(\frac{3}{4} e^{3\tau} + \frac{1}{4} e^{-\tau} \right) \mathrm{d}\tau \\ &= A e^{-t} \left(\frac{3}{4} \tau - \frac{1}{16} e^{-4t} \right) \Bigg|_0^t + B e^{-4t} \left(\frac{1}{4} e^{3\tau} - \frac{1}{4} e^{-\tau} \right) \Bigg|_0^t \\ &= A e^{-t} \left(\frac{3}{4} t - \frac{1}{16} e^{-4t} + \frac{1}{16} \right) + B e^{-4t} \left(\frac{1}{4} e^{3t} - \frac{1}{4} e^{-t} \right) \\ &= A \left(\frac{3}{4} t e^{-t} - \frac{1}{16} e^{-5t} + \frac{1}{16} e^{-t} \right) + B \left(\frac{1}{4} e^{-t} - \frac{1}{4} e^{-5t} \right) \\ &= e^{-t} \left(\frac{1}{16} A \right) + t e^{-t} \left(\frac{3}{4} A \right) + e^{-5t} \left(-\frac{1}{16} A - \frac{1}{4} B \right) \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{16} A &= 0 \\ \frac{3}{4} A &= 0 \\ -\frac{1}{16} A - \frac{1}{4} B &= 0 \end{aligned}$$

Therefore, solving,

$$\begin{aligned} A &= 0 \\ B &= 0 \end{aligned}$$

Exercise 4.
It is given that the impulse response of system 1 is

$$g_1(t) = \left(e^{-t} + e^{-2t} \right) \delta_{-1}(t)$$

It is given that system 1 is of minimal order, i.e. a minimal order system that satisfies the requirements.

(1) Find the ODE that represents system 1.

(2) It is now given that the initial conditions of system 1 are

$$y_1(0^-) = 2$$
$$y_1{}^{(1)}(0^-) = 1$$

Find the unit step response of system 1.

Another system, system 2, is given. It is known that the particular solution of system 2 to a unit step input is

$$y_{p_2}(t) = \left(-2 + e^{-t} + e^{-4t} + te^{-4t}\right) \delta_{-1}(t)$$

It is given that system 2 is of minimal order.

(3) Find the impulse response of system 2, $g_2(t)$.
(4) The initial conditions of system 2 are given to be

$$y_2(0^-) = 2$$
$$y_1{}^{(1)}(0^-) = 3$$
$$y_1{}^{(2)}(0^-) = 1$$

Find the homogeneous solution of system 2.

Solution 4.

(1)

$$g_1(t) = \left(e^{-t} + e^{-2t}\right) \delta_{-1}(t)$$

Therefore,

$$\lambda_1 = -1$$
$$\lambda_2 = -2$$

Therefore, the characteristic equation is

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$
$$\therefore \lambda^2 + 3\lambda + 2 = 0$$

Therefore, the corresponding homogeneous ODE is

$$y_1{}''(t) + 3y_1{}'(t) + 2y_1(t) = 0$$

Therefore, the ODE is

$$y_1{}''(t) + 3y_1{}'(t) + 2y_1(t) = b_1 u_1{}'(t) + b_0 u(t)$$

For y_δ, the initial conditions are

$$y_\delta(0^+) = 0$$
$$y_\delta{}'(0^+) = 1$$

Therefore, solving for y_δ,

$$y_\delta{}''(t) + 3y_\delta{}'(t) + 2y_\delta(t) = 0$$

Therefore,

$$y_\delta(t) = Ae^{-t} + Be^{-2t}$$
$$y_\delta{}'(t) = -Ae^{-t} - 2Be^{-2t}$$

Therefore, substituting the initial conditions,

$$0 = A + B$$
$$1 = -A - 2B$$

Therefore,

$$A = 1$$
$$B = -1$$

Therefore,

$$y_\delta(t) = e^{-t} - e^{-2t}$$
$$\therefore y_\delta{}'(t) = -e^{-t} + 2e^{-2t}$$

As $g_1(t)$ is the impulse response,

$$\begin{aligned} g_1(t) &= b_1 y_\delta{}'(t) + b_0 y_\delta(t) \\ &= b_1\left(-e^{-t} + 2e^{-2t}\right) + b_0\left(e^{-t} - e^{-2t}\right) \\ &= e^{-t}(b_0 - b_1) + e^{-2t}(2b_1 - b_0) \end{aligned}$$

Therefore, comparing with the given impulse response,

$$b_0 - b_1 = 1$$
$$2b_1 - b_0 = 1$$

Therefore,

$$b_0 = 3$$
$$b_1 = 2$$

Therefore, the ODE is

$$y_1{}''(t) + 3y_1{}'(t) + 2y_1(t) = 2u_1{}'(t) + 3u(t)$$

(2)

$$y_1{}''(t) + 3y_1{}'(t) + 2y_1(t) = 2u_1{}'(t) + 3u(t)$$
$$y_1(0^-) = 2$$
$$y_1{}'(0^-) = 1$$

Therefore, the homogeneous ODE is

$$y_1{}''(t) + 3y_1{}'(t) + 2y_1(t) = 0$$

Therefore, the characteristic equation is

$$\lambda^2 + 3\lambda + 2 = 0$$

Therefore,

$$\lambda_1 = -1$$
$$\lambda_2 = -2$$

Therefore,

$$\begin{aligned} y_{h1}(t) &= Ae^{-t} + Be^{-2t} \\ \therefore {y_{h1}}'(t) &= -Ae^{-t} - 2Be^{-2t} \end{aligned}$$

Substituting the initial conditions,

$$\begin{aligned} 2 &= A + B \\ 1 &= -A - 2B \end{aligned}$$

Therefore, solving,

$$\begin{aligned} A &= 5 \\ B &= -3 \end{aligned}$$

Therefore,

$$y_{h1}(t) = 5e^{-t} - 3e^{-2t}$$

As $g_1(t)$ is the impulse response,

$$\begin{aligned} y_{p1}(t) &= g_1(t) * u(t) \\ &= \int\limits_0^t g_1(t)u(t-\tau) \\ &= \int\limits_0^t \left(e^{-\tau} + e^{-2\tau}\right) \mathrm{d}\tau \\ &= -e^{-t} - \frac{1}{2}e^{-2t} + \frac{3}{2} \end{aligned}$$

Therefore,

$$\begin{aligned} y_1(t) &= y_{h1}(t) + y_{p1}(t) \\ &= \left(e3^{-t} - \frac{7}{2}e^{-2t} + \frac{3}{2}\right) \delta_{-1}(t) \end{aligned}$$

(3)

$$y_{p2}(t) = \left(-2 + e^{-t} + e^{-4t} + te^{-4t}\right) \delta_{-1}(t)$$

Therefore, as the impulse response is the derivative of the particular solution part of the step response,

$$\begin{aligned} g_2(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \left(-2 + e^{-t} + e^{-4t} + te^{-4t}\right) \delta_{-1}(t) \\ &= \left(-e^{-t} - 3e^{-4t} - 4te^{-4t}\right) \delta_{-1}(t) \end{aligned}$$

(4)

$$g_2(t) = \left(-e^{-t} - 3e^{-4t} - 4te^{-4t}\right) \delta_{-1}(t)$$

Therefore,

$$\lambda_1 = -1$$
$$\lambda_2 = -4$$
$$\lambda_3 = -4$$

Therefore, the characteristic equation is

$$(\lambda + 1)(\lambda + 4)^2 = 0$$
$$\therefore \lambda^3 + 9\lambda^2 + 24\lambda + 16 = 0$$

Therefore, the homogeneous ODE is

$$y'''(t) + 9y''(t) + 24y'(t) + 16y(t) = 0$$

Therefore, the homogeneous solution is

$$y_h(t) = Ae^{-t} + Be^{-4t} + Cte^{-4t}$$
$$\therefore {y_h}'(t) = -Ae^{-t} + (C - 4B)e^{-4t} - 4Cte^{-4t}$$
$$\therefore {y_h}''(t) = Ae^{-t} + (16B - 8C)e^{-4t} + 16te^{-4t}$$

Substituting the initial conditions,

$$2 = A + B$$
$$3 = -A + C - 4B$$
$$1 = A + 16B - 8C$$

Therefore, solving,

$$A = \frac{19}{3}$$
$$B = -\frac{13}{3}$$
$$C = -8$$

Therefore, the homogeneous solution is

$$y_{h2}(t) = \left(\frac{19}{3}e^{-t} - \frac{13}{3}e^{-4t} - 8te^{-4t}\right)\delta_{-1}(t)$$