INTRODUCTION TO PROBABILITY AND STATISTICS
ASSIGNMENT 5

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Exercise 1.
If X is a binomial random variable with expected value 6 and variance 2.4, find $\mathrm{P}(X = 5)$.

Solution 1.

$$\begin{aligned}
\mathrm{E}[X] &= 6 \\
\therefore np &= 6 \\
\mathrm{V}(X) &= 2.4 \\
\therefore np(1-p) &= 2.4
\end{aligned}$$

Therefore,

$$\begin{aligned}
1 - p &= 0.4 \\
\therefore p &= 0.6
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{P}(X = i) &= {}^{n}\mathrm{C}_{i}(p)^{i}(1-p)^{n-i} \\
\therefore (X = 5) &= {}^{10}\mathrm{C}_{5}(0.6)^{5}(0.4)^{5}
\end{aligned}$$

Exercise 2.
Each member of a seven judge panel independently makes a correct decision with probability 0.7. If the panel's decision is made by majority rule, what is the probability that the panel makes the correct decision? Given that four of the judges agreed, what is the probability that the panel made the correct decision?

Solution 2.
Let A be the decision that the panel makes the correct decision.
Let B be the decision that four of the judges agree.
Therefore,

$$\mathrm{P}(A) = \sum_{i=4}^{7} {}^{7}\mathrm{C}_{i}(0.7)^{i}(0.3)^{7-i}$$

Date: Sunday 13th March, 2016.

Therefore,

$$\begin{aligned}
\mathrm{P}(A|B) &= \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \\
&= \frac{{}^7\mathrm{C}_4 (0.7)^4 (0.3)^3}{{}^7\mathrm{C}_4 (0.7)^4 (0.3)^3 + {}^7\mathrm{C}_4 (0.3)^4 (0.7)^3}
\end{aligned}$$

Exercise 3.
A player bets on a number from 1 to 6. Three dice are then rolled, and if the number bet by the player appears i times, where $i = 1, 2, 3$, then the player wins i units. If the number bet by the player does not appear on any of the dice, then the player loses 1 unit. Is this game fair to the player?

Solution 3.
Let X be the player's winnings.
Let Y be the number of times the number the player bet on appeared. Therefore,

$$Y \sim \mathrm{Bin}\left(3, \frac{1}{6}\right)$$

Therefore,

$$\begin{aligned}
\mathrm{P}(X = -1) &= \mathrm{P}(Y = 0) \\
&= {}^3\mathrm{C}_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^3 \\
&= \frac{125}{216} \\
\mathrm{P}(X = 1) &= \mathrm{P}(Y = 1) \\
&= {}^3\mathrm{C}_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^2 \\
&= \frac{75}{216} \\
\mathrm{P}(X = 2) &= \mathrm{P}(Y = 2) \\
&= {}^3\mathrm{C}_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^1 \\
&= \frac{15}{216} \\
\mathrm{P}(X = 3) &= \mathrm{P}(Y = 3) \\
&= {}^3\mathrm{C}_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^0 \\
&= \frac{1}{216}
\end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{E}[X] &= (-1)\left(\frac{125}{216}\right) + (1)\left(\frac{75}{216}\right) + (2)\left(\frac{15}{216}\right) + (3)\left(\frac{1}{216}\right) \\ &= -\frac{17}{216} \end{aligned}$$

Therefore, as the expected value of the winnings is less than 0, the game is not fair towards the player.

Exercise 4.
Surveys indicated that 40% of the Zulu tribe members use the toothpaste 'Zebra'. A marketing company wants to interview a user of that toothpaste. Let X be the number of persons that the company would contact until a person who uses 'Zebra' will be found.

(1) What is the distribution of X?
(2) What is the expectation and variance of X?
(3) What is the probability that the company will interview at most 3 persons until a 'Zebra' user is found?
(4) Let Y be the number of clients that the company would contact until 2 persons who 'Zebra' will be found. What is the distribution of Y?
(5) What is the expectation and variance of Y?

Solution 4.

(1) X is a geometric random variable, with probability 0.4 for success. Therefore,

$$X \sim \mathrm{Geo}(0.4)$$

(2)

$$X \sim \mathrm{Geo}(0.4)$$

Therefore,

$$\begin{aligned} \mathrm{E}[X] &= \frac{1}{p} \\ &= \frac{1}{0.4} \\ &= 2.5 \\ \mathrm{V}(X) &= \frac{1-p}{p^2} \\ &= \frac{1-0.4}{(0.4)^2} \\ &= \frac{0.6}{0.16} \\ &= 3.75 \end{aligned}$$

(3)

$$X \sim \mathrm{Geo}(0.4)$$

Therefore,

$$\begin{aligned}
P(X \leq 3) &= 1 - P(X > 3) \\
&= 1 - \sum_{k=4}^{\infty} (1-p)^{k-1} p \\
&= 1 - (1-p)^3 \\
&= 1 - (0.6)^3 \\
&= 1 - 0.216 \\
&= 0.784
\end{aligned}$$

(4)

$$X \sim \mathrm{NB}(2, 0.4)$$

(5)

$$X \sim \mathrm{NB}(2, 0.4)$$

Therefore,

$$\begin{aligned}
E[X] &= \frac{r}{p} \\
&= \frac{2}{0.4} \\
&= 5 \\
V(X) &= \frac{r(1-p)}{p^2} \\
&= \frac{(2)(0.6)}{(0.4)^2} \\
&= \frac{0.12}{0.16} \\
&= 7.5
\end{aligned}$$

Exercise 5.
In the Israeli Parliament, the Knesset, 24 out of 120 MPs are women.

(1) If you randomly choose chairpersons for the 12 parliamentary committees, what are the chances that exactly 3 of them are women, given that
  (a) Each MP can be a chairperson of any number of committees.
  (b) Each MP can be a chairperson of at most one committee.
(2) Which method is better for promoting gender equality?

Solution 5.
(1) Let X be the number of parliamentary committees which have women as chairpersons such that a MP can chair any number of committees.

Therefore,

$$\begin{aligned} X &\sim \text{Bin}\left(12, \frac{24}{120}\right) \\ &\sim \text{Bin}\left(12, \frac{1}{5}\right) \end{aligned}$$

Therefore,

$$\begin{aligned} \text{P}(X = 3) &= {}^{12}\text{C}_3 \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^9 \\ &= 0.2362232013 \end{aligned}$$

(2) Let X be the number of parliamentary committees which have women as chairpersons such that a MP can chair at most one committee. Therefore,

$$X \sim \text{HG}\left(12, 120, 24\right)$$

Therefore,

$$\begin{aligned} \text{P}(X = 3) &= \frac{{}^{12}\text{C}_3 {}^{120-24}\text{C}_{12-3}}{{}^{120}\text{C}_{12}} \\ &= \frac{{}^{12}\text{C}_3 {}^{96}\text{C}_9}{{}^{120}\text{C}_{12}} \\ &= 0.02705523278 \end{aligned}$$

Exercise 6.

Alice likes hippopotamuses and walks daily with Bob to the zoo. The walking time to the zoo is distributed uniformly between 15 and 20 minutes with intervals of 1 minute.

(1) What is the chance that on a random day, Alice and Bob would not walk more than 17 minutes?
(2) What is the expectation and variance of the walking time to the zoo?
(3) What is the probability that today and tomorrow the walking time would exceed 19 minutes?