INTRODUCTION TO PROBABILITY AND STATISTICS ASSIGNMENT 7

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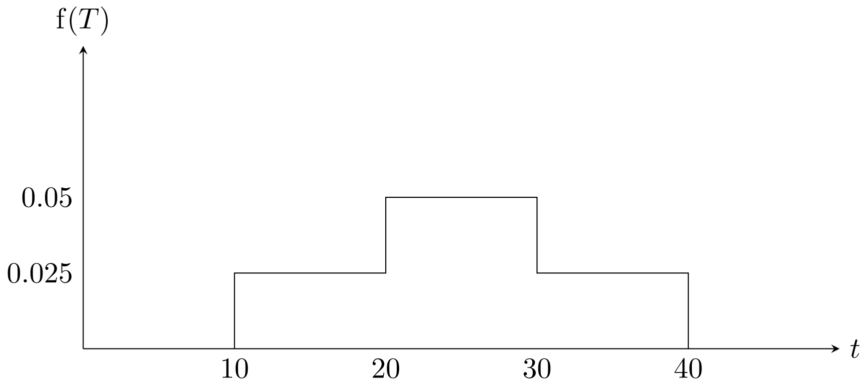
Exercise 1.
The number of minutes of playing time of a certain high school basketball player in a randomly chosen game is a random variable whose probability density function is given.



Find the probability that the player plays

(1) over 15 minutes.
(2) between 20 and 35 minutes.
(3) less than 30 minutes.
(4) more than 36 minutes.

Solution 1.
(1)

$$\begin{aligned} \mathrm{P}(T > 15) &= \int\limits_{15}^{\infty} \mathrm{f}(T) \,\mathrm{d}t \\ &= 0.875 \end{aligned}$$

(2)

$$\begin{aligned} \mathrm{P}(20 < T < 35) &= \int\limits_{20}^{35} \mathrm{f}(T) \,\mathrm{d}t \\ &= 0.625 \end{aligned}$$

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(3)

$$\begin{aligned}
\mathrm{P}(T < 30) &= \int_{30}^{\infty} \mathrm{f}(T)\,\mathrm{d}t \\
&= 0.75
\end{aligned}$$

(4)

$$\begin{aligned}
\mathrm{P}(T > 36) &= \int_{36}^{\infty} \mathrm{f}(T)\,\mathrm{d}t \\
&= 0.1
\end{aligned}$$

Exercise 2.
For some constant c, the random variable X has probability density function

$$\mathrm{f}(x) = \begin{cases} cx^4 & ; \quad 0 < x < 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Find $\mathrm{E}[X]$ and $\mathrm{V}(X)$.

Solution 2.
As $\mathrm{f}(x)$ is a probability density function,

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \mathrm{f}(x)\,\mathrm{d}x \\
&= \int_{0}^{2} cx^4 \\
&= c\left(\frac{32}{5}\right)
\end{aligned}$$

Therefore,

$$c = \frac{5}{32}$$

Therefore,

$$\begin{aligned}
\mathrm{E}[X] &= \int_{-\infty}^{\infty} x\,\mathrm{f}(x)\,\mathrm{d}x \\
&= \frac{5}{32}\int_{0}^{2} x^5\,\mathrm{d}x \\
&= \frac{5}{32}\frac{64}{6} \\
&= \frac{5}{3}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{E}\left[g(X)\right] &= \int_{-\infty}^{\infty} g(x)\,\mathrm{f}(x)\,\mathrm{d}x \\
\therefore \mathrm{E}\left[X^2\right] &= \frac{5}{32}\int_0^2 x^2 x^4 \\
&= \frac{5}{32}\int_0^2 x^6 \\
&= \frac{5}{32}\frac{128}{7} \\
&= \frac{20}{7}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{V}(X) &= \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2 \\
&= \frac{20}{7} - \frac{25}{9} \\
&= \frac{5}{63}
\end{aligned}$$

Exercise 3.
Your company must make a scaled bid for a construction project. If you succeed in winning the contract by having the lowest bid, then you plan to pay another firm $\$100,000$ to do the work. If you believe that the minimum bid of the other participating companies can be modelled as the value of a random variable that is uniformly distributed on $(70, 140)$, how much should you bid to maximize your profit?

Exercise 4.
At a certain bank, the amount of time that a customer spends being served by a teller is an exponential random variable with mean of 5 minutes. If there is a customer in service when you enter the bank, what is the probability that he will still be with the teller after an additional 4 minutes?

Exercise 5.
Alice is leaving work in a uniformly distributed time between 7 and 9, i.e.

$$X \sim \mathrm{U}(7,9)$$

The time it takes her to commute home is

$$Y = 1 + \frac{1}{X}$$

Find the density and distribution of Y.

Exercise 6.

A randomly chosen IQ test taker obtains a score that is approximately a normal random variable with mean 100 and standard deviation 15. Find the probability that the score of such a person is

(1) above 125.
(2) between 90 and 110.

Solution 6.

(1) Let X be the obtained score.
Let

$$\begin{aligned} Z &= \frac{X-\mu}{\sigma} \\ &= \frac{X-100}{15} \end{aligned}$$

Therefore, Z is a standard normal random variable. Therefore,

$$\begin{aligned} \mathrm{P}(X > 125) &= \mathrm{P}\left(Z > \frac{125-100}{15}\right) \\ &= \mathrm{P}\left(\frac{5}{3}\right) \\ &= 1 - \Phi\left(\frac{5}{3}\right) \end{aligned}$$

(2) Let X be the obtained score.
Let

$$\begin{aligned} Z &= \frac{X-\mu}{\sigma} \\ &= \frac{X-100}{15} \end{aligned}$$

Therefore, Z is a standard normal random variable. Therefore,

$$\begin{aligned} \mathrm{P}(90 < X < 100) &= \mathrm{P}\left(\frac{90-100}{15} < Z < \frac{110-100}{15}\right) \\ &= \mathrm{P}\left(\frac{-2}{3} < Z < \frac{2}{3}\right) \\ &= \Phi\left(\frac{2}{3}\right) - \Phi\left(-\frac{2}{3}\right) \\ &= \Phi\left(\frac{2}{3}\right) - \left(1 - \Phi\left(\frac{2}{3}\right)\right) \\ &= 2\Phi\left(\frac{2}{3}\right) - 1 \end{aligned}$$

Exercise 7.

The life of a certain type of automobile tyre is normally distributed with mean $34,000$ miles and standard deviation $4,000$ miles.

(1) What is the probability that such a tyre lasts over $40,000$ miles?

(2) What is the probability that such a tyre lasts between 30,000 and 35,000 miles?
(3) Given that such a tyre has survived 30,000 miles, what is the conditional probability that the tyre survives another 10,000 miles?

Solution 7.

(1) Let X be the life of the tyre.
Let

$$\begin{aligned} Z &= \frac{X-\mu}{\sigma} \\ &= \frac{X-34000}{4000} \end{aligned}$$

Therefore, Z is a standard normal random variable. Therefore,

$$\begin{aligned} \mathrm{P}(X > 40000) &= \mathrm{P}\left(Z > \frac{40000-34000}{4000}\right) \\ &= \mathrm{P}\left(Z > \frac{3}{2}\right) \\ &= 1 - \Phi\left(\frac{3}{2}\right) \end{aligned}$$

(2) Let X be the life of the tyre.
Let

$$\begin{aligned} Z &= \frac{X-\mu}{\sigma} \\ &= \frac{X-34000}{4000} \end{aligned}$$

Therefore, Z is a standard normal random variable. Therefore,

$$\begin{aligned} \mathrm{P}(30000 < X < 35000) &= \mathrm{P}\left(\frac{30000-34000}{4000} < Z < \frac{35000-34000}{4000}\right) \\ &= \mathrm{P}\left(-1 < Z < \frac{1}{4}\right) \\ &= \Phi\left(\frac{1}{4}\right) - \Phi(-1) \\ &= \Phi\left(\frac{1}{4}\right) - \big(1 - \varphi(1)\big) \\ &= \Phi\left(\frac{1}{4}\right) + \Phi(1) - 1 \end{aligned}$$

(3) Let X be the life of the tyre.
Let

$$\begin{aligned} Z &= \frac{X-\mu}{\sigma} \\ &= \frac{X-34000}{4000} \end{aligned}$$

Therefore, Z is a standard normal random variable. Therefore,

$$\begin{aligned} \mathrm{P}(X > 30000) &= \mathrm{P}\left(Z > \frac{30000-34000}{4000}\right) \\ &= \mathrm{P}(Z > -1) \\ &= 1 - \Phi(1) \\ \mathrm{P}(X > 40000) &= \mathrm{P}\left(Z > \frac{40000-34000}{4000}\right) \\ &= \mathrm{P}\left(Z > \frac{3}{2}\right) \\ &= 1 - \Phi\left(\frac{3}{2}\right) \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{P}(X > 40000 | X > 30000) &= \frac{\mathrm{P}(X > 40000 \cap X > 30000}{\mathrm{P}(X > 30000)} \\ &= \frac{\mathrm{P}(X > 40000)}{\mathrm{P}(X > 30000)} \\ &= \frac{1 - \Phi\left(\frac{3}{2}\right)}{1 - \Phi(1)} \end{aligned}$$

Exercise 8.
The annual rainfall in Cleveland, Ohio, is approximately a normal random variable with mean 40.2 inches and standard deviation 8.4 inches. Let A_i be the event that the rainfall in the next ith year exceeds 44 inches, and fume that all A_i are independent. Find the probability that

(1) next year's rainfall will exceed 44 inches?
(2) the yearly rainfall in exactly 3 of the next 7 years will exceed 44 inches?

Solution 8.
(1) Let X be the rainfall in a year.
Let

$$\begin{aligned} Z &= \frac{X - \mu}{\sigma} \\ &= \frac{X - 30.2}{8.4} \end{aligned}$$

Therefore, Z is a standard normal random variable. Therefore,

$$\begin{aligned}
\mathrm{P}(X > 44) &= \mathrm{P}\left(Z > \frac{44 - 30.2}{8.4}\right) \\
&= \mathrm{P}\left(Z > \frac{13.8}{8.4}\right) \\
&= \mathrm{P}\left(Z > \frac{23}{14}\right) \\
&= 1 - \Phi\left(\frac{23}{14}\right)
\end{aligned}$$

(2) For any year, as the rainfall in every year is independent of the previous years, the probability that the rainfall will exceed 44 inches is

$$\mathrm{P}(X > 44) = 1 - \Phi\left(\frac{23}{14}\right)$$

Therefore, the probability that the rainfall will exceed 44 inches in exactly 3 of the next 7 years is $\frac{7}{3}\left(1 - \Phi\left(\frac{23}{14}\right)\right)^3\left(\Phi\left(\frac{23}{14}\right)\right)^4$.