INTRODUCTION TO PROBABILITY AND STATISTICS ASSIGNMENT 6

AAKASH JOG
ID : 989323563

Exercise 1.
On average, 5.2 hurricanes hit a certain region in a year. What is the probability that there will be 3 or fewer hurricanes hitting this year?

Solution 1.
Let X be the number of hurricanes hitting the region in a year. Therefore,

$$X \sim \text{Poi}(5.2)$$

Therefore,

$$\begin{aligned}
\mathrm{P}(X \leq 3) &= \sum_{i=0}^{3} \frac{e^{-\lambda}\lambda^i}{i!} \\
&= \sum_{i=0}^{3} \frac{e^{-5.2}(5.2)^i}{i!}
\end{aligned}$$

Exercise 2.
The number of eggs laid on a tree leaf by an insect of a certain type is a Poisson random variable with parameter λ. However, such a random variable can be observed only if it is positive, since if it is zero, then we cannot know that such an insect was on the leaf. Let Y denote the observed number of eggs, then

$$\mathrm{P}(Y = i) = \mathrm{P}(X = i | X > 0)$$

where X is a Poisson random variable with parameter λ. Find $\mathrm{E}[Y]$.

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Solution 2.

$$\begin{aligned}
\mathrm{E}[Y] &= \sum_{i=0}^{\infty} i\,\mathrm{P}(X = i | X > 0) \\
&= \sum_{i=0}^{\infty} i\frac{\mathrm{P}(X = i \cap X > 0)}{\mathrm{P}(X > 0)} \\
&= \sum_{i=1}^{\infty} i\frac{\mathrm{P}(X = i)}{\mathrm{P}(X > 0)} \\
&= \frac{\mathrm{E}[X]}{\mathrm{P}(X > 0)} \\
&= \frac{\lambda}{1 - e^{-\lambda}}
\end{aligned}$$

Exercise 3.
A professor with ADHD never ends his classes on time. Let T be the time in minutes that the class continues beyond its scheduled end. The probability distribution function of T is

$$\mathrm{f}(t) = \begin{cases} ct(5-t)^2 & ; \quad 0 \le t \le 5 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

(1) Find c.
(2) Find the cumulative distribution function of T.
(3) A student bets with her friends that in 12 out of 13 independent classes, the class will continue at least 2 minutes after its scheduled end. What are her chances to win the bet?

Solution 3.
(1)

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \mathrm{f}(t)\,\mathrm{d}t \\
&= \int_{0}^{5} ct(5-t)^2\,\mathrm{d}t \\
&= \frac{625c}{12}
\end{aligned}$$

Therefore,

$$c = \frac{12}{625}$$

(2)

$$
\begin{aligned}
\mathrm{F}_T(t) &= \int\limits_{-\infty}^{t} \mathrm{f}(s)\,\mathrm{d}s \\
&= \begin{cases} 0 & ; \quad t < 0 \\ \frac{12}{625}\left(\frac{25t^2}{2} - \frac{10t^3}{3} + \frac{t^4}{4}\right) & ; \quad 0 \le t \le 5 \\ 1 & ; \quad 5 < t \end{cases}
\end{aligned}
$$

(3)

$$
\begin{aligned}
\mathrm{P}(T \ge 2) &= \int\limits_{0}^{2} \mathrm{f}(t)\,\mathrm{d}t \\
&= \int\limits_{0}^{2} \frac{12}{625} t(5-t)^2\,\mathrm{d}t \\
&= \frac{328}{625}
\end{aligned}
$$

Exercise 4.
X, the percentage of correct answers of a student in a test is distributed according to

$$
\mathrm{f}(t) = \begin{cases} ct(100-t) & ; \quad 0 \le t \le 100 \\ 0 & ; \quad \text{otherwise} \end{cases}
$$

(1) What is the value of c, given that $\mathrm{f}(t)$ is a legitimate probability distribution function?
(2) Find $\mathrm{F}_X(t)$.
(3) Calculate the probability that a student will fail, i.e. get less than 55.
(4) Calculate the probability of failure in the test if
   (a) The student's grade is higher than 54.
   (b) The student's grade is higher than 40.

Solution 4.

(1)

$$
\begin{aligned}
1 &= \int\limits_{-\infty}^{\infty} \mathrm{f}(t)\,\mathrm{d}t \\
&= \int\limits_{0}^{100} ct(100-t) \\
&= -c\left(-50t^2 + \frac{t^3}{3}\right)\Bigg|_{0}^{100} \\
&= \frac{500000c}{3}
\end{aligned}
$$

Therefore,

$$
c = \frac{3}{500000}
$$

(2)

$$
\begin{aligned}
\mathrm{F}_X(x) &= \int\limits_{-\infty}^{x} \mathrm{F}_X(x)\,\mathrm{d}x \\
&= \begin{cases} 0 & x < 0 \\ -\frac{3}{500000}\left(-50x^2 + \frac{x^3}{3}\right) & ; \quad 0 \leq t \leq 100 \\ 1 & 100 < x \end{cases}
\end{aligned}
$$

(3)

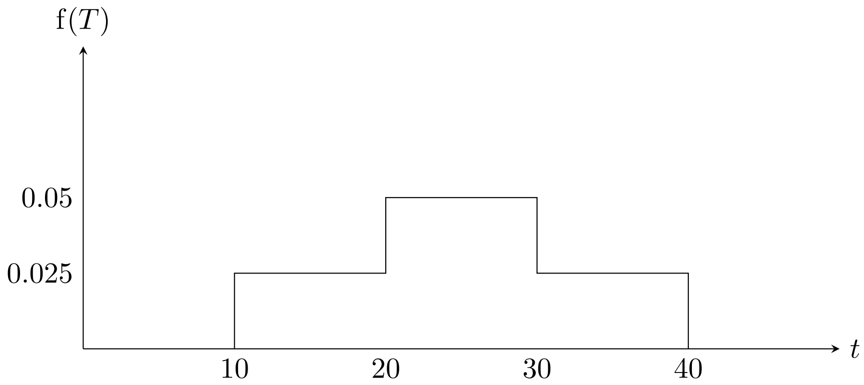
$$
\begin{aligned}
\mathrm{P}(X < 55) &= \int\limits_{0}^{55} \mathrm{f}(t)\,\mathrm{d}t \\
&= \int\limits_{0}^{55} \frac{3}{500000} t(100-t)\,\mathrm{d}t \\
&= \frac{2299}{4000}
\end{aligned}
$$

(4)

$$
\begin{aligned}
\mathrm{P}(X < 55 | X > 54) &= \frac{\mathrm{P}(54 < X < 55)}{\mathrm{P}(X > 54)} \\
&= \frac{\frac{7439}{500000}}{\frac{6877}{15625}} \\
&= \frac{7439}{220064}
\end{aligned}
$$

$$\begin{aligned}
\mathrm{P}(X < 55 | X > 40) &= \frac{\mathrm{P}(40 < X < 55)}{\mathrm{P}(X > 54)} \\
&= \frac{\frac{891}{4000}}{\frac{81}{125}} \\
&= \frac{11}{32}
\end{aligned}$$

Exercise 5.
The number of minutes of playing time of a certain high school basketball player in a randomly chosen game is a random variable whose probability density function is given.



Find the probability that the player plays

(1) over 15 minutes.
(2) between 20 and 35 minutes.
(3) less than 30 minutes.
(4) more than 36 minutes.

Solution 5.
(1)

$$\begin{aligned}
\mathrm{P}(T > 15) &= \int_{15}^{\infty} \mathrm{f}(T)\,\mathrm{d}t \\
&= 0.875
\end{aligned}$$

(2)

$$\begin{aligned}
\mathrm{P}(20 < T < 35) &= \int_{20}^{35} \mathrm{f}(T)\,\mathrm{d}t \\
&= 0.625
\end{aligned}$$

(3)

$$\begin{aligned}\mathrm{P}(T < 30) &= \int\limits_{30}^{\infty} \mathrm{f}(T)\,\mathrm{d}t \\ &= 0.75\end{aligned}$$

(4)

$$\begin{aligned}\mathrm{P}(T > 36) &= \int\limits_{36}^{\infty} \mathrm{f}(T)\,\mathrm{d}t \\ &= 0.1\end{aligned}$$

Exercise 6.

For some constant c, the random variable X has the following probability density function.

$$\mathrm{f}(x) = \begin{cases} ax + bx^2 & ; \quad 0 < x < 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

If

$$\mathrm{E}[X] = 0.6$$

find

(1) $\mathrm{P}(X < 12)$.
(2) $\mathrm{V}(X)$.

Exercise 7.

Your company must make a scaled bid for a construction project. If you succeed in winning the contract by having the lowest bid, then you plan to pay another firm $\$100,000$ to do the work. If you believe that the minimum bid of the other participating companies can be modelled as the value of a random variable that is uniformly distributed on $(70, 140)$, how much should you bid to maximize your profit?

Exercise 8.

At a certain bank, the amount of time that a customer spends being served by a teller is an exponential random variable with mean of 5 minutes. If there is a customer in service when you enter the bank, what is the probability that he will still be with the teller after an additional 4 minutes?