LINEAR ALGEBRA : HOMEWORK 9

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1.

a.

$$\begin{aligned}T(f_1(x)+f_2(x)) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(f_1(x)+f_2(x)\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(f_1(x)\right)+\frac{\mathrm{d}}{\mathrm{d}x}\left(f_2(x)\right)\\ &= T(f_1(x))+T(f_2(x))\end{aligned}$$

$$\begin{aligned}T(\alpha f(x)) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha f(x)\right)\\ &= \alpha\frac{\mathrm{d}}{\mathrm{d}x}f(x)\\ &= \alpha T(f(x))\end{aligned}$$

Therefore, T is linear.

b. $\{1, x, x^2, x^3\}$ is a basis for V.

c. $\{1, x, x^2\}$ is a basis for the range of T with input V.

d.

$$\begin{aligned}T(1,x,x^2,x^3) &= A(1,x,x^2,x^3)\\ &= (0,1,2x,3x^2)\\ \therefore A\begin{pmatrix}1&0&0&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix} &= \begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&0&0\end{pmatrix}\\ \therefore A &= \begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&0&0\end{pmatrix}\end{aligned}$$

e.

$$\begin{aligned}T(k) &= 0\\ \therefore \ker T &= \mathbb{R}\end{aligned}$$

f.

$$\dim(\text{range}) + \dim(\text{kernel}) = \dim V$$

is valid for all vector spaces V and all linear transformations.

2.

a.

$$[T]_E = P[T]_B P^{-1}$$

$$\begin{aligned}b_1 &= 1e_1 + 2e_2\\ b_2 &= 2e_1 - 3e_2\end{aligned}$$

Therefore,

$$\begin{aligned}P &= \begin{pmatrix}1&2\\2&-3\end{pmatrix}\\ \therefore P^{-1} &= \begin{pmatrix}3/7&2/7\\2/7&-1/7\end{pmatrix}\\ \therefore [T]_E &= \begin{pmatrix}1&2\\2&-3\end{pmatrix}\begin{pmatrix}2&1\\-1&3\end{pmatrix}\begin{pmatrix}3/7&2/7\\2/7&-1/7\end{pmatrix}\\ &= \begin{pmatrix}2&-1\\1&3\end{pmatrix}\end{aligned}$$

b.

$$\begin{aligned}T(x,y) &= \begin{pmatrix}2&-1\\1&3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\\ &= \begin{pmatrix}2x-y\\x+3y\end{pmatrix}\end{aligned}$$

c.

$$\operatorname{im}(T) = \operatorname{span}\left\{\begin{pmatrix}2\\7\end{pmatrix}, \begin{pmatrix}-1\\3\end{pmatrix}\right\}$$

$$\begin{aligned}2x-y &= 0\\ x+3y &= 0\end{aligned}$$

The matrix is

$$\begin{pmatrix}2&-1\\1&3\end{pmatrix} \to \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Therefore, $\ker T = \{\mathbb{O}\}$.

3.

a.

$$\begin{aligned}T\begin{pmatrix}x\\y\end{pmatrix} &= \begin{pmatrix}x+y\\y\end{pmatrix}\\ T &: \mathbb{R}^2 \to \mathbb{R}^2\\ B = B' &= \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}\end{aligned}$$

i.

$$T(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$T(e_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\therefore [T]_{B_0,B_0} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$b_1 = 1e_1 + 1e_2$$
$$b_2 = 0e_1 + 1e_2$$

$$[T]_{B,B'} = P^{-1}[T]_{B_0,B_0}P$$

Therefore,

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$
$$\therefore P^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$
$$\therefore [T]_{B,B'} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

ii.

$$\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ -x \end{pmatrix}$$
$$\therefore \operatorname{im} T = \left\{ \begin{pmatrix} 2x+y \\ -x \end{pmatrix} \right\}$$

iii.

$$\ker T = \{0\}$$

iv.

$$\dim(\operatorname{im} T) = 2$$

v.

$$\dim(\ker T) = 0$$

b.

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x-4y+9z \\ 5x+3y+2z \end{pmatrix}$$
$$T : \mathbb{R}^3 \to \mathbb{R}^2$$
$$B = B' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

i.

$$T(e_1) = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$
$$T(e_2) = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$
$$T(e_3) = \begin{pmatrix} 9 \\ 2 \end{pmatrix}$$
$$\therefore [T]_B = \begin{pmatrix} 2 & -4 & 9 \\ 5 & 3 & 2 \end{pmatrix}$$

ii.

$$\operatorname{im} T = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} -4 \\ 3 \end{pmatrix}, \begin{pmatrix} 9 \\ 2 \end{pmatrix} \right\}$$

iii.

$$\ker T = \{2x - 4y + 9z = 5x + 3y + 2z = 0\}$$

iv.

$$\dim(\operatorname{im} T) = 2$$

c.

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x+4y \\ 5x-2y \\ x+7z \\ 4x \end{pmatrix}$$
$$T : \mathbb{R}^3 \to \mathbb{R}^4$$
$$B = B' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

i.

$$T(e_1) = \begin{pmatrix} 3 \\ 5 \\ 1 \\ 4 \end{pmatrix}$$
$$T(e_2) = \begin{pmatrix} 4 \\ -2 \\ 0 \\ 0 \end{pmatrix}$$
$$T(e_3) = \begin{pmatrix} 0 \\ 0 \\ 7 \\ 0 \end{pmatrix}$$
$$\therefore [T]_{B_0,B_0} = \begin{pmatrix} 3 & 4 & 0 \\ 5 & -2 & 0 \\ 1 & 0 & 7 \\ 4 & 0 & 0 \end{pmatrix}$$

ii.

$$\operatorname{im} T = \operatorname{span}\left\{ \begin{pmatrix} 3 \\ 5 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 7 \\ 0 \end{pmatrix} \right\}$$

iii.

$$\ker T = \{0\}$$

iv.

$$\dim(\operatorname{im} T) = 4$$

v.

$$\dim(\operatorname{im} T) = 0$$

d.

$$T\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = 2x + 3y - 7z + w$$

$$T : \mathbb{R}^4 \to \mathbb{R}$$

$$B = B' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

i.

$$\begin{aligned} T(e_1) &= 2 \\ T(e_2) &= 3 \\ T(e_3) &= -7 \\ T(e_4) &= 1 \\ \therefore [T]_B &= \begin{pmatrix} 2 & 3 & -7 & 1 \end{pmatrix} \end{aligned}$$

ii.

$$\operatorname{im} T = \operatorname{span}\{2, 3, -7, 1\}$$

iii.

$$\ker T = \{2x + 3y - 7z + w = 0\}$$

iv.

$$\dim(\operatorname{im} T) = 1$$

v.

$$\dim(\ker T) = 1$$