LINEAR ALGEBRA : HOMEWORK 11

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Part 1. Eigenvalues and Eigenvectors Continued, Diagonalization

1.

a.

$$
\begin{aligned}
A &= \begin{pmatrix} 7 & -1 & -4 \\ 14 & 1 & -12 \\ 8 & -1 & -5 \end{pmatrix} \\
\therefore p_A(\lambda) &= \begin{vmatrix} 7-\lambda & -1 & -4 \\ 14 & 1-\lambda & -12 \\ 8 & -1 & -5-\lambda \end{vmatrix} \\
&= (7-\lambda)((1-\lambda)(-5-\lambda)-12) \\
&\quad + (14(-5-\lambda)+96) \\
&\quad - 4(-14-8(1-\lambda)) \\
&= -(\lambda+1)(\lambda^2-4\lambda+5) \\
\therefore \lambda &= -1
\end{aligned}
$$

Therefore, as $p_A(\lambda)$ does not split completely, A is not diagonalizable.

$$
\begin{aligned}
V_{-1} &= \mathrm{N}\begin{pmatrix} 8 & -1 & -4 \\ 14 & 2 & -12 \\ 8 & -1 & -4 \end{pmatrix} \\
&= \mathrm{N}\begin{pmatrix} 8 & -1 & -4 \\ 14 & 2 & 12 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \right\}
\end{aligned}
$$

b.

$$
\begin{aligned}
p_A(\lambda) &= -(\lambda+1)(\lambda^2-4\lambda+5) \\
&= -(\lambda+1)(\lambda-2+i)(\lambda-2-i) \\
\therefore \lambda &= -1, 2+i, 2-i
\end{aligned}
$$

$$
\begin{aligned}
V_{-1} &= \operatorname{span}\left\{\begin{pmatrix}2\\4\\3\end{pmatrix}\right\}\\
V_{2+i} &= \mathrm{N}\begin{pmatrix}5-i & -1 & -4\\ 14 & -1-i & -12\\ 8 & -1 & -7-i\end{pmatrix}\\
&= \left\{\begin{pmatrix}1\\1-i\\1\end{pmatrix}\right\}\\
V_{2-i} &= \mathrm{N}\begin{pmatrix}5+i & -1 & -4\\ 14 & -1+i & -12\\ 8 & -1 & -7+i\end{pmatrix}\\
&= \left\{\begin{pmatrix}1\\1+i\\1\end{pmatrix}\right\}
\end{aligned}
$$

Therefore, for all three eigenvalues, the algebraic and geometric multiplicities are 1. Therefore, A is diagonalizable.

$$
\begin{aligned}
D &= \begin{pmatrix}-1 & 0 & 0\\ 0 & 2+i & 0\\ 0 & 2-i & 0\end{pmatrix}\\
P &= \begin{pmatrix}2 & 1 & 1\\ 4 & 1-i & 1+i\\ 3 & 1 & 1\end{pmatrix}
\end{aligned}
$$

2. For the following linear transformation find the eigenvalues and a basis for the eigenspace of each eigenvalue. Determine whether the transformation is diagonalizable.

$$
\begin{aligned}
&T(x,y,z) = (2x+y, y-z, 2y+4z)\\
&T : \mathbb{R}^3 \to \mathbb{R}^3
\end{aligned}
$$

$$
\begin{aligned}
P_T(\lambda) &= \left|[T]_E - \lambda I\right|\\
&= \left|\big([T(1,0,0)]_E \quad [T(0,1,0)]_E \quad [T(0,0,1)]_E\big) - \lambda I\right|\\
&= \left|\big([(2,0,0)]_E \quad [(1,1,2)]_E \quad [(0,-1,4)]_E\big) - \lambda I\right|\\
&= \left|\begin{pmatrix}2 & 1 & 0\\ 0 & 1 & -1\\ 0 & 2 & 4\end{pmatrix} - \begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}\right|\\
&= \begin{vmatrix}2-\lambda & 1 & 0\\ 0 & 1-\lambda & -1\\ 0 & 2 & 4-\lambda\end{vmatrix}\\
&= (2-\lambda)((1-\lambda)(4-\lambda)+2)\\
&= (2-\lambda)(\lambda^2 - 5\lambda + 6)\\
&= -(\lambda-2)^2(\lambda-3)\\
\therefore \lambda &= 2, 3
\end{aligned}
$$

$$
\begin{aligned}
V_1 &= \mathrm{N}\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{pmatrix} \\
&= \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}
\end{aligned}
$$

$$
\begin{aligned}
V_2 &= \mathrm{N}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{pmatrix} \\
&= \mathrm{N}\begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}\right\}
\end{aligned}
$$

For eigenvalue 2, the geometric multiplicity is not equal to the algebraic multiplicity. Therefore, the transformation is not diagonalizable.

3.

$$
\begin{aligned}
\det(A - I) &= 0 \\
Ax &= 3x
\end{aligned}
$$

Therefore, 1 and 3 are eigenvalues of A.
As $\operatorname{rank}(A + I) = 2$, there is a zero row in $(A - I)$. Therefore,

$$\det(A + I) = 0$$

Therefore, -1 is an eigenvalue of A.

a. As

$$\lambda = -1, 1, 3$$

$$P_A(\lambda) = (\lambda + 1)(\lambda - 1)(\lambda - 3)$$

b.

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

As $A \sim D$, the trace of A is equal to the trace of D.
Therefore, the trace of A is 3.

c. As $A \sim D$, $\det A = \det D$.
Therefore,

$$\det A = -3$$

d. As -3 is not an eigenvalue of A,

$$\det(A + 3I) \neq 0$$

Therefore, $(A + 3I)$ is invertible.

Part 2. Orthogonality

1. WHICH OF THE FOLLOWING VECTOR SPACES IS AN INNER PRODUCT SPACE?

a.
$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle = 8x_1x_2$$

As $8x_1x_2$ may be negative, it is not an inner product space.

b.
$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle = {x_1}^2 + 7{x_2}^2$$
$$\therefore \langle v_1, v_1 \rangle \geq 0$$

The equality holds if and only if $v_1 = \mathbb{O}$.

$$\begin{aligned} \langle v_1 + v_2, v_3 \rangle &= (x_1 + y_1)z_1 + 7(x_2 + y_2)z_2 \\ &= x_1z_1 + y_1z_1 + 7x_2z_2 + 7y_2z_2 \\ &= \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle \end{aligned}$$

$$\begin{aligned} \langle \alpha v_1, v_2 \rangle &= \alpha x_1y_1 + 7\alpha x_2y_2 \\ &= 7\langle v_1, v_2 \rangle \end{aligned}$$

$$\begin{aligned} \overline{\langle v_2, v_1 \rangle} &= x_1y_1 + 7x_2y_2 \\ &= \langle v_1, v_2 \rangle \end{aligned}$$

Therefore, it is an inner product space.

c. Let
$$A = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$
$$B = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$$
$$\therefore AB = \begin{pmatrix} x_{11}y_{11} + x_{12}y_{21} & x_{11}y_{12} + x_{12}y_{22} \\ x_{21}y_{11} + x_{22}y_{21} & x_{21}y_{12} + x_{22}y_{22} \end{pmatrix}$$
$$\therefore \operatorname{trace}(AB) = x_{11}y_{11} + x_{12}y_{21} + x_{21}y_{12} + x_{22}y_{22}$$

$$\begin{aligned} \langle A, B \rangle &= \operatorname{trace}(AB) \\ &= x_{11}y_{11} + x_{12}y_{21} + x_{21}y_{12} + x_{22}y_{22} \end{aligned}$$

Therefore,
$$\langle A, A \rangle = {x_{11}}^2 + 2x_{12}x_{21} + {x_{22}}^2$$

$\operatorname{trace}(A^2)$ may be negative. Therefore, it is not an inner product space.