Recitation 3

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Contents

1 Matrix Determinant

The determinant function maps each square matrix $A \in M_n(\mathbb{F})$ to a scalar $|A| = \det(A) \in \mathbb{F}$.
For $n = 1$, we define $\det((a)) = a$.
Let A be a square matrix of dimensions $n \times n$.
The (i, j) minor of A, M_{ij}, is the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing the i^{th} row and j^{th} column of A.
The determinant can be developed using the i^{th} row as

$$\det(A) = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} M_{ik}$$

The determinant can also be developed using the j^{th} column as

$$\det(A) = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} M_{kj}$$

1.1 Properties

1.1.1 If B is formed by multiplying one row (or column) of A by a scalar c, then, $\det(B) = c \cdot \det(A)$

1.1.2 If B is formed by replacing two rows (or columns) of A by each other, then, $\det(B) = -\det(A)$

1.1.3 If B is formed by adding one row (or column), multiplied by a scalar, to another row (or column), then, $\det(B) = \det(A)$

1.1.4 If A has atleast one row or column having all elements equal to zero, then, $\det(A) = 0$.

1.1.5 The determinant of an upper-triangular or lower-triangular matrix is equal to the product of the elements on the principal diagonal.

1.1.6 $\det(A^T) = \det(A)$

1.1.7 $\det(A \cdot B) = \det(A) \cdot \det(B)$

1.1.8 Matrix A is invertible iff $\det(A) \neq 0$. If so, $\det(A^{-1}) = \dfrac{1}{\det(A)}$

1.2 Examples

1.2.1 Let A, B, P be invertible matrices, satisfying $B = P^{-1}AP$. Show that $|A^{-1}B| = 1$.

Multiplying the given equation by A^{-1} from the left,

$$\begin{aligned}
A^{-1}B &= A^{-1}P^{-1}AP \\
\therefore |A^{-1}B| &= |A^{-1}P^{-1}AP| \\
&= |A^{-1}||P^{-1}||A||P| \\
&= \frac{|A||P|}{|A||P|} \\
&= 1
\end{aligned}$$

1.3 Cramer's Rule

Let $A \in M_n(\mathbb{F})$ and $b \in \mathbb{F}^n$. If $|A| \neq 0$, the system $Ax = b$ has a unique solution, whose individual values are given by

$$x_j = \frac{|A_j|}{|A|}$$

where A_j is the matrix formed by replacing the j^{th} column of A by the vector b.

2 Adjoint Matrix

A square matrix of dimensions $n \times n$ is said to be adjoint to $A \in M_n(\mathbb{F})$ if $(\mathrm{adj}(A))_{ij} = (-1)^{i+j} M_{ji}$.

$$A \cdot \mathrm{adj}(A) = |A| \cdot I$$

If $|A| \neq 0$

$$A^{-1} = \frac{\mathrm{adj}(A)}{|A|}$$