Recitation 4

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Contents

1 Linear Vector Spaces

1.1 Examples

1. $\mathbb{R}^n = \{(x_1, \dots, x_n); x_1, \dots, x_n \in \mathbb{R}\}$

2. $\mathbb{C}^n = \{(x_1, \dots, x_n); x_1, \dots, x_n \in \mathbb{C}\}$

3. $C([0,1]) = \{f : [0,1] \to \mathbb{R}; f$ is continuous$\}$
   where $(f+g)(x) = f(x) + g(x), (\alpha f)(x) = \alpha f(x); \alpha \in \mathbb{R}, f \in C([0,1])$

4. Matrices over $\mathbb{F}$ with matrix addition and scalar multiplication as per standard definitions.

5. $\mathbb{R}_n[x] = \{p(x) = p_0 + p_1 x + \dots + p_n x^n; p_0, \dots, p_n \in \mathbb{R}\}$
   where $(p+q)(x) = (p_0 + q_0) + \dots + (p_n + q_n)x^n, (\alpha p)(x) = \alpha(p(x)) = (\alpha p_0) + \dots + (\alpha p_n)x^n$

1.2 Exercises

Example 1. Is

$$W_1 = \{(a,b,c) \in \mathbb{R}^3; a+b+c = 0\}$$

a subspace of $\mathbb{R}^3$?

Solution.

$$0+0+0 = 0 \Rightarrow (0,0,0) \in W_1$$

$$\begin{aligned} v, u &\in W_1 \\ \therefore v_1 + v_2 + v_3 &= u_1 + u_2 + u_3 = 0 \\ \therefore v_1 + u_1 + v_2 + u_2 + v_3 + u_2 &= 0 \\ \therefore v + u = (v_1 + u_1, v_2 + u_2, v_3 + u_3) &\in W_1 \end{aligned}$$

$$\begin{aligned} \alpha &\in \mathbb{R} \\ v = (v_1, v_2, v_3) &\in W_1 \\ \therefore v_1 + v_2 + v_3 &= 0 \\ \therefore \alpha v &= (\alpha v_1, \alpha v_2, \alpha v_3) \\ \alpha v_1 + \alpha v_2 + \alpha v_3 &= 0 \\ \therefore \alpha v &\in \mathbb{R} \end{aligned}$$

Example 2. Is

$$W_2 = \{(a, b, c) \in \mathbb{R}^3; a \geq 0\}$$

a subspace of $\mathbb{R}^3$?

Solution. W_2 is not a linear subspace of $\mathbb{R}^3$, as for $\alpha = -1, v = (1, 0, 0)$, $\alpha v = (-1, 0, 0) \notin W_2$

Example 3. Is

$$W_3 = \{p(x) \in \mathbb{R}_3[x]; p(0) = 1\}$$

a subspace of $\mathbb{R}^3$?

Solution.

$$0(0) = 0 \neq 1$$
$$\therefore 0 \notin W_3$$

Hence, W_3 is not a linear subspace of $R_3[x]$.

Example 4. Show that the solutions space of homogeneous linear system is a linear subspace.

Solution. Let A be the matrix representing the homogeneous matrix, with n variables over $\mathbb{F}$.

$$N(A) = \{x; x \in \mathbb{F}^n, Ax = 0\}$$

$$\begin{aligned} x &= 0 \in N(A) \\ \therefore A \cdot 0 &= 0 \\ \therefore 0 &\in N(A) \end{aligned}$$

$$\begin{aligned} x, y &\in N(A) \\ Ax = Ay &= 0 \\ \therefore A(x+y) &= Ax + Ay = 0 \\ \therefore x + y &\in N(A) \end{aligned}$$

$$\begin{aligned} \lambda &\in \mathbb{F} \\ x &\in N(A) \\ \therefore Ax &= 0 \\ \therefore A(\lambda x) &= \lambda(Ax) \\ &= \lambda \cdot 0 \\ &= 0 \\ \therefore \lambda x &\in N(A) \end{aligned}$$

Therefore, $N(A)$ is a linear subspace.

1.3 Linear Combinations

Let V be a linear vector space over $\mathbb{F}$ and $K \subset V$ be a finite set of vectors from V.

$$k = \{v_1, \dots, v_n\}$$

Every expression of the form

$$\sum_{i=1}^{n} \alpha_i v_i = 0; \alpha_1, \dots, \alpha_n \in \mathbb{F}$$

is called a linear combination of K.
K is said to be linearly dependant if $\exists \alpha_1, \dots, \alpha_n$, not all zeros, s.t.

$$\sum_{i=1}^{n} \alpha_i v_i = 0$$

otherwise, K is said to be linearly dependant, i.e.

$$\alpha_1 = \cdots = \alpha_n = 0$$