Recitation 12

Wednesday 7th January, 2015

Contents

1 Linear Transformation Diagonalization

Example 1. Let $V = M_{2\times 2}(\mathbb{R})$ and $T : V \to V$.

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & a-2b \\ c & a \end{pmatrix}$$

Find $[T]_E$, all eigenvalues, the eigenspace of the maximal eigenvalue. Determine whether T is diagonalizable.

Solution.

$$\begin{aligned}
[T]_E &= \big([T(e_1)]_E \quad [T(e_2)]_E \quad [T(e_3)]_E \quad [T(e_4)]_E\big) \\
&= \left(\left[\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right]_E \quad \left[\begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}\right]_E \quad \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]_E \quad \left[\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right]_E\right) \\
&= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
P_T(\lambda) = |[T]_E - \lambda I| \\
&= \begin{vmatrix} -\lambda & 0 & 0 & 1 \\ 1 & -2-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} \\
&= (1-\lambda)\begin{pmatrix} -\lambda & 0 & 1 \\ 1 & -2-\lambda & 0 \\ 1 & 0 & \lambda \end{pmatrix} \\
&= -(1-\lambda)(2+\lambda)(\lambda^2-1) \\
\therefore \lambda &= 1, -1, -2
\end{aligned}$$

$$
\begin{aligned}
V_1 &= \mathrm{N}\begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \\
&= \mathrm{N}\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} w \\ w/3 \\ z \\ w \end{pmatrix} \\
&= \mathrm{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \\ 3 \end{pmatrix} \right\}
\end{aligned}
$$

For each eigenvalue, the algebraic multiplicity is equal to the geometeric multiplicity. Therefore T is diagonalizable.

Example 2. Let $A \in M_{2\times 2}(\mathbb{R})$, s.t.

$$
\begin{aligned}
A\begin{pmatrix} 1 \\ 3 \end{pmatrix} &= \begin{pmatrix} -2 \\ -6 \end{pmatrix} \\
A\begin{pmatrix} 2 \\ 5 \end{pmatrix} &= \begin{pmatrix} 2 \\ 5 \end{pmatrix}
\end{aligned}
$$

Find A and the eigenvalues and eigenspaces of A^{-1}.

Solution.

$$Av_1 = -2v_1$$

Therefore, -2 is an eigenvalue of A, and v_1 is an eigenvector of A, corresponding to -2.

$$Av_2 = 1v_2$$

Therefore, 1 is an eigenvalue of A, and v_2 is an eigenvector of A, corresponding to 1.
Therefore, the characteristic polynomial is

$$p_A(\lambda) = (\lambda - 1)(\lambda + 2)$$

Therefore,

$$V_{-2} = \text{span}\left\{\begin{pmatrix}1\\3\end{pmatrix}\right\}$$
$$V_1 = \text{span}\left\{\begin{pmatrix}2\\5\end{pmatrix}\right\}$$

Therefore, A is diagonalizable, and its diagonal form is

$$D = \begin{pmatrix}-2 & 0\\0 & 1\end{pmatrix}$$

and the corresponding P is

$$\begin{pmatrix}1 & 2\\3 & 5\end{pmatrix}$$

$$\begin{aligned}
A &= PDP^{-1}\\
\therefore A &= \begin{pmatrix}1 & 2\\3 & 5\end{pmatrix}\begin{pmatrix}-2 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}-5 & 2\\3 & -1\end{pmatrix}\\
&= \begin{pmatrix}16 & -6\\45 & -17\end{pmatrix}
\end{aligned}$$

If λ is an eigenvalue of an invertible matrix A, then $\lambda \neq 0$ and eigenvector v of A, corresponding to λ is also an eigenvector of A^{-1} corresponding to λ_{-1}.

Therefore,

Eigenvalue	Eigenvector
$-\frac{1}{2}$	$\begin{pmatrix}1\\3\end{pmatrix}$
1	$\begin{pmatrix}2\\5\end{pmatrix}$

2 Inner Product Spaces

Example 3. Determine whether the following is an inner product.

$$\begin{aligned}
V &= \mathbb{R}^2\\
\langle x, y\rangle &= x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2
\end{aligned}$$

Solution. Let

$$
\begin{aligned}
x &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
y &= \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\
z &= \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\langle x, x \rangle &= \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle & = {x_1}^2 - x_1x_2 - x_2x_1 + 3{x_2}^2 \\
&= {x_1}^2 - 2x_1x_2 + {x_2}^2 + 2{x_2}^2 \\
&= (x_1 - x_2)^2 + 2{x_2}^2 \\
&\geq 0
\end{aligned}
$$

$$
\langle x, x \rangle = 0 \iff x_1 = x_2 = 0
$$

$$
\begin{aligned}
\langle x + y, z \rangle &= (x_1 + y_1)z_1 - (x_1 + y_1)z_2 - (x_2 + y_2)z_2 + 3(x_2 + y_2)z_2 \\
&= \langle x, z \rangle + \langle y, z \rangle
\end{aligned}
$$

$$
\langle \alpha x, y \rangle = \alpha \langle x, y \rangle
$$

$$
\langle x, y \rangle = \langle y, x \rangle
$$

Therefore, it is an inner product.