Recitation 11

Sunday 4^{th} January, 2015

Contents

1 Algebraic and Geometric Multiplicity

Example 1.

$$A = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

Find the eigenvalues and algebraic multiplicity and eigenspaces with geometric multiplicity.

Solution.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -1-\lambda & 2 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{vmatrix} \quad = (-1-\lambda)((1-\lambda)^2 - 4) \\ &= -(\lambda+1)^2(\lambda-3) \end{aligned}$$

Therefore,

Eigenvalue	Algebraic Multiplicity	Eigenspace	Geometric Multiplicity
-1	2	$\text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$	1
3	1	$\text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\right\}$	1

Example 2. Check if the following matrix is diagonalizable, and if it is, diagonalize it.

$$A = \begin{pmatrix} 4 & 0 & -3 \\ -3 & 1 & 3 \\ 6 & 0 & -5 \end{pmatrix}$$

Solution.

$$A = \begin{pmatrix} 4 & 0 & -3 \\ -3 & 1 & 3 \\ 6 & 0 & -5 \end{pmatrix}$$

$$\begin{aligned} \therefore p_A(x) &= |A - \lambda I| \\ &= \begin{vmatrix} 4-\lambda & 0 & -3 \\ -3 & 1-\lambda & 3 \\ 6 & 0 & -5-\lambda \end{vmatrix} \\ &= (4-\lambda)(1-\lambda)(-5-\lambda) + 3(1-\lambda)6 \\ &= (1-\lambda)((4-\lambda)(-5-\lambda) + 18) \\ &= -(\lambda-1)^2(\lambda+2) \\ \therefore \lambda &= 1, -2 \end{aligned}$$

$$\begin{aligned} V_1 &= \mathrm{N} \begin{pmatrix} 3 & 0 & -3 \\ -3 & 0 & 3 \\ 6 & 0 & -6 \end{pmatrix} \\ &= \mathrm{N} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \mathrm{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \end{aligned}$$

Therefore,

Eigenvalue	Algebraic Multiplicity	Eigenspace	Geometric Multiplicity
1	2	$\mathrm{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$	2
-1	1	$\mathrm{span} \left\{ \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \right\}$	1

Therefore, the geometric multiplicity is 3. Hence, A is diagonalizable.
Therefore,

$$P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Example 3. Check if the following matrix is diagonalizable, and if it is, diagonalize it.

$$A = \begin{pmatrix} 4 & 0 & -3 \\ 1 & 1 & 1 \\ 6 & 0 & 5- \end{pmatrix}$$

Solution.

$$\begin{aligned} A &= \begin{pmatrix} 4 & 0 & -3 \\ 1 & 1 & 1 \\ 6 & 0 & 5- \end{pmatrix} \\ \therefore p_A(x) &= |A - \lambda I| \\ &= \begin{vmatrix} 4-\lambda & 0 & -3 \\ 1 & 1-\lambda & 1 \\ 6 & 0 & -5-\lambda \end{vmatrix} \\ &= (4-\lambda)(1-\lambda)(-5-\lambda) + 3(1-\lambda)6 \\ &= (1-\lambda)((4-\lambda)(-5-\lambda)+18) \\ &= -(\lambda-1)^2(\lambda+2) \end{aligned}$$

Therefore,

Eigenvalue	Algebraic Multiplicity	Eigenspace	Geometric Multiplicity
1	2	$\text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$	1
-1	1		1

Hence, by the explicit criterion for diagonalization, B cannot be diagonalized.

Example 4. Let $V = M_{2\times 2}(\mathbb{R})$ and $T : V \to V$.

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & a-2b \\ c & a \end{pmatrix}$$

Find $[T]_E$, all eigenvalues, the eigenspace of the maximal eigenvalue. Determine whether T is diagonalizable.

Solution.

$$\begin{aligned} [T]_E &= \big([T(e_1)]_E \quad [T(e_2)]_E \quad [T(e_3)]_E \quad [T(e_4)]_E\big) \\ &= \left(\left[\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right]_E \quad \left[\begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}\right]_E \quad \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]_E \quad \left[\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right]_E\right) \end{aligned}$$