Recitation 13

Wednesday 14^{th} January, 2015

Contents

1 Orthogonality

Example 1. Find scalars $a, b, c \in \mathbb{R}$, s.t.

$$B = \left\{ v_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$$

is an orthogonal basis of $\mathbb{R}^3$ with scalar product.

Solution.

$$\begin{aligned} \langle v_1, v_2 \rangle &= 0 \\ \langle v_2, v_3 \rangle &= 0 \\ \langle v_3, v_1 \rangle &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle &= 0 \\ \left\langle \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} a &= -c \\ b &= 0 \end{aligned}$$

Therefore,

$$v_3 = \begin{pmatrix} a \\ 0 \\ -a \end{pmatrix}$$

Example 2. Find an orthonormal basis of the inner product space $V = \text{span}\{B\}$ over $\mathbb{C}$.

$$B = \left\{ b_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, b_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$$

$$\langle x, y \rangle = x_1 \overline{y_1} + x_2 \overline{y_2}$$

Solution.

$$
\begin{aligned}
\widetilde{v_1} &= \begin{pmatrix} 1 \\ i \end{pmatrix} \\
\widetilde{v_2} &= b_2 - \frac{\langle b_2, b_1 \rangle}{\|b_1\|^2} \widetilde{v_1} \\
&= \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 1 \\ i \end{pmatrix} \right\|^2} \begin{pmatrix} 1 \\ i \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ i \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ -i \end{pmatrix} \\
\therefore \widetilde{B} &= \left\{ \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\} \\
\therefore B^0 &= \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}
\end{aligned}
$$

Example 3. Do there exist $a, b, c \in \mathbb{R}$, s.t. U is orthogonal?

$$
U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & c \\ 0 & b & \frac{1}{\sqrt{3}} \\ a & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}
$$

Solution.

$$
\begin{aligned}
&U^t \cdot U = I \\
&\therefore I = \begin{pmatrix} \frac{1}{2} + a^2 & \frac{1}{\sqrt{12}} - \frac{a}{\sqrt{6}} & -\frac{c}{\sqrt{2}} + \frac{a}{\sqrt{3}} \\ \frac{1}{\sqrt{12}} - \frac{a}{\sqrt{6}} & \frac{1}{6} + b^2 + \frac{1}{6} & -\frac{c}{\sqrt{6}} + \frac{b}{\sqrt{3}} - \frac{1}{\sqrt{18}} \\ -\frac{c}{\sqrt{2}} + \frac{a}{\sqrt{3}} & -\frac{c}{\sqrt{6}} + \frac{b}{\sqrt{3}} - \frac{1}{\sqrt{18}} & c^2 + \frac{2}{3} \end{pmatrix}
\end{aligned}
$$

Therefore,

$$\begin{aligned} \frac{1}{2} + a^2 &= 1 \\ \frac{1}{3} + b^2 &= 1 \\ \frac{2}{3} + c^2 &= 1 \\ \frac{1}{\sqrt{12}} - \frac{a}{\sqrt{6}} &= 0 \\ -\frac{c}{\sqrt{2}} + \frac{a}{\sqrt{3}} &= 0 \\ -\frac{c}{\sqrt{6}} + \frac{b}{\sqrt{3}} - \frac{1}{\sqrt{18}} &= 0 \end{aligned}$$

Solving,

$$\begin{aligned} a &= \frac{1}{\sqrt{2}} \\ b &= \sqrt{\frac{2}{3}} \\ c &= \frac{1}{\sqrt{3}} \end{aligned}$$

Example 4. Find a unitary matrix $U \in M_{3\times3}(\mathbb{C})$ that has the vector

$$b_1 = \begin{pmatrix} \frac{1+i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \end{pmatrix}$$

as the first row.

Solution. Completing to a basis of $\mathbb{C}^3$, let

$$\begin{aligned} b_2 &= (0, 1, 0) \\ b_3 &= (0, 0, 1) \end{aligned}$$

Using Gram - Schmidt Process, the orthonormal basis is

$$B^0 = \left\{ \begin{pmatrix} {}^{1+i}/_{\sqrt{3}} \\ -{}^{1}/_{\sqrt{3}} \\ 0 \end{pmatrix}, \begin{pmatrix} {}^{1+i}/_{3} \\ {}^{2}/_{3} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$