Recitation 10

Wednesday 31^{st} December, 2014

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1 Eigenvalues, Eigenvectors and Eigenspaces

1. Eigenvectors corresponding to different eigenvlues are linearly independent.

2. If v is an eigenvector of A corresponding to λ, then v is also an eigenvector of A^k corresponding to λ^k.

3. If v is an eigenvector of A corresponding to eigenvalue λ and $p(x) = \sum_{j=0}^{k} a_j x^j$, then v is also an eigenvector of $p(A)$ corresponding to the eigenvalue $p(\lambda)$.

4. If A is invertible, then 0 is not an eigenvalue of A.

5. For all eigenvectors v corresponding to the eigenvalue λ of A, v is also an eigenvector of A, corresponding to the eigenvalue λ^{-1}.

Example 1. Find eigenvalues and eigenspaces of

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Solution.

$$\begin{aligned} P_A(\lambda) &= \det \begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} \\ &= (2-\lambda)^2 - 1 \\ &= \lambda^2 - 4\lambda + 3 \\ &= (\lambda - 1)(\lambda - 3) \end{aligned}$$

Therefore, 1 and 4 are the eigenvalues of A.

$$\begin{aligned} V_1 &= \mathrm{N}(A - 1 \cdot I) \\ &= \mathrm{N} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ \therefore V_1 &= \left\{ \begin{pmatrix} x \\ x \end{pmatrix} \right\} \\ &= \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \end{aligned}$$

$$
\begin{aligned}
V_3 &= \mathrm{N}(A - 3 \cdot I) \\
&= \mathrm{N}\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\therefore V_3 &= \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} \right\} \\
&= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}
\end{aligned}
$$

Example 2. Find eigenvalues and eigenspaces of

$$
B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}
$$

Solution.

$$
\begin{aligned}
P_B(\lambda) &= \det(A - \lambda I) \\
&= \begin{vmatrix} \lambda & 0 & 0 \\ 0 & -\lambda & -2 \\ 0 & 1 & 3-\lambda \end{vmatrix} \\
&= \lambda(\lambda^2 - 3\lambda + 2) \\
&= \lambda^3 - 3\lambda^2 + 2\lambda \\
&= \lambda(\lambda - 1)(\lambda - 2)
\end{aligned}
$$

Therefore, 0, 1, 2 are eigenvalues of B.

$$
\begin{aligned}
V_0 &= \mathrm{N}(B - 0 \cdot I) \\
&= \mathrm{N}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \\
&= \left\{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}
\end{aligned}
$$

$$
\begin{aligned}
V_1 &= \mathrm{N}(B - 1 \cdot I) \\
&= \mathrm{N}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{pmatrix} \\
&= \mathrm{N}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \left\{ \begin{pmatrix} 0 \\ -2z \\ z \end{pmatrix} \right\} \\
&= \operatorname{span}\left\{ \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\}
\end{aligned}
$$

$$
\begin{aligned}
V_2 &= \mathrm{N}(B - 2 \cdot I) \\
&= \mathrm{N}\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix} \\
&= \left\{ \begin{pmatrix} 0 \\ -z \\ z \end{pmatrix} \right\} \\
&= \operatorname{span}\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}
\end{aligned}
$$

Example 3. Let A be a square matrix of order 3 with the eigenvalues

$$
\begin{aligned}
\lambda_1 &= 0 \\
\lambda_2 &= 1 \\
\lambda_3 &= 2
\end{aligned}
$$

Let v_1, v_2, v_3 be the corresponding eigenvectors.

1. Find a basis for $N(A)$.
2. Find a solution for $Av = v_2 + v_3$.

Solution. As v_1, v_2, v_3 are eigenvectors corresponding to different eigenvalues, they are linearly independent. Hence they form a basis of $\mathbb{R}^3$.

Let $v \in \mathrm{N}(A)$.
Therefore,

$$\begin{aligned} v &= \alpha v_1 + \beta v_2 + \gamma v_3 \\ \therefore Av &= A(\alpha v_1 + \beta v_2 + \gamma v_3) \end{aligned}$$

As $v \in \mathrm{N}(A)$, $Av = 0$

$$\begin{aligned} \therefore 0 &= A(\alpha v_1 + \beta v_2 + \gamma v_3) \\ &= \alpha A v_1 + \beta A v_2 + \gamma A v_3 \end{aligned}$$

As $Av_1 = \lambda_1 v_1$,

$$\begin{aligned} 0 &= \alpha \lambda_1 v_1 + \beta \lambda_2 v_2 + \gamma \lambda_3 v_3 \\ &= \beta v_2 + 2\gamma v_3 \end{aligned}$$

As v_2 and v_3 are linearly independent, $\beta = \gamma = 0$

$$\begin{aligned} \therefore v &= \alpha v_1 \\ \therefore \mathrm{N}(A) &= \operatorname{span}\{v_1\} \end{aligned}$$

$$\begin{aligned} v &= \alpha v_1 + \beta v_2 + \gamma v_3 \\ Av &= v_2 + v_3 \\ \therefore \alpha A v_1 + \beta A v_2 + \gamma A v_3 &= v_2 + v_3 \\ \therefore (\beta - 1) v_2 + (2\gamma - 1) v_3 &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \beta &= 1 \\ \gamma &= \frac{1}{2} \end{aligned}$$

Therefore,

$$v = v_2 + \frac{1}{2} v_3$$