Review Session 1

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Example 1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + y \\ y - z \\ 2y + 4z \end{pmatrix}$$

Is T diagonalizable?

Solution. Let $B = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}$.
Therefore,

$$T(e_1) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$$T(e_2) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$T(e_3) = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$$

Therefore,

$$\begin{aligned} T(e_1) &= 2e_1 \\ T(e_2) &= e_1 + e_2 + 2e_3 \\ T(e_3) &= -e_2 + 4e_3 \\ \therefore [T]_B &= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix} \end{aligned}$$

Therefore,

$$
\begin{aligned}
p_T(\lambda) &= \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 2 & 4-\lambda \end{vmatrix} \\
&= (2-\lambda)\big((1-\lambda)(4-\lambda)+2\big) \\
&= (2-\lambda)(6-5\lambda+\lambda^2) \\
&= -(\lambda-2)(\lambda-2)(\lambda-3) \\
\therefore \lambda &= 2, 3
\end{aligned}
$$

$$
\begin{aligned}
V_2 &= \mathrm{N}\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{pmatrix} \\
&= \mathrm{N}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}
\end{aligned}
$$

Therefore, $\dim V_2 = 1$. Hence the transformation is not diagonalizable.

Example 2. Let

$$
A = \begin{pmatrix} 1 & b & b^2 \\ 0 & a & 2ab \\ 0 & 0 & a^2 \end{pmatrix}
$$

Find all (a, b) such that A is diagonalizable. Find all (a, b) such that A is invertible, and for these (a, b), find A^{-1}.

Solution.

$$
\begin{aligned}
p_A(x) &= \begin{vmatrix} x-1 & b & b^2 \\ 0 & x-a & 2ab \\ 0 & 0 & x-a^2 \end{vmatrix} \\
&= (x-1)(x-a)(x-a^2) \\
\therefore \lambda &= 1, a, a^2
\end{aligned}
$$

If $a \neq 0$, $a \neq 1$, $a \neq -1$, the algebraic and geometric multiplicities are 1. Therefore, A is diagonalizable.
If $a = 0$,

$$\lambda = 0, 1$$

Therefore,

$$\begin{aligned} V_0 &= \mathrm{N}\begin{pmatrix} -1 & -b & -b^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \mathrm{N}\begin{pmatrix} 1 & b & b^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \mathrm{span}\left\{ \begin{pmatrix} -b \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -b^2 \\ 0 \\ 1 \end{pmatrix} \right\} \end{aligned}$$

Therefore, for $a = 0$, A is diagonalizable.

If $a = 1$,

$$\lambda = 1$$

If $b = 0$,

$$\begin{aligned} V_1 &= \mathrm{N}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \end{aligned}$$

If $b \neq 0$,

$$\begin{aligned} V_1 &= \mathrm{N}\begin{pmatrix} 0 & b & b^2 \\ 0 & 0 & 2b \\ 0 & 0 & 0 \end{pmatrix} \\ &= \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \end{aligned}$$

Therefore, for $a = 1$, A is diagonalizable if $b = 0$.

If $a = -1$,

$$\begin{aligned} p_A(x) &= (x-1)^2(x+1) \\ \therefore \lambda &= -1, 1 \end{aligned}$$

Therefore,

$$\begin{aligned} V_1 &= \mathrm{N}\begin{pmatrix} 0 & b & b^2 \\ 0 & -2 & -2b \\ 0 & 0 & 0 \end{pmatrix} \\ &= \mathrm{N}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -b \\ 1 \end{pmatrix}\right\} \end{aligned}$$

Therefore, for $a = -1$, A is diagonalizable.

Therefore, A is diagonalizable for all $(a, b) \in \mathbb{R}^2 \setminus (1, b \neq 0)$.

For A to be invertible, $\det(A) \neq 0$. Therefore, A is invertible if and only if $a \neq 0$.

$$\therefore A^{-1} = \begin{pmatrix} 1 & -{}^{b}/_{a} & {}^{b^2}/_{a^2} \\ 0 & {}^{1}/_{a} & -{}^{2b}/_{a^2} \\ 0 & 0 & {}^{1}/_{a^2} \end{pmatrix}$$