Recitation 6

Wednesday 3$^{\text{rd}}$ December, 2014

Contents

1 Spanning Sets

Example 1. Find a set $k \in \mathbb{R}^4$, s.t.

$$\mathrm{span}(k) = \{x \in \mathbb{R}^4 | Ax = 0\} \qquad ; \quad A = \begin{pmatrix} 1 & 1 & -2 & 1 \\ 1 & -2 & 1 & 2 \\ 0 & 3 & -3 & -1 \end{pmatrix}$$

Solution.

$$\begin{pmatrix} 1 & 1 & -2 & 1 \\ 1 & -2 & 1 & 2 \\ 0 & 3 & -3 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 & 1 \\ - & -3 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore,

$$x + y - 2z + w = 0$$
$$-3y + 3z + w = 0$$

Solving,

$$y = z + \frac{w}{3}$$
$$x = z - \frac{4}{3}w$$

Therefore,

$$\begin{aligned} \mathrm{span}(k) &= \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} | z, w \in \mathbb{R}, x = z - \frac{4}{3}w, y = z + \frac{w}{3} \right\} \\ &= \left\{ \begin{pmatrix} z - \frac{4}{3}w \\ y = z + \frac{w}{3} \\ z \\ w \end{pmatrix} | z, w \in \mathbb{R}, x = z - \frac{4}{3}w, y = z + \frac{w}{3} \right\} \\ &= \mathrm{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 0 \\ 3 \end{pmatrix} \right\} \end{aligned}$$

Example 2. Given

$$k = \{x^2 + 1, 2x, x_2 - 1\} \subseteq \mathbb{R}_2[x]$$

prove

$$\text{span}(k) = \mathbb{R}_2[x]$$

Solution. Proving

$$\text{span}(k) = \mathbb{R}_2[x]$$

is equivalent to proving

$$\begin{gathered}\text{span}(k) \subseteq \mathbb{R}_2[x] \\ \& \\ \mathbb{R}_2[x] \subseteq \text{span}(k)\end{gathered}$$

Let

$$p(x) = ax^2 + bx + c$$

Let

$$\begin{aligned}ax^2 + bx + c &= \alpha(x^2 + 1) + \beta(2x) + \gamma(x^2 - 1) \\ &= (\alpha + \gamma)x^2 + (2\beta)x + (\alpha - \gamma)\end{aligned}$$

This system has a solution. Therefore there exists such a linear combination. Therefore,

$$\mathbb{R}_2[x] \subseteq \text{span}(k)$$

It is obvious that

$$\text{span}(k) \subseteq \mathbb{R}_2[x]$$

Therefore,

$$\text{span}(k) = \mathbb{R}_2[x]$$

2 Subspaces

Example 3. Given subspaces

$$\begin{aligned}W_1 &= \text{span}\{(1, 1, 0, 1), (1, -1, 0, 1)\} \\ W_2 &= \text{span}\{(0, 1, 1, -1), (0, 1, -1, 1)\}\end{aligned}$$

Calculate $W_1 \cap W_2$.

Solution. Let $u \in W_1 \cap W_2$.
Therefore, $u \in W_1$ and $u \in W_2$.
Therefore,

$$\begin{aligned} u &= \alpha v_1 + \beta v_2 = \gamma v_3 + \delta v_4 \\ \therefore \alpha v_1 + \beta v_2 - \gamma v_3 - \delta v_4 &= 0 \end{aligned}$$

Therefore, the matrix of $v_1, v_2, -v_3, -v_4$ is

$$\begin{pmatrix} v_1 & v_2 & -v_3 & -v_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, δ is free.
Therefore,

$$\begin{aligned} \alpha &= \delta \\ \beta &= -\delta \\ \gamma &= -\delta \end{aligned}$$

Therefore,

$$\begin{aligned} u &= \alpha v_1 + \beta v_2 \\ &= \delta(v_1 - v_2) \\ \therefore W_1 \cap W_2 &= \{\delta(v_1 - v_2) | \delta \in \mathbb{R}\} \\ &= \operatorname{span}\{0, 1, 0, 0\} \end{aligned}$$

3 Coordinates

Definition 1. Let V be a finitely generated vector space and $B = \{v_1, \dots, v_n\}$ be a basis of V.
Each $u \in V$ can be uniquely represented as

$$u = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

The scalars $\alpha_1, \ldots, \alpha_n$ are called the coordinates of u according to the basis B.
The vector

$$[u]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

is called the coordinate vector of u according to the basis B.

Example 4. Find a basis and dimension to the subspace

$$W = \operatorname{span}\{1 - x, x + x^2, 1 + x^2\} \subset \mathbb{R}_2[x]$$

Solution. Let

$$\begin{aligned} p_1(x) &= 1 - x \\ p_2(x) &= x + x^2 \\ p_3(x) &= 1 + x^2 \end{aligned}$$

The standard basis of $\mathbb{R}_2[x]$ is $E = \{1, x, x^2\}$.
Therefore,

$$[p_1]_E = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$[p_2]_E = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$[p_3]_E = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Let

$$\tilde{W} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} = \mathrm{R} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Therefore,

$$W = \operatorname{span}\{1 + x^2, x + x^2\}$$

4 Transformation Matrices

Definition 2. Let $B = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_n\}$ be two bases of V. The transformation matrix from the base C to the base B is

$$[B]_C = \begin{pmatrix} [b_1]_C & \ldots & [b_n]_C \end{pmatrix}$$

4.1 Properties

$$[V]_C = [B]_C[V]_B$$
$$[C]_B = [B]_C^{-1}$$
$$[D]_C = [B]_C[D]_B$$