Recitation 8

Wednesday 17th December, 2014

Contents

1 Linear Maps

Example 1. Is the following map linear?

$$T : \mathbb{R}^3 \to \mathbb{R}^2 \qquad ; \quad T((x,y,z)) = (x+2y, 2x+z)$$

Solution.

$$\begin{aligned} T((x_1,y_1,z_1)+(x_2,y_2,z_2)) &= T((x_1+x_2, y_1+y_2, z_1+z_2)) \\ &= (x_1+x_2+2y_1+2y_2, 2x_1+2x_2, z_1, z_2) \\ &= (x_1+2y_1, 2x_1+z_1)+(x_2+2y_2, 2x_2+z_2) \\ &= T((x_1,y_1,z_1))+T((x_2,y_2,z_2)) \end{aligned}$$

$$\begin{aligned} T(\alpha(x,y,z)) &= T((\alpha x, \alpha y, \alpha z)) \\ &= (\alpha x + 2\alpha y, 2\alpha x + \alpha z) \\ &= \alpha T((x,y,z)) \end{aligned}$$

Example 2. Is the following map linear?

$$T : \mathbb{R}^3 \to \mathbb{R}^2 \quad ; T((x,y,z)) = (xy, x^2)$$

Solution.

$$\begin{aligned} T(-1(0,0,1)) &= (0,1) \\ -1 \cdot T((0,0,1)) &= (0,-1) \\ \therefore T(-1(0,0,1)) &\neq -1 \cdot T((0,0,1)) \end{aligned}$$

Hence the map is not linear.

Theorem 1. *If $\{u_1, \ldots, u_n\}$ is a basis of U, we can define a linear map $T : U \to V$ using the images of the elements of the basis, i.e. $\{T(u_1), \ldots, T(u_n)\}$.*

Example 3. Find, a linear map T, if it exists, such that is satisfies the given conditions.

$$
\begin{aligned}
T : \mathbb{R}^3 &\to \mathbb{R}^3 \\
T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right) &= \begin{pmatrix}3\\0\\0\end{pmatrix} \\
T\left(\begin{pmatrix}2\\-1\\2\end{pmatrix}\right) &= \begin{pmatrix}0\\0\\0\end{pmatrix} \\
T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) &= \begin{pmatrix}1\\0\\0\end{pmatrix}
\end{aligned}
$$

Solution.

$$
\left\{ u_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, u_2 = \begin{pmatrix}2\\-1\\2\end{pmatrix}, u_3 = \begin{pmatrix}1\\0\\0\end{pmatrix} \right\}
$$

is linearly independent, and $\dim \mathbb{R}^3 = 3$. Therefore, the above set is a basis of $\mathbb{R}^3$.
Therefore, T exists.

$$
\begin{aligned}
T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) &= \begin{pmatrix}1\\0\\0\end{pmatrix} \\
T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) &= \begin{pmatrix}2\\0\\0\end{pmatrix} \\
T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) &= \begin{pmatrix}0\\0\\0\end{pmatrix} \\
\therefore T\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right) &= T(xe_1 + ye_2 + ze_3) \\
&= \begin{pmatrix}x+2y\\0\\0\end{pmatrix}
\end{aligned}
$$