Recitation 2

Wednesday 5^{th} November, 2014

Contents

1 Row Echelon Form and Soving Systems of Linear Equations

1.1 Solve the following system of linear equations

$$
\begin{aligned}
x + 2y - 3z &= 4 \\
x + 3y + x &= 11 \\
2x + 5y - 4z &= 12
\end{aligned}
$$

Solution:

$$
A = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 3 & 1 \\ 2 & 5 & -4 \end{pmatrix}
$$

$$
x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}
$$

$$
b = \begin{pmatrix} 4 \\ 11 \\ 13 \end{pmatrix}
$$

$Ax = b$ is the matrix form of the system.
The augmented matrix is $(A|b)$.

$$
(A|b) = \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 1 & 3 & 1 & 11 \\ 2 & 5 & -4 & 13 \end{array}\right)
$$

We will try to bring the augmented matrix into reduced REF, i.e. of the form $\left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right)$ In order to transform the matrix to Reduced REF.
We can do one of the following operations at each time.

1. $R_i \to cR_i$
2. $R_i \leftrightarrow R_j$
3. $R_i \to R_i + cR_j$

These elementary operations preserve the set of elements.

1.2 Gaussian Eliminiation

1. We will make sure that in the upper left corner we have an element different from 0. If we don't, we will switch the 1^{st} row with another row.

If all elements in column 1 are zeros, we will ignore this column and consider the next one.

2. We will multiply the first row with a constant such that the first element will be 1
3. We will cancel all other elements in the first column, except the one in the first row, by elementary row operation $R_i \to R_i + cR_j$
4. We will repeat the above steps, ignoring the last row and last column, until we get an upper-triangular matrix.

1.3 Find the solutions of

1.3.1

$$\begin{aligned} x + 2y - 3z &= -1 \\ 3x - y + 2z &= 7 \\ 5x + 3y - 4z &= 2 \end{aligned}$$

Solution:

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{array}\right) \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_1 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & -7 & 11 & 7 \end{array}\right) \xrightarrow{R_2 \to -\frac{1}{7}R_2} \left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & 1 & -\dfrac{11}{7} & -\dfrac{10}{7} \\ 0 & -7 & 11 & 7 \end{array}\right) \xrightarrow{R_3 \to R_3 + 7R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & 1 & -\dfrac{11}{7} & -\dfrac{10}{7} \\ 0 & 0 & 0 & -3 \end{array}\right)$$

$0 \neq -3 \Rightarrow$ No solution

1.3.2

$$\begin{aligned} x + 2y - 3z &= 6 \\ 2x - y + 4z &= 2 \\ 4x + 3y - 2z &= 14 \end{aligned}$$

Solution:

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 2 & -1 & 4 & 2 \\ 4 & 3 & -2 & 14 \end{array}\right) \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & -5 & 10 & -10 \end{array}\right) \xrightarrow{R_2 \to -\frac{1}{5}R_2} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ 0 & -5 & 10 & -10 \end{array}\right) \xrightarrow{R_3 \to R_3 + 5R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{aligned} x + z &= 2 \\ y - 2z &= 2 \end{aligned}$$

z is a free variable and x, y are dependant variables.

$$\therefore x = \begin{pmatrix} 2 - t \\ 2t + 2 \\ t \end{pmatrix}$$

Therefore, the system of equations has infinite number of solutions.

1.4 Check if the following matrices are invertible and find the inverse if it exists.

1.4.1

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$$

1.4.1.1 Solution:

$$\begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 6 & -1 & -1 \end{pmatrix} \xrightarrow[R_2 \to R]{R_1 \to R}$$

$$\begin{pmatrix} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{pmatrix}$$

1.5 Invertible Matrices

A square matrix is invertible iff its reduced REF is I.

$$\begin{aligned} (A|I) &\xrightarrow{\text{Gaussian Elimination}} (I|B) \\ \therefore B \cdot A &= I \\ \therefore B &= A^{-1} \end{aligned}$$