

Theorem 33 (Gram - Schmidt Process).

Step 1 $\tilde{v}_1 = v_1$, denote $w_1 = \text{span}\{\tilde{v}_1\} = \text{span}\{v_1\}$, $B_1 = \{\tilde{v}_1\}$

Step 2 $\tilde{v}_2 = v_2 - \pi_{B_1}(v_2) = v_2 - \frac{\langle v_2, \tilde{v}_1 \rangle}{\langle \tilde{v}_1, \tilde{v}_1 \rangle} \tilde{v}_1$

As $\tilde{v}_2 \perp \tilde{v}_1$, $B_2 = \{\tilde{v}_1, \tilde{v}_2\}$ is an orthogonal set. Denote $W_2 = \text{span}\{\tilde{v}_1, \tilde{v}_2\} = \text{span}\{v_1, v_2\}$.

Step 3 $\tilde{v}_3 = v_3 - \pi_{B_2}(v_3) = v_3 - \frac{\langle v_3, \tilde{v}_1 \rangle}{\langle \tilde{v}_1, \tilde{v}_1 \rangle} \tilde{v}_1 - \frac{\langle v_3, \tilde{v}_2 \rangle}{\langle \tilde{v}_2, \tilde{v}_2 \rangle} \tilde{v}_2$

As $\tilde{v}_3 \in W_2^\perp$, $B_3 = \{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ is an orthogonal set. Denote $W_2 = \text{span}\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\} = \text{span}\{v_1, v_2, v_3\}$.

\vdots

Step n The n^{th} step gives $\tilde{B}_n = \{\tilde{v}_1, \dots, \tilde{v}_n\}$ which is an orthogonal basis of V .

B^0 is obtained by normalization of B_n .

$$v_1^0 = \frac{1}{\|\tilde{v}_1\|}$$

\vdots

$$v_n^0 = \frac{1}{\|\tilde{v}_n\|}$$

Theorem 34 (Bessel's Inequality). Let $\{v_1, \dots, v_m\}$ be an orthonormal set. Let $v \in V$ be any vector. Then

$$\|v\|^2 \geq |\langle v, v_1 \rangle|^2 + \dots + |\langle v, v_m \rangle|^2$$

and the equality holds if and only if $v \in \text{span}\{v_1, \dots, v_m\}$.

Theorem 35 (Cauchy - Schwarz Inequality). Let $u, v \in V$ be any vectors. Then

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$$

and the equality holds if and only if $\{u, v\}$ is linearly dependent.

Theorem 36. Let W be a subspace of V . Then

$$V = W \oplus W^\perp$$

Corollary 36.1. Let B be an orthogonal basis of W . Then $\pi_B(v)$ does not depend on the choice of B .

LINEAR ALGEBRA

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1. MATRICES

Definition 1 (Adjoint matrix).

$$A^* \doteq \overline{A}^t$$

Definition 2 (Row rank). The number of non-zero rows in A_R is called the row rank of A . It is denoted by r .

$$r \leq n$$

Theorem 1 (Gaussian Elimination). Step 1 Find the first non-zero column C_p of A .

Step 2 Denote by a_{ip} the first non-zero entry of C_p .

Step 3 Switch the 1^{st} and i^{th} rows.

Step 4 Multiply the 1^{st} row by $\frac{1}{a_{ip}}$.

Step 5 Using row operations of type III, make all other entries of the p^{th} column zeros.

Step 6 Ignoring the top row and C_p , repeat steps Step 1 to Step 5.

2. VECTOR SPACES

Definition 3 (Subspace). Let $U \subseteq V$.

Axiom 1 $0 \in U$

Axiom 2 If $x, y \in U$, then, $(x + y) \in U$

Axiom 3 If $x \in U, \alpha \in \mathbb{F}$, then, $\alpha x \in U$

Definition 4 (Operations on subspaces).

$$U_1 \cap U_2 = \{x \in V : x \in U_1 \text{ and } x \in U_2\}$$

$$U_1 \cup U_2 = \{x \in V : x \in U_1 \text{ or } x \in U_2\}$$

$$U_1 + U_2 = \{x \in V : x = x_1 + x_2, x_1 \in U_1, x_2 \in U_2\}$$

Definition 5 (Span). $\text{span}(S)$ is the collection of all linear combinations of finite number of vectors of S with coefficients from \mathbb{F} . $\text{span}(S)$ is a subspace of V

Definition 6 (Spanning sets and dimensionality). If V has atleast one finite spanning set, V is said to be finite-dimensional.

Definition 7 (Isomorphic spaces). Let V/\mathbb{F} and W/\mathbb{F} be vector spaces. We say that V is isomorphic to W if there is a map $\varphi : V \rightarrow W$, s.t.

- (1) φ is one-to-one and onto
- (2) $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2); \forall v_1, v_2 \in V$
- (3) $\varphi(\alpha v) = \alpha \varphi(v); \forall v \in V, \forall \alpha \in \mathbb{F}$

Theorem 2. If $S = \{v_1, \dots, v_m\}$ is a spanning set of V , and if S is not a basis of V , a basis B of V can be obtained by removing some elements from S .

Proof. If S is linearly independent, then it is a basis.

Otherwise, if S is linearly dependent, it has an element, WLG, say v_m , which is a linear combination of the others.

$$v_m = \alpha_1 v_1 + \dots + \alpha_{m-1} v_{m-1}$$

Let

$$S' = S - \{v_m\}$$

S' is a spanning set.

Therefore, $\forall v \in V$

$$\begin{aligned} v &= \beta_1 v_1 + \dots + \beta_{m-1} v_{m-1} + \beta_m v_m \\ &= \beta_1 v_1 + \dots + \beta_{m-1} + \beta_m (\alpha_1 v_1 + \dots + \alpha_{m-1} v_{m-1}) \\ &= \gamma_1 v_1 + \dots + \gamma_{m-1} v_{m-1} \end{aligned}$$

If S' is linearly independent, then it is a basis, else the same process above can be repeated till we get a basis.

Therefore, a basis is a smallest spanning set. \square

Theorem 3. If $B_0 = \{v_1, \dots, v_n\}$ is a linearly independent set, and if B_0 is a basis of V , a basis of V can be obtained by adding elements to B_0 .

Theorem 4. Let V be a vector space, s.t. $\dim V = n$.

If B satisfies 2 out of the 3 following conditions, then it is a basis.

- (1) B has n elements.
- (2) B is a spanning set.
- (3) B is linearly dependent.

Theorem 5 (Dimension Theorem).

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$$

Theorem 6.

$$U + W = \text{span}(U \cup W)$$

If $U = \text{span}(B)$ and $W = \text{span}(B')$ then, $U + W = \text{span}(B \cup B')$.

Corollary 27.1. Any orthonormal set is linearly independent.

Corollary 27.2. Any orthonormal set consisting of $n = \dim V$ vectors is an orthonormal basis of V .

Theorem 28. Let $B = \{v_1, \dots, v_n\}$ be an orthonormal basis of V . Let $v \in V$. Let

$$[v]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}. \text{ Then,}$$

$$\alpha_1 = \langle v, v_1 \rangle$$

$$\vdots$$

$$\alpha_n = \langle v, v_n \rangle$$

Theorem 29 (Pythagoras Theorem). Let $B = \{v_1, \dots, v_n\}$ be an orthonormal basis

of V . Let $v \in V$. Let $[v]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$. Then,

$$\|v\|^2 = |\alpha_1|^2 + \dots + |\alpha_n|^2$$

Definition 25 (Unitary matrix). Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let A be an $n \times n$ matrix. A is said to be a unitary matrix if

$$A^* = \overline{A}^t = \overline{A}^t = A^{-1}$$

If $\mathbb{F} = \mathbb{R}$, unitary matrices are called orthogonal matrices.

Theorem 30. Let A be an $n \times n$ matrix. Let v_1, \dots, v_n be the columns of A . Let A be an $n \times n$ matrix. Let r_1, \dots, r_n be the columns of A . Then the following are equivalent.

- (1) A is unitary.
- (2) $\{v_1, \dots, v_n\}$ is an orthonormal basis of \mathbb{F}^n , with respect to standard dot product.
- (3) $\{r_1, \dots, r_n\}$ is an orthonormal basis of \mathbb{F}^n , with respect to standard dot product.

Theorem 31. Let V be an inner product space. Let B be an orthonormal basis of V . Let B' be another basis of V . Let P be the transition matrix from B to B' . Then B' is orthonormal if and only if P is unitary.

Definition 26. Let $S \subset V$ be a set of vectors.

$$S^\perp \doteq \{v \in V \mid \langle u, v \rangle = 0, \forall u \in S\}$$

Theorem 32. S^\perp is a subspace of V .

Theorem 23 (Jordan Theorem). Let $T : V \rightarrow V$ be a linear operator such that $p_T(x)$ splits completely. Then there exists a basis B of V such that $[T]_B$ is of the form

$$[T]_B = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_l \end{pmatrix} J_i = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & 0 & \ddots & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

4. INNER PRODUCT SPACES

Definition 21 (Inner product). Let V be a vector space over \mathbb{R} or \mathbb{C} .

- (1) $\langle \alpha_1 v_1 + \alpha_2 v_2, w \rangle = \alpha_1 \langle v_1, w \rangle + \alpha_2 \langle v_2, w \rangle, \forall v_1, v_2, w \in V, \forall \alpha_1, \alpha_2 \in \mathbb{F}$
- (2) $\langle v, w \rangle = \overline{\langle w, v \rangle}, \forall v, w \in V$
- (3) $\langle v, v \rangle$ is a real non-negative number, $\forall v \in V$

Theorem 24 (Sesquilinearity).

$$\langle v, \beta_1 w_1 + \beta_2 w_2 \rangle = \overline{\beta_1} \langle v, w_1 \rangle + \overline{\beta_2} \langle v, w_2 \rangle$$

Definition 22 (Gram matrix). Let V be an inner product space. Let

$$B = \{v_1, \dots, v_n\}$$

be a basis of V .

$$G_B = \begin{pmatrix} \langle v_1, v_1 \rangle & \dots & \langle v_1, v_n \rangle \\ \vdots & & \vdots \\ \langle v_n, v_1 \rangle & \dots & \langle v_n, v_n \rangle \end{pmatrix}$$

Theorem 25. $\langle v, w \rangle = [v]_B^t G_B \overline{[w]_B}$

Theorem 26. Let B, \tilde{B} be bases of V . Let P be the transition matrix from B to \tilde{B} . Then

$$G_{\tilde{B}} = P^t G_B \bar{P}$$

where \bar{P} is the matrix obtained by replacing all elements of P by their complex conjugates.

Definition 23 (Norm). $\|v\| \doteq \sqrt{\langle v, v \rangle}$

Definition 24 (Orthogonality). $u \perp v \iff \langle u, v \rangle = 0$

Theorem 27. Let S be an orthogonal set such that $0 \notin S$. Then S is linearly independent.

Theorem 7 (Changing a basis). Let $B = \{v_1, \dots, v_n\}$ be a basis of V , s.t. $\dim V = n$. Let $B' = \{v'_1, \dots, v'_n\}$.

As B is a spanning set, all of v'_1, \dots, v'_n can be expressed as a linear combination of v_1, \dots, v_n .

$$v'_1 = \gamma_{11}v_1 + \dots + \gamma_{n1}v_n$$

$$\vdots$$

$$v'_n = \gamma_{1n}v_1 + \dots + \gamma_{nn}v_n$$

Definition 8 (Transition matrix). The matrix

$$C = \begin{pmatrix} \gamma_{11} & \dots & \gamma_{1n} \\ \vdots & & \vdots \\ \gamma_{n1} & \dots & \gamma_{nn} \end{pmatrix}$$

is called the transition matrix from B to B' .

$$B'_{1 \times n} = B_{1 \times n} C_{n \times n}$$

Theorem 8. B' is a basis of V iff C is invertible.

Corollary 8.1. If A has two identical rows, then $\det(A) = 0$.

Theorem 9. If A, B, C are some matrices, and 0 is the zero matrix,

$$\begin{pmatrix} A_{m \times m} & B \\ 0 & C_{n \times n} \end{pmatrix} = \det(A) \cdot \det(C)$$

Theorem 10.

$$\det(AB) = \det(A) \det(B)$$

Theorem 11 (Calculation of determinant). Let A be a $m \times n$ matrix, and let A_{ij} be the matrix obtained by removing the i^{th} row and j^{th} column from A .

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij})$$

Definition 9 (Linear map). Let V and W be vector spaces over the same field \mathbb{F} .

$$\varphi : V \rightarrow W$$

is said to be a linear map if

- (1) $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2); \forall v_1, v_2 \in V$
- (2) $\varphi(\alpha v) = \alpha \varphi(v); \forall v \in V, \forall \alpha \in \mathbb{F}$

Definition 10 (Matrix of a linear map). Let $\varphi : V \rightarrow W$ be a linear map.

$$B = \{v_1, \dots, v_n\}$$

$$B' = \{w_1, \dots, w_m\}$$

be bases of V and W respectively.

Let

$$\begin{aligned}\varphi(v_1) &= \alpha_{11}w_1 + \cdots + \alpha_{m1}w_m \\ &\vdots \\ \varphi(v_n) &= \alpha_{1n}w_1 + \cdots + \alpha_{mn}w_m\end{aligned}$$

The matrix

$$A = [\varphi]_{B,B'} = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix}$$

is called the matrix of φ with respect to the bases B and B' .

Theorem 12.

$$[\varphi(z)]_{B'} = [\varphi]_{B,B'}[z]_B$$

Theorem 13. Let V, W be vector spaces over \mathbb{F} , $\dim(V) = n$, $\dim(W) = m$. Let $\varphi : V \rightarrow W$ be a linear map. Let B, \widetilde{B} be bases of V and let B', \widetilde{B}' be bases of W . Let $A = [\varphi]_{B,B'}$ and $\widetilde{A} = [\varphi]_{\widetilde{B},\widetilde{B}'}$ be the matrices of φ w.r.t. the pairs B, B' and $\widetilde{B}, \widetilde{B}'$. Let P denote the transition matrix from B to \widetilde{B} , and let Q denote the transition matrix from B' to \widetilde{B}' . Then,

$$\widetilde{A}_{m \times n} = Q_{m \times m}^{-1} A_{n \times n} P_{n \times n}$$

Definition 11 (Operations on linear maps).

$$\begin{aligned}(\varphi + \varphi')(v) &\doteq \varphi(v) + \varphi'(v) \\ (\alpha\varphi)(v) &\doteq \alpha\varphi(v)\end{aligned}$$

Definition 12 (Composed map).

$$(\varphi' \circ \varphi)(v) \doteq \varphi'(\varphi(v))$$

Theorem 14 (Matrix of composed map).

$$[\varphi' \circ \varphi]_{B,B''} = [\varphi']_{B',B''}[\varphi]_{B,B'}$$

Definition 13 (Kernel and image). Let $\varphi : V \rightarrow W$ be a linear map.

$$\begin{aligned}\ker \varphi &\doteq \{v \in V : \varphi(v) = \mathbb{O}\} \\ \text{im } \varphi &\doteq \{\phi(v) : v \in V\}\end{aligned}$$

Theorem 15. $\ker \varphi$ is a subspace of V and $\text{im } \varphi$ is a subspace of W .

Theorem 16. Let $\varphi : V \rightarrow W$ be a linear map. Then

$$\dim V = \dim(\ker(\varphi)) + \dim(\text{im}(\varphi))$$

3. LINEAR OPERATORS

Definition 14 (Linear operator). A linear operator or transformation

$$T : V \rightarrow V$$

is a linear map from a vector space V to itself.

Definition 15 (Similarity of matrices).

$$A \sim \tilde{A} \iff \tilde{A} = P^{-1}AP$$

Definition 16 (Diagonalizability). If A is similar to a diagonal matrix, A is said to be diagonalizable. P , s.t. $P^{-1}AP = D$ is called a diagonalizing matrix for A . D is called a diagonal form of A .

Theorem 17 (Explicit criterion for diagonalization). Let A be an $n \times n$ matrix, s.t. $p_A(x)$ splits completely. Then A is diagonalizable if and only if $\forall \lambda_i$ of A , the algebraic multiplicity coincides with the geometric multiplicity.

Theorem 18. Let $\lambda_1, \dots, \lambda_s$ be pairwise distinct eigenvalues of an $n \times n$ matrix A , i.e. $\forall i \neq j, \lambda_i \neq \lambda_j$. Let v_1, \dots, v_s be eigenvalues of A corresponding to $\lambda_1, \dots, \lambda_s$. Then the set $S = \{v_1, \dots, v_s\}$ is linearly independent.

Corollary 18.1. Let $A_{n \times n}$ have n distinct eigenvalues. Then, A is diagonalizable.

Definition 17 (Characteristic Polynomial). Let A be any $n \times n$ matrix.

$$p_A(x) = \det(xI_n - A)$$

is called the characteristic polynomial. Its roots are eigenvalues of A .

Theorem 19. If $A \sim A'$, then $p_A(x) = p_{A'}(x)$.

Definition 18 (Algebraic multiplicity of eigenvalue). Largest possible integer value of k such that $p_A(x)$ is divisible by $(x - \lambda)^k$.

Definition 19 (Eigenspace).

$$V_\lambda = \{V \in \mathbb{F}^n : Av = \lambda v\}$$

Theorem 20. An eigenspace of a matrix is a subspace of the field over which the matrix is defined.

Definition 20 (Geometric multiplicity of eigenvalue). $m = \dim V_\lambda$

Theorem 21. The geometric multiplicity of an eigenvalue is less than or equal to the algebraic multiplicity.

Theorem 22 (Criterion for triangularization). An operator $T : V \rightarrow V$ is triangularizable, if and only if $p_T(x)$ splits completely.