Linear Algebra

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Part I

General Information

1 Contact Information

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2 Grades

Final Exam: 80%
Midterm Exam: 10%
Homework: 10%
Passing Criteria: 60%

Part II
Fields

1 Definition

Definition 1 (Field)**.** The set $\mathbb{F}$ is a field if there are operations $+, \cdot$ satisfying the following properties:

(A1) $\forall a, b \in \mathbb{F}; a + b = b + a$

(A2) $\forall a, b \in \mathbb{F}; (a + b) + c = a + (b + c)$

(A3) There is an element $0 \in \mathbb{F}$ s.t. $a + 0 = 0 + a = a$

(A4) $\forall a \in F, \exists b \in \mathbb{F}$ s.t. $a + b = 0$

(M1) $\forall a, b \in \mathbb{F}, a \cdot b = b \cdot a$

(M2) $\forall a, b \in \mathbb{F}, (a \cdot b) \cdot c = a \cdot (b \cdot c)$

(M3) There is an element $1 \in \mathbb{F}$ s.t. $a \cdot 1 = 1 \cdot a = a(1 \neq 0)$

(M4) $\forall a \in \mathbb{F}, (a \neq 0), \exists b \in \mathbb{F}$ s.t. $a \cdot b = 1$

(AM) $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$

If $\mathbb{F}$ is a field, one can define subtraction and division as follows.

$$a - b \doteq a + (-b)$$
$$\frac{a}{b} \doteq a \cdot \frac{1}{b}$$

1.1 Examples of Fields

1. $\mathbb{R}$
2. $\mathbb{C}$
3. $\mathbb{F}_p$

1.2 Examples of Non-fields (Rings)

1. $\mathbb{Z}$, as M4 is not satisfied.

If we define $\mathbb{F}_2 = 0, 1; 0 + 0 = 0; 0 + 1 = 1 + 0 = 1$; then, necessarily, $1 + 1 = 0$, otherwise, 1 will have no additive inverse.

2 Examples

Example 1. Let p be a prime number.
$\mathbb{F}_p$ is defined as follows.

$$\forall m \in \mathbb{Z}, m = a \centerdot p + \overline{m}$$

The operations + and $\centerdot$ are defined as

$$\overline{a} + \overline{b} = \overline{(a + b)}$$
$$\overline{a} \centerdot \overline{b} = \overline{(a \centerdot b)}$$

1. $\mathbb{F}_p$ is a field.
2. If $\mathbb{F}$ is a set of q elements, we can define on $\mathbb{F}$ a structure of a field iff $q = p^t$, where p is prime, $t \geq 1$.

Example 2. For a field of 4 elements $\{0,\ 1\ \alpha,\ \beta\}$, the addition and multiplication tables are as follows.

$+$	0	1	α	β
0	0	1	α	β
1	1	0	β	α
α	α	β	0	1
β	β	α	0	1

Part III
Matrices

1 Definition

Definition 2 (Matrix)**.** Let $\mathbb{F}$ be a field, $m, n \geq 1$.
Then, $A(m \times n)$ is a table consisting of m rows and n columns, filled by elements of $\mathbb{F}$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

2 Addition of Matrices

Definition 3 (Addition of matrices)**.** Let A, B be $m \times n$ matrices over $\mathbb{F}$. Then, $C = A + B$ is defined as follows.

$$c_{ij} = a_{ij} + bij$$

2.0.1 Properties

1. $A + B = B + A, \forall A, B$ s.t. the sum is defined

2. $(A + B) + C = A + (B + C), \forall A, B, C$ s.t. the sums are defined

3. There is a matrix $\mathbb{O}$, s.t. $A + \mathbb{O} = \mathbb{O} + A = A$

4. For any $A, \exists B$ s.t. $B = -A$

3 Multiplication of a matrix by a scalar

Definition 4 (Multiplication of a matrix by a scalar)**.** Let A be a $m \times n$ matrix over $\mathbb{F}$. Let $\alpha \in \mathbb{F}$ be a scalar. Then, $C = \alpha A$ is defined as follows.

$$c_{ij} = \alpha a_{ij}$$

4 Multiplication of matrices

Definition 5 (Multiplication of matrices)**.** Let A be a $m \times n$ matrix over $\mathbb{F}$.
Let B be a $n \times p$ matrix over $\mathbb{F}$.
Then, $C = AB$ is defined as follows.

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

Example 3. For matrices A, B, of same size, is $AB = BA$?

Solution. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
$\therefore AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
$\therefore AB \neq BA$

Remark 1. $A \neq \mathbb{O}, B \neq \mathbb{O}$, but $AB = \mathbb{O}$.

5 Zero Divisor

Definition 6 (Zero divisor)**.** We say that a square matrix $A \neq \mathbb{O}$ is a zero divisor if either there is a square matrix B s.t. $AB = \mathbb{O}$, or there is a square matrix C, s.t. $CA = \mathbb{O}$.

Remark 2. $\mathbb{O}B = C\mathbb{O} = \mathbb{O}$.

Remark 3. $AC = BC \nRightarrow A = B$. In general, we cannot cancel matrices on either side of an equation.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, C = \mathbb{O}$$

$$\therefore AB = CB = \mathbb{O} \,\&\, B \neq \mathbb{O}$$

But, we cannot cancel B, as $A \neq C$.

6 Theorem ('Good properties of matrix multiplication')

Theorem 1.

$$(AB)C = A(BC) \tag{1.1}$$
$$A(B+C) = AB + AC \tag{1.2}$$
$$(A+B)C = AC + BC \tag{1.3}$$
$$(\alpha A) = \alpha(AB) \tag{1.4}$$

Proof. Denote $AB = D, BC = G, (AB)C = F, A(BC) = H$
We need to prove $F = H$
Let the dimensions of the matrices be as follows.
$A_{m\times n}, B_{n\times p}, C_{p\times q}$
$\therefore F_{m\times q}, H_{m\times q}$

$$d_{ik} = \sum_j a_{ij}b_{jk}$$
$$\therefore g_{jl} = \sum_k b_{jk}bkl$$
$$f_{il} = \sum_k d_{ik}ckl = \sum_k(\sum_j a_{ij}bjk)c_{kl} = \sum_k\sum_j a_{ij}b_{jk}c_{kl}$$
$$h_{il} = \sum_j a_{ij}gjl = \sum_j a_{ij}(\sum_k bjkc_{kl}) = \sum_k\sum_j a_{ij}b_{jk}c_{kl}$$
$$f_{il} = h_{il}$$
$$F = H$$

□

7 Square Matrices

Let A be a square matrix of size $n \times n, n \geq 1$

7.1 Diagonal Matrices

Definition 7 (Diagonal matrix)**.** We say that A is a diagonal matrix if $a_{ij} = 0$, whenever $i \neq j$.

Theorem 2. *Let A and B be diagonal $n \times n$ matrices.*

$$a_{rr} = \alpha_r, b_{rr} = \beta_r$$

Then, $AB = BA = C, C$ is a diagonal matrix with $c_{rr} = a_{rr}b_{rr}$.

7.1.1 Proof

$a_{ij} = \begin{cases} 0, i \neq j \\ \alpha_i, i = j \end{cases}$

$b_{ij} = \begin{cases} 0, i \neq j \\ \beta_i, i = j \end{cases}$

$c_{ik} = \sum_{j=1}^{n} a_{ij}bjk = a_{ii}b_{ik} = \alpha_i b_{ik} = \begin{cases} 0, i \neq k \\ \alpha_i\beta_i, i = k \end{cases}$

Similarly for BA.

7.2 Upper-triangular Matrices

We say that A is an upper-triangular matrix if $a_{ij} = 0$, whenever $i > j$.

7.3 Lower-triangular Matrices

We say that A is a lower-triangular matrix if $a_{ij} = 0$, whenever $i < j$.

Remark

Diagonal matrices are upper-triangular and lower-triangular. Conversely, if a matrix is both upper-triangular and lower-triangular, it is a diagonal matrix.

7.4 Theorem

If A and B are both upper-triangular, then AB and BA are upper-triangular too.

7.4.1 Proof

Denote $C = AB$.

$$\therefore c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

Suppose $i > k$, then, either $i > j$ or $j > k$. So, in each case, atleast one of a_{ij} or b_{jk} is 0.

7.5 Identity Matrix

Let $n \geq 1$. We call I_n the $n \times n$ identity matrix.

$$I_n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

7.6 Theorem

Let I_n be the identity $n \times n$ matrix. Then, for any $n \times n$ matrix B, we have

$$I_n B = B I_n = B$$

7.6.1 Proof

$$I_n = (e_{ij}); e_{ij} = \begin{cases} 0, i \neq j \\ 1, i = j \end{cases}$$

Denote $C = I_n B$. We have

$$c_{ik} = \sum_{j=1}^{n} e_{ij} b_{jk} = e_{ii} b_{ik} = 1 \cdot b_{ik} = b_{ik}$$

$$\therefore C = B \Rightarrow I_n B = B$$

Similarly for $BI_n = B$.

7.7 Inverse of Matrix

Let A be an $n \times n$ matrix. We say that A is invertible if there exist B, C, s.t. $AB = I_n$ and $CA = I_n$

Remark

$A = \mathbb{O}$ is not invertible because $\mathbb{O}B = C\mathbb{O} = \mathbb{O} \neq I_n$

Remark

There are non-zero matrices which are not invertible.
Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
If possible, let there be C s.t. $CA = I_2$.
Let $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
We have $CA = I$.
$\therefore (CA)B = IB$
$\therefore C(AB) = B$
$\therefore C\mathbb{O} = B$
$\therefore \mathbb{O} = B$
But, $B \neq 0$. Therefore, C does not exist.

7.7.1 If $AB = I_n$ and $CA = I_n$, then $B = C$

$$\begin{aligned} C &= CI \\ &= C(AB) \\ &= (CA)B \\ &= IB \\ &= B \end{aligned}$$

7.7.2 Inverse of a Matrix

If A is invertible, i.e. if there exists B, s.t. $AB = BA = I$, then, B is called the inverse of A, and is denoted by A^{-1}.

7.7.3 If $AB = I$, then $BA = I$.

7.7.4 If A is invertible, then A cannot be a zero divisor.

If possible, let A be a zero divisor.
Therefore, either $AB = \mathbb{O}$, for some $B \neq \mathbb{O}$; or $CA = \mathbb{O}$, for some $C \neq \mathbb{O}$

Case I: $AB = \mathbb{O}$

$$\begin{aligned} AB &= \mathbb{O} \\ \therefore A^{-1}(AB) &= A^{-1}\mathbb{O} \\ \therefore (A^{-1}A)B &= \mathbb{O} \\ \therefore IB &= \mathbb{O} \\ \therefore B &= \mathbb{O} \end{aligned}$$

This contradicts the assumption $B \neq \mathbb{O}$

Case II: $CA = \mathbb{O}$

$$\begin{aligned} CA &= \mathbb{O} \\ \therefore (CA)A^{-1} &= \mathbb{O}A^{-1} \\ \therefore C(A^{-1}A) &= \mathbb{O} \\ \therefore CI &= \mathbb{O} \\ \therefore C &= \mathbb{O} \end{aligned}$$

This contradicts the assumption $C \neq \mathbb{O}$

7.7.5 If A and B are invertible, then $A + B$ may or may not be invertible.

If $A = B$, then $A + B = 2A$ is invertible.
If $A = -B$, then $A + B = \mathbb{O}$ is not invertible.

7.7.6 If A and B are invertible, then AB must be invertible.

$$
\begin{aligned}
(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\
&= AIA^{-1} \\
&= AA^{-1} \\
&= I
\end{aligned}
$$

Similarly, $(B^{-1}A^{-1})(AB) = I$

$\therefore (AB)^{-1} = B^{-1}A^{-1}$

8 Transpose of a Matrix

Let A be a $m \times n$ matrix, $A = (a_{ij})_{1\leq i\leq m;1\leq j\leq n}$

$B = A^t$ is defined as follows.

$$b_{ji} = a_{ij}$$

8.1 Properties of A^t

1. $(A+B)^t = A^t + B^t$
2. $(\alpha A)^t = \alpha A^t$
3. $(AB)^t = B^t A^t$
4. If A is invertible, then, A^t must be invertible, and $(A^t)^{-1} = (A^{-1})^t$

9 Adjoint Matrix

$$A^* \doteq \overline{A}^t$$

For example,

$$A = \begin{pmatrix} 1 & 1+i & 2-1 \\ i & -5i & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & -i \\ 1-i & 5i \\ 2+i & 3 \end{pmatrix}$$

9.0.1 Properties of Adjoint Matrices

1. $(A+B)^* = A^* + B^*$
2. $(\alpha A)^* = \overline{\alpha} A^*$
3. $(AB)^* = B^* A^*$
4. If A is invertible, then A^* is invertible, and $(A^*)^{-1} = (A^{-1})^*$

10 Row Operations on Matrices

10.1 Elementary Row Operations

Let A be a $m \times n$ matrix with rows $a_1, \dots a_m$. We define 3 types of elementary row operations.

I $a_i \leftrightarrow a_j$ (Switch of the i^{th} and j^{th} rows.)

II $a_i \to \alpha a_i (\alpha \neq 0)$ (Multiplication of a row by a non-zero scalar.)

III $a_i \to a_i + \alpha a_j (j \neq i)$ (Addition of a row multiplied by a scalar, and another row.)

$E_{\text{I}}, E_{\text{II}}, E_{\text{III}}$ are matrices obtained from the identity matrix by applying elementary row operations I, II, III, respectively. These matrices are called elementary matrices.

10.2 Theorems

Let $e_i = \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix}$ be a $1 \times m$ matrix.
Let A be any $m \times n$ matrix.
Then, $e_i A =$ the i^{th} row of A.

10.2.1 $E_{\text{I}} A =$ the matrix obtained from A by an elementary row operation I

Proof
Let A be any $m \times n$ matrix.

$$\therefore E_{\mathrm{I}}A = \begin{pmatrix} e_1 A \\ \vdots \\ e_j A \\ \vdots \\ e_i A \\ \vdots \\ e_m A \end{pmatrix}$$

10.2.2 $E_{\mathbf{II}}A =$ the matrix obtained from A by an elementary row operation II

Proof

Let A be any $m \times n$ matrix.

$$\therefore E_{\mathrm{I}}A = \begin{pmatrix} e_1 A \\ \vdots \\ \alpha e_i A \\ \vdots \\ e_m A \end{pmatrix}$$

10.2.3 $E_{\mathbf{III}}A =$ the matrix obtained from A by an elementary row operation III

Proof

Let A be any $m \times n$ matrix.

$$\therefore E_{\mathrm{I}}A = \begin{pmatrix} e_1 A \\ \vdots \\ a_{i1} + \alpha a_{j1} \cdots + a_{in} + \alpha a_{jn} \\ \vdots \\ e_j A \\ \vdots \\ e_m A \end{pmatrix}$$

$$= \begin{pmatrix} 1^{\text{st}} \text{ row of } A \\ \vdots \\ i^{\text{th}} \text{ row of } A + \alpha(j^{\text{th}}) \text{ row of } A \\ \vdots \\ j^{\text{th}} \text{ row of } A \\ \vdots \\ m^{\text{th}} \text{ row of } A \end{pmatrix}$$

10.2.4 All elementary matrices are invertible, moreover, the inverses of $E_{\mathbf{I}}, E_{\mathbf{II}}, E_{\mathbf{III}}$ are also elementary matrices of the same type.

$$E_{\mathrm{I}}^{-1} = E_{\mathrm{I}}$$
$$\Leftrightarrow E_{\mathrm{I}}^2 = I_m$$

$$
\begin{aligned}
E_{\mathrm{I}}^2 &= E_{\mathrm{I}} E_{\mathrm{I}} \\
&= \begin{pmatrix} e_1 E_{\mathrm{I}} \\ \vdots \\ e_j E_{\mathrm{I}} \\ \vdots \\ e_i E_{\mathrm{I}} \\ \vdots \\ e_m E_{\mathrm{I}} \end{pmatrix} \\
&= \begin{pmatrix} 1^{\text{st}} \text{ row of } A \\ \vdots \\ j^{\text{th}} \text{ row of } A \\ \vdots \\ i^{\text{th}} \text{ row of } A \\ \vdots \\ m^{\text{th}} \text{ row of } A \end{pmatrix} \\
&= \begin{pmatrix} e_1 \\ \vdots \\ e_j \\ \vdots \\ e_i \\ \vdots \\ e_m \end{pmatrix} = I_m
\end{aligned}
$$

Similarly for E_{II}, to get the inverse, α is replaced by $\dfrac{1}{\alpha}$

$$E_{\text{II}} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \alpha & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

$$\therefore E_{\text{II}}^{-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dfrac{1}{\alpha} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Similarly for E_{III}, to get the inverse, α is replaced by $-\alpha$

$$E_{\text{III}} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & \alpha & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

$$\therefore E_{\text{III}}^{-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -\alpha & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

10.3 Row-equivalent of a Matrix

A matrix A' is a row-equivalent of A, if A' is obtained for A, by a finite sequence of elementary row operations.

11 Row Echelon Form of a Matrix

11.1 Definition

Let A be an $m \times n$ matrix.
Denote the i^{th} row of A by a_i.
The leading entry of a non-zero row a_i is its first non-zero entry.
Denote the column where the leading entry occurs by l_i.

$a_{ij} = 0$ if $j < l(i)$
$a_{ij} \neq 0$ if $j = l(i)$

We say that A is in row echelon form(REF) if the following conditions hold.

1. The non-zero rows are at the top of A. ($r =$ the number of non-zero rows)
2. The leading entries go right as we go down, i.e. $l(1) < l_2 < \cdots < l(r)$
3. All leading entries equal 1, i.e. if $j = l(i)$, then, $a_{ij} = 1$
4. Any column which contains a leading entry must have all other entries equal to 0, i.e. if $j = l(i)$, then, $a_{kj} = 0; \forall k \neq i$

11.2 Notation

The REF of A will be denoted by A_R.

12 Row Rank of a Matrix

The number of non-zero rows in A_R is called the row rank of A. It is denoted by r.

$r \leq n$

13 Gauss Theorem

Any $m \times n$ matrix A can be brought to REF by a sequence of elementary row operations.

13.1 Elimination Algorithm

Step 1 Find the first non-zero column C_p of A.

Step 2 Denote by a_{ip} the first non-zero entry of C_p.

Step 3 Switch the 1^{st} and i^{th} rows.

Step 4 Multiply the 1^{st} row by $\dfrac{1}{a_{ip}}$.

Step 5 Using row operations of type III, make all other entries of the p^{th} column zeros.

Step 6 Ignoring the top row and C_p, repeat steps Step 1 to Step 5.

13.1.1 Example

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 4 & 7 \\ 0 & -1 & 7 & 6 \end{pmatrix} \xrightarrow{R_1 \to R_2} \begin{pmatrix} 0 & -1 & 4 & 7 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 7 & 6 \end{pmatrix} \xrightarrow{R_1 \to -R_1} \begin{pmatrix} 0 & 1 & -4 & -7 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 7 & 6 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_1}$$

$$\begin{pmatrix} 0 & 1 & -4 & -7 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 3 & -1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 0 & 1 & -4 & -7 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \xrightarrow{R_2 \to \frac{R_2}{3}} \begin{pmatrix} 0 & 1 & -4 & -7 \\ 0 & 0 & 1 & -\dfrac{1}{3} \\ 0 & 0 & 0 & -1 \end{pmatrix} \xrightarrow{R_1 \to R_1 + 4R_2}$$

$$\begin{pmatrix} 0 & 1 & 0 & -\dfrac{25}{3} \\ 0 & 0 & 1 & -\dfrac{1}{3} \\ 0 & 0 & 0 & -1 \end{pmatrix} \xrightarrow{R_3 \to -R_3} \begin{pmatrix} 0 & 1 & 0 & -\dfrac{25}{3} \\ 0 & 0 & 1 & -\dfrac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \to R_1 + \frac{25}{3} R_3} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\dfrac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 + \frac{1}{3} R_3}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

13.2 Row Spaces of Matrices

Definition 8 (Row space of a matrix)**.** Let A be a $m \times n$ matrix over $\mathbb{F}$. $\mathrm{R}(A)$ is defined as

$$\mathrm{R}(A) = \operatorname{span} v_1, \ldots, v_m$$

where $v_1, \ldots, v_m$ are rows of A.
$\mathrm{R}(A)$ a subspace of the vector space of all rows of length n, is called the row space of A.

Definition 9 (Row rank of a matrix)**.** $\dim \mathrm{R}(A)$ is called the row-rank of A, and is denoted by $\mathrm{rr}(A)$.

Theorem 3. *Let P be a $l \times m$ matrix. Then*

1. $\mathrm{R}(PA) \subseteq \mathrm{R}(A)$
2. *If P is an invertible $m \times m$ matrix, then* $\mathrm{R}(PA) = \mathrm{R}(A)$

Corollary 3.1.

$$A' \overset{\mathrm{R}}{\sim} A \implies \mathrm{R}(A') = \mathrm{R}(A)$$

Theorem 4. *If A is in REF, and if r is the number of non-zero rows in A, then*

$$\mathrm{rr}(A) = r$$

Corollary 4.1. *The following are equivalent*

1. $A \overset{R}{\sim} A'$
2. *There is an invertible matrix P,* s.t. $A' = PA$
3. $\mathrm{R}(A) = \mathrm{R}(A')$
4. *A and A' have the same REF*

13.3 Column Equivalence

Definition 10 (Elementary column operations, column equivalence, column echelon form, column space and column rank)**.** If A is a $m \times n$ matrix, we can define elementary column operations, column equivalence ($A \overset{C}{\sim}$) and column echelon form (CEF), the column space of A ($\mathrm{C}(A)$), and the column rank of A ($\mathrm{cr}(A)$).

Theorem 5.

$$\mathrm{cr}(A) = \mathrm{rr}(A) = r$$

Proof. Let $r = \mathrm{rr}(A) = \dim \mathrm{R}(A)$.
Choose r rows of A which form a basis of $\mathrm{R}(A)$, WLG, say $v_1, \ldots, v_r$.
Let

$$X_{r\times n} = \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix}$$

$$\mathrm{span}(X) = \mathrm{R}(A)$$

Hence, any row of A can be expressed as a linear combination of $v_1, \ldots, v_r$

$$v_i = \sum_{j=1}^{r} y_{ij} v_j$$

Let

$$Y_{m\times r} = (y_{ij})$$

Therefore,

$$A = YX$$

Considering each column of A as a linear combination of columns of Y,

$$\mathrm{C}(A) \subseteq \mathrm{C}(Y)$$
$$\therefore \mathrm{cr}(A) \leq \mathrm{cr}(Y) \leq r = \mathrm{rr}(A)$$
$$\therefore \mathrm{cr}(A) \leq \mathrm{rr}(A)$$

Similarly,

$$\mathrm{rr}(A) \leq \mathrm{cr}(A) \therefore \mathrm{cr}(A) = \mathrm{rr}(A)$$

□

Corollary 5.1. *The following are equivalent*

1. $A \overset{C}{\sim} A'$
2. *There is an invertible matrix* Q, s.t. $A' = QA$
3. $\mathrm{C}(A) = \mathrm{C}(A')$
4. *A and A' have the same CEF*

Part IV

Linear Systems

1 Definition

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Here, all x_i are taken to be unknowns, and all a_{ij}, b_i are given.
A solution to such a system is a collection $d_1, \ldots, d_n$, s.t. after replacing x_i by d_i, we get equalities.
We assume that all a_{ij}, b_i belond to $\mathbb{F}$, and we are looking for solutions $d_i \in \mathbb{F}$.
Given such a system, we define $A_{m\times n} = (a_{ij}), b_{m\times 1} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, x_{n\times 1} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$
Then, we can write the system as

$$Ax = b$$

A solution to this system is $d_n = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$, s.t. $Ad = b$

Let D be the set of all $d = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$
D may be empty, infinite, or a singleton set.

2 Equivalent Systems

Two systems $Ax = b$ and $A'x = b'$ are called equivalent, if every solution of the first system is also a solution of the second system, and vice versa.

3 Solution of a System of Equations

We want to bring a given system

$$Ax = b$$

to the form

$$A_R x = b_R$$

using elementary row operations.
We denote the augmented or extended matrix of the system as follows.

$$\overline{A}_{m\times(n+1)} = (A_{m\times n}|b_{m\times 1})$$

Then apply Gaussian elimination method to $\overline{A}$, in order to get the matrix

$$(A_R|b_R)$$

As A_R is obtained from A using elementary row operations,

$$A_R = E_n \dots E_2 E_1 A$$

where every E_i is an elementary matrix.
Let $P = E_n \dots E_2 E_1$. P is invertible, as it is a product of elementary matrices.

$$\begin{aligned} A_R &= PA \\ \therefore A_R d &= PAd \\ &= Pb \\ &= b_R \end{aligned}$$

Conversely, let d be a solution to

$$\begin{aligned} A_R d &= b_R \\ \therefore PAd &= b_R \\ \therefore P^{-1}(PAd) &= P^{-1} b_R \\ \therefore Ad &= b \end{aligned}$$

If we have a system $Ax = b$, we may and will assume that A is in REF, i.e. $A = A_R, b = b_R$.
Let $l(1), \dots l(r)$ denote the numbers of the columns containing leading entries.

Let $b = \begin{pmatrix} b_1 \\ \vdots \\ b_r \\ b_{r+1} \\ \vdots \\ b_m \end{pmatrix}$

Therefore,

$$\begin{aligned} 1 \cdot x_{l(1)} + \ldots &= b_1 \\ 1 \cdot x_{l(2)} + \ldots &= b_2 \\ &\vdots \\ 1 \cdot x_{l(r)} &= b_r \\ 0 &= b_{r+1} \\ &\vdots \\ 0 &= b_m \end{aligned}$$

4 Homogeneous Systems

4.1 Definition

A system of the form

$$Ax = \mathbb{O}$$

is called a homogeneous system.

Remark

Any homogeneous system is consistent and has a trivial solution $x = \mathbb{O}$

4.2 Solutions of Homogeneous Systems

If r = number of non-zero rows, let $t = n - r$ = number of free variables. If $t > 0$, denote the numbers of the columns that <u>do not</u> contain leading entries by $z(1), \ldots, z(t)$

4.2.1 Example

$$A = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore,

$$\begin{aligned} m &= 4 \\ n &= 6 \\ r &= 3 \\ t &= 3 \\ l(1) &= 2 \\ l(2) &= 4 \\ l(3) &= 5 \\ z(1) &= 1 \\ z(2) &= 3 \\ z(3) &= 6 \end{aligned}$$

Therefore,

$$\begin{aligned} x_2 + 2x_3 - 3x_6 &= 0 \\ x_4 - x_6 &= 0 \\ x_5 + 7x_6 &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} &x_2 = -2x_3 + 3x_6 \\ &x_4 = x_6 \\ &x_5 = -7x_6 \end{aligned}$$

$$\begin{pmatrix} x_2 \\ x_4 \\ x_5 \end{pmatrix} = C_{3\times 3} \begin{pmatrix} x_1 \\ x_3 \\ x_6 \end{pmatrix}$$

where $C_{3\times 3} = \begin{pmatrix} 0 & -2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -7 \end{pmatrix}$

The free variables x_1, x_3, x_6 can be considered as parameters, $x_1 = \gamma_1, x_2 =$

$\gamma_2, x_3 = \gamma_3$.
Therefore,

$$x_2 = -2\gamma_3 + 3\gamma_6$$
$$x_4 = \gamma_6$$
$$x_5 = -7\gamma_6$$

4.2.2 General Solution

4.2.2.1 Case I: $t = 0$

If $t = 0$, there are no free variables, and the system has a unique trivial solution.

4.2.2.2 Case II: $t > 0$

$$\begin{pmatrix} x_{l(1)} \\ x_{l(2)} \\ \vdots \\ x_{l(r)} \end{pmatrix} = C_{r \times t} \begin{pmatrix} x_{z(1)} \\ x_{z(2)} \\ \vdots \\ x_{z(t)} \end{pmatrix}$$

C is filled by coefficients of the equations obtained after shifting the terms containing all z_i to the RHS.

4.3 Properties

4.3.1 For a homogeneous system $Ax = 0$, if c and d are solutions, then $c + d$ is also a solution.

$$\begin{aligned} Ac &= \mathbb{O} \\ Ad &= \mathbb{O} \\ \therefore A(c+d) &= Ac + Ad \\ &= \mathbb{O} + \mathbb{O} \\ &= \mathbb{O} \end{aligned}$$

4.3.2 For a homogeneous system $Ax = 0$, if c is a solution and $\alpha \in \mathbb{F}$, then, αc is a solution too.

$$\begin{aligned} Ac &= \mathbb{O} \\ \therefore A(\alpha c) &= \alpha(Ac) \\ &= \alpha\mathbb{O} \\ &= \mathbb{O} \end{aligned}$$

4.4 Fundamental Solutions

We define t fundamental solutions or basic solutions, $v_1, \ldots v_t$.
We define t columns, each of length n as follows.
For the i^{th} column v_i, we set

$$\begin{aligned} x_{z(1)} &= 0 \\ x_{z(i)} &= 1 \\ &\vdots \\ x_{z(t)} &= 0 \end{aligned}$$

and for $x_{l(1)}, \ldots, x_{l(r)}$,

$$\begin{pmatrix} x_{l(1)} \\ \vdots \\ x_{l(r)} \end{pmatrix} = C \begin{pmatrix} x_{z(1)} \\ \vdots \\ x_{z(t)} \end{pmatrix} = i^{\text{th}}\text{column of } C$$

4.4.1 Theorem: Any solution d of the system $Ax = \mathbb{O}$ can be obtained from the basic solutions $v_1, \ldots, v_t$ as a linear combination of the basic solutions, $d = \alpha_1 v_1 + \ldots \alpha_t v_t$

One can choose another collection $v'_1, \ldots, v'_t$ s.t. any solution of $Ax = \mathbb{O}$ can be obtained as a linear combination of $v'_1, \ldots, v'_t$. In such a case, we get another form of the general solution.

4.5

$$r \leq \min m, n$$

If $r = n$, i.e. $t = 0$, the system has a unique solution.
If $r < n$, i.e. $t > 0$, the system has more than one solutions. Its general solution can be expressed as in terms of t parameters, where each free variable serves as a parameter, whose value can be any element of $\mathbb{F}$.
If $m < n$, then $r < n$. Therefore, the system has more than one solution.

5 Non-Homogeneous Systems

5.1 Definition

Consider a system $Ax = b; b \neq \mathbb{O}$. The extended matrix is defined as

$$\widetilde{A} = (A|b) = \begin{pmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{pmatrix}$$

5.2 Solutions of Non-Homogeneous Systems

Let $\widetilde{r}$ be the number of non-zero rows in the REF of $\widetilde{A}$, i.e. $\widetilde{A_R}$.

5.2.1 Case I: $\widetilde{r} = r$

$$b'_{r+1} = \dots = b'_m = 0$$

5.2.1.1 Case a: $r = n$, i.e. $t = 0$

Therefore,

$$\begin{aligned} x_1 &= b'_1 \\ \dots \\ x_r &= b'_r \end{aligned}$$

Hence, the system has a unique solution.

5.2.1.2 Case b: $r < n$, i.e. $t > 0$

Therefore,

$$\begin{aligned} x_l(1) &= b'_1 + c_{11}x_{z(1)} + \dots + c_{1t}x_{z(t)} \\ &\vdots \\ x_l(r) &= b'_1 + c_{r1}x_{z(1)} + \dots + c_{rt}x_{z(t)} \end{aligned}$$

5.2.2 Case II: $\widetilde{r} > r$

In this case, the $(r+1)^{\text{th}}$ row represents an equation of the form $0 = 1$. Therefore, the system is inconsistent.

5.3 General Solution

The general solution of $Ax = b$ can be expressed by adding the general solution of $Ax = b$ and any particular solution of $Ax = b$.

If c is a solution of $Ax = \mathbb{O}$, and d is a solution of $Ax = b$, then $c + d$ is a solution of $Ax = b$.
Conversely, if d and d' are solutions of $Ax = b$, then, $c = d' - d$ is a solution of $Ax = \mathbb{O}$.

Part V
Vector Spaces

1 Definition

Let $\mathbb{F}$ be a field. A vector space V, over $\mathbb{F}$, is a set on which there are two operations, denoted by $+$ and $\cdot$, where

$+$ is the addition of elements of V

$\cdot$ is the multiplication of an element of V by an element of $\mathbb{F}$

s.t. the sum of elements of V lies in V, and the product of an element of V by an element of $\mathbb{F}$ lies in V, and the following properties hold.

(A1) $x + y = y + x; \forall x, y \in V$

(A2) $(x + y) + x = x + (y + z); \forall x, y, z \in V$

(A3) $\exists \mathbb{O} \in V$, s.t. $\mathbb{O} + x = x + \mathbb{O} = x; \forall x \in V$

(A4) $\forall x \in V, \exists y \in V$, s.t. $x + y = \mathbb{O}$. (y is denoted as $-x$.)

(M1) $\alpha(x + y) = \alpha x + \alpha y; \forall \alpha \in \mathbb{F}, \forall x, y \in V$

(M2) $(\alpha + \beta)x = \alpha x + \beta y; \forall \alpha, \beta \in \mathbb{F}, \forall x \in V$

(M3) $(\alpha\beta)x = \alpha(\beta x) = \beta(\alpha x); \forall \alpha, \beta \in \mathbb{F}, \forall x \in V$

(M4) $1 \cdot x = x; \forall x \in V$

Elements of V are called vectors, and elements of $\mathbb{F}$ are called scalars.

1.1 Examples

1.1.1 Geometric Vectors in Plane

1.1.2 Arithmetic Vector Space

Let $\mathbb{F}$ be a field, and $n \geq 1 \in \mathbb{Z}$.
Let $V = \mathbb{F}^n$ be a set of ordered n-tuples.
We define

$$(\alpha_1, \ldots, \alpha_n) + (\beta_1, \ldots, \beta_n) = (\alpha_1 + \beta_2, \ldots, \alpha_n + \beta_n)$$
$$\alpha(\alpha_1, \ldots, \alpha_n) = (\alpha\alpha_1, \ldots, \alpha\alpha_n)$$

1.1.3

Let $\mathbb{F}$ be a field, and $m, n \geq 1 \in \mathbb{Z}$.
Let $V = \mathbb{F}^{mn}$ be the set of all $(m \times n)$ matrices over $\mathbb{F}$, i.e. a set of ordered mn-tuples. For $X, Y \in V$, we use the usual definitions of $X + Y$ and αX from algebra of matrices.

2 Properties

1. $\alpha\mathbb{O} = \mathbb{O}; \forall \alpha \in F$
2. $\alpha(-x) = -(\alpha x)$
3. $x - y \doteq x + (-y)$
4. $0x = \mathbb{O}; \forall x \in V$
5. $(-1)x = -x; \forall x \in V$
6. $(\alpha - \beta) = \alpha x - \beta x; \forall \alpha, \beta \in F, \forall x \in V$

2.0.4 Proof of 1

$$\begin{aligned} \alpha\mathbb{O} &= \alpha(\mathbb{O} + \mathbb{O} \\ &= \alpha\mathbb{O} + \alpha\mathbb{O} \end{aligned}$$

For $\alpha\mathbb{O} \exists y$ s.t. $\alpha\mathbb{O} + y = \mathbb{O}$.
Therefore,

$$\begin{aligned} \alpha\mathbb{O} + y &= (\alpha\mathbb{O} + \alpha\mathbb{O}) + y \\ \therefore \mathbb{O} &= \alpha\mathbb{O} + (\mathbb{O} + y) \\ &= \alpha\mathbb{O} + \mathbb{O} \\ &= \alpha\mathbb{O} \end{aligned}$$

3 Subspaces

Let V be a vector space over $\mathbb{F}$. Let $U \subseteq V$. U is called a subspace of V if the following properties hold.

Axiom 1 $\mathbb{O} \in U$

Axiom 2 If $x, y \in U$, then, $(x + y) \in U$

Axiom 3 If $x \in U, \alpha \in \mathbb{F}$, then, $\alpha x \in U$

3.1 Examples

Example 4. Let V be the set of all geometric vectors in plane.
If U_1 is the set of all vectors along the x-axis, U_2 is the singleton set of a specific vector along the x-axis, and U_3 is the set of all vectors along the x-axis and a specific vector not along the x-axis. Which of U_1, U_2, U_3 are subspaces of V?

Solution. U_1 is a subspace of V as it satisfies all three axioms.
U_2 is not a subspace of V as it does not satisfy any of the three axioms.
U_3 is not a subspace of V as it does not satisfy Axiom 3

Example 5.

$$\begin{aligned} \mathbb{F} &= \mathbb{R} \\ V &= \mathbb{C} = \{\alpha + \beta i; \alpha, \beta \in \mathbb{R}\} \end{aligned}$$

where $+$ is addition in $\mathbb{C}$ and $\cdot$ is multiplication by real scalars.

$$\begin{aligned} U_1 &= \{\alpha + 0i\} \\ U_2 &= \{0 + \beta i\} \end{aligned}$$

Which of U_1, U_2, U_3 are subspaces of V?

Solution. Both U_1 and U_2 are subspaces of V, as they satisfy all three axioms.

Example 6. Let $V = \mathbb{F}$, where $+$ is addition in $\mathbb{F}$, and $\cdot$ is multiplication in $\mathbb{F}$.

$$\begin{aligned} U_1 &= \{\alpha + 0i\} \\ U_2 &= \{0 + \beta i\} \end{aligned}$$

Which of U_1, U_2 are subspaces of V?

Solution. Neither U_1 nor U_2 are subspaces of V.

Example 7. Let $V = \{f : [0, 1] \to \mathbb{R}\}$, where $+$ and $\cdot$ is defined as follows.

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) \\ (\alpha f)(x) &= \alpha f(x) \end{aligned}$$

$\mathbb{O}$ is the function with graph $x = 0$.

$$U = \{\text{all continuous functions}[0,1] \to \mathbb{R}\}$$

Is U is subspace?

Solution. $\mathbb{O} \in \mathbb{R}$. Therefore, Axiom 1 is satisfied. Similarly, Axiom 2 and Axiom 3 are also satisfied.

3.2 Operations on Subspaces

Let V/F be a vector space, and U_1, U_2 be subspaces of V.

$$\begin{aligned}
U_1 \cap U_2 &= \{x \in V : x \in U_1 \text{ and } x \in U_2\} \\
U_1 \cup U_2 &= \{x \in V : x \in U_1 \text{ or } x \in U_2\} \\
U_1 + U_2 &= \{x \in V : x = x_1 + x_2, x_1 \in U_1, x_2 \in U_2\}
\end{aligned}$$

Example 8. Let V be a set of geometric vectors in 3D space.
Let U_1 be the xy-plane, and U_2 be the yz-plane. If $U_1 \cap U_2$ a subspace of V?

Solution.

$$\begin{aligned}
\mathbb{O} \in U_1, \mathbb{O} \in U_2 &\Rightarrow \mathbb{O} \in U_1 \cap U_2 \\
x, y \in U_1 \cap U_2 &\Rightarrow x, y \in U_1, x, y \in U_2 \\
&\Rightarrow x + y \in U_1, x + y \in U_2 \\
&= x + y \in U_1 \cap U_2
\end{aligned}$$

Similarly, if $x \in U_1 \cap U_2, \alpha in \mathbb{F}$, then, $\alpha x \in U_1 \cap U_2$. Therefore, $U_1 \cap U_2$ is a subspace of V.

4 Spans

Definition 11 (Span)**.** Let $V/\mathbb{F}$ be a vector space. Let $S \subset V$ be non-empty.

$$\operatorname{span}(S) = \{x \in V : x = \alpha_1 v_1 + \cdots + \alpha_m v_m, \alpha_1, \ldots, \alpha_m \in \mathbb{F}, v_1, \ldots, v_m \in S\}$$

$\operatorname{span}(S)$ is the collection of all linear combinations of finite number of vectors of S with coefficients from $\mathbb{F}$

Theorem 1. $\operatorname{span}(S)$ *is a subspace of* V

Proof.

$$\mathbb{O} = 0v \Rightarrow \mathbb{O} \in \text{span}(S)$$

$$\begin{aligned} x, y \in \text{span}(S) &\Rightarrow x = \alpha_1 v_1 + \cdots + \alpha_m v_m, \beta_1 w_1 + \cdots + \beta_m w_m \\ &\Rightarrow x + y = \alpha_1 v_1 + \cdots + \alpha_m v_m + \beta_1 w_1 + \cdots + \beta_m w_m \in \text{span}(S) \end{aligned}$$

$$\begin{aligned} x \in \text{span}(S), \alpha \in \mathbb{F} &\Rightarrow \alpha_1 v_1 + \cdots + \alpha_m v_m \\ &\Rightarrow \alpha x = \alpha(\alpha_1 v_1 + \cdots + \alpha_m v_m) \\ &\Rightarrow \alpha x = \alpha\alpha_1 v_1 + \cdots + \alpha\alpha_m v_m \in \text{span}(S) \end{aligned}$$

□

Definition 12 (Spanning sets and dimensionality)**.** Let $V/\mathbb{F}$ be a vector space. A set $S \subseteq V$ is said to be a spanning set, if $\text{span}(S) = V$.
If V has atleast one finite spanning set, V is said to be finite-dimensional. Otherwise, V is said to be infinite-dimensional.

Remark 4. V may have many finite spanning sets, of different sizes

Definition 13 (Basis of a vector space)**.** Let $V/\mathbb{F}$ be a vector space. We say that $B = \{v_1, \ldots, v_n\} \subset V$ is a basis of V if every vector $v \in V$ can be expressed in a unique way

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n \quad ; \alpha_1, \ldots, \alpha_n \in \mathbb{F}$$

that is, as a linear combination of elements of B.

Definition 14 (Isomorphic spaces)**.** Let $V/\mathbb{F}$ and $W/\mathbb{F}$ be vector spaces. We say that V is isomorphic to W if there is a map $\varphi : V \to W$, s.t.

1. φ is one-to-one and onto
2. $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2); \forall v_1, v_2 \in V$
3. $\varphi(\alpha v) = \alpha\varphi(v); \forall v \in V, \forall \alpha \in \mathbb{F}$

Theorem 2. *If a vector space $V/\mathbb{F}$ has a basis $B = \{v_1, \ldots, v_n\}$ consisting of n elements, then it is isomorphic to the space*

$$W = \mathbb{F}^n = \left\{ \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \right\}$$

Proof. Let $B' = \{e_1, \ldots, e_n\}$, where

$$e_1 = \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$$

B' is a basis of Q, as any $w = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in W$ can be expressed in a unique way

$$w = \alpha_1 e_1 + \cdots + \alpha_n e_n$$

Let $\varphi : V \to W$,

$$\begin{aligned} \varphi(v_1) &= e_1 \\ &\vdots \\ \varphi(v_n) &= e_n \end{aligned}$$

For any $v = \alpha_1 v_1 + \cdots + \alpha_n v_n \in V$,

$$\varphi(v) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Therefore,

$$\begin{aligned} \varphi(\alpha_1 v_1 + \cdots + \alpha_n v_n) &= \alpha_1 e_1 + \alpha_n e_n \\ &= \alpha_1 \varphi(v_1) + \cdots + \alpha_n \varphi(v_n) \end{aligned}$$

If $v \neq v'$,

$$\begin{aligned} v &= \alpha_1 v_1 + \cdots + \alpha_n v_n \\ v' &= \alpha_1' v_1 + \cdots + \alpha_n' v_n \end{aligned}$$

Hence φ is one-to-one.

For any $w = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in W$.

Let $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$.
Therefore,

$$\varphi(v) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = w$$

Therefore, φ is onto. □

5 Linear Dependence

Definition 15 (Linearly dependent subsets)**.** Let $V/\mathbb{F}$ be a vector space. Let $S \subseteq V$ be a finite subset. S is said to be linearly dependent if there exist scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$, not all equal to zero, s.t.

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = \mathbb{O}$$

Otherwise, S is said to be linearly independent if all $\alpha_1 = \cdots = \alpha_n = 0$.

Example 9. Is $S = \{v_1, \dots, v_l, v, \alpha v\}$ linearly dependent?

Solution.

$$(0)v_1 + \cdots + (0)v_l + (-\alpha)v + (1)\alpha v = \mathbb{O}$$

Therefore, as not all coefficients are zero, S is linearly dependent.

Example 10. Is $S = \{v_1, \dots, v_l, \mathbb{O}\}$ linearly dependent?

Solution.

$$(0)v_1 + \cdots + (0)v_l + (1)\mathbb{O} = \mathbb{O}$$

Therefore, as not all coefficients are zero, S is linearly dependent.

Theorem 3. *Any basis $B = \{v_1, \dots, v_n\}$ of a vector space V is linearly independent.*

Proof. Let

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = \mathbb{O}$$

Also,

$$(0)v_1 + \cdots + (0)v_n = \mathbb{O} \tag{3.1}$$

Therefore, there are two representations of $v = \mathbb{O}$ as linear combinations of elements of B. By the definition of basis, they must coincide.
Therefore,

$$\begin{aligned} \alpha_1 &= 0 \\ &\vdots \\ \alpha_n &= 0 \end{aligned}$$

Hence, B is linearly independent. $\square$

5.1 Properties of Linearly Dependent and Independent Sets

Theorem 4. *If $S \subseteq S'$ and S is linearly dependent, then S' is also linearly dependent.*

Theorem 5. *If $S \subseteq S'$ and S' is linearly independent, then S is also linearly independent.*

Theorem 6. *Let $S = \{v_1, \ldots, v_n\}$. S is linearly dependent iff one of the v_is is a linear combination of the others.*

Proof of statement. Suppose

$$\begin{aligned} & v_n = \alpha_1 v_1 + \cdots + \alpha_{n-1} v_{n-1} \\ & \therefore \alpha_1 v_1 + \cdots + \alpha_{n-1} v_{n-1} + (-1) v_n = \mathbb{O} \end{aligned}$$

Therefore, S is linearly dependent. $\square$

Proof of converse. Suppose

$$\alpha_1 v_1 + \cdots + \alpha_{n-1} v_{n-1} + \alpha_n v_n = \mathbb{O}$$

not all of α_is are 0. WLG, let $\alpha_n \neq 0$

$$\therefore v_n = -\frac{\alpha_1}{\alpha_m} v_1 - \cdots - \frac{\alpha_{n-1}}{\alpha_m} v_{m-1}$$

$\square$

Theorem 7. *Let $S = \{v_1, \ldots, v_m\}$. Let $w \in V$. Suppose w is a linear combination of v_is*

$$w = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

Then, such an expression is unique iff S is linearly dependent.

Proof of statement. Let

$$w = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

be unique.
If possible, let

$$\beta_1 v_1 + \cdots + \beta_n v_n = \mathbb{O}$$

not all β_is are zero.
Then,

$$(\alpha_1 + \beta_1)v_1 + \cdots + (\alpha_n \beta_n)v_n = w$$

This is another expression for w, and contradicts the assumption. $\square$

Proof of converse. If possible, let S be linearly independent. Assume

$$w = \alpha_1' v_1 + \cdots + \alpha_n' v_n$$

Therefore,

$$(\alpha_1 - \alpha_1')v_1 + \cdots + (\alpha_n - \alpha_n')v_n = \mathbb{O}$$

Therefore, S is linearly dependent, which contradicts the assumption. $\square$

Theorem 8 (Main Lemma on Linear Independence)**.** *Suppose V is spanned by n vectors.*
Let $S = \{v_1, \ldots, v_m\} \subset V$. Suppose $m > n$.
Then, S is linearly dependent.

Proof. Let $E = \{w_1, \ldots, w_n\}$ be a spanning set for V, $V = \operatorname{span}(E)$.
Therefore, all elements of S can be represented as linear combinations of elements of E.

$$\begin{aligned} v_1 &= \beta_{11} w_1 + \cdots + \beta_{1n} w_n \\ &\vdots \\ v_m &= \beta_{m1} w_1 + \cdots + \beta m n w_n \end{aligned}$$

Let

$$\begin{aligned} \alpha_1 v_1 + \cdots + \alpha_m v_m &= \mathbb{O} \\ \therefore \alpha_1(\beta_{11} w_1 + \cdots + \beta_{1n} w_n) + \cdots + \alpha_m(\beta_{m1} w_1 + \cdots + \beta_{mn} w_n) &= \mathbb{O} \\ \therefore (\alpha_1 \beta_{11} + \cdots + \alpha_m \beta_{m1}) w_1 + \cdots + (\alpha_1 \beta_{1n} + \cdots + \alpha_m \beta_{mn}) &= \mathbb{O} \end{aligned}$$

Therefore

$$\begin{aligned}\alpha_1\beta_{11} + \cdots + \alpha_m\beta_{m1} &= 0\\ &\vdots\\ \alpha_1\beta_{1n} + \cdots + \alpha_m\beta_{mn} &= 0\end{aligned}$$

These equations form a homogeneous linear system with respect to $\alpha_1, \ldots, \alpha_m$. As $m > n$, the system has a non-zero solution. Therefore not all α_is are zero. Hence S is linearly dependent. □

Definition 16 (Alternative definition of a basis)**.** $B = \{v_1, \ldots, v_n\}$ is said to be a basis of V if B is a spanning set and B is linearly independent.

Theorem 9. *If B and B' are bases of V, then they contain the same number of elements.*

Proof. If possible, let B contain n elements $\{v_1, \ldots, v_n\}$, and B' contain m elements $\{w_1, \ldots, w_m\}$, $m > n$.
Therefore, B is a spanning set and B' contains more elements than n, hence by Main Lemma on Linear Independence, B' is linearly dependent. Also, B' is a basis, so it is linearly independent.
This is a contradiction. □

Definition 17 (Dimension of a vector space)**.** Let $V/\mathbb{F}$ be a finite-dimensional vector space. The number of elements in any basis B of V is called the dimension of V.

$$n = \dim V$$

Remark 5. If V and W are vector spaces over $\mathbb{F}$, s.t.

$$\dim V = \dim W$$

then, V is isomorphic to W

Theorem 10. *If $S = \{v_1, \ldots, v_m\}$ is a spanning set of V, and if S is not a basis of V, a basis B of V can be obtained by removing some elements from S.*

Proof. If S is linearly independent, then it is a basis.
Otherwise, if S is linearly dependent, it has an element, WLG, say v_m, which is a linear combination of the others.

$$v_m = \alpha_1 v_1 + \cdots + \alpha_{m-1} v_{m-1}$$

Let

$$S' = S - \{v_m\}$$

S' is a spanning set.
Therefore, $\forall v \in V$

$$\begin{aligned} v &= \beta_1 v_1 + \cdots + \beta_{m-1} v_{m-1} + \beta_m v_m \\ &= \beta_1 v_1 + \cdots + \beta_{m-1} + \beta_m(\alpha_1 v_1 + \cdots + \alpha_{m-1} v_{m-1}) \\ &= \gamma_1 v_1 + \cdots + \gamma_{m-1} v_{m-1} \end{aligned}$$

If S' is linearly independent, then it is a basis, else the same process above can be repeated till we get a basis.
Therefore, a basis is a smallest spanning set. □

Theorem 11. *If $B_0 = \{v_1, \ldots, v_n\}$ is a linearly independent set, and if B_0 is a basis of V, a basis of V can be obtained by adding elements to B_0.*

Theorem 12. *Let V be a vector space, s.t. $\dim V = n$.*
If B satisfies 2 out of the 3 following conditions, then it is a basis.

1. *B has n elements.*
2. *B is a spanning set.*
3. *B is linearly dependent.*

Theorem 13 (Dimension Theorem)**.**

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$$

Theorem 14.

$$U + W = \operatorname{span}(U \cup W)$$

If

$$\begin{aligned} U &= \operatorname{span}(B) \\ W &= \operatorname{span}(B') \end{aligned}$$

then,

$$U + W = \operatorname{span}(B \cup B')$$

Proof. Let $v \in U + W$.
Then,

$$\begin{aligned} v &= u + w \quad ; u \in U, w \in W \\ u &\in U \cup W \\ w &\in U \cup W \\ \therefore v &\in \text{span}(U \cup W) \end{aligned}$$

Let

$$v \in \text{span}(U \cup W) \therefore v = \alpha_1 v_1 + \cdots + \alpha_k v_k \quad ; v_i \in U \cup W$$

Let

$$\begin{aligned} v_1, \ldots, v_l &\in U \\ v_{l+1}, \ldots, v_k &\in W \end{aligned}$$

Therefore,

$$\begin{aligned} v &= (\alpha_1 v_1 + \cdots + \alpha_l v_l) + (\alpha_{l+1} v_{l+1} + \cdots + \alpha k v_k) \\ \therefore v &\in U + W \end{aligned}$$

□

5.2 Changing a Basis

Let $B = \{v_1, \ldots, v_n\}$ be a basis of V, s.t. $\dim V = n$. Let $B' = \{v_1', \ldots, v_n'\}$. As B is a spanning set, all of $v_1', \ldots, v_n'$ can be expressed as a linear combination of $v_1, \ldots, v_n$.

$$\begin{aligned} v_1' &= \gamma_{11} v_1 + \cdots + \gamma_{n1} v_n \\ &\vdots \\ v_n' &= \gamma_{1n} v_1 + \cdots + \gamma_{nn} v_n \end{aligned}$$

Definition 18 (Transition matrix)**.** The matrix

$$C = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{pmatrix}$$

is called the transition matrix from B to B'.

If B and B' are considered as row vectors of length n filled by vectors,

$$\begin{aligned} v'_1 &= \gamma_{11}v_1 + \cdots + \gamma_{n1}v_n \\ &\vdots \\ v'_n &= \gamma_{1n}v_1 + \cdots + \gamma_{nn}v_n \end{aligned}$$

can be written as

$$B'_{1\times n} = B_{1\times n}C_{n\times n}$$

Theorem 15. *B' is a basis of V iff C is invertible.*

Proof of statement. Let $B' = BC$ be a basis.
B' is a basis, and hence is a spanning set. Therefore, any vector from B can be expressed as a linear combination of elements of B'.
Therefore,

$$\begin{aligned} B &= B'Q \\ &= BCQ \end{aligned}$$

Also,

$$B = BI$$

Therefore,

$$I = CQ$$

Similarly,

$$\begin{aligned} B' &= BC \\ &= B'QC \end{aligned}$$

Also,

$$B' = B'I$$

Therefore,

$$I = QC$$

Therefore,

$$CQ = QC = I$$

Hence C is invertible. $\square$

Proof of converse. Let $B' = BC$ and C be invertible. Therefore, B' is a basis iff B' is a spanning set.
Let $z \in V$. As B is a spanning set,

$$z = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

Therefore,

$$z = Bg$$

where

$$g = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$$\begin{aligned} \therefore z &= Bg \\ &= B(Ig) \\ &= B(CC^{-1})g \\ &= (BC)(C^{-1}g) \end{aligned}$$

Let $C^{-1}g = f$

$$\therefore z = B'f$$

Therefore, z can be expressed as a linear combination of vectors from B'. □

Remark 6. Let B be a basis of V. If

$$BP = BQ$$

where P and Q are $n \times n$ matrices, then

$$P = Q$$

Example 11. Let $B = \{e_1, e_2\}$ and $B' = \{e_1', e_2'\}$, where

$$\begin{aligned} e_1' &= e_1 + e_2 \\ e_2' &= -e_1 + e_2 \end{aligned}$$

Solution.

$$\begin{aligned} e_1' &= e_1 + e_2 \\ e_2' &= -e_1 + e_2 \\ \therefore C &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

$$e_1 = \frac{1}{2}e_1' - \frac{1}{2}e_2'$$
$$e_2 = \frac{1}{2}e_1' + \frac{1}{2}e_2'$$
$$\therefore C^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

5.3 Representation of Vectors in a Basis

Let V be a vector space of dimension n. Let $B = \{v_1, \dots, v_n\}$ be a basis of V.
Let $z \in V$.
z can be written as a unique linear combination of elements of B.

$$z = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

The representation of z w.r.t B can be represented as

$$[z]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

5.3.1 Properties of Representations

1. $[z_1 + z_2]_B = [z_1]_B + [z_2]_B$
2. $[\alpha z]_B = \alpha [z]_B$
3. $[z_1]_B = [z_2]_B \iff z_1 = z_2$
4. $\forall \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{F}^n, \exists z \in V, \text{ s.t. } [z]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$

6 Determinants

6.1 Definition

Definition 19 (Determinants)**.** Given an $n \times n$ matrix A, $n \geq 1$, $\det(A)$ is defined as follows.

$$\begin{aligned}
n &= 1 & \det(a) &= a \\
n &= 2 & \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= a_{11}a_{22} - a_{12}a_{21} \\
&\vdots \\
n &= n
\end{aligned}$$

The determinant of a $n \times n$ matrix is the summation of $n!$ summands. Each summand is the product of n elements, each from a different row and column.

Summand	Permutation	Number of Elementary Permutations[1]	Parity
$a_{11}a_{22}a_{33}$	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	0	even
$a_{12}a_{23}a_{31}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	$2\ ((1,2,3) \to (2,1,3) \to (2,3,1))$	even
$a_{13}a_{21}a_{32}$	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	$2\ ((1,2,3) \to (1,3,2) \to (3,1,2))$	even
$a_{13}a_{22}a_{31}$	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$	$1\ ((1,2,3) \to (3,2,1)$	odd
$a_{12}a_{21}a_{33}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$	$1\ ((1,2,3) \to (2,1,3)$	odd
$a_{11}a_{23}a_{32}$	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$	$1\ ((1,2,3) \to (1,3,2)$	odd

6.2 Properties

Theorem 16. *If A, A' are matrices s.t. all rows except the i^{th} row are identical, and A'' is obtained by addition of i^{th} row of A and i^{th} row of A',*

[1] Any permutation can be represented as a result of a series of elementary permutations, i.e. permutations of 2 elements only. The parity of a particular permutation depends of the parity of the number of elementary functions required for it.

then

$$\det(A'') = \det(A) + \det(A')$$

Theorem 17. *If A' is obtained from A by switching two rows, then*

$$\det(A') = -\det(A)$$

Theorem 18. *If A' is obtained from A by multiplication of a row by a scalar α, then*

$$\det(A') = \alpha \det(A)$$

Theorem 19. *If A' is obtained from A by adding to the i^{th} row the j^{th} row multiplied by a scalar α, then*

$$\det(A') = \det(A)$$

Corollary 19.1 (Corollary of Property 2)**.** *If A has two identical rows, then $\det(A) = 0$.*

Theorem 20. *The determinant of upper triangular and lower triangular matrices is the product of the elements on the principal diagonal.*

Theorem 21.

$$\det(A^t) = \det(A)$$

Corollary 21.1. *In all above theorems, the properties which are applicable to rows, are also applicable to columns.*

Theorem 22. *If A, B, C are some matrices, and $\mathbb{O}$ is the zero matrix,*

$$\begin{pmatrix} A_{m\times m} & B \\ \mathbb{O} & C_{n\times n} \end{pmatrix} = \det(A) \cdot \det(C)$$

Theorem 23.

$$\det(AB) = \det(A)\det(B)$$

Corollary 23.1. *If A is invertible, then*

$$\det(A) \neq 0$$

Proof. A is invertible.
Therefore, $\exists P$, s.t.

$$\begin{aligned} PA &= I \\ \therefore \det(PA) &= \det(I) \\ \therefore \det(P)\det(A) &= 1 \\ \therefore \det(A) &\neq 0 \end{aligned}$$

□

Theorem 24. *If*

$$\det(A) \neq 0$$

then A is invertible.

Proof. If possible let A be non invertible.
Let the REF of A be A_R.
As A is non invertible, A_R has a zero row. Therefore,

$$\det(A_R) = 0$$

But

$$\det(A) = 0$$

This is not possible as elementary row operations cannot change a non-zero determinant to zero.
Therefore, A is invertible. □

Theorem 25.

$$\det(A) \neq 0$$

iff the rows of A are linearly independent iff the columns of A are linearly independent.

Proof. If possible, let the rows of A be linearly dependent.
Therefore, either all of them are zeros, or one row is the linear combination of the others.
Case 1 (All rows are zeros).

$$\therefore \det(A) = 0$$

Case 2 (One row is a linear combination of the others). Let

$$
\begin{aligned}
v_n &= \alpha_1 v_1 + \cdots + \alpha_{n-1} v_{n-1} \\
\therefore A &= \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix}
\end{aligned}
$$

$v_n \to v_n - \alpha_1 v_1 + \cdots + \alpha_{n-1} v_{n-1}$

$$
\begin{aligned}
\therefore A' &= \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ \mathbb{O} \end{pmatrix} \\
\therefore \det(A') &= 0 \\
\therefore \det(A) &= 0
\end{aligned}
$$

This contradicts $\det(A) \neq 0$. Therefore, the rows of A must be linearly independent.

If $v_1, \ldots, v_n$ are linearly independent,

$$
\begin{aligned}
\dim \mathrm{R}(A) &= n \\
\therefore r &= n
\end{aligned}
$$

Therefore, there are no zero rows in REF of A. Hence A is invertible.

$$\therefore \det(A) \neq 0$$

□

6.3 Practical Methods for Computing Determinants

6.4 Expansion along a row/ column

Let A be a $m \times n$ matrix, and let A_{ij} be the matrix obtained by removing the i^{th} row and j^{th} column from A.

$$\det(A) = \sum_{j=1}^{n}(-1)^{i+j} a_{ij} \det(A_{ij})$$

6.5 Determinant Rank

Definition 20 (Determinant rank)**.** Let A be any $m \times n$ matrix. Consider all square sub-matrices of A and compute their determinants. If there is an $r \times r$ sub-matrix of A s.t. its determinant is non-zero, but the determinants of all $(r+1) \times (r+1)$ sub-matrices of A are zero, then, r is called the determinant rank of A.

Theorem 26. *The determinant rank of A is equal to the rank of A.*

7 Linear Maps

7.1 Definition

Definition 21 (Linear map)**.** Let V and W be vector spaces over the same field $\mathbb{F}$.

$$\varphi : V \to W$$

is said to be a linear map if

1. $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2); \forall v_1, v_2 \in V$
2. $\varphi(\alpha v) = \alpha\varphi(v); \forall v \in V, \forall \alpha \in \mathbb{F}$

7.2 Properties

1. $\varphi(\mathbb{O}) = \mathbb{O}$
2. $\varphi(-v) = -\varphi(v)$

7.3 Matrix of a Linear Map

Definition 22 (Matrix of a linear map)**.** Let $\varphi : V \to W$ be a linear map. Let

$$n = \dim V$$
$$m = \dim W$$

Let

$$B = \{v_1, \dots, v_n\}$$
$$B' = \{w_1, \dots, w_m\}$$

be bases of V and W respectively.
Let

$$\varphi(v_1) = \alpha_{11}w_1 + \dots + \alpha_{m1}w_m$$
$$\vdots$$
$$\varphi(v_n) = \alpha_{1n}w_1 + \dots + \alpha_{mn}w_m$$

The matrix

$$A = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \dots & \alpha_{mn} \end{pmatrix}$$

is called the matrix of φ with respect to the bases B and B'.
It is denoted as

$$A = [\varphi]_{B,B'}$$

Theorem 27. *Let*

$$\varphi : V \to W$$

be a linear map.
Let B and B' be bases of V and W respectively, and let

$$A = [\varphi]_{B,B'}$$

be the matrix of φ with respect to B and B'. Then, $\forall x \in V$,

$$[\varphi(z)]_{B'} = A[z]_B$$

Proof. Let

$$B = \{v_1, \dots, v_n\}$$
$$B' = \{w_1, \dots, w_m\}$$

Case 3 ($z \in B$). WLG, let $z = v_i$. Then,

$$[z]_B = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

i.e. all rows except the i^{th} row are 0.
Let this vector be e_i.
Therefore,

$$A[z]_B = Ae_i$$

is the i^{th} column of A.

$$[\varphi(z)]_{B'} = [\varphi(v_i)]_{B'}$$

is the i^{th} row in the formulae of $\varphi(v_1), \dots, \varphi(v_n)$.
Therefore, it is the i^{th} column of A.

Case 4 ($z \in V$ is an arbitrary vector). Let

$$z = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

Therefore,

$$\begin{aligned}
[\varphi(z)]_{B'} &= [\varphi(\alpha_1 v_1 + \cdots + \alpha_n v_n)]_{B'} \\
&= [\alpha_1 \varphi(v_2) + \cdots + \alpha_n \varphi(v_n)]_{B'} \\
&= \alpha_1 [\varphi(v_1)]_{B'} + \cdots + \alpha_n [\varphi(v_n)]_{B'} \\
&= \alpha_1 \cdot (1^{\text{st}}\text{column of } A) + \cdots + \alpha_n c_n \cdot (n^{\text{th}}\text{column of } A) \\
&= A[z]_B
\end{aligned}$$

$\square$

7.4 Change of Bases

Theorem 28. *Let V, W be vector spaces over $\mathbb{F}$, $\dim(V) = n$, $\dim(W) = m$. Let $\varphi : V \to W$ be a linear map. Let B, $\widetilde{B}$ be bases of V and let B' and $\widetilde{B'}$ be bases of W. Let $A = [\varphi]_{B,B'}$ and $\widetilde{A} = [\varphi]_{\widetilde{B},\widetilde{B'}}$ be the matrices of φ w.r.t.*

the pairs B, B' and $\widetilde{B}$, $\widetilde{B'}$. Let P denote the transition matrix from B to $\widetilde{B}$, and let Q denote the transition matrix from B' to $\widetilde{B'}$. Then,

$$\widetilde{A}_{m\times n} = Q^{-1}_{m\times m} A_{m\times n} P_{n\times n}$$

Proof. $\forall z \in V$,

$$[\varphi(z)]_{B'} = A[z]_B \tag{28.1}$$
$$[\varphi(z)]_{\widetilde{B'}} = A[z]_{\widetilde{B}} \tag{28.2}$$

We have

$$[z]_B = P[z]_{\widetilde{B}} \tag{28.3}$$
$$[\varphi(z)]_{B'} = Q[\varphi(z)]_{\widetilde{B'}} \tag{28.4}$$

Therefore,

(28.1) in (28.4) $\implies$

$$A[z]_B = Q[\varphi(z)]_{\widetilde{B'}} \tag{28.5}$$

(28.3) in (28.5) $\implies$

$$AP[z]_{\widetilde{B}} = Q[\varphi(z)]_{\widetilde{B'}} \tag{28.6}$$

Multiplying on the left by Q^{-1},

$$Q^{-1}AP[z]_{\widetilde{B}} = [\varphi(z)]_{\widetilde{B'}}$$
$$\therefore [\varphi(z)]_{\widetilde{B'}} = Q^{-1}AP[z]_{\widetilde{B}}$$

Comparing with (28.2),

$$\widetilde{A} = Q^{-1}AP$$

□

7.5 Operations on Linear Maps

Definition 23 (Operations on linear maps)**.** Let

$$\varphi : V \to W$$
$$\varphi' : V \to W$$

be linear maps.

$$\varphi + \varphi' : V \to W$$

is defined as

$$(\varphi + \varphi')(v) = \varphi(v) + \varphi'(v)$$

and

$$\alpha\varphi : V \to W$$

is defined as

$$(\alpha\varphi)(v) = \alpha\varphi(v)$$

Definition 24 (Composed map)**.** Let

$$\begin{aligned}\varphi &: V \to W \\ \varphi' &: W \to U\end{aligned}$$

be linear maps.

$$(\varphi' \circ \varphi) : V \to U$$

is defined as

$$(\varphi' \circ \varphi)(v) = \varphi'(\varphi(v))$$

Theorem 29 (Matrix of composed map)**.** *Let $\varphi : V \to W$, $\varphi' : W \to U$ be linear maps. Let $(\varphi \circ \varphi') : V \to U$ be the composed map. Let* $\dim V = n$, $\dim W = m$, $\dim U = l$. *Let B, B', B'' be bases of V, W, U respectively. Let $A = [\varphi]_{B,B'}$, $A' = [\varphi']_{B',B''}$ be the matrices of φ, φ'. Let $A'' = [\varphi' \circ \varphi]_{B,B''}$ be the matrix of the composed map. Then,*

$$A'' = A'A$$

Proof. Let $z \in V$.

$$\begin{aligned}[(\varphi' \circ \varphi)(z)]_{B''} &= [\varphi'(\varphi(z))]_{B''} \\ &= A'[\varphi(z)]_{B'} \\ &= A'A[z]_B\end{aligned}$$

By definition,

$$[(\varphi' \circ \varphi)(z)]_{B''} = A''[z]_B$$

Therefore,

$$A'' = A'A$$

□

7.6 Kernel and Image

Definition 25 (Kernel and image)**.** Let $\varphi : V \to W$ be a linear map.

$$\begin{aligned}\ker\varphi &\doteq \{v \in V : \varphi(v) = \mathbb{O}\}\\ \operatorname{im}\varphi &\doteq \{\phi(v) : v \in V\}\end{aligned}$$

Theorem 30. $\ker\varphi$ *is a subspace of* V *and* $\operatorname{im}\varphi$ *is a subspace of* W*.*

Proof.

$$\begin{aligned}\varphi(\mathbb{O}) &= \mathbb{O}\\ \therefore \mathbb{O} &\in \ker\varphi\end{aligned}$$

If $v_1, v_2 \in \ker\varphi$, then

$$\begin{aligned}\varphi(v_1 + v_2) &= \varphi(v_1) + \varphi(v_2)\\ &= \mathbb{O} + \mathbb{O}\\ &= \mathbb{O}\end{aligned}$$

$$\therefore v_1 + v_2 \in \ker V$$

If $v \in \ker\varphi$, $\alpha \in \mathbb{F}$, then

$$\begin{aligned}\varphi(\alpha v) &= \alpha\varphi(v)\\ &= \alpha\mathbb{O}\\ &= \mathbb{O} \therefore \alpha v \in \ker\varphi\end{aligned}$$

Therefore, $\ker\varphi$ is a subspace of W.

$$\begin{aligned}\varphi(\mathbb{O}) &= \mathbb{O}\\ \therefore \mathbb{O} &\in \operatorname{im}\varphi\end{aligned}$$

If $w_1, w_2 \in \operatorname{im}\varphi$, then

$$\begin{aligned}w_1 &= \varphi(v_1)\\ w_2 &= \varphi(v_2)\\ \therefore w_1 + w_2 &= \varphi(v_1) + \varphi(v_2)\\ &= \varphi(v_1 + v_2)\\ \therefore w_1 + w_2 &\in \operatorname{im}\varphi\end{aligned}$$

If $w \in W$, $\alpha \in \mathbb{F}$, then

$$\begin{aligned}\alpha w &= \alpha\phi(v)\\ &= \varphi(\alpha v)\\ \therefore \alpha w &\in \operatorname{im}\varphi\end{aligned}$$

Therefore, $\operatorname{im}\varphi$ is a subspace of W. $\square$

7.6.1 Dimensions of Kernel and Image

Theorem 31. *Let $\varphi : V \to W$ be a linear map. Then*

$$\dim(\ker(\varphi)) + \dim(\operatorname{im}(\varphi))$$

Proof. Let $\ker \varphi = U$, $U \subseteq V$.
Let $B_0 = \{v_1, \dots, v_k\}$ be a basis of U.
Completing B_0 to a basis B of V,

$$B = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$$

Let

$$\begin{aligned} w_{k+1} &= \varphi(v_{k+1}) \\ &\vdots \\ w_n &= \varphi(v_n) \end{aligned}$$

Therefore, we need to prove that B' is a basis of $W' = \operatorname{im}(\varphi)$, by proving that B' is a spanning set and that B' is linearly independent.
Take $w \in \operatorname{im}(\varphi)$, so that there is $v \in V$ s.t. $\varphi(v) = w$.
Representing v as a linear combination of elements of B,

$$\begin{aligned} v &= \alpha_1 v_1 + \dots + \alpha_k v_k + \alpha_{k+1} v_{k+1} + \dots + \alpha_n v_n \\ \therefore w &= \varphi(v) \\ &= \varphi(\alpha_1 v_1 + \dots + \alpha_k v_k + \alpha_{k+1} v_{k+1} + \dots + \alpha_n v_n) \\ &= \alpha_1 \varphi(v_1) + \dots + \alpha_k \varphi(v_k) + \alpha_{k+1} \varphi(v_{k+1}) + \dots + \alpha_n \varphi(v_n) \\ &= \alpha_{k+1} \varphi(v_{k+1}) + \dots + \alpha_n \varphi(v_n) \\ &= \alpha_{k+1} w_{k+1} + \dots + \alpha_n w_n \\ &\in \operatorname{span}(B') \end{aligned}$$

Therefore, B' is a spanning set for W'.
Let

$$\beta_{k+1} w_{k+1} + \dots + \beta_n w_n = \mathbb{O}$$

Therefore, B' is linearly independent iff

$$\beta_{k+1} = \dots = \beta_n = 0$$

As φ is a linear map,

$$\begin{aligned} &\varphi(\beta_{k+1} v_{k+1} + \dots + \beta_n v_n) = \mathbb{O} \\ &\therefore \beta_{k+1} v_{k+1} + \dots + \beta_n v_n \in \ker \varphi \end{aligned}$$

Therefore, it can be expressed as a linear combination of vectors of B_0, which is a basis of $\ker\varphi$.
Let

$$\begin{aligned} \beta_{k+1}v_{k+1} + \cdots + \beta_n v_n &= \alpha_{k+1}v_{k+1} + \cdots + \alpha_n v_n \\ \therefore \alpha_{k+1}v_{k+1} + \cdots + \alpha_n v_n - \beta_{k+1}v_{k+1} - \cdots - \beta_n v_n &= \mathbb{O} \end{aligned}$$

As $\{v_1, \dots, v_n\}$ is a basis of V, all coefficients must be 0
Therefore,

$$\beta_{k+1}v_{k+1} = \cdots = \beta_n v_n = 0$$

Hence, as B' is a spanning set of $\operatorname{im}\varphi$ and also linearly independent, B' is a basis of $\operatorname{im}\varphi$.
Therefore,

$$\begin{aligned} \dim(\operatorname{im}\varphi) &= \text{ size of } B' \\ &= n - k \\ &= n - \dim(\ker\varphi) \\ \therefore \dim(\operatorname{im}\varphi) + \dim(\ker\varphi) &= \dim V \end{aligned}$$

□

Corollary 31.1.

$$\dim(\operatorname{im}\varphi) = r$$

where r is the rank of A

Corollary 31.2. *Let $A_{m\times n}$ be a matrix of rank r. Let $\mathrm{C}(A)$ be the column space of A, and let $\dim \mathrm{C}(A)$ be the column rank of A. Then*

$$\dim \mathrm{C}(A) = r$$

Proof. Define

$$\varphi : \mathbb{F}^n \to \mathbb{F}^m$$

s.t. $A = [\varphi]_{B,B'}$, where B is the standard basis of $\mathbb{F}$.

$$\begin{aligned} B &= \left\{ \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \right\} \\ &= \{e_1, \dots, e_n\} \end{aligned}$$

$\forall v \in \mathbb{F}^n$, we have

$$[\varphi(v)]_{B'} = A[v]_B$$

If $v = e_i$,

$$[\varphi(e_i)] = Ae_i$$

which is the i^{th} column of A. So, the space spanned by $\{\varphi(e_1), \ldots, \varphi(e_n)\}$ is equal to $\mathrm{C}(A)$. But it is also in $\operatorname{im}\varphi$.
Therefore,

$$\operatorname{im}\varphi = \mathrm{C}(A)$$

and

$$\begin{aligned} \dim(\operatorname{im}\varphi) &= \dim(\mathrm{C}(A)) \\ \therefore r &= \dim(\mathrm{C}(A)) \end{aligned}$$

□

Remark 7. Let $\varphi : V \to W$ be a linear map. Let $w \in \operatorname{im}(\varphi)$, so that there is $v \in V$ s.t. $\varphi(v) = w$. Then any v' s.t. $\varphi(v') = w$ can be written down as $v' = v + v_0$ where $v_0 \in \ker\varphi$.

Part VI
Linear Operators

1 Definition

Definition 26 (Linear operator)**.** A linear operator or transformation

$$T : V \to V$$

is a linear map from a vector space V to itself.

2 Similar Matrices

Let B and $\widetilde{B}$ be bases of V. Let A and $\widetilde{A}$ be the representing matrices

$$\begin{aligned} A &= [T]_B \\ \widetilde{A} &= [T]_{\widetilde{B}} \end{aligned}$$

Both these are $n \times n$ matrices, where $n = \dim V$. Let P denote the transition matrix from B to $\widetilde{B}$. Then,

$$\widetilde{A} = P^{-1}AP$$

Definition 27 (Similarity of matrices)**.** Let A, $\widetilde{A}$ be $n \times n$ matrices. A is said to be similar to $\widetilde{A}$, denoted as $A \sim \widetilde{A}$, if there exists an invertible $n \times n$ matrix P, s.t. $\widetilde{A} = P^{-1}AP$.

2.1 Properties of Similar Matrices

1. $A \sim A$
2. If $A \sim \widetilde{A}$, then $\widetilde{A} \sim A$
3. If $A \sim \widetilde{A}$ and $\widetilde{A} \sim \widetilde{\widetilde{A}}$, then $A \sim \widetilde{\widetilde{A}}$
4. If $A \sim \widetilde{A}$, then $\det(A) = \det(\widetilde{A})$
5. If $A \sim I$, then $A = I$

3 Diagonalization

Given a square matrix $A_{n \times n}$, decide whether or not A is similar to some diagonal matrix D. If it is, find D, and P s.t. $P^{-1}AP = D$.
Alternatively,
Given an operator $T : V \to V$, decide whether or not there exists a basis B of V, s.t. $[T]_B$ is a diagonal matrix D. If it exists, find D, and B, s.t. $[T]_B = D$.

Definition 28 (Diagonalizability)**.** If A is similar to a diagonal matrix, A is said to be diagonalizable. P, s.t. $P^{-1}AP = D$ is called a diagonalizing matrix for A. D is called a diagonal form of A.

4 Eigenvalues and Eigenvectors

Definition 29 (Eigenvalue and eigenvector)**.** Let A be a $n \times n$ matrix over $\mathbb{F}$. $\lambda \in \mathbb{F}$ is said to be an eigenvalue of A, if $\exists v \in \mathbb{F}, v \neq 0$, such that

$$Av = \lambda v$$

v is called an eigenvector corresponding to λ.

Definition 30 (Alternate definition of eigenvalue and eigenvector)**.** Let $T : V \to V$ be a linear operator, where V is a vector space over $\mathbb{F}$. $\lambda \in \mathbb{F}$ is said to be an eigenvalue of A, if $\exists v \in V, v \neq 0$, such that

$$T(v) = \lambda v$$

v is called an eigenvector corresponding to λ.

Definition 31 (Spectrum)**.** The collection of all eigenvalues of a matrix, or a linear operator is called the spectrum.

Theorem 1. *Let A be a $n \times n$ matrix. $\lambda \in \mathbb{F}$ is an eigenvalue of A iff*

$$\det(\lambda I_n - A) = 0$$

Proof. λ is an eigenvalue of A

$$\begin{aligned}
&\iff \exists v \in \mathbb{F}^n, v \neq 0, \text{ s.t. } Av = \lambda v \\
&\iff \exists v \in \mathbb{F}^n, v \neq 0, \text{ s.t. } (\lambda I - A)v = \mathbb{O} \\
&\iff v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\
&\iff (\lambda I - A) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0 \text{ has a non-zero solution} \\
&\iff \text{there are free variables} \\
&\iff \det(\lambda I - A) = 0
\end{aligned}$$

□

Theorem 2 (General criterion for diagonalization)**.** *Let A be a $n \times n$ matrix. A is diagonalizable if and only if there exists a basis $B = \{v_1, \ldots, v_n\}$ of $\mathbb{F}^n$ consisting of eigenvectors of A. In such a case, the diagonal entries of D are eigenvalues of A, and B can be chosen as consisting of the columns of P, where $P^{-1}AP = D$.*

Corollary 2.1. *If A has no eigenvalues, then it is not diagonalizable.*

Theorem 3. *Let $\lambda_1, \ldots, \lambda_s$ be pairwise distinct eigenvalues of an $n \times n$ matrix A, i.e. $\forall i \neq j, \lambda_i \neq \lambda_j$. Let $v_1, \ldots, v_s$ be eigenvalues of A corresponding to $\lambda_1, \ldots, \lambda_s$. Then the set $S = \{v_1, \ldots, v_s\}$ is linearly independent.*

Proof. If possible, let S be linearly dependent. Let S' denote a linearly dependent subset of S of smallest possible size, say l. WLG, let $S' = \{v_1, \dots, v_l\}$. Hence, $\exists \alpha_1, \dots, \alpha_l \in \mathbb{F}$, s.t.

$$\alpha_1 v_1 + \cdots + \alpha_l v_l = \mathbb{O} \tag{3.1}$$

Multiplying (3.1) on both sides by A,

$$\alpha_1 A v_1 + \cdots + \alpha_l A v_l = \mathbb{O} \tag{3.2}$$
$$\alpha_1 \lambda_1 v_1 + \cdots + \alpha_l \lambda_l v_l = \mathbb{O} \tag{3.3}$$

Multiplying (3.1) on both sides by λ_l

$$\alpha_1 \lambda_1 v_1 + \cdots + \alpha_l A v_l = \mathbb{O} \tag{3.4}$$

Subtracting (3.4) from (3.3)

$$\alpha_1(\lambda_1 - \lambda_l)v_1 + \cdots + \alpha_{l-1}(\lambda_{l-1} - \lambda_l)v_{l-1} = \mathbb{O} \tag{3.5}$$

Solving,

$$\alpha_1 = \alpha_l = 0$$

This is a contradiction. □

Corollary 3.1. *Let $A_{n\times n}$ have n distinct eigenvalues. Then, A is diagonalizable.*

Proof. Let $v_1, \dots, v_n$ be eigenvectors of A, corresponding to $\lambda_1, \dots, \lambda_n$. As they are distinct, by the above theorem, they are linearly independent. The number of elements in the set $\{v_1, \dots, v_n\}$ is n. Therefore, the set is a basis. Hence, according to General criterion for diagonalization, A is diagonalizable. □

5 Characteristic Polynomial

Definition 32 (Characteristic Polynomial)**.** Let A be any $n \times n$ matrix.

$$p_A(x) = \det(xI_n - A)$$

is called the characteristic polynomial.

5.1 Properties

1. The roots of $p_A(x)$ are the eigenvalues of A.
2. $\deg p_A(x) = n$
3. The coefficient of x^n is 1.
4. The constant term is $\alpha_0 = (-1)^n \det(A)$.
5. The coefficient of x^{n-1} is $\alpha_{n-1} = -(a_{11} + \cdots + a_{nn})$.

Theorem 4. *If $A \sim A'$, then $p_A(x) = p_{A'}(x)$.*

Proof.

$$\begin{aligned}
A' &= P^{-1}AP \\
\therefore p_{A'}(x) &= \det(xI - A') \\
&= \det(xI - P^{-1}AP) \\
&= \det(P^{-1}(xI)P - P^{-1}AP) \\
&= \det(P^{-1}(xI - A)P) \\
&= \cancel{\det(P^{-1})}\det(xI - A)\cancel{\det(P)} \\
&= \det(xI - A) \\
&= p_A(x)
\end{aligned}$$

□

Definition 33 (Alternative definition of characteristic polynomial)**.** Let $T : V \to V$ be a linear operator. The characteristic polynomial of T is defined as the characteristic polynomial of any representing matrix of T.

Theorem 5. *Let $f(x)$, $g(x)$ be polynomials. Then $\exists q(x), r(x)$, s.t.*

$$f(x) = g(x)q(x) + r(x)$$

and $\deg r(x) < \deg g(x)$.

Definition 34 (Remainder)**.** If

$$f(x) = g(x)q(x) + r(x)$$

$r(x)$ is called the remainder after division of $f(x)$ by $g(x)$. If $r(x) = \mathbb{O}$, $f(x)$ is said to be divisible by $g(x)$.

Corollary 5.1. *Let $f(x)$ be a polynomial and let α be a root of f. Then $f(x)$ is divisible by $(x-\alpha)$.*

Definition 35 (Algebraic multiplicity of eigenvalue)**.** Let A be a $n \times n$ matrix, and let $p_A(x)$ be the characteristic polynomial of A, and let λ be an eigenvalue of A. The algebraic multiplicity of λ is defined as the largest possible integer value of k such that $p_A(x)$ is divisible by $(x-\lambda)^k$.

Definition 36 (Eigenspace)**.** Let A be a $n \times n$ matrix, and let λ be an eigenvalue of A. The eigenspace of A corresponding to λ is defined as

$$V_\lambda = \{v \in \mathbb{F}^n ; Av = \lambda v\}$$

Theorem 6. *An eigenspace of a matrix is a subspace of the field over which the matrix is defined.*

Definition 37 (Geometric multiplicity of eigenvalue)**.** $m = \dim V_\lambda$ is called the geometric multiplicity of λ.

Theorem 7. *Let λ be an eigenvalue of $A_{n\times n}$. Let k be the algebraic multiplicity of λ and let m be the geometric multiplicity of λ. Then*

$$m \leq k$$

Proof.

$$m = \dim V_\lambda$$

Therefore, let $B_0 = \{v_1, \dots, v_m\}$ be a basis of V_λ.
Completing B_0 to $B = \{v_1, \dots, v_m, v_{m+1}, \dots, v_n\}$, a basis of $\mathbb{F}^n$.
Let $P_{n\times n}$ be a matrix with columns $v_1, \dots, v_n$.

$$P = \begin{pmatrix} v_1 & \dots & v_m & v_{m+1} & \dots & v_n \end{pmatrix}$$

P is invertible as $v_1, \dots, v_n$ are linearly independent.
Consider $A' = P^{-1}AP$.

$$\begin{aligned}
\therefore P^{-1}AP &= P^{-1}A \begin{pmatrix} v_1 & \dots & v_m & v_{m+1} & \dots & v_n \end{pmatrix} \\
&= P^{-1} \begin{pmatrix} Av_1 & \dots & Av_m & Av_{m+1} & \dots & Av_n \end{pmatrix} \\
&= P^{-1} \begin{pmatrix} \lambda v_1 & \dots & \lambda v_m & \star & \dots & \star \end{pmatrix} \\
&= \begin{pmatrix} P^{-1}(\lambda v_1) & \dots & P^{-1}(\lambda v_m) & \star & \dots & \star \end{pmatrix} \\
&= \begin{pmatrix} \lambda e_1 & \dots & \lambda e_m) & \star & \dots & \star \end{pmatrix} \\
&= \begin{pmatrix} \lambda I_m & \star \\ 0 & \widetilde{A} \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
p_{A'}(x) &= \det(xI_n + A') \\
&= \det\left(\begin{pmatrix} xI_m & 0 \\ 0 & xI_{n-m} \end{pmatrix} - \begin{pmatrix} \lambda I_m & \star \\ 0 & \widetilde{A} \end{pmatrix}\right) \\
&= \det\begin{pmatrix} (x-\lambda)I_m & \star \\ 0 & xI_{n-m} - \widetilde{A} \end{pmatrix} \\
&= \det((x-\lambda)I_m) \cdot \det(xI_{n-m} - \widetilde{A}) \\
&= (x-\lambda)^m \cdot p_{\widetilde{A}}(x)
\end{aligned}
$$

As $A \sim \widetilde{A}$,

$$
p_A(x) = p_{A'} = (x-\lambda)^m p_{\widetilde{A}}(x)
$$

By the definition of Algebraic multiplicity of eigenvalue, $k \geq m$. $\square$

Theorem 8. *If a matrix $A_{n\times n}$ is diagonalizable, then its characteristic polynomial $p_A(x)$ can be represented as a product of linear factors.*

$$
p_A(x) = (x-\lambda_1)^{k_1} \ldots (x-\lambda_s)^{k_s}
$$

where k_i is the algebraic multiplicity of λ_i and $\lambda_1, \ldots, \lambda_s$ are pairwise distinct.

Proof. As A is diagonalizable, let $A \sim D$,

$$
\begin{pmatrix}
\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \ddots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_s & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \ddots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_s
\end{pmatrix}
$$

Then,

$$
\begin{aligned}
p_A(x) &= p_D(x) \\
&= \det \begin{pmatrix}
x-\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x-\lambda_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \ddots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x-\lambda_s & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \ddots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x-\lambda_s
\end{pmatrix} \\
&= (x-\lambda_1)^{k_1} \dots (x-\lambda_s)^{k_s}
\end{aligned}
$$

□

Theorem 9 (Explicit criterion for diagonalization)**.** *Let A be an $n \times n$ matrix, s.t. $p_A(x)$ splits completely. Then A is diagonalizable if and only if $\forall \lambda_i$ of A, the algebraic multiplicity coincides with the geometric multiplicity.*

Proof of statement. If p_A splits completely, then $k_1 + \dots + k_n = n$.
If A is diagonalizable, then by the General criterion for diagonalization, there is $B = \{v_1, \dots, v_n\}$, a basis of $\mathbb{F}^n$, s.t. each v_i is an eigenvector of A.
Dividing $v_1, \dots, v_n$ into s groups corresponding to $\lambda_1, \dots, \lambda_s$, to each λ_i, there correspond at most $m_i = \dim V_{\lambda_i}$ eigenvectors, as they are a part of a basis and hence linearly independent.
Therefore,

$$n \leq m_1 + \dots + m_s$$

As p_A splits completely,

$$n = k_1 + \dots + k_s$$

Also, $k_i \geq m_i$

$$\therefore k_1 + \dots + k_s = m_1 + \dots + m_s$$

moreover, $\forall i$, s.t. $1 \leq i \leq s$,

$$k_i = m_i$$

□

Proof of converse.

$\forall i$, s.t. $1 \leq 1 \leq s$

$$\therefore k_i = m_i$$

As $k_1 + \cdots + k_s = n$,

$$m_1 + \cdots + m_s = n$$

Let the bases of the eigenspaces $V_{\lambda_1}, \ldots, V_{\lambda_s}$ be $B_1, \ldots, B_s$.

$$|B_1| = m_1$$
$$\vdots$$
$$|B_s| = m_s$$

Let $B = B_1 \cup \cdots \cup B_s$. $|B| = n$.
It is enough to prove that B is linearly independent.
Let

$$B = \{v_1, v_2, \ldots, w_1, w_2, \ldots, u_1, u_2, \ldots\}$$

Suppose

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \beta_1 w_1 + \beta_2 w_2 + \cdots + \gamma_1 u_1 + \gamma_2 u_2 + \ldots = \mathbb{O}$$

If possible, let at least one coefficient be non-zero. WLG, let $\alpha_1 \neq 0$.
Hence, as $v_1, v_2, \ldots$ form B_1 which is a basis of v_{λ_1},

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \neq \mathbb{O}$$

Let

$$w = \beta_1 w_1 + \beta_2 w_2 + \ldots$$
$$\ldots$$
$$u = \gamma_1 u_1 + \gamma_2 u_2 + \ldots$$

Therefore,

$$v + w + \cdots + u = \mathbb{O}$$

where $v \neq \mathbb{O}$ and $v \in V_{\lambda_1}, w \in V_{\lambda_2}, \ldots, u \in V_{\lambda_s}$.
But as $\lambda_1, \ldots, \lambda_s$ are pairwise distinct, $v, w, \ldots, u$ are linearly independent.
This is a contradiction. Therefore, B is a basis. Hence, as B consists of eigenvectors of A, by the General criterion for diagonalization, A is diagonalizable. □

Theorem 10 (Criterion for triangularization). *An operator $T : V \to V$ is triangularizable, i.e. there is a basis B of V such that $[T]_B$ is upper triangular, if and only if $p_T(x)$ splits completely.*

Theorem 11 (Jordan Theorem). *Let $T : V \to V$ be a linear operator such that $p_T(x)$ splits completely. Then there exists a basis B of V such that $[T]_B$ is of the form*

$$[T]_B = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_l \end{pmatrix}$$

where each J_i is of the form

$$\begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

where λ is some eigenvalue of T.

Part VII
Inner Product Spaces

1 Definition

Definition 38 (Inner product). Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. Let V be a vector space over $\mathbb{F}$. An inner product on V is a function in two vector arguments with scalar values which associates to two given vectors $v, w \in V$ their product $< v, w > \in \mathbb{F}$ so that the following properties are satisfied.

1. $\langle \alpha_1 v_1 + \alpha_2 v_2, w\rangle = \alpha_1 \langle v_1, w\rangle + \alpha_2 \langle v_2, w\rangle$, $\forall v_1, v_2, w \in V$, $\forall \alpha_1, \alpha_2 \in \mathbb{F}$

2. $\langle v, w\rangle = \overline{\langle w, v\rangle}$, $\forall v, w \in V$

3. $\langle v, v\rangle$ is a real non-negative number, $\forall v \in V$

Example 12. The dot product of two vectors is defined as follows. Is it an inner product?

$$V = \mathbb{F}^n$$

$$\left\langle \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \right\rangle = \alpha_1 \overline{\beta_1} + \cdots + \alpha_n \overline{\beta_n}$$

Solution. All three axioms are satisfied by this product. Hence, it is an inner product.

Theorem 1 (Sesquilinearity)**.**

$$\langle v, \beta_1 w_1 + \beta_2 w_2 \rangle = \overline{\beta_1} \langle v, w_1 \rangle + \overline{\beta_2} \langle v, w_2 \rangle$$

$\forall v, w_1, w_2 \in V, \beta_1, \beta_2 \in \mathbb{F}$

Definition 39 (Length)**.** The length of a vector

$$v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

is defined to be

$$\|v\| = \sqrt{\alpha_1^2 + \cdots + \alpha_n^2}$$

Example 13. Let V be the vector space consisting of all continuous functions $f : [a, b] \to \mathbb{R}$.

$$\langle f, g \rangle = \int\limits_a^b f(x) g(x) \, \mathrm{d}x$$

Solution. All three axioms are satisfied by this product. Hence, it is an inner product.

2 Computation of Inner Products

Definition 40 (Gram matrix)**.** Let V be an inner product space. Let

$$B = \{v_1, \dots, v_n\}$$

be a basis of V.

$$G_B = \begin{pmatrix} \langle v_1, v_1 \rangle & \dots & \langle v_1, v_n \rangle \\ \vdots & & \vdots \\ \langle v_n, v_1 \rangle & \dots & \langle v_n, v_n \rangle \end{pmatrix}$$

is called the Gram matrix of the inner product with respect to B.

Example 14. Find the Gram matrix of $V = \mathbb{F}^n$ with standard dot product with respect to

$$B = \left\{ \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

Solution.

$$\begin{aligned} G_B &= \begin{pmatrix} \langle e_1, e_1 \rangle & \dots & \langle e_1, e_n \rangle \\ \vdots & & \vdots \\ \langle e_n, e_1 \rangle & \dots & \langle e_n, e_n \rangle \end{pmatrix} \\ &= \begin{pmatrix} 1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1 \end{pmatrix} \end{aligned}$$

Example 15. Find the Gram matrix of $V = \mathbb{F}^n$ with standard dot product with respect to

$$B = \left\{ \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 7 \end{pmatrix} \right\}$$

Solution.

$$\begin{aligned} G_B &= \begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle \end{pmatrix} \\ &= \begin{pmatrix} 25 & 46 \\ 46 & 85 \end{pmatrix} \end{aligned}$$

Theorem 2.

$$\langle v, w \rangle = [v]_B^t G_B \overline{[w]}_B$$

Proof. Let

$$B = \{v_1, \ldots, v_n\}$$

be a basis of V.
The Gram matrix is

$$G_B = \Big(\langle v_i, v_j \rangle\Big) = \Big(g_{ij}\Big)$$

To compute $\langle v, w \rangle$, find

$$[v]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$
$$[w]_B = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\begin{aligned}
\langle v, w \rangle &= \langle \alpha_1 v_1 + \cdots + \alpha_n v_n, \beta_1 v_1 + \cdots + \beta_n v_n \rangle \\
&= \alpha_1 \overline{\beta_1} \langle v_1, v_1 \rangle + \cdots + \alpha_1 \overline{\beta_n} \langle v_1, v_n \rangle \\
&\quad + \alpha_2 \overline{\beta_1} \langle v_2, v_1 \rangle + \cdots + \alpha_2 \overline{\beta_n} \langle v_2, v_n \rangle \\
&\quad + \ldots \\
&\quad + \alpha_n \overline{\beta_1} \langle v_n, v_1 \rangle + \cdots + \alpha_n \overline{\beta_n} \langle v_n, v_n \rangle \\
&= \alpha_1 g_{11} \overline{\beta_1} + \cdots + \alpha_1 g_{1n} \overline{\beta_n} \\
&\quad + \alpha_2 g_{21} \overline{\beta_1} \rangle + \cdots + \alpha_2 g_{2n} \overline{\beta_n} \\
&\quad + \ldots \\
&\quad + \alpha_n g_{n1} \overline{\beta_1} + \cdots + \alpha_n g_{nn} \overline{\beta_n} \\
&= [v]_B^t G_B \overline{[w]_B}
\end{aligned}$$

□

2.1 Change of Basis

Theorem 3. *Let B, $\widetilde{B}$ be bases of V. Let P be the transition matrix from B to $\widetilde{B}$. Then*

$$G_{\widetilde{B}} = P^t G_B \overline{P}$$

where $\overline{P}$ is the matrix obtained by replacing all elements of P by their complex conjugates.

Proof.

$$[v]_B = P[v]_{\widetilde{B}}$$

$$\begin{aligned}
\langle v, w\rangle &= [v]_B^t G_B \overline{[w]_B} \\
&= (P[v]_{\widetilde{B}})^t G_B \overline{(P[w]_{\widetilde{B}})} \\
&= [v]_{\widetilde{B}}^t (P^t G_B \overline{P}) \overline{[w]_{\widetilde{B}}}
\end{aligned}$$

Also,

$$\langle v, w\rangle = [v]_{\widetilde{B}}^t G_{\widetilde{B}} \overline{[w]_{\widetilde{B}}}$$

Therefore,

$$G_{\widetilde{B}} = P^t G_B \overline{P}$$

□

3 Norms

3.1 Definition

Definition 41 (Norm)**.** Let V be a vector space over $\mathbb{F}$ with inner product. $\forall v \in V$,

$$\|v\| \doteq \sqrt{\langle v, v\rangle}$$

$\|v\|$ is called the norm of v.

3.2 Properties

1. Positivity
   $\|v\| \geq 0, \forall v \in V$
   $\|v\| = 0 \iff v = \mathbb{O}$

2. Homogeneity
   $\|\alpha v\| = |\alpha|\|v\|, \forall v \in V, \forall \alpha \in \mathbb{F}$

3. Triangle Inequality
   $\|u + v\| \leq \|u\| + \|v\|, \forall u, v \in V$

4 Orthogonality

4.1 Definition

Definition 42 (Orthogonality)**.** A vector $u \in V$ is said to be orthogonal to $v \in V$ if

$$\langle u, v \rangle = 0$$

It is denoted as $u \perp v$.

4.2 Properties

1. If $u \perp v$, then $v \perp u$.

2. If $u \perp v$, $\alpha, \beta \in \mathbb{F}$, then $\alpha u \perp \beta v$.

3. $\mathbb{O} \perp v$, $\forall v \in V$.

5 Orthogonal and Orthonormal Bases

Let V be a vector space over $\mathbb{F}$ with an inner product. Let $S \subset V$.

Definition 43 (Orthogonal set)**.** S is said to be orthogonal if any two distinct vectors from S are orthogonal.

Definition 44 (Orthonormal set)**.** S is said to be orthonormal if it is orthogonal and the norm of every vector is 1.

Definition 45 (Orthogonal basis)**.** S is said to be an orthogonal basis of V if it is orthogonal and a basis of V.

Definition 46 (Orthonormal basis)**.** S is said to be an orthonormal basis of V if it is orthonormal and a basis of V.

Theorem 4. *Let S be an orthogonal set such that $\mathbb{O} \notin S$. Then S is linearly independent.*

Proof. Let

$$\begin{aligned} \alpha_1, \dots, \alpha_m &\in \mathbb{F} \\ v_1, \dots v_m &\in S \end{aligned}$$

Let

$$\alpha_1 v_1 + \cdots + \alpha_m v_m = \mathbb{O}$$

S is linearly independent if and only if

$$\alpha_1 = \cdots = \alpha_m = 0$$

$$\alpha_1 v_1 + \cdots + \alpha_m v_m = \mathbb{O}$$

Multiplying both sides by v_1,

$$\begin{aligned} \langle \alpha_1 v_1 + \cdots + \alpha_m v_m, v_1 \rangle &= \langle \mathbb{O}, v_1 \rangle \\ \therefore \alpha_1 \langle v_1, v_1 \rangle + \cdots + \alpha_m \langle v_m, v_1 \rangle &= 0 \end{aligned}$$

As $v_1, \dots, v_m$ are orthogonal,

$$\langle v_2, v_1 \rangle = \cdots = \langle v_m, v_1 \rangle$$

$$\therefore \alpha_1 \langle v_1, v_1 \rangle = 0$$

As $v_1 \neq \mathbb{O}$

$$\begin{aligned} \langle v_1, v_1 \rangle &\neq 0 \\ \therefore \alpha_1 &= 0 \end{aligned}$$

Similarly,

$$\alpha_2 = \cdots = \alpha_m = 0$$

$\square$

Corollary 4.1. *Any orthonormal set is linearly independent.*

Corollary 4.2. *Any orthonormal set consisting of $n = \dim V$ vectors is an orthonormal basis of V.*

Example 16. Is the set

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

orthonormal?

Solution. The norm of the elements of S is not 1. Hence S is not orthonormal.

Theorem 5. *Let $B = \{v_1, \ldots, v_n\}$ be an orthonormal basis of V. Let $v \in V$. Let $[v]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$. Then,*

$$\begin{aligned} \alpha_1 &= \langle v, v_1 \rangle \\ &\vdots \\ \alpha_n &= \langle v, v_n \rangle \end{aligned}$$

Proof.

$$\begin{aligned} v &= \alpha_1 v_1 + \cdots + \alpha_n v_n \\ \therefore \langle v, v_1 \rangle &= \langle \alpha_1 v_1 + \cdots + \alpha_n v_n, v_1 \rangle \\ &= \alpha_1 \langle v_1, v_1 \rangle + \cdots + \alpha_n \langle v_n, v_1 \rangle \\ &= \alpha_1 \end{aligned}$$

Similarly, in general, $\forall 1 \leq i \leq n$,

$$\langle v, v_i \rangle = \alpha_i$$

□

Theorem 6 (Pythagoras Theorem)**.** *Let $B = \{v_1, \ldots, v_n\}$ be an orthonormal basis of V. Let $v \in V$. Let $[v]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$. Then,*

$$\|v\|^2 = |\alpha_1|^2 + \cdots + |\alpha_n|^2$$

Proof.

$$\begin{aligned}
\|v\|^2 &= \langle v, v\rangle \\
&= \langle \alpha_1 v_1 + \cdots + \alpha_n v_n, \alpha_1 v_1 + \cdots + \alpha_n v_n\rangle \\
&= \alpha_1 \overline{\alpha_1} + \cdots + \alpha_n \overline{\alpha_n} \\
&= |\alpha_1|^2 + \ldots |\alpha_n|^2
\end{aligned}$$

□

6 Unitary Matrices

Definition 47. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let A be an $n \times n$ matrix. A is said to be a unitary matrix if

$$A^* = \overline{A}^t = \overline{A^t} = A^{-1}$$

If $\mathbb{F} = \mathbb{R}$, unitary matrices are called orthogonal matrices.

1. I is a unitary matrix.
2. If A_1 and A_2 are unitary matrices, then $(A_1 A_2)^* = A_2^* A_1^*$.
3. If A is unitary, A^{-1} is also unitary.

Theorem 7. *Let A be an $n \times n$ matrix. Let $v_1, \ldots, v_n$ be the columns of A. Let A be an $n \times n$ matrix. Let $r_1, \ldots, r_n$ be the columns of A. Then the following are equivalent.*

1. *A is unitary.*
2. *$\{v_1, \ldots, v_n\}$ is an orthonormal basis of $\mathbb{F}^n$, with respect to standard dot product.*
3. *$\{r_1, \ldots, r_n\}$ is an orthonormal basis of $\mathbb{F}^n$, with respect to standard dot product.*

Proof. As A is unitary, A^t is also unitary.

$$\begin{aligned}
(A^t)^* &= (\overline{A^t})^t \\
&= (A^*)^t \\
&= (A^{-1})^t \\
&= (A^t)^{-1}
\end{aligned}$$

$$
\begin{aligned}
& A \text{ is unitary} \\
\iff & A^* = A^{-1} \\
\iff & AA^* = I \\
\iff & A\overline{A}^t = I \\
\iff & (A\overline{A}^t)_{ik} = I_{ik} \\
& = \sum_{j=1}^{n} a_{ij}\overline{a_{ik}} \\
& = r_i \cdot \overline{r_k} \iff \{r_1, \dots, r_n\} \text{ is an orthonormal basis}
\end{aligned}
$$

□

Theorem 8. *Let V be an inner product space. Let B be an orthonormal basis of V. Let B' be another basis of V. Let P be the transition matrix from B to B'. Then B' is orthonormal if and only if P is unitary.*

Proof of statement.

$$G_{B'} = P^t G_B \overline{P}$$

If B' is orthonormal,

$$
\begin{aligned}
\therefore I &= P^t I \overline{P} \\
&= P^t \overline{P}
\end{aligned}
$$

Therefore, P is unitary. □

Proof of converse. If P is unitary,

$$G_{B'} = P^t G_B \overline{P}$$

As B is orthonormal,

$$
\begin{aligned}
G_B &= I \\
\therefore G_{B'} &= P^t \overline{P}
\end{aligned}
$$

As P is unitary,

$$
\begin{aligned}
P^t \overline{P} &= I \\
\therefore G_{B'} &= I
\end{aligned}
$$

Therefore, B' is orthonormal. □

7 Projections

7.1 Definition

Definition 48. Let $S \subset V$ be a set of vectors.

$$S^{\perp} \doteq \{v \in V | \langle u, v\rangle = 0 \forall u \in S\}$$

Theorem 9. *$S^{\perp}$ is a subspace of V.*

Proof.

$$\langle u, \mathbb{O}\rangle = 0 \therefore \mathbb{O} \in S^{\perp}$$

If $v_1, v_2 \in S^{\perp}$,

$$\begin{aligned}\langle u, v_1 + v_2\rangle &= \langle u, v_1\rangle + \langle u, v_2\rangle \\ &= 0 + 0 \\ &= 0\end{aligned}$$

If $v \in S^{\perp}$,

$$\begin{aligned}\langle u, \alpha v\rangle &= \overline{\alpha}\langle u, v\rangle \\ &= 0\end{aligned}$$

□

Theorem 10.

$$S^{\perp} = \text{span}(S)^{\perp}$$

Proof. Let $v \in S^{\perp}$, $u \in \text{span}(S)$.
Let $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$, $u_1, \ldots, u_m \in S$.
Therefore,

$$\begin{aligned}u &= \alpha_1 u_1 + \cdots + \alpha_m v_m \\ \therefore \langle u, v\rangle &= \langle \alpha_1 u_1 + \cdots + \alpha_m v_m, v\rangle \\ &= \alpha_1\langle u_1, v\rangle + \cdots + \alpha_m\langle u_m, v\rangle \\ &= \alpha_1 \cdot 0 + \cdots + \alpha_m \cdot 0 \\ &= 0\end{aligned}$$

Therefore, $v \in S^{\perp}$.
Therefore, $S^{\perp} \subset \text{span}(S)^{\perp}$.
$S \subset \text{span}(S)$. Therefore, let $v \in \text{span}(S)^{\perp}$. Then,

$$\langle u, v\rangle = 0$$

for all $u \in \text{span}(S)$.
Hence for all $u \in S$,

$$\langle u, v \rangle = 0$$

Therefore, $\text{span}(S)^{\perp} \subset S^{\perp}$. $\square$

Definition 49 (Projection)**.** Let V be an inner product space. Let W be a subspace of V. Let $v \in V$. Let $B = \{w_1, \ldots, w_m\}$ be a basis of W. The projection of v onto W is defined as follows.

$$\pi_B(v) = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \cdots + \frac{\langle v, w_m \rangle}{\langle w_m, w_m \rangle} w_m$$

7.2 Properties

1. $\pi_B(v) \in W$
2. $\pi_B(v) = v \iff v \in W$
3. $v - \pi_B(v) \in W^{\perp}$

7.3 Gram - Schmidt Process

Input	Any basis $B = \{v_1, \ldots, v_n\}$ of V.
Intermediate Output	Orthogonal basis $\widetilde{B} = \{\widetilde{v_1}, \ldots, \widetilde{v_n}$ of V
Final Output	Orthonormal basis $B^0 = \{{v_1}^1, \ldots, {v_n}^0\}$ of V

Step 1 $\widetilde{v_1} = v_1$, denote $w_1 = \text{span}\{\widetilde{v_1}\} = \text{span}\{v_1\}$, $B_1 = \{\widetilde{v_1}\}$

Step 2 $\widetilde{v_2} = v_2 - \pi_{B_1}(v_2) = v_2 - \dfrac{\langle v_2, \widetilde{v_1} \rangle}{\langle \widetilde{v_1}, \widetilde{v_1} \rangle} \widetilde{v_1}$

As $\widetilde{v_2} \perp \widetilde{v_1}$, $B_2 = \{\widetilde{v_1}, \widetilde{v_2}\}$ is an orthogonal set. Denote $W_2 = \text{span}\{\widetilde{v_1}, \widetilde{v_2}\} = \text{span}\{v_1, v_2\}$.

Step 3 $\widetilde{v_3} = v_3 - \pi_{B_2}(v_3) = v_3 - \dfrac{\langle v_2, \widetilde{v_1} \rangle}{\langle \widetilde{v_1}, \widetilde{v_1} \rangle} \widetilde{v_1} - \dfrac{\langle v_3, \widetilde{v_2} \rangle}{\langle \widetilde{v_2}, \widetilde{v_2} \rangle}$

As $\widetilde{v_3} \in {W_2}^{\perp}$, $B_3 = \{\widetilde{v_1}, \widetilde{v_2}, \widetilde{v_3}\}$ is an orthogonal set. Denote $W_2 = \text{span}\{\widetilde{v_1}, \widetilde{v_2}, \widetilde{v_3}\} = \text{span}\{v_1, v_2, v_3\}$.

$\vdots$

Step n The n^{th} step gives $\widetilde{B_n} = \{\widetilde{v_1}, \dots, \widetilde{v_n}\}$ which is an orthogonal basis of V.

B^0 is obtained by normalization of $\widetilde{B_n}$.

$$\begin{aligned} {v_1}^0 &= \frac{1}{\|\widetilde{v_1}\|} \\ &\vdots \\ {v_n}^0 &= \frac{1}{\|\widetilde{v_n}\|} \end{aligned}$$

Example 17.

$$\begin{aligned} B &= \{v_1, v_2, v_3\} \\ &= \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \end{aligned}$$

Solution.

$$
\begin{aligned}
\widetilde{v_1} &= v_1 \\
&= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\
\widetilde{v_2} &= v_2 - \frac{\langle v_2, \widetilde{v_1} \rangle}{\langle \widetilde{v_1}, \widetilde{v_1} \rangle} \\
&= \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\
\widetilde{v_3} &= v_3 - \frac{\langle v_3, \widetilde{v_1} \rangle}{\langle \widetilde{v_1}, \widetilde{v_1} \rangle} \widetilde{v_1} - \frac{\langle v_3, \widetilde{v_2} \rangle}{\langle \widetilde{v_2}, \widetilde{v_2} \rangle} \widetilde{v_2} \\
&= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - 0 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
\therefore \widetilde{B_3} &= \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}
\end{aligned}
$$

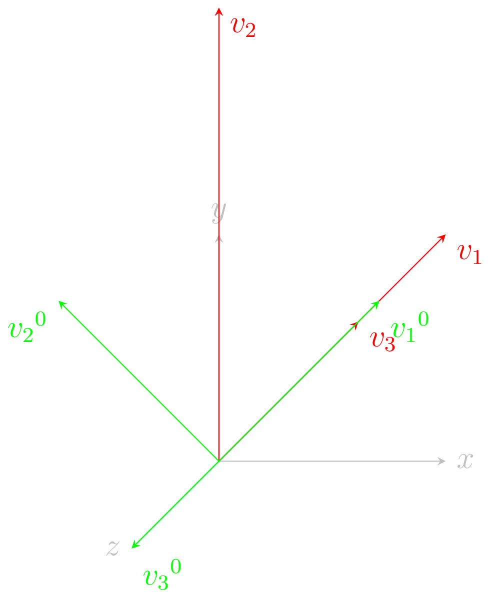
Therefore, normalizing $\widetilde{B_3}$,

$$v_1{}^0 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$v_2{}^0 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$v_3{}^0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore B^0 = \left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$



7.4 Inequalities

Theorem 11 (Bessel's Inequality)**.** *Let* $\{v_1, \dots, v_m\}$ *be an orthonormal set. Let* $v \in V$ *be any vector. Then*

$$\|v\|^2 \geq |\langle v, v_1 \rangle|^2 + \cdots + |\langle v, v_m \rangle|^2$$

and the equality holds if and only if $v \in \operatorname{span}\{v_1, \dots, v_m\}$.

Proof. $\{v_1, \ldots, v_m\}$ can be completed to an orthonormal basis

$$B = \{v_1, \ldots, v_m, v_{m+1}, \ldots, v_n\}$$

Using Pythagoras Theorem,

$$\begin{aligned} \|v\|^2 &= |\langle v, v_1\rangle|^2 + \cdots + |\langle v, v_m\rangle|^2 + |\langle v, v_{m+1}\rangle|^2 + \cdots + |\langle v, v_n\rangle|^2 \\ \therefore \|v\|^2 &\geq |\langle v, v_1\rangle|^2 + \cdots + |\langle v, v_m\rangle|^2 \end{aligned}$$

The equality holds if and only if

$$|\langle v, v_{m+1}\rangle|^2 + \cdots + |\langle v, v_n\rangle|^2 = 0$$

if and only if

$$\begin{aligned} |\langle v, v_{m+1}\rangle|^2 &= 0 \\ &\vdots \\ |\langle v, v_n\rangle|^2 &= 0 \end{aligned}$$

If $v \in \text{span}\{v_1, \ldots, v_m\}$,

$$v = \alpha_1 v_1 + \cdots + \alpha_m v_m$$

Therefore,

$$\begin{aligned} \langle v, v_{m+1}\rangle &= \langle \alpha_1 v_1 + \cdots + \alpha_m v_m, v_{m+1}\rangle \\ &= \alpha_1 \langle v_1, v_{m+1}\rangle + \cdots + \alpha_m \langle v_m, v_{m+1}\rangle \end{aligned}$$

as the basis is orthonormal, $\langle v_i, v_{m+1}\rangle$

$$\therefore \langle v, v_{m+1}\rangle = 0$$

Similarly,

$$\begin{aligned} |\langle v, v_{m+2}\rangle|^2 &= 0 \\ &\vdots \\ |\langle v, v_n\rangle|^2 &= 0 \end{aligned}$$

Conversely, if

$$\begin{aligned} |\langle v, v_{m+1}\rangle|^2 &= 0 \\ &\vdots \\ |\langle v, v_n\rangle|^2 &= 0 \end{aligned}$$

let

$$
\begin{aligned}
v &= \alpha_1 v_1 + \cdots + \alpha_m v_m + \alpha_{m+1} v_{m+1} + \alpha_n v_n \\
\therefore 0 &= \langle v, v_{m+1} \rangle \\
\therefore 0 &= \langle \alpha_1 v_1 + \cdots + \alpha_m v_m + \alpha_{m+1} v_{m+1} + \alpha_n v_n, v_{m+1} \rangle
\end{aligned}
$$

All $\langle v_i, v_{m+1} \rangle$ except $\langle v_{m+1}, v_{m+1} \rangle$ are 0.
Therefore,

$$
|\langle v, v_{m+1} \rangle|^2 + \cdots + |\langle v, v_n \rangle|^2 = 0
$$

□

Theorem 12 (Cauchy - Schwarz Inequality). *Let $u, v \in V$ be any vectors. Then*

$$
|\langle u, v \rangle| \leq \|u\| \cdot \|v\|
$$

and the equality holds if and only if $\{u, v\}$ is linearly dependent.

Proof. If $u = \mathbb{O}$, the equality holds.
Let $u \neq \mathbb{O}$.
Let

$$
\begin{aligned}
u^0 &= \frac{1}{\|u\|} \\
\|u^0\| &= 1
\end{aligned}
$$

Applying Bessel's Inequality to the orthonormal set $\{u^0\}$,

$$
\begin{aligned}
\|v\|^2 &\geq |\langle v, u^0 \rangle|^2 \\
|\langle v, u^0 \rangle|^2 &= \left| \left\langle v, \frac{1}{\|u\|} u \right\rangle \right|^2 \\
&= \left| \frac{1}{\|u\|} \langle v, u \rangle \right|^2 \\
&= \left(\frac{1}{\|u\|} |\langle v, u \rangle| \right)^2 \\
&= \frac{1}{\|u\|^2} |\langle v, u \rangle|^2 \\
\therefore \|v\|^2 &\geq \frac{1}{\|u\|^2} |\langle v, u \rangle|^2
\end{aligned}
$$

By Bessel's Inequality, the equality holds if and only if

$$
v \in \operatorname{span}\{u^0\} = \operatorname{span}\{u\}
$$

Therefore, v and u are linearly independent. □

8 Angle

Definition 50 (Angle)**.** Let V be a vector space over $\mathbb{R}$ with inner product $\langle , \rangle$. Let $u, v \in V$, $u \neq \mathbb{O}$, $v \neq \mathbb{O}$. The angle between u and v is defined as

$$\cos\varphi \doteq \frac{\langle u, v\rangle}{\|u\| \cdot \|v\|}$$

9 Triangle Inequality

Theorem 13 (Triangle Inequality Theorem)**.** *Let $u, v \in V$. Then*

$$\|u + v\| \leq \|u\| + \|v\|$$

Proof.

$$\begin{aligned}
\|u + v\|^2 &= \langle u + v, u + v\rangle \\
&= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle \\
&= \|u\|^2 + \langle u, v\rangle + \overline{\langle u, v\rangle} + \|v\|^2 \\
&= \|u\|^2 + 2\Re\big(\langle u, v\big) + \|v\|^2
\end{aligned}$$

As $\Re(z) \leq |z|$,

$$\|u + v\|^2 \leq \|u\|^2 + 2\left|\langle u, v\rangle\right| + \|v\|^2$$

Hence, by Cauchy - Schwarz Inequality,

$$\begin{aligned}
\|u + v\|^2 &\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \\
\therefore \|u + v\|^2 &\leq \big(\|u\| + \|v\|\big)^2 \\
\therefore \|u + v\| &\leq \|u\| + \|v\|
\end{aligned}$$

□

10 Orthogonal Decomposition

Theorem 14. *Let W be a subspace of V. Then*

$$V = W \oplus W^{\perp}$$

Proof. Let B be an orthogonal basis of V. Consider a projection $\pi_B(v)$. Therefore,

$$v = \pi_B(v) + \big(v - \pi_B(v)\big)$$

$$\begin{aligned} \pi_B(v) &\in W \\ v - \pi_B(v) &\in W^\perp \end{aligned}$$

Therefore,

$$V = W + W^\perp$$

If possible, let $u \in W \cap W^\perp$. Therefore, $u \in W$ and $u \in W^\perp$. By the definition of orthogonality,

$$\begin{aligned} \langle u \in W, u \in W^\perp \rangle &= 0 \\ \therefore u &= 0 \end{aligned}$$

Therefore,

$$V = W \oplus W^\perp$$

□

Corollary 14.1. *Let B be an orthogonal basis of W. Then $\pi_B(v)$ does not depend on the choice of B.*

Proof. As B is an orthogonal basis of W,

$$v = \pi_B(v) + \big(v - \pi_B(v)\big)$$

Let B' be another orthogonal basis of W. Therefore,

$$v = \pi_{B'}(v) + \big(v - \pi_{B'}(v)\big)$$

Therefore,

$$\begin{aligned} \pi_B(v) &\in W \\ \pi_{B'}(v) &\in W \end{aligned}$$

and

$$\begin{aligned} v - \pi_B(v) &\in W^\perp \\ v - \pi_{B'}(v) &\in W^P\perp \end{aligned}$$

As

$$V = W \oplus W^{\perp}$$

such a representation is unique. Therefore,

$$\pi_B(v) = \pi_{B'}(v)$$

$\square$

Theorem 15. *Let $u, v \in V$, s.t. $u \perp v$. Then*

$$\|u \pm v\|^2 = \|u\|^2 + \|v\|^2$$

Proof.

$$\begin{aligned} \|u \pm v\|^2 &= \|u\|^2 + \|v\|^2 \\ &= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle \\ &= \langle u, u\rangle + \langle v, v\rangle \\ &= \|u\|^2 + \|v\|^2 \end{aligned}$$

$\square$

11 Distance

Definition 51 (Distance)**.** Let $u, v \in V$. The distance $\mathrm{d}(u, v)$ from u to v is defined as

$$\mathrm{d}(u, v) \doteq \|u - v\|$$

Theorem 16. *Let $u, v \in V$. Then*

$$\mathrm{d}(u, v) \geq 0$$

and the equality holds if and only if $u = v$.

Theorem 17. *Let $u, v \in V$. Then*

$$\mathrm{d}(u, v) = \mathrm{d}(v, u)$$

Theorem 18. *Let $u, v \in V$. Then*

$$\mathrm{d}(u, v) + \mathrm{d}(v, w) \geq \mathrm{d}(u, w)$$

Theorem 19. *The projection $\pi_W(v)$ is the vector in W closest to v, i.e.*

$$\mathrm{d}\left(v, \pi_W(v)\right) = \min_{w \in W} \mathrm{d}(v, w)$$

Proof. Let $v \in V$. For any vector $w \in W$,

$$\begin{aligned} \left(\mathrm{d}(v, u)\right)^2 &= \|v - w\|^2 \\ &= \left\|\left(v = \pi_W(v)\right) + \left(\pi_W(v) - w\right)\right\|^2 \\ &= \left\|v - \pi_W(v)\right\|^2 + \left\|\pi_W(v) - w\right\|^2 \\ &\geq \left\|v - \pi_W(v)\right\|^2 \\ \therefore \left(\mathrm{d}(v, u)\right)^2 &\geq \mathrm{d}\left(v, \pi_W(v)\right)^2 \end{aligned}$$

□

12 Adjoint Map

Definition 52 (Linear functional)**.** A linear functional $\varphi : V \to \mathbb{F}$ is a linear map, with $\mathbb{F}$ considered as a 1 dimensional vector space over itself.

Theorem 20 (Riesz Representation Theorem)**.** *Let V be an inner product space, s.t. $n = \dim V$. Let $\varphi : V \to \mathbb{F}$ be any linear functional. Then there exists a unique vector $u \in V$, dependent on φ, s.t. $\forall v \in V$,*

$$\varphi(v) = \langle v, u \rangle$$

Proof. If possible, let $u_1, u_2 \in V$, s.t. $\forall v \in V$,

$$\varphi(v) = \langle v, u_1 \rangle = \langle v, u_2 \rangle$$

Therefore,

$$\langle v, u_1 - u_2 \rangle = 0$$

Let $v = u_1 - u_2$. Therefore,

$$\begin{aligned} \langle v, u_1 - u_2 \rangle &= \langle u_1 - u_2, u_1 - u_2 \rangle \\ \therefore \langle u_1 - u_2, u_1 - u_2 \rangle &= 0 \\ \therefore u_1 - u_2 &= 0 \\ \therefore u_1 &= u_2 \end{aligned}$$

Therefore, u, if it exists, is unique.

Let

$$\begin{aligned} B &= \{v_1, \dots, v_2\} \\ \widetilde{B} &= \{1\} \end{aligned}$$

be orthonormal bases of V and $\mathbb{F}$ respectively.
Let

$$\begin{aligned} A &= [\varphi]_{B,\widetilde{B}} \\ &= \begin{pmatrix} \alpha_1 & \dots & \alpha_n \end{pmatrix} \end{aligned}$$

be the representation matrix.
Therefore,

$$[\varphi(v)]_{\widetilde{B}} = A[v]_B$$

Let

$$\begin{aligned} v &= \beta_1 v_1 + \dots + \beta_n v_n \\ \therefore [v]_B &= \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \end{aligned}$$

Therefore,

$$\begin{aligned} [\varphi(v)]_{\widetilde{B}} &= \begin{pmatrix} \alpha_1 & \dots & \alpha_n \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \\ &= \alpha_1 \beta_1 + \dots + \alpha_n \beta_n \\ &= \beta_1 \alpha_1 + \dots + \beta_n \alpha_n \\ &= \beta_1 \overline{\overline{\alpha_1}} + \dots + \beta_n \overline{\overline{\alpha_n}} \\ &= \begin{pmatrix} \beta_1 & \dots & \beta_n \end{pmatrix} \overline{\begin{pmatrix} \overline{\alpha_1} \\ \vdots \\ \overline{\alpha_n} \end{pmatrix}} \\ &= \left\langle \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}, \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \right\rangle \end{aligned}$$

Let $u \in V$, s.t.

$$[u]_B = \begin{pmatrix} \overline{\alpha_1} \\ \vdots \\ \overline{\alpha_n} \end{pmatrix}$$

$$[\varphi(v)]_{\widetilde{B}} = \langle v, u \rangle$$

and

$$[\varphi(v)]_{\widetilde{B}} = \varphi(v) \cdot 1$$
$$\therefore \varphi(v) = \langle v, u \rangle$$

$\square$