LINEAR ALGEBRA : COMPENDIUM

AAKASH JOG

1. Matrices

Definition 1 (Adjoint matrix)**.**

$$A^* \doteq \overline{A}^t$$

Definition 2 (Row rank)**.** The number of non-zero rows in A_R is called the row rank of A. It is denoted by r.

$$r \leq n$$

Theorem 1 (Gaussian Elimination)**.**

Step 1 *Find the first non-zero column C_p of A.*
Step 2 *Denote by a_{ip} the first non-zero entry of C_p.*
Step 3 *Switch the 1^{st} and i^{th} rows.*
Step 4 *Multiply the 1^{st} row by $\frac{1}{a_{ip}}$.*
Step 5 *Using row operations of type III, make all other entries of the p^{th} column zeros.*
Step 6 *Ignoring the top row and C_p, repeat steps Step 1 to Step 5.*

2. Vector Spaces

Definition 3 (Subspace)**.** Let $U \subseteq V$.

Axiom 1 $\mathbb{O} \in U$
Axiom 2 If $x, y \in U$, then, $(x + y) \in U$
Axiom 3 If $x \in U, \alpha \in \mathbb{F}$, then, $\alpha x \in U$

Definition 4 (Operations on subspaces)**.**

$$U_1 \cap U_2 = \{x \in V : x \in U_1 \text{ and } x \in U_2\}$$

$$U_1 \cup U_2 = \{x \in V : x \in U_1 \text{ or } x \in U_2\}$$

$$U_1 + U_2 = \{x \in V : x = x_1 + x_2, x_1 \in U_1, x_2 \in U_2\}$$

Date: Monday 26^{th} January, 2015.

Definition 5 (Span)**.** span(S) is the collection of all linear combinations of finite number of vectors of S with coefficients from $\mathbb{F}$. span(S) is a subspace of V

Definition 6 (Spanning sets and dimensionality)**.** If V has atleast one finite spanning set, V is said to be finite-dimensional.

Definition 7 (Isomorphic spaces)**.** Let $V/\mathbb{F}$ and $W/\mathbb{F}$ be vector spaces. We say that V is isomorphic to W if there is a map $\varphi : V \to W$, s.t.

(1) φ is one-to-one and onto
(2) $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2); \forall v_1, v_2 \in V$
(3) $\varphi(\alpha v) = \alpha\varphi(v); \forall v \in V, \forall \alpha \in \mathbb{F}$

Theorem 2. *If $S = \{v_1, \dots, v_m\}$ is a spanning set of V, and if S is not a basis of V, a basis B of V can be obtained by removing some elements from S.*

Proof. If S is linearly independent, then it is a basis.
Otherwise, if S is linearly dependent, it has an element, WLG, say v_m, which is a linear combination of the others.

$$v_m = \alpha_1 v_1 + \dots + \alpha_{m-1} v_{m-1}$$

Let

$$S' = S - \{v_m\}$$

S' is a spanning set.
Therefore, $\forall v \in V$

$$\begin{aligned} v &= \beta_1 v_1 + \dots + \beta_{m-1} v_{m-1} + \beta_m v_m \\ &= \beta_1 v_1 + \dots + \beta_{m-1} + \beta_m(\alpha_1 v_1 + \dots + \alpha_{m-1} v_{m-1}) \\ &= \gamma_1 v_1 + \dots + \gamma_{m-1} v_{m-1} \end{aligned}$$

If S' is linearly independent, then it is a basis, else the same process above can be repeated till we get a basis.
Therefore, a basis is a smallest spanning set. □

Theorem 3. *If $B_0 = \{v_1, \dots, v_n\}$ is a linearly independent set, and if B_0 is a basis of V, a basis of V can be obtained by adding elements to B_0.*

Theorem 4. *Let V be a vector space, s.t.* $\dim V = n$.
If B satisfies 2 out of the 3 following conditions, then it is a basis.

(1) B has n elements.
(2) B is a spanning set.
(3) B is linearly dependent.

Theorem 5 (Dimension Theorem)**.**

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$$

Theorem 6.

$$U + W = \operatorname{span}(U \cup W)$$

If $U = \operatorname{span}(B)$ *and* $W = \operatorname{span}(B')$ *then,* $U + W = \operatorname{span}(B \cup B')$.

Theorem 7 (Changing a basis)**.** *Let* $B = \{v_1, \dots, v_n\}$ *be a basis of* V*, s.t.* $\dim V = n$*. Let* $B' = \{v'_1, \dots, v'_n\}$.
As B *is a spanning set, all of* $v'_1, \dots, v'_n$ *can be expressed as a linear combination of* $v_1, \dots, v_n$.

$$\begin{aligned} v'_1 &= \gamma_{11} v_1 + \dots + \gamma_{n1} v_n \\ &\vdots \\ v'_n &= \gamma_{1n} v_1 + \dots + \gamma_{nn} v_n \end{aligned}$$

Definition 8 (Transition matrix)**.** The matrix

$$C = \begin{pmatrix} \gamma_{11} & \dots & \gamma_{1n} \\ \vdots & & \vdots \\ \gamma_{n1} & \dots & \gamma_{nn} \end{pmatrix}$$

is called the transition matrix from B to B'.

$$B'_{1\times n} = B_{1\times n} C_{n\times n}$$

Theorem 8. B' *is a basis of* V *iff* C *is invertible.*

Corollary 8.1. *If* A *has two identical rows, then* $\det(A) = 0$.

Theorem 9. *If* A*,* B*,* C *are some matrices, and* $\mathbb{O}$ *is the zero matrix,*

$$\begin{pmatrix} A_{m\times m} & B \\ \mathbb{O} & C_{n\times n} \end{pmatrix} = \det(A) \cdot \det(C)$$

Theorem 10.

$$\det(AB) = \det(A)\det(B)$$

Theorem 11 (Calculation of determinant)**.** *Let* A *be a* $m \times n$ *matrix, and let* A_{ij} *be the matrix obtained by removing the* i^{th} *row and* j^{th} *column from* A.

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

Definition 9 (Linear map)**.** Let V and W be vector spaces over the same field $\mathbb{F}$.

$$\varphi : V \to W$$

is said to be a linear map if

(1) $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2); \forall v_1, v_2 \in V$
(2) $\varphi(\alpha v) = \alpha\varphi(v); \forall v \in V, \forall \alpha \in \mathbb{F}$

Definition 10 (Matrix of a linear map)**.** Let $\varphi : V \to W$ be a linear map.

$$B = \{v_1, \dots, v_n\}$$
$$B' = \{w_1, \dots, w_m\}$$

be bases of V and W respectively.
Let

$$\varphi(v_1) = \alpha_{11} w_1 + \dots + \alpha_{m1} w_m$$
$$\vdots$$
$$\varphi(v_n) = \alpha_{1n} w_1 + \dots + \alpha_{mn} w_m$$

The matrix

$$A = [\varphi]_{B,B'} = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \dots & \alpha_{mn} \end{pmatrix}$$

is called the matrix of φ with respect to the bases B and B'.

Theorem 12.

$$[\varphi(z)]_{B'} = [\varphi]_{B,B'} [z]_B$$

Theorem 13. *Let V, W be vector spaces over $\mathbb{F}$, $\dim(V) = n$, $\dim(W) = m$. Let $\varphi : V \to W$ be a linear map. Let B, $\widetilde{B}$ be bases of V and let B' and $\widetilde{B'}$ be bases of W. Let $A = [\varphi]_{B,B'}$ and $\widetilde{A} = [\varphi]_{\widetilde{B},\widetilde{B'}}$ be the matrices of φ w.r.t. the pairs B, B' and $\widetilde{B}$, $\widetilde{B'}$. Let P denote the transition matrix from B to $\widetilde{B}$, and let Q denote the transition matrix from B' to $\widetilde{B'}$. Then,*

$$\widetilde{A}_{m \times n} = Q^{-1}_{m \times m} A_{m \times n} P_{n \times n}$$

Definition 11 (Operations on linear maps)**.**

$$(\varphi + \varphi')(v) \doteq \varphi(v) + \varphi'(v)$$
$$(\alpha\varphi)(v) \doteq \alpha\varphi(v)$$

Definition 12 (Composed map)**.**

$$(\varphi' \circ \varphi)(v) \doteq \varphi'(\varphi(v))$$

Theorem 14 (Matrix of composed map)**.**

$$[\varphi' \circ \varphi]_{B,B''} = [\varphi']_{B',B''} [\varphi]_{B,B'}$$

Definition 13 (Kernel and image)**.** Let $\varphi : V \to W$ be a linear map.

$$\ker \varphi \doteq \{v \in V : \varphi(v) = \mathbb{O}\}$$
$$\operatorname{im} \varphi \doteq \{\phi(v) : v \in V\}$$

Theorem 15. $\ker\varphi$ *is a subspace of* V *and* $\operatorname{im}\varphi$ *is a subspace of* W*.*

Theorem 16. *Let* $\varphi : V \to W$ *be a linear map. Then*

$$\dim V = \dim(\ker(\varphi)) + \dim(\operatorname{im}(\varphi))$$

3. Linear Operators

Definition 14 (Linear operator)**.** A linear operator or transformation

$$T : V \to V$$

is a linear map from a vector space V to itself.

Definition 15 (Similarity of matrices)**.**

$$A \sim \widetilde{A} \iff \widetilde{A} = P^{-1}AP$$

Definition 16 (Diagonalizability)**.** If A is similar to a diagonal matrix, A is said to be diagonalizable. P, s.t. $P^{-1}AP = D$ is called a diagonalizing matrix for A. D is called a diagonal form of A.

Theorem 17 (Explicit criterion for diagonalization)**.** *Let* A *be an* $n \times n$ *matrix, s.t.* $p_A(x)$ *splits completely. Then* A *is diagonalizable if and only if* $\forall \lambda_i$ *of* A*, the algebraic multiplicity coincides with the geometric multiplicity.*

Theorem 18. *Let* $\lambda_1, \ldots, \lambda_s$ *be pairwise distinct eigenvalues of an* $n \times n$ *matrix* A*, i.e.* $\forall i \neq j, \lambda_i \neq \lambda_j$*. Let* $v_1, \ldots, v_s$ *be eigenvalues of* A *corresponding to* $\lambda_1, \ldots, \lambda_s$*. Then the set* $S = \{v_1, \ldots, v_s\}$ *is linearly independent.*

Corollary 18.1. *Let* $A_{n\times n}$ *have* n *distinct eigenvalues. Then,* A *is diagonalizable.*

Definition 17 (Characteristic Polynomial)**.** Let A be any $n \times n$ matrix.

$$p_A(x) = \det(xI_n - A)$$

is called the characteristic polynomial. Its roots are eigenvalues of A.

Theorem 19. *If* $A \sim A'$*, then* $p_A(x) = p_{A'}(x)$*.*

Definition 18 (Algebraic multiplicity of eigenvalue)**.** Largest possible integer value of k such that $p_A(x)$ is divisible by $(x - \lambda)^k$.

Definition 19 (Eigenspace)**.**

$$V_\lambda = \{V \in \mathbb{F}^n; Av = \lambda v\}$$

Theorem 20. *An eigenspace of a matrix is a subspace of the field over which the matrix is defined.*

Definition 20 (Geometric multiplicity of eigenvalue)**.** $m = \dim V_\lambda$

Theorem 21. *The geometric multiplicity of an eigenvalue is less than or equal to the algebraic multiplicity.*

Theorem 22 (Criterion for triangularization)**.** *An operator $T : V \to V$ is triangularizable, if and only if $p_T(x)$ splits completely.*

Theorem 23 (Jordan Theorem)**.** *Let $T : V \to V$ be a linear operator such that $p_T(x)$ splits completely. Then there exists a basis B of V such that $[T]_B$ is of the form*

$$[T]_B = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_l \end{pmatrix} J_i = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

4. Inner Product Spaces

Definition 21 (Inner product)**.** Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$.

(1) $\langle \alpha_1 v_1 + \alpha_2 v_2, w\rangle = \alpha_1 \langle v_1, w\rangle + \alpha_2 \langle v_2, w\rangle$, $\forall v_1, v_2, w \in V$, $\forall \alpha_1, \alpha_2 \in \mathbb{F}$
(2) $\langle v, w\rangle = \overline{\langle w, v\rangle}$, $\forall v, w \in V$
(3) $\langle v, v\rangle$ is a real non-negative number, $\forall v \in V$

Theorem 24 (Sesquilinearity)**.**

$$\langle v, \beta_1 w_1 + \beta_2 w_2\rangle = \overline{\beta_1}\langle v, w_1\rangle + \overline{\beta_2}\langle v, w_2\rangle$$

Definition 22 (Gram matrix)**.** Let V be an inner product space. Let

$$B = \{v_1, \dots, v_n\}$$

be a basis of V.

$$G_B = \begin{pmatrix} \langle v_1, v_1\rangle & \dots & \langle v_1, v_n\rangle \\ \vdots & & \vdots \\ \langle v_n, v_1\rangle & \dots & \langle v_n, v_n\rangle \end{pmatrix}$$

Theorem 25. $\langle v, w\rangle = [v]_B^t G_B \overline{[w]}_B$

Theorem 26. *Let B, $\widetilde{B}$ be bases of V. Let P be the transition matrix from B to $\widetilde{B}$. Then*

$$G_{\widetilde{B}} = P^t G_B \overline{P}$$

where $\overline{P}$ is the matrix obtained by replacing all elements of P by their complex conjugates.

Definition 23 (Norm)**.** $\|v\| \doteq \sqrt{\langle v, v\rangle}$

Definition 24 (Orthogonality)**.** $u \perp v \iff \langle u, v\rangle = 0$

Theorem 27. *Let S be an orthogonal set such that $\mathbb{O} \notin S$. Then S is linearly independent.*

Corollary 27.1. *Any orthonormal set is linearly independent.*

Corollary 27.2. *Any orthonormal set consisting of $n = \dim V$ vectors is an orthonormal basis of V.*

Theorem 28. *Let $B = \{v_1, \dots, v_n\}$ be an orthonormal basis of V. Let $v \in V$. Let $[v]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$. Then,*

$$\begin{aligned} \alpha_1 &= \langle v, v_1 \rangle \\ &\vdots \\ \alpha_n &= \langle v, v_n \rangle \end{aligned}$$

Theorem 29 (Pythagoras Theorem)**.** *Let $B = \{v_1, \dots, v_n\}$ be an orthonormal basis of V. Let $v \in V$. Let $[v]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$. Then,*

$$\|v\|^2 = |\alpha_1|^2 + \cdots + |\alpha_n|^2$$

Definition 25 (Unitary matrix)**.** Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let A be an $n \times n$ matrix. A is said to be a unitary matrix if

$$A^* = \overline{A}^t = \overline{A^t} = A^{-1}$$

If $\mathbb{F} = \mathbb{R}$, unitary matrices are called orthogonal matrices.

Theorem 30. *Let A be an $n \times n$ matrix. Let $v_1, \dots, v_n$ be the columns of A. Let A be an $n \times n$ matrix. Let $r_1, \dots, r_n$ be the columns of A. Then the following are equivalent.*

(1) A is unitary.
(2) $\{v_1, \dots, v_n\}$ is an orthonormal basis of $\mathbb{F}^n$, with respect to standard dot product.
(3) $\{r_1, \dots, r_n\}$ is an orthonormal basis of $\mathbb{F}^n$, with respect to standard dot product.

Theorem 31. *Let V be an inner product space. Let B be an orthonormal basis of V. Let B' be another basis of V. Let P be the transition matrix from B to B'. Then B' is orthonormal if and only if P is unitary.*

Definition 26. Let $S \subset V$ be a set of vectors.

$$S^\perp \doteq \{v \in V | \langle u, v \rangle = 0, \forall u \in S\}$$

Theorem 32. *$S^{\perp}$ is a subspace of V.*

Definition 27 (Projection)**.** Let V be an inner product space. Let W be a subspace of V. Let $v \in V$. Let $B = \{w_1, \dots, w_m\}$ be a basis of W. The projection of v onto W is defined as follows.

$$\pi_B(v) = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \dots + \frac{\langle v, w_m \rangle}{\langle w_m, w_m \rangle} w_m$$

Theorem 33 (Gram - Schmidt Process)**.**

Step 1 $\widetilde{v_1} = v_1$, *denote* $w_1 = \operatorname{span}\{\widetilde{v_1}\} = \operatorname{span}\{v_1\}$, $B_1 = \{\widetilde{v_1}\}$

Step 2 $\widetilde{v_2} = v_2 - \pi_{B_1}(v_2) = v_2 - \frac{\langle v_2, \widetilde{v_1} \rangle}{\langle \widetilde{v_1}, \widetilde{v_1} \rangle} \widetilde{v_1}$

As $\widetilde{v_2} \perp \widetilde{v_1}$, $B_2 = \{\widetilde{v_1}, \widetilde{v_2}\}$ *is an orthogonal set. Denote* $W_2 = \operatorname{span}\{\widetilde{v_1}, \widetilde{v_2}\} = \operatorname{span}\{v_1, v_2\}$.

Step 3 $\widetilde{v_3} = v_3 - \pi_{B_2}(v_3) = v_3 - \frac{\langle v_2, \widetilde{v_1} \rangle}{\langle \widetilde{v_1}, \widetilde{v_1} \rangle} \widetilde{v_1} - \frac{\langle v_3, \widetilde{v_2} \rangle}{\langle \widetilde{v_2}, \widetilde{v_2} \rangle}$

As $\widetilde{v_3} \in {W_2}^{\perp}$, $B_3 = \{\widetilde{v_1}, \widetilde{v_2}, \widetilde{v_3}\}$ *is an orthogonal set. Denote* $W_2 = \operatorname{span}\{\widetilde{v_1}, \widetilde{v_2}, \widetilde{v_3}\} = \operatorname{span}\{v_1, v_2, v_3\}$.

$\vdots$

Step n The n^{th} *step gives* $\widetilde{B_n} = \{\widetilde{v_1}, \dots, \widetilde{v_n}\}$ *which is an orthogonal basis of* V.

B^0 *is obtained by normalization of* $\widetilde{B_n}$.

$$\begin{aligned} {v_1}^0 &= \frac{1}{\|\widetilde{v_1}\|} \\ &\vdots \\ {v_n}^0 &= \frac{1}{\|\widetilde{v_n}\|} \end{aligned}$$

Theorem 34 (Bessel's Inequality)**.** *Let $\{v_1, \dots, v_m\}$ be an orthonormal set. Let $v \in V$ be any vector. Then*

$$\|v\|^2 \geq |\langle v, v_1 \rangle|^2 + \dots + |\langle v, v_m \rangle|^2$$

and the equality holds if and only if $v \in \operatorname{span}\{v_1, \dots, v_m\}$.

Theorem 35 (Cauchy - Schwarz Inequality)**.** *Let $u, v \in V$ be any vectors. Then*

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$$

and the equality holds if and only if $\{u, v\}$ is linearly dependent.

Theorem 36. *Let W be a subspace of V. Then*

$$V = W \oplus W^{\perp}$$

Corollary 36.1. *Let B be an orthogonal basis of W. Then $\pi_B(v)$ does not depend on the choice of B.*