Recitation 14

Wednesday 21$^{\text{st}}$ January, 2015

Contents

1 Orthogonal Complement

Example 1. Find the orthogonal complement of

$$U = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

with inner product

$$\langle u, v, \rangle = u^T v$$

over $\mathbb{R}^3$.

Solution.

$$\begin{aligned} U^{\perp} &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle = 0 \right\} \\ &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| x + y = y + z = 0 \right\} \\ &= \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\} \end{aligned}$$

Example 2. Find the orthogonal complement of

$$U = \{q(x) \in V | q(x) + q(-x) = 0\}$$

with inner product

$$\langle p(x), q(x) \rangle = \frac{1}{2} \int_{-1}^{1} p(x) q(x) \, \mathrm{d}x$$

over $\mathbb{R}_2[x]$.

Solution.

$$\begin{aligned} U &= \{a + bx + cx^2 | a + bx + cx^2 + a - bx + cx^2 = 0\} \\ &= \operatorname{span}\{x\} \\ \therefore U^{\perp} &= \{a + bx + cx^2 | \langle a + bx + cx^2, x \rangle = 0\} \\ &= \operatorname{span}\{1, x^2\} \end{aligned}$$

Example 3. Find the orthogonal projection of $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on

$$U = \{s \in V | s = s^T\} \subset M_{2\times 2}(\mathbb{R})$$

with respect to

$$\langle A, B \rangle = \text{trace}\left(A^T \cdot B\right)$$

Solution.

$$\begin{aligned} U &= \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} \right\} \\ &= \text{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \end{aligned}$$

$$\begin{aligned} B &= B_U + B_{U^\perp} \\ \therefore B - B_U &= B_{U^\perp} \end{aligned}$$

Therefore,

$$\begin{aligned} \langle B - B_U, s_1 \rangle &= 0 \\ \langle B - B_U, s_2 \rangle &= 0 \\ \langle B - B_U, s_3 \rangle &= 0 \end{aligned}$$

Solving,

$$B_U = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$$

Example 4. Prove that for the Fibonacci series staring with 0 and 1, the n^{th} element is

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right)$$

Solution.

$$\begin{aligned}
\begin{pmatrix} F_1 \\ F_0 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} F_2 \\ F_1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&\vdots \\
\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}
\end{aligned}$$

If $A = P^{-1}DP$,

$$A^n = PD^nP^{-1}$$

Let

$$\begin{aligned}
A &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\
\therefore p_A(\lambda) &= \lambda^2 - \lambda - 1 \\
\therefore \lambda &= \frac{1 \pm \sqrt{5}}{2} \\
&= \phi_\pm
\end{aligned}$$

Therefore,

$$\begin{aligned}
V_{\phi_+} &= \mathrm{N} \begin{pmatrix} 1 - \phi_+ & 1 \\ 1 & -\phi_+ \end{pmatrix} \\
&= \operatorname{span} \left\{ \begin{pmatrix} \phi_+ \\ 1 \end{pmatrix} \right\} \\
V_{\phi_+} &= \mathrm{N} \begin{pmatrix} 1 - \phi_- & 1 \\ 1 & -\phi_- \end{pmatrix} \\
&= \operatorname{span} \left\{ \begin{pmatrix} \phi_- \\ 1 \end{pmatrix} \right\}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
D &= \begin{pmatrix} \phi_+ & 0 \\ 0 & \phi_- \end{pmatrix} \\
P &= \begin{pmatrix} \phi_+ & \phi_- \\ 1 & 1 \end{pmatrix} \\
\therefore P^{-1} &= \frac{1}{\phi_+ - \phi_-} \begin{pmatrix} 1 & -\phi_- \\ -1 & \phi_+ \end{pmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -\phi_- \\ -1 & \phi_+ \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
A^n &= PD^nP^{-1} \\
&= \frac{1}{\sqrt{5}} \begin{pmatrix} {\phi_+}^{n+1} - {\phi_-}^{n+1} & {\phi_+}^{n} \cdot \phi_- - \phi_+ \cdot {\phi_+}^{n+1} \\ {\phi_+}^{n} - {\phi_-}^{n} & {\phi_+}^{n} \cdot \phi_- - \phi_+ \cdot {\phi_-}^{n} \end{pmatrix}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} &= \frac{1}{\sqrt{5}} A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{pmatrix} {\phi_+}^{n+1} - {\phi_-}^{n+1} \\ {\phi_+}^{n} - {\phi_-}^{n} \end{pmatrix} \\
\therefore F_n &= \frac{1}{\sqrt{5}} \left({\phi_+}^{n} - {\phi_-}^{n} \right) \\
&= \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)
\end{aligned}
$$

Example 5. Find the projection of $a = (1, 2, 3)$ on the subspace

$$
U = \left\{ (a_1, a_2, a_3) | a_1 + a_2 + a_3 = 0 \right\}
$$

Solution.

$$
\begin{aligned}
U &= \left\{ (-a_2 - a_3, a_2, a_3) \right\} \\
&= \left\{ -a_2(1, -1, 0) - a_3(1, 0, -1) \right\} \\
&= \text{span} \left\{ (1, -1, 0), (1, 0, -1) \right\}
\end{aligned}
$$

$$a = a_U + a_{u^\perp}$$
$$a_{U^\perp} = a - a_U$$

Therefore,

$$\langle a - a_U, u_1 \rangle = 0$$
$$\langle a - a_U, u_2 \rangle = 0$$

Solving,

$$a_U = (-1, 0, 1)$$