Linear Algebra : Homework 9

Aakash Jog
ID : 989323563

1

a

$$[T]_E = P[T]_B P^{-1}$$

$$b_1 = 1e_1 + 2e_2$$
$$b_2 = 2e_1 - 3e_2$$

Therefore,

$$P = \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix}$$

$$\therefore P^{-1} = \begin{pmatrix} 3/7 & 2/7 \\ 2/7 & -1/7 \end{pmatrix}$$

$$\therefore [T]_E = \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 3/7 & 2/7 \\ 2/7 & -1/7 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$$

b

$$T(x, y) = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 2x - y \\ x + 3y \end{pmatrix}$$

c

$$\text{im}(T) = \text{span}\left\{ \begin{pmatrix} 2 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\}$$

$$2x - y = 0$$
$$x + 3y = 0$$

The matrix is

$$\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore, $\ker T = \{\mathbb{O}\}$.

2

a

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ y \end{pmatrix}$$
$$T : \mathbb{R}^2 \to \mathbb{R}^2$$
$$B = B' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

i

$$T(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$T(e_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore [T]_{B_0, B_0} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$b_1 = 1e_1 + 1e_2$$
$$b_2 = 0e_1 + 1e_2$$

$$[T]_{B,B'} = P^{-1}[T]_{B_0, B_0} P$$

Therefore,

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\therefore P^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\therefore [T]_{B,B'} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

ii

$$\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ -x \end{pmatrix}$$

$$\therefore \text{im} \, T = \left\{ \begin{pmatrix} 2x + y \\ -x \end{pmatrix} \right\}$$

iii

$$\ker T = \{0\}$$

iv

$$\dim(\operatorname{im} T) = 2$$

v

$$\dim(\ker T) = 0$$

b

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - 4y + 9z \\ 5x + 3y + 2z \end{pmatrix}$$

$$T : \mathbb{R}^3 \to \mathbb{R}^2$$

$$B = B' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

i

$$T(e_1) = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$T(e_2) = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$T(e_3) = \begin{pmatrix} 9 \\ 2 \end{pmatrix}$$

$$\therefore [T]_B = \begin{pmatrix} 2 & -4 & 9 \\ 5 & 3 & 2 \end{pmatrix}$$

ii

$$\operatorname{im} T = \operatorname{span} \left\{ \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} -4 \\ 3 \end{pmatrix}, \begin{pmatrix} 9 \\ 2 \end{pmatrix} \right\}$$

iii

$$\ker T = \{2x - 4y + 9z = 5x + 3y + 2z = 0\}$$

iv

$$\dim(\operatorname{im} T) = 2$$

c

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x + 4y \\ 5x - 2y \\ x + 7z \\ 4x \end{pmatrix}$$

$$T : \mathbb{R}^3 \to \mathbb{R}^4$$

$$B = B' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

i

$$T(e_1) = \begin{pmatrix} 3 \\ 5 \\ 1 \\ 4 \end{pmatrix}$$

$$T(e_2) = \begin{pmatrix} 4 \\ -2 \\ 0 \\ 0 \end{pmatrix}$$

$$T(e_3) = \begin{pmatrix} 0 \\ 0 \\ 7 \\ 0 \end{pmatrix}$$

$$\therefore [T]_{B_0, B_0} = \begin{pmatrix} 3 & 4 & 0 \\ 5 & -2 & 0 \\ 1 & 0 & 7 \\ 4 & 0 & 0 \end{pmatrix}$$

ii

$$\operatorname{im} T = \operatorname{span} \left\{ \begin{pmatrix} 3 \\ 5 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 7 \\ 0 \end{pmatrix} \right\}$$

iii

$$\ker T = \{0\}$$

iv

$$\dim(\operatorname{im} T) = 4$$

v

$$\dim(\operatorname{im} T) = 0$$

d

$$T \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = 2x + 3y - 7z + w$$

$$T : \mathbb{R}^4 \to \mathbb{R}$$

$$B = B' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

i

$$T(e_1) = 2$$
$$T(e_2) = 3$$
$$T(e_3) = -7$$
$$T(e_4) = 1$$
$$\therefore [T]_B = \begin{pmatrix} 2 & 3 & -7 & 1 \end{pmatrix}$$

ii

$$\operatorname{im} T = \operatorname{span}\{2, 3, -7, 1\}$$

iii

$$\ker T = \{2x + 3y - 7z + w = 0\}$$

iv

$$\dim(\operatorname{im} T) = 1$$

v

$$\dim(\ker T) = 1$$