Recitation 9

Wednesday 24th December, 2014

Contents

1 Kernel and Image

If $\{u_1, \ldots, u_n\}$ is a basis of U, then $\mathrm{im}\,(T) = \mathrm{span}\{T(u_1), \ldots, T(u_n)\}$

Example 1. Find a basis for $\mathrm{im}\,(T)$.

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \qquad\qquad T(x, y) = (x - y, y - x)$$

Solution.

$$\begin{aligned}\mathrm{im}\,(T) &= \mathrm{span}\{T(e_1), T(e_2)\}\\ &= \mathrm{span}\{T(1, 0), T(0, 1)\}\\ &= \mathrm{span}\{(1, -1), (-1, 1)\}\\ &= \mathrm{span}\{(1, -1)\}\end{aligned}$$

Example 2. Find a basis for $\mathrm{im}\,(T)$.

$$T : \mathbb{R}_2[x] \to M_2(\mathbb{R}) \qquad\qquad T(ax^2 + bx + c) = \begin{pmatrix} b & c \\ -c & a \end{pmatrix}$$

Solution.

$$\begin{aligned}\mathrm{im}\,(T) &= \mathrm{span}\{T(e_1), T(e_2), T(e_3)\}\\ &= \mathrm{span}\{T(1), T(x), T(x^2)\}\\ &= \mathrm{span}\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}\end{aligned}$$

Example 3. Let

$$T : \mathbb{R}^5 \to \mathbb{R}^5$$

Is it possible that $\dim(\mathrm{im}\,(T)) = \dim(\ker(T))$?

Solution.

$$\dim(\mathrm{im}\,(T)) + \dim(\ker(T)) = \dim(V)$$

Therefore, if $\dim(\mathrm{im}\,(T)) = \dim(\ker(T))$, $\dim(\mathrm{im}\,(T)) = \dim(\ker(T)) = \dfrac{5}{2}$. Therefore such a case is not possible.

2 Linear Maps

Definition 1 (Isomorphism)**.** A linear map T is called an isomorphism, if it is one-to-one and onto.

The representation matrix is invertible if and only if the map is an isomorphism.

Example 4.

$$T : \mathbb{R}^3 \to M_{2\times 2}(\mathbb{R}) \qquad\qquad T((x, y, x)) = \begin{pmatrix} y & z \\ -z & x \end{pmatrix}$$

Find the representation matrix of T corresponding to the bases

$$B = \left\{ u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$C = \left\{ v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, u_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, u_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, u_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

Further, find $T(1, -2, 1)$.

Solution.

$$\begin{aligned} [T]_{B,C} &= \big([T(u_1)]_C \quad [T(u_2)]_C \quad [T(u_3)]_C \big) \\ &= \left(\left[\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right]_C \quad \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]_C \quad \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]_C \right) \\ &= \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 0 & -1/2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{aligned}$$

$$[T(1, -2, 1)] = [T]_{B,C}[(1, -2, 1)]_B$$