LINEAR ALGEBRA : HOMEWORK 9

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1.

a.

$$T(f_1(x) + f_2(x)) = \frac{\mathrm{d}}{\mathrm{d}x}(f_1(x) + f_2(x))$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}(f_1(x)) + \frac{\mathrm{d}}{\mathrm{d}x}(f_2(x))$$

$$= T(f_1(x)) + T(f_2(x))$$

$$T(\alpha f(x)) = \frac{\mathrm{d}}{\mathrm{d}x}(\alpha f(x))$$

$$= \alpha \frac{\mathrm{d}}{\mathrm{d}x}f(x)$$

$$= \alpha T(f(x))$$

Therefore, T is linear.

b. $\{1, x, x^2, x^3\}$ is a basis for V.

c. $\{1, x, x^2\}$ is a basis for the range of T with input V.

d.

$$T(1, x, x^2, x^3) = A(1, x, x^2, x^3)$$

$$= (0, 1, 2x, 3x^2)$$

$$\therefore A \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\therefore A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

e.

$$T(k) = 0$$

$$\therefore \ker T = \mathbb{R}$$

f.

$$\dim(\text{range}) + \dim(\text{kernel}) = \dim V$$

is valid for all vector spaces V and all linear transformations.

2.

a.

$$[T]_E = P[T]_B P^{-1}$$

$$b_1 = 1e_1 + 2e_2$$

$$b_2 = 2e_1 - 3e_2$$

Therefore,

$$P = \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix}$$

$$\therefore P^{-1} = \begin{pmatrix} 3/7 & 2/7 \\ 2/7 & -1/7 \end{pmatrix}$$

$$\therefore [T]_E = \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3/7 & 2/7 \\ 2/7 & -1/7 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$$

b.

$$T(x, y) = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 2x - y \\ x + 3y \end{pmatrix}$$

c.

$$\mathrm{im}(T) = \mathrm{span}\left\{\begin{pmatrix} 2 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix}\right\}$$

$$2x - y = 0$$

$$x + 3y = 0$$

The matrix is

$$\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore, $\ker T = \{\mathbb{O}\}$.

3.

a.

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ y \end{pmatrix}$$

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

$$B = B' = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$

i.

$$T(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$T(e_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore [T]_{B_0,B_0} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$b_1 = 1e_1 + 1e_2$$
$$b_2 = 0e_1 + 1e_2$$

$$[T]_{B,B'} = P^{-1}[T]_{B_0,B_0}P$$

Therefore,

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\therefore P^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\therefore [T]_{B,B'} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

ii.

$$\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ -x \end{pmatrix}$$

$$\therefore \operatorname{im} T = \left\{ \begin{pmatrix} 2x + y \\ -x \end{pmatrix} \right\}$$

iii.

$$\ker T = \{0\}$$

iv.

$$\dim(\operatorname{im} T) = 2$$

v.

$$\dim(\ker T) = 0$$

vi. T is one-to-one and onto.

b.

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - 4y + 9z \\ 5x + 3y + 2z \end{pmatrix}$$

$$T : \mathbb{R}^3 \to \mathbb{R}^2$$

$$B = B' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

i.

$$T(e_1) = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$T(e_2) = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$T(e_3) = \begin{pmatrix} 9 \\ 2 \end{pmatrix}$$

$$\therefore [T]_B = \begin{pmatrix} 2 & -4 & 9 \\ 5 & 3 & 2 \end{pmatrix}$$

ii.

$$\operatorname{im} T = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} -4 \\ 3 \end{pmatrix}, \begin{pmatrix} 9 \\ 2 \end{pmatrix} \right\}$$

iii.

$$\ker T = \{2x - 4y + 9z = 5x + 3y + 2z = 0\}$$

iv.

$$\dim(\operatorname{im} T) = 2$$

v. T is onto but not one-to-one as the system of equations for the kernel has more than one solutions.

c.

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x + 4y \\ 5x - 2y \\ x + 7z \\ 4x \end{pmatrix}$$

$$T : \mathbb{R}^3 \to \mathbb{R}^4$$

$$B = B' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

i.

$$T(e_1) = \begin{pmatrix} 3 \\ 5 \\ 1 \\ 4 \end{pmatrix}$$

$$T(e_2) = \begin{pmatrix} 4 \\ -2 \\ 0 \\ 0 \end{pmatrix}$$

$$T(e_3) = \begin{pmatrix} 0 \\ 0 \\ 7 \\ 0 \end{pmatrix}$$

$$\therefore [T]_{B_0,B_0} = \begin{pmatrix} 3 & 4 & 0 \\ 5 & -2 & 0 \\ 1 & 0 & 7 \\ 4 & 0 & 0 \end{pmatrix}$$

ii.

$$\operatorname{im} T = \operatorname{span}\left\{ \begin{pmatrix} 3 \\ 5 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 7 \\ 0 \end{pmatrix} \right\}$$

iii.

$$\ker T = \{0\}$$

iv.

$$\dim(\operatorname{im} T) = 4$$

v.

$$\dim(\operatorname{im} T) = 0$$

d.

$$T \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = 2x + 3y - 7z + w$$

$$T : \mathbb{R}^4 \to \mathbb{R}$$

$$B = B' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

i.

$$T(e_1) = 2$$
$$T(e_2) = 3$$
$$T(e_3) = -7$$
$$T(e_4) = 1$$
$$\therefore [T]_B = \begin{pmatrix} 2 & 3 & -7 & 1 \end{pmatrix}$$

ii.

$$\operatorname{im} T = \operatorname{span}\{2, 3, -7, 1\}$$

iii.

$$\ker T = \{2x + 3y - 7z + w = 0\}$$

iv.

$$\dim(\operatorname{im} T) = 1$$

v.

$$\dim(\ker T) = 1$$