NUMERICAL ANALYSIS : ASSIGNMENT 4

AAKASH JOG
ID : 989323563

Exercise 1.
Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = e^{-x}$. Consider $n+1$ sample point $\{x_0, \dots, x_n\}$ in $[0,1]$, defined by

$$x_k = kh$$
$$h = \frac{1}{n}$$

where $k \in \{0, \dots, n\}$.
Consider the piecewise linear interpolation $l(x)$ based on this data.

(1) Bound $\left|e(x)\right| = \left|f(x) - l(x)\right|$ for $x \in [0,1]$.
(2) How many samples do we need in order to guarantee an interpolation error bounded by 10^{-8}?

Solution 1.
(1)

$$\begin{aligned}
\left|e(x)\right| &\le \frac{h^2}{8} \max_{t\in[0,1]} \left|f''(x)\right| \\
&\le \frac{1}{8n^2} \left|\left(e^{-x}\right)''\right| \\
&\le \frac{1}{8n^2} \left|e^{-x}\right| \\
&\le \frac{1}{8n^2}
\end{aligned}$$

(2)

$$\begin{aligned}
\left|e(x)\right| &\le \frac{1}{8n^2} \\
\therefore 10^{-8} &\le \frac{1}{8n^2} \\
\therefore n^2 &\ge \frac{10^8}{8} \\
\therefore n &\ge \frac{10^4}{2\sqrt{2}} \\
\therefore n &\ge 3536
\end{aligned}$$

Therefore, we need 3536 samples to guarantee an interpolation error bounded by 10^{-8}.

Date: Tuesday 10^{th} November, 2015.

Exercise 2.
Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \cos(x)$. Consider $n+1$ sample point $\{x_0, \ldots, x_n\}$ in $[0,1]$, defined by

$$\begin{aligned} x_k &= kh \\ h &= \frac{1}{n} \end{aligned}$$

where $k \in \{0, \ldots, n\}$, where n is even. Consider the piecewise quadratic interpolation $l(x)$ based on this data. Namely, we approximate $f(x)$ at each $x \in [x_{2k}, x_{2k+2}]$ by the quadratic interpolating polynomial based on the samples $\{x_{2k}, x_{2k+1}, x_{2k+2}\}$.

(1) Bound $\left|e(x)\right| = \left|f(x) - l(x)\right|$ for $x \in [0, \pi]$.
(2) How many samples do we need in order to guarantee an interpolation error bounded by 9×10^{-5}?

Solution 2.
(1)

$$\begin{aligned} \left|e(x)\right| &\leq \frac{\sqrt{8}}{27} \frac{\pi^3}{4n^3} \frac{1}{3!} \max_{t \in [0,\pi]} \left|f'''(x)\right| \\ &\leq \frac{\sqrt{2}\pi^3}{486} \max_{t \in [0,\pi]} \left|\left(\cos(x)\right)'''\right| \\ &\leq \frac{\sqrt{2}\pi^3}{486} \end{aligned}$$

Exercise 3.
Let f be infinitely differentiable in $[a, b]$. Assume that there exists $M > 0$ such that

$$\max_{a \leq x \leq b} \left|f^{(k)}(x)\right| < M^k$$

for any $k = \mathbb{N}$. Show that the error in the interpolating polynomial satisfies

$$\begin{aligned} \lim_{n \to \infty} \left|e_n(x)\right| &= \lim_{n \to \infty} \left|f(x) - p_n(x)\right| \\ &= 0 \end{aligned}$$

Solution 3.

$$\begin{aligned} \lim_{n \to \infty} \left|e_n(x)\right| &= \lim_{n \to \infty} \left|f(x) - p_n(x)\right| \\ &= \lim_{n \to \infty} \left|\frac{f^{(n+1)}(c)}{(n+1)!}\right| \left|\prod_{i=0}^{n} (x - x_i)\right| \\ &\leq \lim_{n \to \infty} \frac{M^{n+1}}{(n+1)!} (b-a)^{n+1} \end{aligned}$$

$$\therefore \lim_{n \to \infty} \left|e(x)\right| = 0$$

$\square$

Exercise 4.

Let $f(x) = x^{\frac{99}{97}}$ in $[-h, h]$. Bound the linear interpolation error based on the samples $\{-h, h\}$, and show that the error is of order $\mathrm{O}\left(h^{\frac{99}{97}}\right)$.

Solution 4.

$$\begin{aligned}
\left|e(x)\right| &\leq \left|\frac{f''(c)}{2!}\right| \left|(x+h)(x-h)\right| \\
&\leq \frac{1}{2}\left|\frac{99}{97} \cdot \frac{2}{97} \cdot c^{-\frac{95}{97}}\right| \left|x^2 - h^2\right| \\
&\leq \left|\frac{99}{(97)^2} c^{-\frac{95}{97}}\right| h^2 \\
&\leq \left|\frac{99}{(97)^2} c^{-\frac{95}{97}}\right| h^{\frac{99}{97}}
\end{aligned}$$

Therefore, $e(x)$ is of order $\mathrm{O}\left(h^{\frac{99}{97}}\right)$.