NUMERICAL ANALYSIS : ASSIGNMENT 8

AAKASH JOG
ID : 989323563

Exercise 1.
Let

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\
B &= \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}
\end{aligned}
$$

Calculate the condition numbers of A and B with respect to the 1, 2, and ∞ norms,

Solution 1.

$$
A^{-1} = \begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix}
$$

Therefore,

$$
\begin{aligned}
\|A\|_1 &= \max\left\{(1+3), (2+4)\right\} \\
&= 6 \\
{\|A\|^{-1}}_1 &= \max\left\{(2+1.5), (1+0.5)\right\} \\
&= 3.5
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\operatorname{cond}(A)_1 &= \|A\|_1 \left\|A^{-1}\right\|_1 \\
&= 21
\end{aligned}
$$

$$
\begin{aligned}
A^{\mathsf{T}} &= \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \\
\therefore A^{\mathsf{T}} A &= \begin{pmatrix} 10 & 14 \\ 14 & 20 \end{pmatrix}
\end{aligned}
$$

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Therefore,

$$\begin{aligned}\rho(A) &= \frac{3345}{112}\\ \therefore \|A\|_2 &= \sqrt{\frac{3345}{112}}\\ &= 5.4650\end{aligned}$$

Therefore,

$$\begin{aligned}{A^{-1}}^{\mathsf{T}} &= \begin{pmatrix}-2 & 1.5\\ 1 & -0.5\end{pmatrix}\\ \therefore {A^{-1}}^{\mathsf{T}}A^{-1} &= \begin{pmatrix}6.25 & -2.75\\ -2.75 & 1.25\end{pmatrix}\end{aligned}$$

Therefore,

$$\begin{aligned}\rho\left(A^{-1}\right) &= \frac{3345}{448}\\ \therefore \|A\|_2 &= \sqrt{\frac{3345}{448}}\\ &= 2.7325\end{aligned}$$

Therefore,

$$\begin{aligned}\operatorname{cond}(A)_2 &= \|A\|_2 \left\|A^{-1}\right\|_2\\ &= 14.9331125\end{aligned}$$

$$\begin{aligned}\|A\|_\infty &= \max\left\{(1+2),(3+4)\right\}\\ &= 7\\ \|A\|^{-1}{}_\infty &= \max\left\{(2+1),(1.5+0.5)\right\}\\ &= 3\end{aligned}$$

Therefore,

$$\begin{aligned}\operatorname{cond}(A)_\infty &= \|A\|_\infty \left\|A^{-1}\right\|_\infty\\ &= 21\end{aligned}$$

$$B^{-1} = \begin{pmatrix}\frac{8}{21} & -\frac{3}{21} & \frac{1}{21}\\ -\frac{3}{21} & \frac{9}{21} & -\frac{3}{21}\\ \frac{1}{21} & -\frac{3}{21} & \frac{8}{21}\end{pmatrix}$$

Therefore,

$$\begin{aligned}\|B\|_1 &= 5\\ \left\|B^{-1}\right\|_1 &= \frac{5}{7}\end{aligned}$$

Therefore,

$$\operatorname{cond}(B)_1 = \frac{25}{7}$$

Therefore,

$$\|B\|_2 = 3 + \sqrt{2}$$

$$\left\|B^{-1}\right\|_2 = \frac{3 + \sqrt{17}}{14}$$

Therefore,

$$\operatorname{cond}(B)_1 = \frac{\left(3 + \sqrt{2}\right)\left(3 + \sqrt{17}\right)}{14}$$

Therefore,

$$\|B\|_\infty = 5$$

$$\left\|B^{-1}\right\|_\infty = \frac{5}{7}$$

Therefore,

$$\operatorname{cond}(B)_\infty = \frac{25}{7}$$

Exercise 2.
Let

$$A = \begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix}$$

$$b = \begin{pmatrix} 1.005 \\ 0.995 \end{pmatrix}$$

$$e_b = \begin{pmatrix} 0.05 \\ -0.05 \end{pmatrix}$$

Let x be the solution to the system

$$Ax = b$$

Assume that the right hand side of the system is noisy, and we actually solve the system

$$A\tilde{x} = \tilde{b}$$

where

$$\tilde{b} = b - e_b$$

Note that this is not the question from class. In class we assumed that the RHS is accurate, but the solution to the system is calculated inaccurately by a numerical method. In this exercise we assume that we know the RHS up to an error, and we calculate the accurate solution of the system with a noisy RHS.

(1) Calculate the condition number of A with respect to the ∞ norm.
(2) Calculate the relative error in the RHS, in the ∞ norm.
(3) Without solving the system, find a bound on the relative error in the solution x, in the ∞ norm.

(4) Calculate x and $\tilde{x}$, and calculate the relative error, in the ∞, in the solution x. How does this error relate to the bound from 3?

Solution 2.

(1)

$$A = \begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix}$$
$$\therefore A^{-1} = \begin{pmatrix} -98 & 99 \\ 99 & -100 \end{pmatrix}$$

Therefore,

$$\begin{aligned}\|A\|_\infty &= \max\left\{(100+99), (99+98)\right\} \\ &= 199 \\ \left\|A^{-1}\right\|_\infty &= \max\left\{(98+99), (99+100)\right\} \\ &= 199\end{aligned}$$

Therefore,

$$\begin{aligned}\operatorname{cond}(A) &= 199^2 \\ &= 39601\end{aligned}$$

(2) The relative error in the RHS, in the ∞ norm, is

$$\begin{aligned}\frac{\|e_b\|_\infty}{\|b\|_\infty} &= \frac{0.05}{1.005} \\ &\approx 0.05\end{aligned}$$

(3)

$$\begin{aligned}\frac{\|e_x\|_\infty}{\|x\|_\infty} &\leq \operatorname{cond}(A)\frac{\|e_b\|_\infty}{\|b\|_\infty} \\ &= \left(199^2\right)(0.05) \\ &\approx 1980\end{aligned}$$

(4)

$$\begin{aligned}x &= A^{-1}b \\ &= \begin{pmatrix} -98 & 99 \\ 99 & -100 \end{pmatrix}\begin{pmatrix} 1.005 \\ 0.995 \end{pmatrix}\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{x} &= A^{-1}\tilde{b} \\ &= A^{-1}(b - e_b) \\ &= \begin{pmatrix} -98 & 99 \\ 99 & -100 \end{pmatrix}\begin{pmatrix} 1.005 - 0.05 \\ 0.995 + 0.05 \end{pmatrix} \\ &= \begin{pmatrix} 9.865 \\ -9.955 \end{pmatrix}\end{aligned}$$

Therefore,

$$\frac{\|x - \tilde{x}\|_\infty}{\|x\|_\infty} = 664$$

Therefore, the error is approximately $\frac{1}{3}$ of the bound from 3.

Exercise 3.

Let A be a matrix, and x_0 be a vector. Consider the iterative method

$$x_{n+1} = Ax_n$$

(1) Prove

$$x_n = A^n x_0$$

(2) Assume that $\|A\| < 1$ in some norm. Prove

$$\lim_{n\to\infty} x_n = \overrightarrow{0}$$

You may use the proposition

$$\begin{aligned} &\lim_{n\to\infty} x_n = \overrightarrow{0} \\ \iff &\lim_{n\to\infty} \|x_n\| = 0 \end{aligned}$$

(3) Let

$$A = \begin{pmatrix} 0 & 2 \\ \frac{1}{3} & 0 \end{pmatrix}$$

Calculate the 1, 2, and ∞ norms of A. Can you use section 2, in order to prove or disprove $\lim_{n\to\infty} x_n = \overrightarrow{0}$?

(4) Calculate A^2, and the ∞ norm of A^2.

(5) Use A^2 and $\left\|A^2\right\|_\infty$ to show

$$\lim_{n\to\infty} x_n = \overrightarrow{0}$$

Solution 3.

(1)

$$\begin{aligned} x_{n+1} &= Ax_n \\ &= A(Ax_{n-1}) \\ &\vdots \\ &= A^n x_0 \end{aligned}$$

(2)

$$\begin{aligned} \lim_{n\to\infty} \|x_n\| &= \lim_{n\to\infty} \|A^n x_0\| \\ &= \lim_{n\to\infty} \|A\|^n \|x_0\| \\ &= 0 \end{aligned}$$

Therefore, by the proposition,

$$\lim_{n\to\infty} x_n = \overrightarrow{0}$$

(3)

$$A = \begin{pmatrix} 0 & 2 \\ \frac{1}{3} & 0 \end{pmatrix}$$

Therefore,

$$\|A\|_1 = 2$$

Therefore,

$$\|A\|_2 = 2$$

Therefore,

$$\|A\|_\infty = 2$$

As $\|A\| > 1$, section 2 cannot be used. Hence, the limit is zero if and only if $x_n = \overrightarrow{0}$.

(4)

$$A = \begin{pmatrix} 0 & 2 \\ \frac{1}{3} & 0 \end{pmatrix}$$

$$\therefore A^2 = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}$$

Therefore,

$$\left\|A^2\right\|_\infty = \frac{2}{3}$$

Therefore, using section 2,

$$\lim_{n\to\infty} x_n = \overrightarrow{0}$$