NUMERICAL ANALYSIS : ASSIGNMENT 1

AAKASH JOG
ID : 989323563

Exercise 2.
Reformulate the following functions, if needed, to avoid loss of significant digits.

(1) $f(x) = \sqrt{1+x} - \sqrt{x}$ for $x >> 1$.
(2) $f(x) = \sqrt{1+x} - \sqrt{x}$ for $x \approx 0$.
(3) $f(x) = \sqrt{1+x} - 1$ for $x >> 1$.
(4) $f(x) = \sqrt{1+x} - 1$ for $x \approx 0$.
(5) $f(x) = \ln(x+1) - \ln(x)$ for $x >> 1$.
(6) $f(x) = \ln(x+1) - \ln(x)$ for $x \approx 0$.

Solution 2.

(1) As $x >> 1$,

$$\begin{aligned} f(x) &= \sqrt{1+x} - \sqrt{x} \\ &= \sqrt{x} - \sqrt{x} \\ &= 0 \\ &= f(x+1) \end{aligned}$$

Therefore, there is a loss of significant digits.
Therefore, the function needs to be reformulated.

$$\begin{aligned} f(x) &= \sqrt{1+x} - \sqrt{x} \\ &= \frac{1}{\sqrt{1+x} + \sqrt{x}} \end{aligned}$$

This expression can never be zero, hence, there is no loss of significant digits.

(2) As $x \approx 0$,

$$\begin{aligned} f(x) &= \sqrt{1+x} - \sqrt{x} \\ &= \sqrt{1} \\ &= 1 \\ &= f(x+1) \end{aligned}$$

Therefore, there is a loss of significant digits.
Therefore, the function needs to be reformulated.

$$f(x) = \left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \dots\right) - \sqrt{x}$$

Date: Tuesday 20^{th} October, 2015.

As this expression is unique for every x, there is no loss of significant digits.

(3) As $x >> 1$,

$$\begin{aligned} f(x) &= \sqrt{1+x} - 1 \\ &= \sqrt{x} - 1 \\ f(x+1) &= \sqrt{2+x} - 1 \\ &= \sqrt{x} - 1 \\ \therefore f(x) &= f(x+1) \end{aligned}$$

Therefore, there is a loss of significant digits.
Therefore, the function needs to be reformulated.

$$\begin{aligned} f(x) &= \sqrt{1+x} - 1 \\ &= \frac{x}{\sqrt{1+x}+1} \\ \therefore f(x+1) &= \frac{x+1}{\sqrt{2+x}+1} \\ &\neq f(x) \end{aligned}$$

Hence, in this formulation, there is no loss of significant digits.

(4) As $x \approx 0$,

$$\begin{aligned} f(x) &= \sqrt{1+x} - 1 \\ &= 0 \end{aligned}$$

Therefore, there is a loss of significant digits.
Therefore, the function needs to be reformulated.

$$\begin{aligned} f(x) &= \sqrt{1+x} - 1 \\ &= \left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \dots\right) + 1 \\ &= \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \dots \end{aligned}$$

Hence, in this formulation, there is no loss of significant digits.

(5) As $x >> 1$,

$$\begin{aligned} f(x) &= \ln(x+1) - \ln(x) \\ &= \ln\left(\frac{x+1}{x}\right) \\ &= \ln(1) \\ &= f(x+1) \end{aligned}$$

Therefore, there is a loss of significant digits.
Therefore, the function needs to be reformulated.

$$\begin{aligned} f(x) &= \ln\left(\frac{x+1}{x}\right) \\ &= \ln\left(1+\frac{1}{x}\right) \end{aligned}$$

Hence, in this formulation, there is no loss of significant digits.

(6) As $x \approx 0$,

$$\begin{aligned} f(x) &= \ln(x+1) - \ln(x) \\ &= \ln\left(\frac{x+1}{x}\right) \\ &= \ln(1) \\ &= f(x+1) \end{aligned}$$

Therefore, there is a loss of significant digits.
Therefore, the function needs to be reformulated.

$$f(x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right) - \ln(x)$$

Hence, in this formulation, there is no loss of significant digits.

Exercise 3.
Let $\tilde{x}$ and $\tilde{y}$ be approximations of the numbers x and y, respectively.
Show

(1)

$$\left|e_{\frac{x}{y}}\right| \leq \frac{|x||e_y| + |y||e_x|}{|y||\tilde{y}|}$$

Solution 3.

(1)

$$
\begin{aligned}
\left|e_{\frac{x}{y}}\right| &= \left|\frac{x}{y} - \frac{\tilde{x}}{\tilde{y}}\right| \\
&= \left|\frac{x\tilde{y} - \tilde{x}y}{y\tilde{y}}\right| \\
&= \left|\frac{x(y - e_y) - (x - e_x)y}{y\tilde{y}}\right| \\
&= \left|\frac{xy - xe_y - xy + ye_x}{y\tilde{y}}\right| \\
&= \left|\frac{xe_y - ye_x}{y\tilde{y}}\right| \\
&= \frac{|xe_y - ye_x|}{|y\tilde{y}|} \\
&\leq \frac{|x||e_y| + |y||e_x|}{|y||\tilde{y}|}
\end{aligned}
$$

$\square$