NUMERICAL ANALYSIS : ASSIGNMENT 4

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Exercise 1.
Consider the stopping criterion

$$\left|\frac{f(x_n)}{f'(x_n)}\right| < \varepsilon$$

Show that in Newton's method this criterion is equivalent to

$$|x_{n+1} - x_n| < \varepsilon$$

Solution 1.

$$\left|\frac{f(x_n)}{f'(x_n)}\right| < \varepsilon$$

By Newton's method,

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ \therefore x_{n+1} - x_n &= -\frac{f(x_n)}{f'(x_n)} \\ \therefore |x_{n+1} - x_n| &= \left|\frac{f(x_n)}{f'(x_n)}\right| \\ \therefore |x_{n+1} - x_n| &\leq \varepsilon \end{aligned}$$

□

Exercise 2.
Let $x_{n+1} = g(x_n)$ be a fixed point method, and assume that $x_n \to a$ as $n \to \infty$, where $g(a) = a$.

(1) Let $e_n = a - x_n$. Show

$$x_{n+1} - x_n = e_n - e_{n+1}$$

(2) Show that

$$e_{n+1} = g'(c_n)e_n$$

for some c_n between a and x_n. Hint: Use the mean value theorem.

(3) Show that

$$e_n \approx \frac{x_{n+1} - x_n}{1 - g'(x_n)}$$

and formulate a corresponding stopping criterion.

(4) Formulate a stopping criterion based on the above.

Date: Tuesday 10^{th} November, 2015.

(5) When is the stopping criterion

$|x_{n+1} - x_n| < \varepsilon$

a good approximation to the theoretical stopping criterion $|e_n| < \varepsilon$?

Solution 2.

(1)

$$\begin{aligned} x_{n+1} - x_n &= (a - e_{n+1}) - (a - e_n) \\ &= e_n - e_{n+1} \end{aligned}$$

□

(2) By Lagrange's Mean Value theorem, $\exists c_n \in (a, x_n)$, such that

$$\begin{aligned} g'(c_n) &= \frac{g(x_n) - g(a)}{x_n - a} \\ &= \frac{x_{n+1} - a}{x_n - a} \\ &= \frac{e_{n+1}}{e_n} \\ \therefore e_{n+1} &= g'(c_n)e_n \end{aligned}$$

□

(3)

$$\begin{aligned} g'(c_n) &= \frac{g(a) - g(x_n)}{a - x_n} \\ \therefore 1 - g'(c_n) &= 1 - \left(\frac{g(a) - g(x_n)}{a - x_n}\right) \\ \therefore 1 - g'(c_n) &= 1 - \left(\frac{g(a) - g(x_n)}{e_n}\right) \\ \therefore e_n(x) &= \frac{x_{n+1} - x_n}{1 - g'(x_n)} \end{aligned}$$

(4) The corresponding stopping criterion is

$|e_n| < \varepsilon$

(5) If $g'(a) = 0$ or $g'(a) = 2$,

$$\big|e_n(x)\big| = \left|\frac{x_{n+1} - x_n}{1}\right|$$

Therefore, this is a good approximation to the theoretical stopping criterion.

Exercise 3.

Let

$$f(x) = e^{-x} - \frac{1}{2}$$

(1) Show that f has a root in $[0, 1]$.
(2) Show that Newton's method converges to the root a of f.

(3) (a) How many points are required in order to guarantee an absolute error bounded by 10^{-2}?
(b) How many points are required in order to guarantee an absolute error bounded by 10^{-7}?
(4) How many iterations are performed if the above stopping criterion, to guarantee an error bounded by 10^{-2} and 10^{-7}?
(5) Cheat and calculate $\ln(2)$, i.e. the root of f. What is a better estimate for the required number of iterations, the worst case bound of section 3 or the error estimate of section 4? Explain why one of them is better than the other. Base you explanation on the notion of the rate of convergence.

Solution 3.
(1)

$$\begin{aligned} f(0) &= 1 - \frac{1}{2} \\ &> 0 \end{aligned}$$

$$\begin{aligned} f(1) &= \frac{1}{e} - \frac{1}{2} \\ &< 0 \end{aligned}$$

Therefore, by the intermediate value theorem, $\exists c \in [0,1]$ such that $f(c) = 0$. Hence f has a root in $[0,1]$. □

(2)

$$\begin{aligned} g(x) &= x - \frac{f(x)}{f'(x)} \\ &= x + 1 - \frac{e^x}{2} \end{aligned}$$

As $g'(x) = 1 - \frac{e^x}{2}$, $g : [0,1] \to [0,1]$. Therefore, as $n \to \infty$, $x_n \to a$. □

(3) (a) By the fixed point theorem,

$$\begin{aligned} |e_n| &\leq \left(\frac{1}{2}\right)^2 |e_0| \\ &\leq 2^{-n} \end{aligned}$$

Therefore,

$$\begin{aligned} 10^{-2} &\geq 2^{-n} \\ \therefore 10^2 &\leq 2^n \\ \therefore n &\geq 2\log_2 10 \\ \therefore n &\geq 7 \end{aligned}$$

Therefore, 7 points are required in order to guarantee an error bounded by 10^{-2}.

(b)

$$\begin{aligned}
10^{-7} &\geq 2^{-n} \\
\therefore 10^{7} &\leq 2^{n} \\
\therefore n &\geq 7\log_2 10 \\
\therefore n &\geq 24
\end{aligned}$$

Therefore, 24 points are required in order to guarantee an error bounded by 10^{-7}.

(4)

$$\begin{aligned}
x_0 &= 0.5 \\
x_1 &= 0.6756393 \\
x_2 &= 0.6929948 \\
x_3 &= 0.6931471 \\
x_4 &= 0.6931471
\end{aligned}$$

Therefore, to guarantee an error bounded by 10^{-2} and 10^{-7}, 2 and 3 iterations, respectively, are required.

(5)

$$\ln(2) = 0.6931471$$

Therefore, the error estimate of section 4 is better, as for it, $n = 3$ and $\varepsilon = 10^{-7}$. In Newton's method, $g'(a) = 0$. Hence, the convergence is expected to be faster.

Exercise 4.

We want to approximate a root a of $f(x) = x - e^{-x^2}$, in $[0, 1]$.

(1) Show that there exists such a root.
(2) We define the iterative method

$$\begin{aligned}
x_{n+1} &= g(x_n) \\
&= e^{-{x_n}^2}
\end{aligned}$$

Show that any root of f is a fixed point of g and vice-versa.
(3) Show that the root of f is unique in $[0, 1]$, and that the iterative method converges to it.
(4) Let $x_0 \in [0, 1]$ be an initial guess. How many iterations are required in order to guarantee an error bounded by 0.05?

Solution 4.

(1)

$$\begin{aligned}
f(0) &= 0 - 1 \\
&< 0
\end{aligned}$$

$$\begin{aligned}
f(1) &= 1 - \frac{1}{e} \\
&> 0
\end{aligned}$$

Therefore, by the intermediate value theorem, there exists such a root a in $[0, 1]$.

(2)

$$\begin{aligned} x_{n+1} &= g(x_n) \\ &= e^{-{x_n}^2} \end{aligned}$$

Therefore,

$$\begin{aligned} g(x) &= x \\ \iff x &= e^{-x^2} \\ \iff x - e^{-x^2} &= 0 \\ \iff f(x) &= 0 \end{aligned}$$

(3)

$$\begin{aligned} g(x) &= e^{-x^2} \\ \therefore g'(x) &= -2xe^{-x^2} \end{aligned}$$

Therefore,

$$\begin{aligned} g(0) &= 1 \\ &\leq 1 \\ g(1) &= \frac{1}{e} \\ &\leq 1 \end{aligned}$$

$$\begin{aligned} \left|g'(x)\right| &= \left|-2xe^{-x^2}\right| \\ &= 2xe^{-x^2} \\ &\leq \sqrt{\frac{2}{e}} \\ &< 1 \end{aligned}$$

Therefore, by the fixed point theorem, $x_n \to a$, as $n \to \infty$, and a is unique.

(4)

$$\begin{aligned} |e_n| &\leq k^n |e_0| \\ &\leq \left(\sqrt{\frac{2}{e}}\right)^n |e_0| \\ &\leq \left(\sqrt{\frac{2}{e}}\right)^n \end{aligned}$$

Therefore,

$$\left(\sqrt{\frac{2}{e}}\right)^n < 5 \times 10^{-2} \therefore n \qquad \geq 20$$

Therefore, at least 20 iterations are required to guarantee an error bounded by 0.05.