# Numerical Analysis : Review Session

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# Contents

**Exercise 1.**
Let $f : [0,1] \to \mathbb{R}$. Consider the partition $0 < \frac{1}{N} < \dots < \frac{N-1}{N} < 1$, for $N \in \mathbb{N}$. Consider the space $PC$, i.e. the space of piecewise constant functions on this partition. Therefore, $g \in PC$, if it is constant on each $\left[\frac{n}{N}, \frac{n+1}{N}\right]$, where $n = 0, \dots, N-1$. Find a formula for $g \in PC$, closest to $f$, in the sense that $\int\limits_0^1 |f(x) - g(x)|^2 \,\mathrm{d}x$ is minimal.

**Solution 1.**
Let

$$\varphi_n(x) = \begin{cases} 1 & ; \quad \frac{n}{N} \le x \le \frac{n+1}{N} \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Therefore, the set of all $\varphi_n(x)$ form a basis of the space. Therefore, let

$$g(x) = \sum_{n=0}^{N-1} c_n \varphi_n$$

where $c_n \in \mathbb{R}$. Therefore, the Gram matrix is

$$\begin{pmatrix} \langle \varphi_0, \varphi_0 \rangle & \dots & \langle \varphi_0, \varphi_{N-1} \rangle \\ \vdots & & \\ \langle \varphi_{N-1}, \varphi_0 \rangle & \dots & \langle \varphi_{N-1}, \varphi_{N-1} \rangle \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ c_{N-1} \end{pmatrix} = \begin{pmatrix} \langle f, \varphi_0 \rangle \\ \vdots \\ \langle f, \varphi_{N-1} \rangle \end{pmatrix}$$

Therefore, as

$$\int\limits_0^1 |f(x) - g(x)|^2 \,\mathrm{d}x = \|f - g\|^2$$

Therefore,

$$\langle \alpha, \beta \rangle = \int\limits_0^1 \alpha(x) \overline{\beta(x)} \,\mathrm{d}x$$

Therefore,

$$\langle \varphi_n, \varphi_m \rangle = \int\limits_0^1 \varphi_n(x) \varphi_m(x) \,\mathrm{d}x$$

If $n \neq m$,

$$\int_0^1 \varphi_n(x)\varphi_m(x)\,\mathrm{d}x = 0$$

If $n = m$,

$$\begin{aligned}\int_0^1 \varphi_n(x)\varphi_m(x)\,\mathrm{d}x &= \int_0^1 \varphi_n(x)\varphi_n(x)\,\mathrm{d}x \\ &= \int_{\frac{n}{N}}^{\frac{n+1}{N}} 1\,\mathrm{d}x \\ &= \frac{1}{N}\end{aligned}$$

Therefore,

$$\langle \varphi_n, \varphi_m \rangle = \begin{cases} 0 & ; \quad n \neq m \\ \frac{1}{N} & ; \quad n = m \end{cases}$$

Therefore, Similarly,

$$\langle f, \varphi_n \rangle = \int_{\frac{n}{N}}^{\frac{n+1}{N}} f(x)\,\mathrm{d}x$$

Therefore,

$$\begin{pmatrix} \frac{1}{N} & \dots & 0 \\ \vdots & & \\ 0 & \dots & \frac{1}{N} \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ c_{N-1} \end{pmatrix} = \begin{pmatrix} \int_0^{\frac{1}{N}} f(x)\,\mathrm{d}x \\ \vdots \\ \int_{\frac{N-1}{N}}^{1} f(x)\,\mathrm{d}x \end{pmatrix}$$

$$\therefore \begin{pmatrix} c_0 \\ \vdots \\ c_{N-1} \end{pmatrix} = \begin{pmatrix} N & \dots & 0 \\ \vdots & & \\ 0 & \dots & N \end{pmatrix} \begin{pmatrix} \int_0^{\frac{1}{N}} f(x)\,\mathrm{d}x \\ \vdots \\ \int_{\frac{N-1}{N}}^{1} f(x)\,\mathrm{d}x \end{pmatrix}$$

Therefore,

$$g(x) = \sum_{n=0}^{N-1} N \left( \int\limits_{\frac{n}{N}}^{\frac{n+1}{N}} f(t)\,\mathrm{d}t \right) \varphi_n(x)$$

**Exercise 2.**
Consider the following system.

$$\begin{pmatrix} 1 & \frac{1}{4} \\ p & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

1. Write the Gauss-Seidel iteration scheme.
2. Find all values of $p$ for which the method converges, for any $x^{(0)}$.
3. Given that $p = \frac{1}{4}$, how many iterations are required in order to reduce the initial error $\left\| x - x^{(0)} \right\|$ by six orders of magnitude, i.e. $10^{-6}$.

**Solution 2.**
1. For Gauss-Seidel,

$$x^{(n+1)} = Bx^{(n)} + d$$

where

$$B = -(L+D)^{-1}U$$
$$d = (L+D)^{-1}b$$

Therefore,

$$L + D = \begin{pmatrix} 1 & 0 \\ p & \frac{1}{2} \end{pmatrix}$$
$$\therefore (L+D)^{-1} = \begin{pmatrix} 1 & 0 \\ p & \frac{1}{2} \end{pmatrix}^{-1} = 2 \begin{pmatrix} \frac{1}{2} & 0 \\ -p & 1 \end{pmatrix}$$

Therefore,

$$B = \begin{pmatrix} -1 & 0 \\ 2p & -2 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{4} \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -\frac{1}{4} \\ 0 & \frac{p}{2} \end{pmatrix}$$
$$d = \begin{pmatrix} 1 & 0 \\ -2p & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 - 2p \end{pmatrix}$$

Therefore,

$$x^{(n+1)} = \begin{pmatrix} 0 & -\frac{1}{4} \\ 0 & \frac{p}{2} \end{pmatrix} x^{(n)} + \begin{pmatrix} 1 \\ 2-2p \end{pmatrix}$$

2. For the method to converge,

$$\rho(B) < 1$$

where $\rho$ is the spectral radius of $B$.
Therefore,

$$\begin{aligned} \det(\lambda I - B) &= \det \begin{pmatrix} \lambda & \frac{1}{4} \\ 0 & \lambda - \frac{p}{2} \end{pmatrix} \\ &= \lambda \left( \lambda - \frac{p}{2} \right) \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= \frac{p}{2} \end{aligned}$$

Therefore, as

$$\rho(B) < 1$$

Therefore,

$$\begin{aligned} |\lambda_2| &< 1 \\ \therefore |p| &< 2 \end{aligned}$$

3.

$$\begin{aligned} e^{(n)} &= x - x^{(n)} \\ &= B^n e^{(0)} \end{aligned}$$

Therefore,

$$\begin{aligned} \left\| e^{(n)} \right\| &= \left\| B^n e^{(0)} \right\| \\ &\leq \|B^n\| \left\| e^{(0)} \right\| \\ &\leq \|B\|^n \left\| e^{(0)} \right\| \end{aligned}$$

The infinity norm of $B$ is

$$\|B\|_\infty = \frac{1}{4}$$

Therefore,

$$\left\|e^{(n)}\right\| \leq 2^{-2n} \left\|e^{(0)}\right\|$$

Therefore, for the error to be less than $10^{-6}$,

$$2^{-2n} < 10^{-6}$$

Therefore,

$$n = 10$$

**Exercise 3.**
Given

$$\begin{aligned} (x_i, f(x_i)) &= ((1,3),(2,16),(3,11)) \\ \left|f^{(3)}(x)\right| &< 5 \\ \left|f^{(4)}(x)\right| &< 4 \\ \left|f^{(5)}(x)\right| &< 3 \end{aligned}$$

1. What is the degree of the interpolation polynomial?

2. Find an upper bound for the error.

3. What is the degree of the interpolation polynomial after adding the sample $(4, 27)$?

4. Let $l_i(x)$ be the Lagrange polynomials based on the four points above. Find the degree of

$$Q(x) = l_1(x) + 2l_2(x) + 3l_3(x) + 4l_4(x)$$

**Solution 3.**

1.

$$
\begin{aligned}
f[x_0] &= 3 \\
f[x_1] &= 16 \\
f[x_2] &= 11
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f[x_0, x_1] &= \frac{3-16}{1-2} \\
&= 13 \\
f[x_1, x_2] &= \frac{16-11}{2-3} \\
&= -5
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f[x_0, x_1, x_2] &= \frac{13-(-5)}{1-3} \\
&= -9
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
p(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\
&= 3 + 13(x-1) - 9(x-1)(x-2)
\end{aligned}
$$

Therefore, the degree of the interpolating polynomial is 2.

2.

$$
\begin{aligned}
e &= f[x_0, x_1, x_2, x](x - x_1)(x - x_2)(x - x_3) \\
&= f[1, 2, 3, x](x-1)(x-2)(x-3) \\
&\leq \frac{f^{(3)}(\xi)}{3!}(x-1)(x-2)(x-3)
\end{aligned}
$$

Therefore, according to the given conditions,

$$
|e| \leq \frac{5}{6}\,|(x-1)(x-2)(x-3)|
$$

3.

$$
\begin{aligned}
f[x_0] &= 3 \\
f[x_1] &= 16 \\
f[x_2] &= 11 \; f[x_3] &= 27
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f[x_0, x_1] &= \frac{3 - 16}{1 - 2} \\
&= 13 \\
f[x_1, x_2] &= \frac{16 - 11}{2 - 3} \\
&= -5 \\
f[x_2, x_3] &= \frac{11 - 27}{3 - 4} \\
&= 16
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f[x_0, x_1, x_2] &= \frac{13 - (-5)}{1 - 3} \\
&= -9 \\
f[x_1, x_2, x_3] &= \frac{-5 - 16}{2 - 4} \\
&= 10.5
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f[x_0, x_1, x_2, x_3] &= \frac{-9 - 10.5}{1 - 4} \\
&\neq 0
\end{aligned}
$$

Therefore, the degree of the interpolation polynomial is 3.

4.

$$
\begin{aligned}
l_1(x) &= \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} \\
l_2(x) &= \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} \\
l_3(x) &= \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} \\
l_4(x) &= \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}
\end{aligned}
$$

Therefore, solving, the degree is 1.

**Exercise 4.**
Consider

$$x_{n+1} = x_n + A\frac{{x_n}^2 - 2}{x_n} + B\frac{{x_n}^2 - 2}{{x_n}^3}$$

1. What is the fixed point?
2. Find $A$ and $B$, such that the rate of convergence is maximal.

**Solution 4.**
1. For the fixed point,

$$x = g(x)$$

Therefore,

$$x = x + A\frac{x - 2}{x} + B\frac{x^2 - 2}{x^3}$$

Therefore,

$$A\frac{x^2 - 2}{x} + B\frac{x^2 - 2}{x^3} = 0$$

Therefore, solving,

$$x = \pm\sqrt{2}$$

or

$$x = \pm\sqrt{-\frac{A}{B}}$$

2.

$$g(x) = A\frac{x^2 - 2}{x} + B\frac{x^2 - 2}{x^3} + x$$

$$\therefore g'(x) = 1 + A\frac{2x^2 - \left(x^2 - 2\right)}{x^2} + B\frac{2x^4 - \left(x^2 - 2\right)3x^2}{x^6}$$

Therefore, if $g'(x) = 0$, for $x = \pm\sqrt{2}$, solving,

$$1 + 2A + B = 0$$

Therefore, if $g'(x) = 0$, for $x = \pm\sqrt{2}$, solving,

$$2A + 5B = 0$$

Therefore, solving,

$$A = -\frac{5}{8}$$
$$B = \frac{1}{4}$$