NUMERICAL ANALYSIS : ASSIGNMENT 7

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Exercise 1.
Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{pmatrix}$$

(1) Find the LU decomposition of A.
(2) Solve

$$\begin{aligned} Ax &= b \\ &= \begin{pmatrix} 7 \\ 28 \\ 69 \end{pmatrix} \end{aligned}$$

using

$$\begin{aligned} Ax &= LUx \\ &= b \end{aligned}$$

Solution 1.
(1)

$$\begin{aligned} & \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{pmatrix} \\ \xrightarrow{R_2 \to R_2 - 2R_1} & \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 3 & 9 & 27 \end{pmatrix} \\ \xrightarrow{R_3 \to R_3 - 3R_1} & \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 6 & 24 \end{pmatrix} \\ \xrightarrow{R_3 \to R_3 - 3R_2} & \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 6 \end{pmatrix} \end{aligned}$$

Date: Tuesday 1^{st} December, 2015.

Therefore,

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 6 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{pmatrix}$$

(2)

$$\begin{aligned} Ax &= b \\ \therefore LUx &= b \end{aligned}$$

Let

$$y = Ux$$

Therefore,

$$Ly = b$$

$$\therefore \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 28 \\ 69 \end{pmatrix}$$

Therefore

$$\begin{aligned} y_1 &= 7 \\ 2y_1 + y_2 &= 28 \\ 3y_1 + 3y_2 + y_3 &= 69 \end{aligned}$$

Therefore, solving,

$$\begin{aligned} y_1 &= 7 \\ y_2 &= 14 \\ y_3 &= 6 \end{aligned}$$

Therefore,

$$y = Ux$$

$$\therefore \begin{pmatrix} 7 \\ 14 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Therefore,

$$\begin{aligned} x_1 + x_2 + x_3 &= 7 \\ 2x_2 + 6x_3 &= 14 \\ 6x_3 &= 6 \end{aligned}$$

Therefore, solving

$$\begin{aligned} x_1 &= 2 \\ x_2 &= 4 \\ x_3 &= 1 \end{aligned}$$

Therefore,

$$x = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

Exercise 2.
In this exercise we demonstrate the importance of row permutations even in the case

$$a_{m,m}{}^{(m)} \neq 0$$

where $a_{m,m}{}^{(m)}$ is the mth entry in the diagonal at step m. We show that row pivoting is important when $a_{m,m}^{(m)}$ is small.
Consider the system

$$\begin{aligned} Ax &= \begin{pmatrix} 10^{-5} & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

We want to solve the system using floating point arithmetic in base 10, and mantissa with 4 digits.
Note that the accurate solution to this system is

$$\begin{aligned} x_1 &= \frac{-1}{1 - 10^{-5}} \\ &\approx -1 \\ x_2 &= \frac{1}{1 - 10^{-5}} \\ &\approx 1 \end{aligned}$$

(1) Solve the system using Gauss elimination without pivoting, in floating point arithmetic. Namely, at each step, only use 4 digits. Is the solution a good approximation to the accurate solution?
(2) Solve the system using Gauss elimination with pivoting, in floating point arithmetic. Namely, at each step, only use 4 digits. Is the solution a good approximation to the accurate solution?

Solution 2.
(1)

$$\begin{aligned} &\begin{pmatrix} 0.000 & 1.000 \\ 1.000 & 1.000 \end{pmatrix} \\ \xrightarrow{R_1 \to R_1 - R_2} &\begin{pmatrix} -1.000 & 0.000 \\ 1.000 & 1.000 \end{pmatrix} \\ \xrightarrow{R_2 \to R_2 + R_1} &\begin{pmatrix} -1.000 & 0.000 \\ 0.000 & 1.000 \end{pmatrix} \end{aligned}$$

Therefore,

$$U = \begin{pmatrix} -1.000 & 0.000 \\ 0.000 & 1.000 \end{pmatrix}$$
$$L = \begin{pmatrix} 1.000 & 0.000 \\ 0.000 & 1.000 \end{pmatrix}$$

Therefore

$$Ax = b$$
$$\therefore LUx = b$$

Let

$$y = Ux$$

Therefore,

$$Ly = B$$
$$\therefore \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Therefore,

$$y_1 = 1$$
$$y_2 = 0$$

Therefore,

$$y = Ux$$
$$\therefore \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Therefore,

$$-x_1 = 1$$
$$x_2 = 0$$

Therefore, solving,

$$x_1 = -1$$
$$x_2 = 0$$

Therefore, this solution is not a good approximation to the accurate solution.

(2) Let

$$V = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} 1.000 & 1.000 \\ 0.000 & 1.000 \end{pmatrix}$$
$$\to \begin{pmatrix} 1.000 & 1.000 \\ 0.000 & 1.000 \end{pmatrix}$$

Therefore,

$$U = \begin{pmatrix} 1.000 & 1.000 \\ 0.000 & 1.000 \end{pmatrix}$$
$$L = \begin{pmatrix} 1.000 & 0.000 \\ 0.000 & 1.000 \end{pmatrix}$$

Using V,

$$B \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let

$$y = Ux$$

Therefore,

$$Ly = B$$
$$\therefore \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Therefore,

$$y_1 = 0$$
$$y_2 = 1$$

Therefore,

$$y = Ux$$
$$\therefore \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Therefore,

$$x_1 + x_2 = 0$$
$$x_2 = 1$$

Therefore, solving,

$$x_1 = -1$$
$$x_2 = 1$$

Therefore, this solution is a good approximation to the accurate solution.

Exercise 3.
For $v \in \mathbb{R}^n$, define

$$\|v\|_\infty = \max_{1 \le k \le n} |v_k|$$
$$\|v\|_1{}' = \frac{1}{n} \sum_{k=1}^{n} |v_k|$$

A different dose of medicine is given to a patient at days 1, 2, . . . , 100. There is a theoretical "dose vector" $v \in \mathbb{R}^{100}$ that holds the optimal dose v_k, to give each day k, in order to best treat the patient. In any day k, a dose that is too far from v_k kills the patient. Let $\tilde{v}$ be a calculated approximation of v. We demand

$$\begin{aligned} \|e\| &= \|v - \tilde{v}\| \\ &< \varepsilon \end{aligned}$$

for some small ε. What norm should we use? Explain your answer.

Solution 3.
If

$$\|v\|_1{}' = \frac{1}{n}\sum_{k=1}^{n}|v_k|$$

is used, the norm is the average of the dosages. Hence, the error is smaller than the infinity norm. Therefore, this norm should be used.