Numerical Analysis

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1 Lecturer Information

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2 Required Reading

1. S. D. Conte and C. de Boor, Elementary Numerical Analysis, 1972

3 Floating Point Representation

Exercise 1.
Represent 9.75 in base 2.

Solution 1.

$$\begin{aligned}
9.75 &= 8 + 1 + \frac{1}{2} + \frac{1}{4} \\
&= 2^3 + 2^0 + 2^{-1} + 2^{-2} \\
&= 2^3 \left(2^0 + 2^{-3} + 2^{-4} + 2^{-5}\right) \\
&= \left(2^{11} \left(1 + 0.001 + 0.0001 + 0.00001\right)\right)_2 \\
&= \left(2^{11} \left(1.00111\right)\right)_2
\end{aligned}$$

Definition 1 (Double precision floating point representation)**.** A floating point representation which uses 64 bits for representation of a number is called a double precision floating point representation.
The standard form of double precision representation is

$$a = \underbrace{\pm}_{1\text{ bit}} \underbrace{1}_{1\text{ bit}} . \underbrace{\cdots}_{52\text{ bits}} \times w^{\underbrace{\pm}_{1\text{ bit}} \underbrace{\cdots}_{10\text{ bits}}}$$

Theorem 1 (Range of double precision floating point representation)**.** *The largest number which can be represented with double precision floating point representation is approximately* 10^{307} *and the smallest number which can be represented is approximately* 10^{-307}*.*

Proof. As the exponent has 10 bits for representation,

$$-\left(10^{10} - 1\right) \le \text{exponent} \le \left(10^{10} - 1\right)$$

Therefore,

$$-1023 \le \text{exponent} \le 1023$$

Therefore, the smallest number, in terms of absolute value, which can be represented, is

$$1.\underbrace{0 \cdots 0}_{52\text{ bits}} \times 2^{-1024} \approx 10^{-307}$$

Therefore, the smallest number which can be represented is approximately 10^{-307}, and the largest number which can be represented is approximately 10^{307}. $\square$

Definition 2 (Overflow)**.** If a result is larger than the largest number which can be represented, it is called overflow.

Definition 3 (Underflow)**.** If a result is smaller than the smallest number which can be represented, it is called underflow.

Definition 4 (Least significant digit)**.**

$$1 = 1.\underbrace{0 \cdots 0}_{52 \text{ zeros}} \times 2^0$$

Let 1_ε be the smallest number larger than 1, which can be represented in double precision floating point representation.
Therefore,

$$\begin{aligned} 1 &= 1.\underbrace{0 \cdots 0}_{51 \text{ zeros}} 1 \times 2^0 \\ &= 1 + 2^{-52} \\ &\approx 1 + 2 \times 10^{-16} \end{aligned}$$

Therefore,

$$\begin{aligned} 1 - 1_\varepsilon &= 2^{-52} \\ &\approx 2 \times 10^{-16} \end{aligned}$$

This number is called the least significant digit, or the machine precision. It is the maximum possible error in representation. It is represented by ε.

Definition 5 (Error)**.** Let the DPFP representation of a number x be $\widetilde{x}$. The absolute error in representation is defined as

$$\begin{aligned} \text{absolute error} &= |x - \widetilde{x}| \\ &= 0.0 \cdots 01 \times 2^{\text{exponent}} \end{aligned}$$

The relative error in representation is defined as

$$\begin{aligned} \delta &= \frac{|x - \widetilde{x}|}{x} \\ &= 0.0 \cdots 01 \\ &< \varepsilon \end{aligned}$$

The maximum error, $2^{-52} \approx 2 \times 10^{-16}$, is called the machine precision.
In general,

$$\widetilde{x} \,\widetilde{\star}\, \widetilde{y} = (x \star y)(1 + \delta)$$

where δ is the relative error, ε is the machine precision, $\delta < \varepsilon$, and $\star$ is an operator.

3.1 Loss of Significant Digits in Addition and Subtraction

Exercise 2.
Represent $\pi + \frac{1}{30}$ in base 10 with 4 digits.

Solution 2.

$$\pi \approx 3.14159$$

Approximating by ignoring the last digits,

$$\tilde{\pi} = 3.141$$

Similarly,

$$\widetilde{\frac{1}{30}} = 3.333 \times 10^{-2}$$

Therefore, adding,

$$\begin{aligned}\tilde{\pi} + \widetilde{\frac{1}{30}} &= 3.141 + 0.03333 \\ &= 3.174\end{aligned}$$

Therefore,

$$\begin{aligned}\delta &= \left| \frac{\left(\tilde{\pi} + \widetilde{\frac{1}{30}}\right) - \left(\pi + \frac{1}{30}\right)}{\pi + \frac{1}{30}} \right| \\ &= 0.0003\end{aligned}$$

Therefore, $\delta < \varepsilon = 0.001$

Exercise 3.
Given

$$\begin{aligned}a &= 1.435234 \\ b &= 1.429111\end{aligned}$$

Find the relative error.

Solution 3.

$$\begin{aligned} a &= 1.435234 \\ b &= 1.429111 \end{aligned}$$

Therefore,

$$a - b = 0.0061234$$

Approximating by ignoring the last digits,

$$\begin{aligned} \widetilde{a} &= 1.435 \\ \widetilde{b} &= 1.429 \end{aligned}$$

Therefore,

$$\widetilde{a} - \widetilde{b} = 0.006$$

Therefore,

$$\delta = \left| \frac{(a-b) - \left(\widetilde{a} - \widetilde{b}\right)}{a-b} \right|$$

Therefore,

$$\begin{aligned} \delta &> 10^{-3} \\ \therefore \delta &> \varepsilon \end{aligned}$$

Exercise 4.
Solve

$$x^2 + 10^8 x + 1 = 0$$

Solution 4.

$$x = \frac{-10^8 \pm \sqrt{10^{16} - 4}}{2}$$

Therefore,

$$x_- \approx -10^8$$

Therefore, by Vietta Rules,

$$\begin{aligned} x_1 x_2 &= \frac{c}{a} \\ x_1 + x_2 &= -\frac{b}{a} \end{aligned}$$

Therefore,

$$\begin{aligned} x_+ x_- &= 1 \\ \therefore x_+ &= \frac{1}{x_-} \\ &\approx -10^{-8} \end{aligned}$$

In MATLAB, this can be executed as x = **roots**([1,10^8,1])
This gives the result

$$x_+ = -7.45 \times 10^{-9}$$

Therefore, the absolute error is

$$\begin{aligned} |\widetilde{x} - x| &= \left| -7.45 \times 10^{-9} - \left(-10^{-8} \right) \right| \\ &= 2.55 \times 10^{-9} \end{aligned}$$

Therefore,

$$\begin{aligned} \delta &= \left| \frac{\widetilde{x} - x}{x} \right| \\ &= \left| \frac{2.55 \times 10^{-9}}{10^{-8}} \right| \\ &= 0.255 \\ &= 25\% \end{aligned}$$

The algorithm used by MATLAB is

if $b \geq 0$ **then**
    $x_1 = \frac{-b-\sqrt{b^2-4ac}}{2a}$
    $x_2 = \frac{x}{ax_1}$
else
    $x_2 = \frac{-b+\sqrt{b^2-4ac}}{2a}$
    $x_1 = \frac{c}{ax_2}$
end if

This is done to avoid subtraction of numbers close to each other, and hence avoid the possible error.

4 Series of Approximations

4.1 Order of Convergence

Definition 6. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a series. $\{\alpha_n\}$ is said to converge to α, denoted as $\alpha_n \to \alpha$, if $\forall \varepsilon > 0$, $\varepsilon \in \mathbb{R}$, $\exists n_0(\varepsilon) \in \mathbb{N}$, such that $\forall n \in \mathbb{N}$, $n > n_0(\varepsilon)$, $|\alpha_n - \alpha| < \varepsilon$.

Usually, the series $\{\alpha_n\}$ is compared to a simpler series such as $\frac{1}{n}, \frac{1}{n^\beta}, \dots$.

Definition 7. α_n is said to be "big-O" of β_n, and is said to behave like β_n, if $\exists k \in \mathbb{R}$, $k > 0$, $\exists n_0 \in \mathbb{N}$, $n_0 > 0$, such that $\forall n > n_0$,

$$|\alpha_n| \leq k|\beta_n|$$

It is denoted as

$$\alpha_n = \mathrm{O}(\beta_n)$$

Definition 8. α_n is said to be "small-O" of β_n if

$$\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$$

It is denoted as

$$\alpha_n = \mathrm{o}(\beta_n)$$

Exercise 5.
Find the order of convergence of

$$\alpha_n = 2n^3 + 3n^2 + 4n + 5$$

Solution 5.

$$\begin{aligned} \alpha_n &= 2n^3 + 3n^2 + 4n + 5 \\ &\leq (2 + 3 + 4 + 5)n^3 \\ \therefore \alpha_n &\leq 14n^3 \end{aligned}$$

Therefore, comparing to the standard form,

$$\begin{aligned} k &= 14 \\ \beta_n &= n^3 \end{aligned}$$

Therefore, as $\forall n \geq 1$, $|a_n| \leq 14|\beta_n|$,

$$\alpha_n = \mathrm{O}(\beta_n)$$

Also,

$$\begin{aligned}\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} &= \lim_{n\to\infty} \frac{2n^3 + 2n^2 + 4n + 5}{n^3} \\ &= 2\end{aligned}$$

Therefore, as the limits is not zero,

$$\alpha_n \neq \mathrm{o}(\beta_n)$$

However, $\forall \delta > 0$,

$$\alpha_n = \mathrm{o}\left(n^{3+\delta}\right)$$

5 Representation of Polynomials

5.1 Power series

Definition 9 (Power series representation of polynomials)**.**

$$P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$$

This representation may lead to loss of significant digits.

Exercise 6.

Let $P(x)$ represent a straight line.

$$\begin{aligned}P(6000) &= \frac{1}{3} \\ P(6001) &= -\frac{2}{3}\end{aligned}$$

If only 5 decimal digits are used, show that there is a loss of significant digits, if the power series representation of the polynomial is used.

Solution 6.
$P(x)$ represents a straight line. Therefore,

$$P(x) = ax + b$$

Therefore,

$$6000a + b = \frac{1}{3}$$
$$6001a + b = -\frac{2}{3}$$

Therefore,

$$\begin{pmatrix} 6000 & 1 \\ 6001 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$$
$$\therefore \begin{pmatrix} a \\ b \end{pmatrix}$$
$$= \frac{1}{|A|} \begin{pmatrix} 1 & -1 \\ -6001 & 6000 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$$
$$= -\begin{pmatrix} 1 \\ -6000.3 \end{pmatrix}$$
$$= \begin{pmatrix} -1 \\ 6000.3 \end{pmatrix}$$

Therefore,

$$a = -1$$
$$b = 6000.3$$

Therefore,

$$P(x) = -x + 6000.3$$

Substituting 6000 and 6001 in this expression,

$$P(6000) = 0.3$$
$$P(6001) = 0.7$$

However, the most accurate values of $P(6000)$ and $P(6001)$, using 5 decimal digits only, should be

$$P(6000) = 0.33333$$
$$P(6001) = -0.66666$$

Therefore, there is a loss of significant digits.

5.2 Shifted Power Series

Definition 10 (Shifted power series representation of polynomials)**.**

$$P_n(x) = a_0 + a_1(x - c) + \cdots + a_n(x - c)^n$$

This representation is a power series shifted by c. Hence, this representation does not lead to loss of significant digits.

Exercise 7.
Let $P(x)$ be a straight line.

$$P(6000) = \frac{1}{3}$$
$$P(6001) = -\frac{2}{3}$$

If only 5 decimal digits are used, show that there is no loss of significant digits, if the shifted power series representation of the polynomial is used, with $c = 6000$.

Solution 7.
$P(x)$ represents a straight line. Therefore,

$$P(x) = a(x - 6000) + b$$

Therefore,

$$b = \frac{1}{3}$$
$$a + b = -0.66666$$
$$\therefore a = -0.99999$$

Therefore,

$$P(x) = -0.99999(x - 6000) + 0.33333$$

Substituting 6000 and 6001 in this expression,

$$P(6000) = 0.33333$$
$$P(6001) = -0.66666$$

Therefore, there is no loss of significant digits, as the values of $P(6000)$ and $P(6001)$ are the most accurate values possible, using 5 decimal digits.

5.3 Newton's Form

Definition 11 (Newton's form of representation of polynomials)**.**

$$P_n(x) = a_0 + a_1(x - c_1) + \cdots + a_n(x - c_1)\ldots(x - c_n)$$

The number of multiplications needed to calculate $P_n(x)$ is

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

The number of additions or subtractions needed to calculate $P_n(x)$ is

$$\sum_{i=1}^{n} i + n = \frac{n(n+1)}{2} + n$$

Therefore, the total number of operations needed to calculate $P_n(x)$ is $\mathrm{O}(n^2)$.

5.4 Nested Newton's Form

Definition 12 (Nested Newton's form of representation of polynomials)**.**

$$P_n(x) = a_0 + (x - c_1)\Big(a_1 + (x - c_2)\left(a_2 + (x - c_3)\left(\ldots\right)\right)\Big)$$

The number of multiplications needed to calculate $P_n(x)$ is

$$\sum_{i=1}^{n} 1 = n$$

The number of additions or subtractions needed to calculate $P_n(x)$ is

$$\sum_{i=1}^{n} 2 = 2n$$

Therefore, the total number of operations needed to calculate $P_n(x)$ is big-O of $\mathrm{O}(n)$.

5.5 Properties of Polynomials

Theorem 2. *For a polynomial in shifted power series form,*

$$P_n(x) = P_n(c) + (x - c)q_{n-1}(x)$$

Proof.

$$
\begin{aligned}
P_n(x) &= a_0 + a_1(x-c) + \cdots + a_n(x-c)^n \\
&= a_0 + (x-c)\left(a_1 + a_2(x-2) + \cdots + a_n(x-c)^{n-1}\right) \\
&= a_0 + (x-c)q_{n-1}(x) \\
&= P_n(c) + (x-c)q_{n-1}(x)
\end{aligned}
$$

□

Theorem 3. *If c is a root of $P_n(x)$, i.e., if*

$$P_n(c) = 0$$

then

$$P_n(x) = (x-c)q_{n-1}(x)$$

If $c_1 \neq c_2$ are roots of $P_n(x)$, then

$$P_n(x) = (x-c_1)(x-c_2)r_{n-2}(x)$$

Similarly, if $P_n(x)$ has n different roots, then

$$P_n(x) = A(x-c_1)\dots(x-c_n)$$

where $A \in \mathbb{R}$.
If $P_n(x)$ has $n+1$ different roots, then

$$P_n(x) = A(x-c_1)\dots(x-c_n)(x-c_{n+1})$$

where $A = 0$.

Theorem 4. *If $p(x)$ and $q(x)$ are polynomials of degree at most n, that satisfy*

$$
\begin{aligned}
p(x_i) &= f(x_i) \\
q(x_i) &= f(x_i)
\end{aligned}
$$

for $i \in \{0, \dots, n\}$, then

$$p_n(x) \equiv q_n(x)$$

This means that there exists a unique polynomial with degree n which passes through $n+1$ points, i.e. $n+1$ points define a unique n degree polynomial.

Proof. Let

$$d_n(x) = p_n(x) - q_n(x)$$

Therefore, $d_n(x)$ is a polynomial of degree at most n, which has $n+1$ roots. Therefore,

$$d_n(x) \equiv 0$$

Therefore,

$$p_n(x) \equiv q_n(x)$$

$\square$

6 Interpolation

Theorem 5 (Weierstrass Approximation Theorem). *Let $f(x) \in \mathrm{C}[a,b]$, i.e. it is continuous on $[a,b]$. Let $\varepsilon > 0$. Then there exists a polynomial $P(x)$ defined on $[a,b]$, such that $\forall x \in [a,b]$,*

$$|f(x) - P(x)| < \varepsilon$$

Definition 13 (Interpolating polynomial). $p(x)$ is said to be the interpolating polynomial of $f(x)$, if for all sample points x_i,

$$f(x_i) = p(x_i)$$

Theorem 6. *Let $f(x)$ such that $\forall i \in \{0, \dots, n\}$,*

$$f(x_i) = y_i$$

Then, there exists a unique polynomial $p(x)$ of degree at most n, which interpolates $f(x)$ at all sample points x_i.

6.1 Direct Method

Definition 14 (Van der Monde matrix). Let

$$p(x) = \sum_{i=0}^{n} a_i x^i$$

Let

$$f(x_i) = y_i$$

Therefore, as

$$p(x_i) = f(x_i)$$

the constraints are

$$\begin{aligned} a_0 + a_1 x_0 + \cdots + a_n {x_0}^n &= y_0 \\ a_1 + a_1 x_1 + \cdots + a_n {x_1}^n &= y_1 \\ &\vdots \\ a_n + a_1 x_n + \cdots + a_n {x_n}^n &= y_n \end{aligned}$$

Therefore,

$$\begin{pmatrix} 1 & x_0 & {x_0}^2 & \dots & {x_0}^n \\ 1 & x_1 & {x_1}^2 & \dots & {x_1}^n \\ 1 & x_2 & {x_2}^2 & \dots & {x_2}^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & {x_n}^2 & \dots & {x_n}^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

The matrix

$$V = \begin{pmatrix} 1 & x_0 & {x_0}^2 & \dots & {x_0}^n \\ 1 & x_1 & {x_1}^2 & \dots & {x_1}^n \\ 1 & x_2 & {x_2}^2 & \dots & {x_2}^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & {x_n}^2 & \dots & {x_n}^n \end{pmatrix}$$

is called the Van der Monde matrix.

Theorem 7. *The Van der Monde matrix is invertible, and hence there exists a unique matrix of coefficients $a_0, \dots, a_n$, and hence the interpolating polynomial $p(x)$ is unique.*

6.2 Lagrange's Interpolation

Definition 15 (Lagrange polynomials)**.** Let

$$L_k(x) = \prod_{i=0;i\neq k}^{n} (x - x_i)$$

Therefore,

$$L_k(x_i) = \begin{cases} 0 & ; \quad i \neq k \\ 1 & ; \quad i = k \end{cases}$$

Let

$$l_k(x) = \frac{L_k(x)}{L_k(x_k)}$$

Therefore,

$$l_k(x_i) = \begin{cases} 0 & ; \quad i \neq k \\ 1 & ; \quad i = k \end{cases}$$

The polynomials $l_i(x)$ are called Lagrange polynomials.

Theorem 8. *Let*

$$p_n(x) = \sum_{i=0}^{n} f(x_i) l_i(x)$$

where $l_i(x)$ are Lagrange polynomials.
Then, $p_n(x)$ is the interpolating polynomial of $f(x)$.

Exercise 8.
Which polynomial of degree 2 interpolates the below data?

x	$f(x)$
1	1
2	3
3	7

Solution 8.

$$L_k(x) = \prod_{i=0;i\neq k}^{n} (x - x_i)$$

Therefore,

$$L_1(x) = (x-2)(x-3)$$
$$L_2(x) = (x-1)(x-3)$$
$$L_3(x) = (x-1)(x-2)$$

Therefore,

$$
\begin{aligned}
L_1(1) &= (1-2)(1-3) \\
&= 2 \\
L_2(2) &= (2-1)(2-3) \\
&= -1 \\
L_3(3) &= (3-1)(3-2) \\
&= 2
\end{aligned}
$$

Therefore,

$$
l_k(x) = \frac{L_k(x)}{L_k(x_k)}
$$

Therefore,

$$
\begin{aligned}
l_1(x) &= \frac{L_1(x)}{L_1(1)} \\
&= \frac{1}{2}(x-2)(x-3) \\
l_2(x) &= \frac{L_2(x)}{L_2(1)} \\
&= -(x-1)(x-3) \\
l_3(x) &= \frac{L_3(x)}{L_3(1)} \\
&= \frac{1}{2}(x-1)(x-2)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
p_2(x) &= \sum f(x_i) l_i(x) \\
&= \frac{1}{2}(x-2)(x-3) - 3(x-1)(x-3) + \frac{7}{2}(x-1)(x-2)
\end{aligned}
$$

Exercise 9.
Given

$$
k(z) = \int\limits_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\sqrt{1-(\sin z)^2(\sin x)^2}}
$$

and

$$\begin{aligned}
k(1) &= 1.5709 \\
k(4) &= 1.5727 \\
k(6) &= 1.5751
\end{aligned}$$

approximate $k(3.5)$.

Solution 9.

$$l_k(x) = \frac{\prod_{i=0;i\neq k}^{n}(x-x_i)}{\prod_{i=0;i\neq k}^{n}(x_k-x_i)}$$

Therefore,

$$\begin{aligned}
l_1(x) &= \frac{(x-4)(x-6)}{(1-4)(1-6)} \\
l_4(x) &= \frac{(x-1)(x-6)}{(4-1)(4-6)} \\
l_6(x) &= \frac{(x-1)(x-4)}{(6-1)(6-4)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
l_1(3.5) &= \frac{(3.5-4)(3.5-6)}{(1-4)(1-6)} \\
&= 0.08333 \\
l_4(3.5) &= \frac{(3.5-1)(3.5-6)}{(4-1)(4-6)} \\
&= 1.04167 \\
l_6(3.5) &= \frac{(3.5-1)(3.5-4)}{(6-1)(6-4)} \\
&= -0.125
\end{aligned}$$

Therefore,

$$\begin{aligned}
p_2(x) &= \sum f(x_i)l_k(x) \\
\therefore p_2(3.5) &= \sum f(x_1)l_k(3.5) \\
&= (1.5709)(0.08333) + (1.5727)(1.04167) + (1.5751)(-0.125) \\
&= 1.57225
\end{aligned}$$

6.3 Hermite Polynomials

Definition 16. Let the given data be of the form $(x_i, f(x_i), f'(x_i))$, where $i = 0, \ldots, n$.
H_{2n+1} is called the Hermite polynomial of $f(x)$.
For H_{2n+1} to be the interpolation polynomial of $f(x)$, the constraints are

$$\begin{aligned} H_{2n+1}(x_i) &= f(x_i) \\ H'_{2n+1}(x_i) &= f'(x_i) \end{aligned}$$

Therefore, the number of constraints are $2n + 2$.
Hence, the polynomial is of degree at most $2n + 1$.

Theorem 9. *Let*

$$H_{2n+1}(x) = \sum_{i=0}^{n} f(x_i)\psi_{n,i}(x) + \sum_{i=0}^{n} f'(x_i)\varphi_{n,i}(x)$$

Let

$$\delta_{ij} = \begin{cases} 0 & ; \quad i \neq j \\ 1 & ; \quad i = j \end{cases}$$

If the polynomials ψ and φ satisfy

$$\begin{aligned} \psi_{n,i}(x_j) &= \delta_{ij} \\ {\psi_{n,i}}'(x_j) &= 0 \\ \varphi_{n,i}(x_j) &= 0 \\ {\varphi'_{n,i}}'(x_j) &= \delta_{ij} \end{aligned}$$

then the polynomial H_{2n+1} is the interpolation polynomial of $f(x)$.

6.4 Newton's Interpolation

Definition 17 (Newton's polynomial)**.** The polynomial

$$p_n(x) = \sum_{i=0}^{n} A_i \prod_{j=0}^{i-1} (x - x_j)$$

is called Newton's polynomial.

Theorem 10. *If $p_k(x)$, constructed based on $x_1, \ldots, x_k$ is known, then $p_{k+1}(x)$, based on $x_1, \ldots, x_{k+1}$ can be constructed as*

$$p_{k+1}(x) = p_k(x) + A_{k+1}(x - x_0) \ldots (x - x_k)$$

Proof. For $i = 0, \ldots, k$,

$$\begin{aligned} p_{k+1}(x_i) &= p_k(x_i) + A_{k+1} \prod_{j=0}^{k} (x_i - x_j) \\ &= p_k(x_i) + 0 \end{aligned}$$

$\forall i = 0, \ldots, k$, $(x_i - x_i) = 0$. Therefore, if $i = j$, $(x_i - x_j) = 0$. Therefore, $\prod (x_i - x_j) = 0$

For $i = k + 1$,

$$\begin{aligned} p_{k+1}(x_{k+1}) &= p_k(x_{k+1}) + A_{k+1} \prod_{j=0}^{k} (x_{k+1} - x_j) \\ &= f(x_{k+1}) \end{aligned}$$

where A_{k+1} can be calculated using $p_k(x_{k+1})$ and $f(x_{k+1})$.
Therefore,
For $n = 1$,

$$\begin{aligned} p_0(x) &= A_0 \\ &= f(x_0) \end{aligned}$$

For $n = 2$,

$$\begin{aligned} p_1(x) &= p_0(x) + A_1(x - x_0) \\ &= f(x_0) - A_1(x - x_0) \\ &= f(x_1) \end{aligned}$$

Therefore,

$$\begin{aligned} A_1 &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ &= f[x_0, x_1] \end{aligned}$$

For $n = 3$,

$$\begin{aligned} p_2(x) &= p_1(x) + A_2(x - x_0)(x - x_1) \\ &= f(x_0) + f[x_0, x_1](x - x_0) \\ &= f(x_0) + f[x_0, x_1](x - x_0) + A_2(x - x_0)(x - x_1) \\ &= f(x_2) \end{aligned}$$

Therefore,

$$
\begin{aligned}
A_2 &= \frac{1}{(x_2 - x_0)(x_2 - x_1)} \left(f(x_2) - f(x_0) - f[x_0, x_1](x_2 - x_0)\right) \\
&= f[x_0, x_1, x_2]
\end{aligned}
$$

and so on.
In general,

$$A_k = f[x_0, \ldots, x_k]$$

□

Definition 18 (Divided difference)**.**

$$
\begin{aligned}
f[x_0, \ldots, x_k] &= \frac{f[x_1, \ldots, x_k] - f[x_0, \ldots, x_{k-1]}}{x_k - x_0} \\
f[x_0] &= f(x_0)
\end{aligned}
$$

is called the kth order divided difference of $f(x)$.

Exercise 10.
Given

$$k(z) = \int\limits_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\sqrt{1 - (\sin z)^2 (\sin x)^2}}$$

and

$$
\begin{aligned}
k(1) &= 1.5709 \\
k(4) &= 1.5727 \\
k(6) &= 1.5751
\end{aligned}
$$

approximate $k(3.5)$.

Solution 10.
For the first order divided differences,

$$k[x_i] = k(x_i)$$

Therefore,

$$\begin{aligned}
k[1] &= k(1) \\
&= 1.5709 \\
k[4] &= k(4) \\
&= 1.5727 \\
k[6] &= k(6) \\
&= 1.5751
\end{aligned}$$

For the second order divided differences,

$$k[x_i, x_j] = \frac{k[i] - k[j]}{i - j}$$

Therefore,

$$\begin{aligned}
k[1,4] &= \frac{k[1] - k[4]}{1 - 4} \\
&= \frac{1.5727 - 1.5709}{3} \\
k[4,6] &= \frac{k[4] - k[6]}{4 - 6} \\
&= \frac{1.5751 - 1.5727}{2}
\end{aligned}$$

For the third order divided differences,

$$k[x_i, x_j, x_k] = \frac{k[i,j] - k[j,k]}{i - k}$$

Therefore,

$$k[1,4,6] = \frac{k[1,4] - k[4,6]}{1 - 6}$$

Hence,

$$\begin{aligned}
A_0 &= k[1] \\
A_1 &= k[1,4] \\
A_2 &= k[1,4,6]
\end{aligned}$$