NUMERICAL ANALYSIS : ASSIGNMENT 10

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Exercise 1.
Let f be 5 times continuously differentiable. Consider the samples $f(-h)$, $f(0)$, $f(h)$.

(1) Find the interpolating polynomial of f based on the samples 0, h. Find a formula for the interpolation error.
(2) Find an approximation of $f'(0)$ based on part 1. Find an error formula for the interpolation error.
(3) Find the interpolating polynomial of f based on the samples $-h$, 0, h. Find a formula for the interpolation error.
(4) Find an approximation of $f'(0)$ based on part 3. Find an error formula for your approximation.
(5) Find an approximation of $f''(0)$ based on part 3. Find an error formula for your approximation.

Solution 1.
(1)

$$
\begin{aligned}
f[x_0] &= f(0) \\
f[x_1] &= f(h)
\end{aligned}
$$

Therefore,

$$f[x_0, x_1] = \frac{f(0) - f(h)}{-h}$$

Therefore,

$$
\begin{aligned}
p_1(x) &= f[x_0] + f[x_0, x_1](x - x_0) \\
&= f(0) + \frac{f(0) - f(h)}{-h} x
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\psi(x) &= \prod (x - x_i) \\
&= (x)(x - h)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f(x) &= p_2(x) + f[x_0, x_1, x_2, x]\psi(x) \\
&= p_2(x) + f[0, h, x](x)(x - h)
\end{aligned}
$$

Therefore,

$$e(x) = f[0, h, x](x)(x - h)$$

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(2)

$$\begin{aligned} f'(x) &\approx {p_1}'(x) \\ \therefore f'(0) &= \frac{f(0) - f(h)}{-h} \end{aligned}$$

Therefore,

$$\begin{aligned} e'(x) &= f[0, h, x, x]\psi(x) + f[0, h, x]\psi'(x) \\ &= f[0, h, x, x](x)(x-h) + f[0, h, x](2x - h) \end{aligned}$$

(3)

$$\begin{aligned} f[x_0] &= f(-h) \\ f[x_1] &= f(0) \\ f[x_2] &= f(h) \end{aligned}$$

Therefore,

$$\begin{aligned} f[x_0, x_1] &= \frac{f(-h) - f(0)}{-h} \\ f[x_1, x_2] &= \frac{f(0) - f(h)}{-h} \end{aligned}$$

Therefore,

$$f[x_0, x_1, x_2] = \frac{f(-h) - 2f(0) + f(h)}{2h^2}$$

Therefore,

$$\begin{aligned} p_2(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &= f(-h) + \frac{f(-h) - f(0)}{-h}(x + h) + \frac{f(-h) - 2f(0) + f(h)}{2h^2}(x + h)x \end{aligned}$$

Therefore,

$$\begin{aligned} \psi(x) &= \prod (x - x_i) \\ &= (x + h)(x)(x - h) \end{aligned}$$

Therefore,

$$\begin{aligned} f(x) &= p_2(x) + f[x_0, x_1, x_2, x]\psi(x) \\ &= p_2(x) + f[-h, 0, h, x](x + h)(x)(x - h) \end{aligned}$$

Therefore,

$$e(x) = f[-h, 0, h, x](x + h)(x)(x - h)$$

(4)

$$\begin{aligned} f'(x) &\approx {p_2}'(x) \\ &= \frac{f(-h) - f(0)}{-h} + \frac{f(-h) - 2f(0) + f(h)}{2h^2}(2x + h) \\ \therefore f'(0) &= \frac{f(-h) - f(0)}{-h} - \frac{f(-h) - 2f(0) + f(h)}{2h^2}h \end{aligned}$$

Therefore,

$$\begin{aligned} e'(x) &= f[-h,0,h,x,x]\psi(x) + f[-h,0,h,x]\psi'(x) \\ &= f[-h,0,h,x,x](x+h)(x)(x-h) + f[-h,0,h,x]\left(3x^2-h^2\right) \end{aligned}$$

(5)

$$\begin{aligned} f''(x) &\approx p_2{}''(x) \\ &= 2\frac{f(-h)-2f(0)+f(h)}{2h^2} \\ \therefore f''(0) &= \frac{f(-h)-2f(0)+f(h)}{h^2} \end{aligned}$$

Therefore,

$$e''(x) = f[-h,0,h,x,x,x](x+h)(x)(x-h) + f[-h,0,h,x,x](6x)$$

Exercise 2.
Let f be 5 times continuously differentiable. Consider the samples $f(-h)$, $f'(-h)$, $f(2h)$.

(1) Find the interpolating polynomial of f based on these samples. Find a formula for the interpolation error.
(2) Find an approximation of $f''(0)$ based on part 1. Find an error formula for the interpolation error.
(3) Find an approximation of $f''(2h)$ based on part 1. Find an error formula for the interpolation error.

Solution 2.
(1)

$$\begin{aligned} f[x_0] &= f(-h) \\ f[x_1] &= f(-h) \\ f[x_2] &= f(2h) \end{aligned}$$

Therefore,

$$\begin{aligned} f[x_0,x_1] &= f'(-h) \\ f[x_1,x_2] &= \frac{f(2h)-f(-h)}{3h} \end{aligned}$$

Therefore,

$$f[x_0,x_1,x_2] = \frac{f(2h)-f(-h)-3hf'(-h)}{9h^2}$$

Therefore,

$$\begin{aligned} p_2(x) &= f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) \\ &= f(-h) + f'(-h)(x+h) + \frac{f(2h)-f(-h)-3hf'(-h)}{9h^2}(x+h)^2 \end{aligned}$$

Therefore, Therefore,

$$\begin{aligned}\psi(x) &= \prod(x - x_i)\\ &= (x+h)^2(x-2h)\end{aligned}$$

Therefore,

$$\begin{aligned}f(x) &= p_2(x) + f[x_0, x_1, x_2, x]\psi(x)\\ &= p_2(x) + f[-h, -h, 2h, x](x+h)^2(x-2h)\end{aligned}$$

Therefore,

$$e(x) = f[-h, -h, 2h, x](x+h)^2(x-2h)$$

(2)

$$\begin{aligned}f''(x) &\approx {p_2}''(x)\\ &= \frac{f(2h) - f(-h) - 3hf'(-h)}{9h^2}(2)\end{aligned}$$

Therefore,

$$f''(0) = 2\frac{f(2h) - f(-h) - 3hf'(-h)}{9h^2}$$

Therefore,

$$\begin{aligned}e''(x) = \quad & f[-h, -h, 2h, x, x, x](x+h)^2(x-2h)\\ &+ 2f[-h, -h, 2h, x, x]\left(3x^2 - 3h^2\right)\\ &+ f[-h, -h, 2h, x](6x)\end{aligned}$$

(3)

$$\begin{aligned}f''(x) &\approx {p_2}''(x)\\ &= \frac{f(2h) - f(-h) - 3hf'(-h)}{9h^2}(2)\end{aligned}$$

Therefore,

$$f''(2h) = 2\frac{f(2h) - f(-h) - 3hf'(-h)}{9h^2}$$

Therefore,

$$\begin{aligned}e''(x) = \quad & f[-h, -h, 2h, x, x, x](x+h)^2(x-2h)\\ &+ 2f[-h, -h, 2h, x, x]\left(3x^2 - 3h^2\right)\\ &+ f[-h, -h, 2h, x](6x)\end{aligned}$$