NUMERICAL ANALYSIS : ASSIGNMENT 6

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Theorem 1 (Fixed point theorem 2)**.** *Let $g(x)$ be a continuously differentiable function. Let a be a fixed point of g. Consider the fixed point iteration $x_{n+1} = g(x_n)$.*
If $\left|g(a)\right| < 1$, then there exists a neighbourhood $(a-\delta, a+\delta)$, where $\delta > 0$, such that for any initial guess $x_0 \in (a-\delta, a+\delta)$ the method converges to a.

Theorem 2 (Rate of convergence)**.** *Consider the method $x_{n+1} = g(x_n)$, with g being k times continuously differentiable. If $g(a) = 0$ and for any $1 \leq k < p$,*

$$g^{(k)}(a) = 0$$

and $g^{(p)}(a) \neq 0$, then the method converges to a in some neighbourhood of a, and the rate of convergence is p.

Exercise 1.
Prove Theorem 2 as follows.

(1) Show that for any $n \in \mathbb{N}$, there exists a point c_n between a and x_n, such that

$$g(x_n) = g(a) + \frac{g^{(p)}(c_n)}{p!}(x_n - a)^p$$

(2) Prove

$$|e_{n+1}| = \left|\frac{g^{(p)}(c_n)}{p!}\right| |e_n|^p$$

(3) Finish proving the theorem. Hint: Show that $c_n \xrightarrow{n\to\infty} a$.

Solution 1.
(1) The Taylor expansion of $g(x_n)$ around a is

$$\begin{aligned} g(x_n) &= g(a) + g'(a)(x_n - a) + \cdots + \frac{g^{(k)}(a)}{k!}(x_n - a)^k + \frac{g^{(p)}(c_n)}{p!}(x_n - a)^p \\ &= g(a) + \frac{g^{(p)}}{p!}(x_n - p)^p \end{aligned}$$

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(2)

$$\begin{aligned} x_{n+1} &= a + \frac{g^{(p)}(c_n)}{p!}(x_n - a)^p \\ \therefore a - x_{n+1} &= -\frac{g^{(p)}(c_n)}{p!}(x_n - a)^p & &= -\frac{g^{(p)}(c_n)}{p!}{e_n}^p \\ \therefore |e_{n+1}| &= \left|\frac{g^{(p)}(c_n)}{p!}\right| |e_n|^p \end{aligned}$$

(3)

$$\lim_{n\to\infty} x_n = a$$

As c_n is between a and x_n,

$$\begin{aligned} \lim_{n\to\infty} c_n &= x_n \\ &= a \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} c_n = a$$

Exercise 2.
In order to find the root of

$$f(x) = x^2 - 2$$

we use the following iterative method.

$$x_{n+1} = x_n + A\frac{{x_n}^2 - 2}{x_n} + B\frac{{x_n}^2 - 2}{{x_n}^3}$$

(1) Find A and B such that the method has a maximal rate of convergence.
(2) Calculate the rate of convergence.

Solution 2.
(1) Let the root of the equation be $f(x) = 0$ be r.

$$\begin{aligned} x_{n+1} &= x_n + A\frac{{x_n}^n - 2}{x_n} + B\frac{{x_n}^2 - 2}{{x_n}^3} \\ &= g(x_n) \end{aligned}$$

Therefore, using the fixed point theorem,

$$\begin{aligned} g(r) &= r \\ \therefore 0 &= 0 \end{aligned}$$

Let

$$\begin{aligned} g'(x) &= 0 \\ \therefore 2A + B &= -1 \end{aligned}$$

Let

$$g''(x) = 0$$
$$\therefore A = -\frac{7}{2}B$$

Therefore, solving,

$$A = -\frac{7}{12}$$
$$B = \frac{1}{6}$$

(2)

$$g'''(x) = -\frac{7}{r^4} + +\frac{r^4 - 20}{r^6}$$
$$\neq 0$$

Therefore, as $g'(x)$ and $g''(x)$ are zero, but $g'''(x) \neq 0$, the rate of convergence is 3.

Exercise 3.
Assume that our machine only performs additions and multiplications, and not division. Let $a > 0$. We approximate $\frac{1}{a}$ using two iterative methods.

$$x_{n+1} = g(x_n)$$
$$g(x) = 2x - ax^2$$

and

$$x_{n+1} = h(x_n)$$
$$h(x) = 3x - 3ax^2 + a^2x^3$$

(1) Show that $\frac{1}{a}$ is a fixed point of both the methods.
(2) Show that there is a neighbourhood $\mathcal{N}_1$ of $\frac{1}{a}$, such that for any $x_0 \in \mathcal{N}_1$, $x_{n+1} = g(x_n) \xrightarrow{n\to\infty} \frac{1}{a}$. Also, show that there is a neighbourhood $\mathcal{N}_2$ of $\frac{1}{a}$ corresponding to h.
(3) For an initial guess $x_0 \in \mathcal{N}_1 \cap \mathcal{N}_2$, which iterative method is faster?

Solution 3.
(1)

$$x_{n+1} = g(x_n)$$
$$g(x) = 2x - ax^2$$

$$x_{n+1} = h(x_n)$$
$$h(x) = 3x - 3ax^2 + a^2x^3$$

For $\frac{1}{a}$ to be a fixed point of both methods,

$$g\left(\frac{1}{a}\right) = \frac{1}{a}$$
$$h\left(\frac{1}{a}\right) = \frac{1}{a}$$

Therefore,

$$\begin{aligned} g\left(\frac{1}{a}\right) &= 2\frac{1}{a} - a\frac{1}{a^2} \\ &= \frac{2}{a} - \frac{1}{a} \\ &= \frac{1}{a} \end{aligned}$$

$$\begin{aligned} h\left(\frac{1}{a}\right) &= 3\frac{1}{a} - 3a\frac{1}{a^2} + a^2\frac{1}{a^3} \\ &= \frac{1}{a} \end{aligned}$$

Therefore, $\frac{1}{a}$ is a fixed point of both methods.

(2) $g(x)$ and $h(x)$ are continuously differentiable function.
$\frac{1}{a}$ is a fixed point of both methods.

$$\begin{aligned} g'\left(\frac{1}{a}\right) &= 2 - 2a\frac{1}{a} \\ &= 0 \end{aligned}$$

$$\begin{aligned} h'\left(\frac{1}{a}\right) &= 3 - 6a\frac{1}{a} + 3a^2\frac{1}{a^2} \\ &= 0 \end{aligned}$$

Therefore, by the Fixed point theorem 2, there exists such a neighbourhood $\mathcal{N}_1$ of $\frac{1}{a}$ corresponding to $g(x)$, and a neighbourhood $\mathcal{N}_2$ of $\frac{1}{a}$ corresponding to $h(x)$. □

Exercise 4.
Let

$$f(x) = x^2$$

(1) Construct Newton's method for finding the root of f.
(2) What is the rate of convergence of the method? You can use the known root.
(3) Does this contradict the theorem that states the rate of convergence of Newton's method is 2?

Solution 4.

(1)

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{{x_n}^2}{2x_n} \\ &= \frac{x_n}{2} \end{aligned}$$

Therefore,

$$\begin{aligned} x_{n+1} &= \frac{x_n}{2} \\ g(x_n) &= x_{n+1} \end{aligned}$$

(2)

$$\begin{aligned} \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^p} &= \lim_{n\to\infty} \frac{|a - x_{n+1}|}{|a - x_n|^p} \\ &= \frac{|x_{n+1}|}{|x_n|^p} \end{aligned}$$

If $p = 1$,

$$\begin{aligned} \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|} &= \frac{1}{2} \\ &\neq 0 \end{aligned}$$

Therefore, the rate of convergence is $p = 1$.

(3) This does not contradict the theorem, as for the theorem, $f'(c) \neq 0$, but in this case, $f'(0) = 0$.

Exercise 5.

Recall the construction of Newton's method. In the construction, at each iteration we approximate f around x_n by a linear polynomial. Then, the approximation of the root at step $n + 1$ is the root of the approximating linear polynomial.

Construct a similar method, that uses an approximation f around x_n by a quadratic polynomial, instead of a linear polynomial. The method should use the valued $f(x_n)$, $f'(x_n)$, $f''(x_n)$ at each iteration n, to construct x_{n+1}.

Solution 5.

Let the root of the equation be r.

Therefore,

$$f(r) = 0$$

Therefore,

$$
\begin{aligned}
0 &= f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{f''(x_n)}{2}(x_{n+1} - x_n)^2 \\
\therefore x_{n+1} - x_n &= -\left(\frac{f'(x_n) + \sqrt{\left(f'(x_n)\right)^2 - 2f(x_n)f''(x_n)}}{f''(x_n)} \right) \\
\therefore x_{n+1} &= x_n - \left(\frac{f'(x_n) + \sqrt{\left(f'(x_n)\right)^2 - 2f(x_n)f''(x_n)}}{f''(x_n)} \right)
\end{aligned}
$$

where $f''(x_n) \neq 0$.

Exercise 6.
We want to find a root a of f, where $f(a) = f'(a) = a$, and $f''(a) \neq 0$. Consider Newton's method

$$
\begin{aligned}
g(x) &= x - \frac{f(x)}{f'(x)} \\
x_{n+1} &= g(x_n)
\end{aligned}
$$

(1) Show that $g'(a) = \frac{1}{2}$.
(2) Prove that there is an interval $\mathcal{N}$ containing a such that for any $x_0 \in \mathcal{N}$, the method converges to a.
(3) What is the rate of convergence of this method?
(4) Define the modified Newton's method

$$
\begin{aligned}
h(x) &= x - 2\frac{f(x)}{f'(x)} \\
x_{n+1} &= h(x_n)
\end{aligned}
$$

Show that the rate of convergence of this method is at least 2.

Solution 6.
(1)

$$
\begin{aligned}
g(x) &= x - \frac{f(x)}{f'(x)} \\
x_{n+1} &= g(x_n)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
g'(x) &= 1 - \frac{f'(x)}{f'(x)} + \frac{f(x)f''(x)}{\left(f'(x)\right)^2} \\
&= \frac{f(x)f''(x)}{\left(f'(x)\right)^2}
\end{aligned}
$$

Therefore,

$$\begin{aligned} g'(a) &= \frac{f(a)f''(a)}{\big(f'(a)\big)^2} \\ &= \frac{af''(a)}{a^2} \\ &= \frac{f''(a)}{a} \\ &= \frac{1}{2} \end{aligned}$$

□

(2)

$$\begin{aligned} \big|g'(a)\big| &= \frac{1}{2} \\ &< 1 \end{aligned}$$

Therefore, and as $g(x)$ is continuously differentiable, by Fixed point theorem 2, there exists such a neighbourhood $\mathcal{N}$ around a.

(3)

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|} \neq 0$$

Therefore, the rate of convergence is 1.

(4)

$$\begin{aligned} h(x) &= x - 2\frac{f(x)}{f'(x)} \\ x_{n+1} &= h(x_n) \end{aligned}$$

Therefore,

$$\begin{aligned} h'(x) &= 1 - 2\left(\frac{f'(x)}{f'(x)} - \frac{f(x)f''(x)}{\big(f'(x)\big)^2}\right) \\ &= -1 + 2\frac{f(x)f''(x)}{\big(f'(x)\big)^2} \\ &= -1 + \frac{f''(x)}{a} \end{aligned}$$

Therefore, as $h'(x) \neq 0$, the rate of convergence of cannot be 1. Hence, it must be at least 2. □