Numerical Analysis : Recitations

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Contents

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1 Instructor Information

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2 Errors

Definition 1 (Error)**.** The absolute error in representation is defined as

$$e_x = x - \widetilde{x}$$

The relative error in representation is defined as

$$\delta = \frac{x - \widetilde{x}}{x}$$

Recitation 1 – Exercise 1.
The dimensions of a field are measured. The length is measured to be $\widetilde{x} = 800$m, with an absolute error bounded by 16. The width is measured to be $\widetilde{y} = 30$m, with an absolute error e_y, such that $|e_y| \leq 6$.

1. Find the approximate bounds for $|\delta_x|$ and $|\delta_y|$.

2. Find the bounds on the absolute error in the calculated area of the field.

Recitation 1 – Solution 1.
1.

$$\begin{aligned} |\delta_x| &= \frac{|e_x|}{|x|} \\ &\leq \frac{16}{|x|} \\ &\approx \frac{16}{800} \\ &= 0.02 \\ \therefore |\delta_x| &\leq 0.02 \end{aligned}$$

$$
\begin{aligned}
|\delta_y| &= \frac{|e_y|}{|y|} \\
&\leq \frac{6}{|y|} \\
&\approx \frac{6}{300} \\
&= 0.02 \\
\therefore |\delta_y| &\leq 0.02
\end{aligned}
$$

2. The measured area of the field is

$$
\begin{aligned}
\widetilde{A} &= \widetilde{x}\widetilde{y} \\
&= 800 \cdot 300 \\
&= 240000
\end{aligned}
$$

The maximum area of the field is

$$
\begin{aligned}
A_{\text{max}} &= (\widetilde{x} + e_{x\text{max}})(\widetilde{y} + e_{y_{\text{max}}}) \\
&= (800 + 16)(300 + 6) \\
&= 249696
\end{aligned}
$$

The maximum area of the field is

$$
\begin{aligned}
A_{\text{min}} &= (\widetilde{x} + e_{x\text{min}})(\widetilde{y} + e_{y_{\text{min}}}) \\
&= (800 - 16)(300 - 6) \\
&= 230496
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|e_{xy}| &\leq (A_{\text{max}} - A_{\text{min}}) \\
&\leq 9696
\end{aligned}
$$

3.

$$
\begin{aligned}
|\delta_{xy}| &= \frac{|e_{xy}}{|xy|} \\
&\leq \frac{9696}{|xy|} \\
&\leq \frac{9696}{230496} \\
&\approx 0.042
\end{aligned}
$$

2.1 Propagation of Error

Recitation 1 – Exercise 2.
Let $\widetilde{x}$, $\widetilde{y}$ be approximations of x, y.

1. Find a formula for the absolute error in $x + y$ in terms of e_x and e_y.

2. Find a formula for δ_{x+y}, δ_{x-y} in terms of δ_x, δ_y, x, y.

3. Let $\delta = \max\{\delta_x, \delta_y\}$. Assuming $x, y > 0$, show

$$|\delta_{x-y}| \leq \frac{x+y}{|x-y|}\delta$$

Recitation 1 – Solution 2.
1.

$$\begin{aligned} e_{x+y} &= (x+y) - (\widetilde{x} + \widetilde{y}) \\ &= (x - \widetilde{x}) + (y - \widetilde{y}) \\ &= e_x + e_y \end{aligned}$$

2.

$$\begin{aligned} \delta_{x+y} &= \frac{e_{x+y}}{x+y} \\ &= \frac{e_x + e_y}{x+y} \\ &= \frac{x\delta_x + y\delta_y}{x+y} \end{aligned}$$

Similarly,

$$\begin{aligned} \delta_{x-y} &= \frac{e_{x-y}}{x-y} \\ &= \frac{e_x - e_y}{x-y} \\ &= \frac{x\delta_x - y\delta_y}{x-y} \end{aligned}$$

3.

$$\begin{aligned}
|\delta_{x-y}| &= \left|\frac{x\delta_x - y\delta_y}{x-y}\right| \\
&\leq \frac{|x||\delta_x| + |y||\delta_y|}{|x-y|} \\
&\leq \frac{x\delta + y\delta}{|x-y|} \\
&= \frac{x+y}{|x-y|}\delta
\end{aligned}$$

Recitation 1 – Exercise 3.
Find a formula for δ_{xy}, in terms of x, y, δ_x, δ_y.

Recitation 1 – Solution 3.

$$\begin{aligned}
\delta_a &= \frac{a - \widetilde{a}}{a} \\
\therefore \widetilde{a} &= a(1 - \delta_a)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\widetilde{x}\widetilde{y} &= (x(1-\delta_x))\Big(y(1-\delta_y)\Big) \\
&= xy(1 - \delta_x - \delta_y + \delta_x\delta_y)
\end{aligned}$$

Also,

$$\widetilde{x}\widetilde{y} = xy(1 - \delta_{xy})$$

Therefore,

$$\delta_{xy} = \delta_x + \delta_y - \delta_x\delta_y$$

3 Interpolation by Polynomials

Theorem 1 (Existence and Uniqueness Theorem)**.** *There exists a unique polynomial $p_n(x)$ which approximates $f(x)$ between the sample points, i.e.*

$$|e_n(x)| = |f(x) - p_n(x)|$$

Recitation 2 – Exercise 1.
Find the interpolation polynomial for the data

x_i	$f(x_i)$
1	3
2	2
4	1

Recitation 2 – Solution 1.
Let

$$p(x) = a_0 + a_1 x + a_2 x^2$$

be the required interpolation polynomial.
Therefore,

$$\begin{aligned}
&p(1) = 3\\
&\therefore 3 = a_0 + a_1 \cdot 1 + a_2 \cdot 1^2\\
&p(2) = 2\\
&\therefore 2 = a_0 + a_1 \cdot 2 + a_2 \cdot 2^2\\
&p(4) = 1\\
&\therefore 1 = a_0 + a_1 \cdot 4 + a_2 \cdot 4^2
\end{aligned}$$

Therefore,

$$\begin{pmatrix} 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \\ 1 & 4 & 4^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Therefore, solving,

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{13}{3} \\ -\frac{3}{2} \\ \frac{1}{6} \end{pmatrix}$$

Therefore,

$$p(x) = \frac{13}{3} - \frac{3}{2}x + \frac{1}{6}x^2$$

Definition 2. Let the sample points be $x_0, \dots, x_{n-1}$.
Lagrange polynomials are n polynomials of degree $n-1$, each of which is 0 at all sample points, except one, at which it is 1.

$$l_i(x) = \frac{(x-x_0)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_{n-1})}{(x_i-x_0)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_{n-1})}$$

Recitation 2 – Exercise 2.
Find the interpolation polynomial for the data

x_i	$f(x_i)$
1	3
2	2
4	1

using Lagrange polynomials.

Recitation 2 – Solution 2.
Let

$$\begin{aligned} x_0 &= 1 \\ x_1 &= 2 \\ x_3 &= 4 \end{aligned}$$

Therefore,

$$\begin{aligned} l_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \\ &= \frac{(x-2)(x-4)}{(1-2)(1-4)} \\ l_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \\ &= \frac{(x-1)(x-4)}{(2-1)(2-4)} \\ l_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\ &= \frac{(x-1)(x-2)}{(4-1)(4-2)} \end{aligned}$$

Therefore,

$$\begin{aligned} p(x) &= f(x_0)l_0(x) + f(x_1)l_1(x) + f(x_2)l_2(x) \\ &= 3l_0(x) + 2l_1(x) + l_2(x) \end{aligned}$$

Recitation 3 – Exercise 1.
Given

x_i	$f(x_i)$
0	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	1

Find the interpolating polynomial in Newton's form.

Recitation 3 – Solution 1.
The interpolating polynomial is

$$p_2(x) = A_0 + A_1(x - x_0) + A_2(x - x_0)(x - x_1)$$

where

$$A_k = f[x_0, \ldots, x_k]$$

Therefore,

$$\begin{aligned} \therefore f[0] &= f(0) \\ &= 0 \\ \therefore f\left[\frac{\pi}{4}\right] &= f\left(\frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2} \\ \therefore f\left[\frac{\pi}{2}\right] &= f\left(\frac{\pi}{2}\right) \\ &= 1 \end{aligned}$$

Therefore,

$$
\begin{aligned}
f\left[0, \frac{\pi}{4}\right] &= \frac{f\left[\frac{\pi}{4}\right] - f[0]}{\frac{\pi}{4} - 0} \\
&= \frac{\frac{\sqrt{2}}{2} - 0}{\frac{\pi}{4}} \\
f\left[\frac{\pi}{4}, \frac{\pi}{2}\right] &= \frac{f\left[\frac{\pi}{2}\right] - f\left[\frac{\pi}{4}\right]}{\frac{\pi}{2} - \frac{\pi}{4}}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f\left[0, \frac{\pi}{4}, \frac{\pi}{2}\right] &= \frac{f\left[\frac{\pi}{4}, \frac{\pi}{2}\right] - f\left[0, \frac{\pi}{4}\right]}{\frac{\pi}{2} - 0} \\
&= \frac{8(1-\sqrt{2})}{\pi^2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A_0 &= 0 \\
A_1 &= \frac{2\sqrt{2}}{\pi} \\
A_2 &= \frac{8(1-\sqrt{2})}{\pi^2}
\end{aligned}
$$

Therefore,

$$
p_2(x) = \frac{2\sqrt{2}}{\pi}x + \frac{8(1-\sqrt{2})}{\pi^2}(x)\left(x - \frac{\pi}{4}\right)
$$

Recitation 3 – Exercise 2.

$\sin\left(\frac{\pi}{3}\right)$ was approximated using Newton's method, at sample points 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, to be

$$
\begin{aligned}
p_2\left(\frac{\pi}{3}\right) &= \frac{2\sqrt{2}}{3} + \frac{8(1-\sqrt{2})}{36} \\
&= 0.8507
\end{aligned}
$$

Find the bounds on the error in this approximation.

Recitation 3 – Solution 2.

$$
\begin{aligned}
|e_n(x)| &= |f(x) - p_n(x)| \\
&\leq \left| \frac{f^{(n+1)}(c)}{(n+1)!} \prod_{j=0}^{n} (x - x_j) \right|
\end{aligned}
$$

where $c \in [\min\{x_0, \ldots, x_n, x\}, \max\{x_0, \ldots, x_n, x\}]$.
Therefore,

$$
\begin{aligned}
|e_2(x)| &\leq \left| \frac{\sin^{(3)}(c)}{3!} \prod_{j=0}^{3} (x - x_j) \right| \\
\therefore \left| e_2\left(\frac{\pi}{3}\right) \right| &\leq \left| \frac{\sin^{(3)}(c)}{3!} \prod_{j=0}^{3} \left(\frac{\pi}{3} - x_j\right) \right| \\
&\leq \left| \frac{\sin^{(3)}(c)}{3!} \left(\frac{\pi}{3} - 0\right) \left(\frac{\pi}{3} - \frac{\pi}{4}\right) \left(\frac{\pi}{3} - \frac{\pi}{2}\right) \right| \\
&\leq \left| \frac{-\cos(c)}{6} \frac{\pi^3}{(3)(12)(6)} \right| \\
&\leq \left| \frac{-\cos(c)\pi^3}{1296} \right|
\end{aligned}
$$

Therefore, as $|\cos(c)|$ is bounded by 0 and 1,

$$
\begin{aligned}
\left| e_2\left(\frac{\pi}{3}\right) \right| &\leq \left| \frac{\pi^3}{1296} \right| \\
&< 0.0242
\end{aligned}
$$

Recitation 4 – Exercise 1.

Find Hermite's interpolating polynomial for the sample points 1, 1, e, for the function $f(x) = \ln(x)$.

Recitation 4 – Solution 1.

$$
f[x_0, \ldots, x_k] = \begin{cases} \frac{f[x_1, \ldots, x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0} & ; \quad x_k \neq x_0 \\ \frac{f^{(k)}(x_0)}{k!} & ; \quad x_k = x_0 \end{cases}
$$

Therefore,

$$
\begin{aligned}
f[1] &= 0 \\
f[1] &= 0 \\
f[e] &= 1
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f[1,1] &= \frac{f'(1)}{1!} \\
&= \left.\frac{1}{x}\right|_{x=1} \\
&= 1 \\
f[1,e] &= \frac{1-0}{e-1} \\
&= \frac{1}{e-1}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f[1,1,e] &= \frac{\frac{1}{e-1} - 1}{e-1} \\
&= \frac{2-e}{(e-1)^2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
p_2(x) &= f[1] + f[e](x-1) + \frac{2-e}{(e-1)^2}(x-1)(x-1) \\
&= 0 + 1(x-1) + \frac{2-e}{(e-1)^2}(x-1)^2
\end{aligned}
$$

4 Fixed Point Iterations and Root Finding

Recitation 5 – Exercise 1.
Show that

$$
\begin{aligned}
e_n &= \alpha - x_n \\
&\approx -\frac{f(x_n)}{f'(x_n)}
\end{aligned}
$$

Recitation 5 – Solution 1.
By Lagrange's Mean Value Theorem, $\exists c \in (a,b)$, such that

$$
f'(c) = \frac{f(b)-f(a)}{b-a}
$$

Let

$$
\begin{aligned}
b &= x_n \\
a &= \alpha
\end{aligned}
$$

Therefore,

$$\frac{f(x_n) - f(\alpha)}{x_n - \alpha} = f'(c_n)$$

where $c_n \in (\alpha, x_n)$.
Therefore,

$$
\begin{aligned}
-e_n &= x_n - \alpha \\
&= \frac{f(x_n)}{f'(c_n)}
\end{aligned}
$$

Therefore, as $\lim\limits_{n\to\infty} x_n = 2$, for $n \to \infty$,

$$c_n = x_n$$

Therefore,

$$e_n = -\frac{f(x_n)}{f'(x_n)}$$

Recitation 5 – Exercise 2.
Let

$$f(x) = e^{-x} - \frac{1}{2}$$

1. Show that f has a root in $[0, 1]$.

2. Show that Newton's method converges to the root α of f, and that α is unique.

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Recitation 5 – Solution 2.

1.

$$\begin{aligned} f(0) &= e^0 - \frac{1}{2} \\ &= \frac{1}{2} \\ f(1) &= \frac{1}{e} - \frac{1}{2} \\ &< \frac{1}{2.7} - \frac{1}{2} \\ &< 0 \end{aligned}$$

Therefore, by the intermediate value theorem, $\exists \alpha$ such that $f(\alpha) = 0$. Hence, f has a root in $[0, 1]$.

2.

$$\begin{aligned} g(x) &= x - \frac{f(x)}{f'(x)} \\ &= x + \frac{e^{-x} - \frac{1}{2}}{e^{-x}} \\ &= x + 1 - \frac{1}{2}e^x \end{aligned}$$

Therefore,

$$g'(x) = 1 - \frac{1}{2}e^x$$

Therefore, as the extrema of g are in $[0, 1]$, $g : [0, 1] \to [0, 1]$.
Similarly, $g'(x)$ is decreasing.
Hence, by the fixed point theorem, as $\lim_{n\to\infty} x_n = \alpha$, α is unique.