# NUMERICAL ANALYSIS : ASSIGNMENT 2

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**Exercise 1.**

(1) Compute the interpolating polynomial for $f(x) = \sqrt{x}$, that interpolates $f$ at the sample points $\left\{\frac{1}{4}, 1, 4\right\}$.
(2) Compute the interpolating polynomial for $g(x) = 2^x$, that interpolates $g$ at the sample points $\{-1, 0, 1\}$.
(3) Approximate the number $\sqrt{2}$ in two ways. First by interpolating polynomial of $f$ from 1, and then by the interpolating polynomial of $g$ from 2.

**Solution 1.**

(1) Let

$$p_f(x) = a_0 + a_1 x + a_2 x^2$$

Therefore,

$$\begin{aligned}
p_f\left(\frac{1}{4}\right) &= a_0 + \frac{1}{4}a_1 + \frac{1}{16}a_2 \\
\therefore \frac{1}{2} &= a_0 + \frac{1}{4}a_1 + \frac{1}{16}a_2 \\
p_f(1) &= a_0 + a_1 + a_2 \\
\therefore 1 &= a_0 + a_1 + a_2 \\
p_f(4) &= a_0 + 4a_1 + 16a_2 \\
\therefore 2 &= a_0 + 4a_1 + 16a_2
\end{aligned}$$

Therefore,

$$\begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{16} \\ 1 & 1 & 1 \\ 1 & 4 & 16 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 2 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{14}{45} \\ \frac{7}{9} \\ -\frac{4}{45} \end{pmatrix}$$

Therefore,

$$p_f(x) = \frac{14}{45} + \frac{7}{9}x - \frac{4}{45}x^2$$

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(2) Let

$$p_g(x) = b_0 + b_1 x + b_2 x^2$$

Therefore,

$$\begin{aligned} p_g(-1) &= b_0 - b_1 + b_2 \\ \therefore \frac{1}{2} &= b_0 - b_1 + b_2 \\ p_g(0) &= b_0 \\ \therefore 1 &= b_0 \\ p_g(1) &= b_0 + b_1 + b_2 \\ \therefore 2 &= b_0 + b_1 + b_2 \end{aligned}$$

Therefore,

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 2 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}$$

Therefore,

$$p_g(x) = 1 + \frac{3}{4}x + \frac{1}{4}x^2$$

(3)

$$\begin{aligned} p_f(x) &= \frac{14}{45} + \frac{7}{9}x - \frac{4}{45}x^2 \\ \therefore \sqrt{2} &= \frac{14}{45} + \frac{7}{9} \cdot 2 - \frac{4}{45} \cdot 4 \\ &= \frac{14}{45} + \frac{14}{9} - \frac{16}{45} \\ &= \frac{14 + 70 - 16}{45} \\ &= \frac{68}{45} \end{aligned}$$

$$\begin{aligned} p_g(x) &= 1 + \frac{3}{4}x + \frac{1}{4}x^2 \\ \therefore \sqrt{2} &= 1 + \frac{3}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{4} \\ &= 1 + \frac{3}{8} + \frac{1}{16} \\ &= \frac{16 + 6 + 1}{16} \\ &= \frac{23}{16} \end{aligned}$$

**Exercise 2.**
Consider the interpolation data

| $x_i$ | $y_i$ |
|---|---|
| $-1$ | $-1$ |
| $0$ | $2$ |
| $1$ | $5$ |

(1) Find the interpolating polynomial using Lagrange polynomials.
(2) Find the interpolating polynomial using Newton's method.
(3) Is the polynomial you got of degree 2? Does this contradict the uniqueness and existence of the interpolating polynomial?

**Solution 2.**
(1)

$$\begin{aligned} x_0 &= -1 \\ x_1 &= 0 \\ x_2 &= 1 \end{aligned}$$

Therefore,

$$\begin{aligned} l_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \\ &= \frac{(x-0)(x-1)}{(-1-0)(-1-1)} \\ &= \frac{(x)(x-1)}{2} \\ l_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \\ &= \frac{(x+1)(x-1)}{(0+1)(0-1)} \\ &= -(x+1)(x-1) \\ l_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\ &= \frac{(x+1)(x-0)}{(1+1)(1-0)} \\ &= \frac{(x)(x+1)}{2} \end{aligned}$$

Therefore,

$$\begin{aligned}
p(x) &= f(x_0)l_0(x) + f(x_1)l_1(x) + f(x_2)l_2(x) \\
&= -\frac{(x)(x-1)}{2} - 2(x+1)(x-1) + 5\frac{(x)(x+1)}{2} \\
&= -\frac{x^2-x}{2} - 2(x^2-1) + 5\frac{x^2+x}{2} \\
&= \frac{4x^2+6x}{2} - 2x^2 + 2 \\
&= 2x^2 + 3x - 2x^2 + 2 \\
&= 3x + 2
\end{aligned}$$

(2)

(3) The polynomial is of degree 1. However this does not contradict the uniqueness and existence of the interpolating polynomial, as all points $(x_i, y_i)$ are collinear. Hence, the polynomial represents a line through these points, and not a parabola. If one of the points would not have lied on the straight line defined by the other two points, the interpolating polynomial would have been of degree 2.

**Exercise 3.**

Let $x_0$, $x_1$, $x_2$ be three points. We want to approximate the function $f(x)$ using a polynomial $p$ of degree up to 2 that satisfies

$$\begin{aligned}
p(x_0) &= f(x_0) \\
p'(x_1) &= f'(x_1) \\
p(x_2) &= f(x_2)
\end{aligned}$$

Assume that $x_0 \neq x_2$, and prove that $p(x)$ exists and is unique if and only if

$$x_1 \neq \frac{x_0 + x_2}{2}$$

Hint: Note that this is not the standard interpolation problem. Write $p(x)$ in its standard form, and derive a system of linear equations with polynomial coefficients as the unknowns. For a system of linear equations, what is the condition which is equivalent to existence and uniqueness of a solution?

**Solution 3.**

Let

$$p(x) = a_0 + a_1 x + a_2 x^2$$

Therefore,

$$\begin{aligned}
p(x_0) &= a_0 + a_1 x_0 + a_2 {x_0}^2 \\
p(x_1) &= a_0 + a_1 x_1 + a_2 {x_1}^2 \\
p(x_2) &= a_0 + a_1 x_2 + a_2 {x_2}^2
\end{aligned}$$

Therefore,

$$V = \begin{pmatrix} 1 & x_0 & {x_0}^2 \\ 1 & x_1 & {x_1}^2 \\ 1 & x_2 & {x_2}^2 \end{pmatrix}$$

Therefore, as the Van der Monde matrix must be invertible,

$$\begin{aligned} &|V| \neq 0 \\ &\therefore 0 \neq \left(x_1{x_2}^2 - x_2{x_1}^2\right) - x_0\left({x_2}^2 - {x_1}^2\right) + {x_0}^2(x_2 - x_1) \\ &\therefore 0 \neq x_1{x_2}^2 - x_2{x_1}^2 - x_0{x_2}^2 + x_0{x_1}^2 + {x_0}^2x_2 - {x_0}^2x_1 \\ &\qquad = (x_0 - x_1)(x_1 - x_2)(x_2 - x_0) \end{aligned}$$

As $x_0 \neq x_2$,

$$(x_2 - x_0) \neq 0$$

Therefore,

$$\begin{aligned} &|V| \neq 0 \\ \iff &(x_0 - x_1)(x_1 - x_2) \neq 0 \end{aligned}$$

Therefore,

$$x_1 \neq \frac{x_0 + x_2}{2}$$

□

**Exercise 4.**
Let $l_k(x)$, $k = 0, \dots, n$ be the Lagrange polynomials with respect to the points $x_0, \dots, x_n$. Show

(1) for any $x \in \mathbb{R}$,

$$\sum_{k=0}^{n} l_k(x) = 1$$

(2) for any $x \in \mathbb{R}$ and $m = 0, \dots, n$,

$$\sum_{k=0}^{n} {x_k}^m l_k(x) = x^m$$

**Solution 4.**
(1)

$$l_k(x) = \frac{(x - x_0)\dots(x - x_{k-1})(x - x_{k+1})\dots(x - x_n)}{(x_k - x_0)\dots(x_k - x_{k-1})(x_k - x_{k+1})\dots(x_k - x_n)}$$

Therefore,

$$p(x) = \sum_{k=1}^{n} f(x_k) l_k(x)$$

Therefore, if $p(x) = f(x) = 1$,

$$1 = \sum_{k=1}^{n} l_k(x)$$

(2) By definition,

$$l_k(x_i) = \begin{cases} 1 & k = i \\ 0 & k \neq i \end{cases}$$

Therefore,

$$x_k{}^m l_k(x_i) = \begin{cases} x_k{}^m & k = i \\ 0 & k \neq i \end{cases}$$

Also,

$$\sum_{k=0}^{n} l_k(x) = 1$$

Therefore,

$$\sum_{k=0}^{n} x_k{}^m l_k(x) = x^m$$