# NUMERICAL ANALYSIS : ASSIGNMENT 3

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**Exercise 1.**
Show that $f[x_0, \dots, x_n]$ does not depend on the order of the points $\{x_0, \dots, x_n\}$.

**Solution 1.**
If $n = 1$,

$$\begin{aligned}
f[x_0, x_1] &= \frac{f(x_0) - f(x_1)}{x_0 - x_1} \\
f[x_1, x_0] &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\
\therefore f[x_0, x_1] &= f[x_1, x_0]
\end{aligned}$$

If possible let

$$f[x_0, \dots, x_k, x_{k+1}, \dots, x_n] = f[x_0, \dots, x_{k+1}, x_k, \dots, x_n]$$

Therefore,

$$\begin{aligned}
f[x_0, \dots, x_{n+1}] &= \frac{f[x_1, \dots, x_k, x_{k+1}, \dots, x_{n+1}] - f[x_0, \dots, x_k, x_{k+1}, \dots, x_n]}{x_{n+1} - x_0} \\
&= \frac{f[x_1, \dots, x_{k+1}, x_k, \dots, x_{n+1}] - f[x_0, \dots, x_{k+1}, x_k, \dots, x_n]}{x_{n+1} - x_0} \\
&= f[x_0, \dots, x_{k+1}, x_k, \dots, x_{n+1}]
\end{aligned}$$

Therefore, by induction, $f[x_0, \dots, x_n]$ does no depend on the order of the points $\{x_0, \dots, x_n\}$. $\square$

**Exercise 3.**
Let $f : [0, 1] \to \mathbb{R}$ be defined as

$$f(x) = e^{-x}$$

Consider the $n+1$ sample points $\{x_0, \dots, x_n\}$ in $[0, 1]$, and the interpolating polynomial $p_n(x)$ of $f(x)$ at these points.

(1) Show that $\forall x \in [0, 1]$,

$$\left|(x - x_0) \dots (x - x_n)\right| \leq 1$$

(2) Prove

$$\begin{aligned}
\left|e_n(x)\right| &= \left|f(x) - p_n(x)\right| \\
&\leq \frac{1}{n!}
\end{aligned}$$

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(3) How many sample points do we need if we require an approximation of $f$ with error less than $10^{-3}$?

**Solution 3.**

(1) As $x \in [0,1]$, and all $x_j \in [0,1]$,

$$|x - x_j| \leq 1$$

for every $j$.
Therefore,

$$\prod (x - x_i) \leq 1$$

$\square$

(2)

$$\begin{aligned}
\left|e_n(x)\right| &= \left|f(x) - p_n(x)\right| \\
&\leq \left| \frac{f^{(n+1)}(c)}{(n+1)!} \prod_{j=0}^{n} (x - x_j) \right| \\
&\leq \left| \frac{(-1)^n f(c)}{n!} \prod_{j=0}^{n} (x - x_j) \right| \\
&\leq \left| \frac{e^{-c}}{n!} \right|
\end{aligned}$$

Therefore, as $c \in [0,1]$,

$$\left|e_n(x)\right| \leq \frac{1}{n!}$$

(3)

$$\begin{aligned}
\left|e_n(x)\right| &\leq \frac{1}{n!} \\
\therefore \frac{1}{n!} &\leq 10^{-3} \\
\therefore n! &\geq 10^3
\end{aligned}$$

Therefore,

$$n \geq 7$$

Therefore, we need at least 7 sample points if we require an approximation of $f$ with error less than $10^{-3}$.