Ordinary Differential Equations

Aakash Jog

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1 First Order ODEs

1.1 Linear Differential Equations with Coefficients Independent of y : $y' + p(x)y = q(x)$

1: Calculate the integrating factor

$$\mu(x) = e^{\int p(x)\,\mathrm{d}x}$$

2: Solve

$$\begin{aligned}\mu(x)y' + \mu(x)p(x)y &= \mu(x)q(x) \\ \therefore (\mu(x)y)' &= \mu(x)q(x)\end{aligned}$$

3:

$$y = \frac{1}{\mu(x)} \int \mu(x)q(x)\,\mathrm{d}t$$

1.2 Exact Differential Equations : $M(x,y)+N(x,y)y' = 0$ and $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

1: Solve

$$\begin{aligned}\psi &= \int M(x,y)\,\mathrm{d}x \\ &= a(x,y) + h(y) \\ \therefore \frac{\partial \psi}{\partial y} &= \frac{\partial a}{\partial y} + h'(y)\end{aligned}$$

2: Compare $\dfrac{\partial \psi}{\partial y}$ and N to find $h'(y)$ and hence $h(y)$.

3:

$$\psi(x,y) = c$$

1.3 Bernoulli Differential Equations : $y'+p(x)y = q(x)y^n$, $y \neq 0, 1$

1: Divide the equation by y^n.

$$y^{-n}y' + p(x)y^{1-n} = q(x)$$

2: Substitute

$$\nu = y^{1-n}$$

3: Differentiate ν

$$\nu' = (1-n)y^{-n}y'$$

4: Substitute

$$y^{1-n} = \nu$$

and

$$y^{-n}y' = \frac{1}{1-n}\nu'$$

5: Solve the linear DE in ν

$$\frac{1}{1-n}\nu' + p(x)\nu = q(x)$$

1.4 Separable Differential Equations : $N(y)y' = M(x)$

1: Separate the variables and integrate

$$\begin{aligned} N(y)\,\mathrm{d}y &= M(x)\,\mathrm{d}x \\ \therefore \int N(y)\,\mathrm{d}y &= \int M(x)\,\mathrm{d}x \end{aligned}$$

1.5 Homogeneous Differential Equations : $y' = f(x, y) = F\left(\frac{y}{x}\right)$

1: Write the function as a function of $\frac{x}{y}$ or $\frac{y}{x}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F\left(\frac{y}{x}\right)$$

2: Let

$$\begin{aligned} \frac{y}{x} &= z \\ \therefore y &= xz \\ \therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= z + x\,\frac{\mathrm{d}z}{\mathrm{d}x} \end{aligned}$$

3: Substitute z and $\dfrac{\mathrm{d}z}{\mathrm{d}x}$

$$z + x\,\frac{\mathrm{d}z}{\mathrm{d}x} = F(z)$$

4: Solve the differential equation in z and x

1.6 Riccati Equations : $y' = f_0(t) + f_1(t)y + f_2(t)y^2$

This method is applicable only if at least one solution is known.

1: Let y_1 be a known solution.

2: Let

$$y = y_1 + \frac{1}{u(t)}$$

3: Differentiate y

$$y' = {y_1}' - \frac{u'}{u^2}$$

4: Substitute $y = y_1 + \frac{1}{u(t)}$ and $y' = {y_1}' - \frac{u'}{u^2}$ in original equation and simplify.

$$
\begin{aligned}
{y_1}' - \frac{u'}{u^2} &= f_0(x) + f_1(x)\left(y_1 + \frac{1}{u}\right) + f_2(x)\left(y_1 + \frac{1}{u}\right)^2 \\
\therefore \cancel{{y_1}'} - \frac{u'}{u^2} &= \cancel{\left(f_0(x) + f_1(x)y_1 + f_2(x){y_1}^2\right)} + f_1(x)\frac{1}{u} + f_2(x)\left(\frac{2y_1}{u} + \frac{1}{u^2}\right) \\
\therefore -\frac{u'}{u^2} &= \frac{f_1(x)}{u} + \frac{2f_2(x)y_1u + f_2(x)}{u^2} \\
\therefore u' &= \left(-f_1(x) - 2f_2y_1\right)u - f_2(x)
\end{aligned}
$$

5: Solve this differential equation in u.

6: Substitute u in

$$y = y_1 + \frac{1}{u(t)}$$

1.7 Non-exact Differential Equations : $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ and $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

1: If $\frac{M_y - N_x}{N}$ is a function of x only,

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\,\mathrm{d}x}$$

2: If $\frac{N_x - M_y}{M}$ is a function of y only,

$$\mu(y) = e^{\int \frac{N_x - M_y}{M}\,\mathrm{d}y}$$

3: If $\frac{y^2(M_y - N_x)}{xM + yN}$ is a function of $\frac{x}{y}$ only,

$$\mu\left(\frac{x}{y}\right) = e^{\int \frac{y^2(M_y - N_x)}{xM + yN}\,\mathrm{d}\left(\frac{x}{y}\right)}$$

4: If $\frac{x^2(N_x - M_y)}{xM + yN}$ is a function of $\frac{y}{x}$ only,

$$\mu\left(\frac{y}{x}\right) = e^{\int \frac{x^2(N_x - M_y)}{xM + yN}\,\mathrm{d}\left(\frac{y}{x}\right)}$$

5: If $\frac{N_x - M_y}{xM - yN}$ is a function of xy only,

$$\mu(xy) = e^{\int \frac{N_x - M_y}{xM - yN}\,\mathrm{d}(xy)}$$

6: If $\frac{M_y - N_x}{z_x N - z_y M}$ is a function of $z(x, y)$ only,

$$\mu(z) = e^{\int \frac{M_y - N_x}{z_x N - z_y M} \, \mathrm{d}z}$$

7: Multiply the equation by μ and solve the exact differential equation

$$\mu M(x, y) \, \mathrm{d}x + \mu N(x, y) \, \mathrm{d}y = 0$$

1.8 Existence and Uniqueness

Definition 1 (Lipschitz function)**.** A function if said to be Lipschitz in y if

$$|f(x) - f(y)| \leq C|x - y|$$

for all x and y in the interval, and where C is independent of x and y.

Theorem 1 (Existence and Uniqueness Theorem)**.** *Let $f(x, y)$ be a continuous function of x, y in an open rectangle D, i.e. not including its boundaries, and Lipschitz in y. Then there exists an interval I such that $x_0 \in I$ and the solution for the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$, exists and is unique in I.*

1.8.1 Showing that a IVP has a unique solution in a particular interval

1: Let the IVP be

$$\begin{aligned} y' &= f(x, y) \\ y(x_0) &= y_0 \end{aligned}$$

2: Show that $f(x, y)$ is continuous in the interval.
3: Show that $f(x, y)$ is Lipschitz in y in the interval, i.e.

$$|f(x) - f(y)| \leq C|x - y|$$

for all x and y in the interval, with C independent of x and y.

2 Second Order ODEs

2.1 Linear Homogeneous Differential Equations with Constant Coefficients : $ay'' + by' + cy = 0$

1: Let

$$\begin{aligned} y &= e^{\lambda t} \\ y' &= \lambda e^{\lambda t} \\ y'' &= \lambda^2 e^{\lambda t} \end{aligned}$$

2: Substitute into the equation

$$\begin{aligned} a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} &= 0 \\ \therefore a\lambda^2 + b\lambda + c &= 0 \end{aligned}$$

3: Solve the quadratic equation in y

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

4: If λ_1 and λ_2 are real and distinct,

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

5: If $\lambda_1 = \lambda_2$,

$$y = c_1 e^{\lambda_1 t} + tc_2 e^{\lambda_1 t}$$

6: If $\lambda_1 = \overline{\lambda}_2 = \alpha + i\beta$,

$$y = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t$$

2.2 Linear Non-homogeneous Differential Equations : $y'' + p(t)y' + q(t)y = g(t)$

1: Solve the corresponding homogeneous differential equation $y'' + p(t)y' + q(t)y = 0$.

2: Let the solution of the corresponding homogeneous differential equation be y_h.

3: Guess a particular solution, $y_p(t)$, using the method of undetermined coefficients or the method of variation of parameters.

4: The solution to the ODE is

$$y = y_h + y_p$$

2.2.1 Method of Undetermined Coefficients

1: Guess a particular solution to the equation.

2:

$g(t)$	$y_p(t)$
$\sum_{i=0}^{n} a_i t^i$	$\sum_{i=0}^{n} A_i t^i$
$ae^{\beta t}$	$Ae^{\beta t}$
$a\cos(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$a\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$a\cos(\beta t) + b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$

3: The general solution to the equation is

$$y = y_h + y_p$$

2.2.2 Method of Variation of Parameters

1: Let $y_1(t)$ and $y_2(t)$ be two solutions to the corresponding homogeneous equation.

2: Solve the equation

$$\begin{pmatrix} y_1(t) & y_2(t) \\ {y_1}'(t) & {y_2}'(t) \end{pmatrix} \begin{pmatrix} {u_1}'(t) \\ {u_2}'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

for ${u_1}'(t)$ and ${u_2}'(t)$.

3:

$$y_p = u_1(t)y_1(t) + u_2(t)y_2(t)$$

2.3 Fundamental Set of Solutions of Linear Second Order Homogeneous ODEs

1: Find the Wronskian

$$\begin{aligned} W(y_1, y_2)(x) &= \begin{vmatrix} y_1(x) & y_2(x) \\ {y_1}'(x) & {y_2}'(x) \end{vmatrix} \\ &= y_1(x){y_2}'(x) - {y_1}'(x)y_2(x) \end{aligned}$$

2: If $W(y_1, y_2)(x) \neq 0$, then $\{y_1, y_2\}$ is a fundamental set of solutions.

2.4 Abel's Theorem

Theorem 2 (Abel's Theorem)**.**

$$W(y_1, y_2)(x) = y_1(x)y_2(x)' - y_1(x)'y_2(x) = Ce^{-\int p(x)\,\mathrm{d}x}$$

Therefore, as $Ce^{-\int p(x)\,\mathrm{d}x}$ can either be always zero or never zero, the Wronskian can also be always zero or never zero. Hence, a set of solutions y_1 and y_2, for which the Wronskian is zero for finite values of x cannot be a fundamental set of solutions.

2.5 Euler's Equations : $ax^2y'' + bxy' + cy = 0$

1: Let

$$\begin{aligned} y &= x^r \\ y' &= rx^{r-1} \\ y'' &= r(r-1)x^{r-2} \end{aligned}$$

2: Substitute into the equation,

$$\begin{aligned} ax^2r(r-1)x^{r-2} + bxrx^{r-1} + cx^r &= 0 \\ \therefore x^r\left(ar(r-1) + br + c\right) &= 0 \\ \therefore ar(r-1) + br + c &= 0 \end{aligned}$$

3: Solve the equation in r,

$$\begin{aligned} ar^2 - ar + br + c &= 0 \\ \therefore r^2(a) - r(b-a) + c &= 0 \\ \therefore r_{1,2} &= \frac{(a-b) \pm \sqrt{(b-a)^2 - 4ac}}{2a} \end{aligned}$$

4: If r_1 and r_2 are real and distinct,

$$y = c_1x^{r_1} + c_2x^{r_2}$$

5: If $r_1 = r_2$,

$$y = c_1x^{r_1} + c_2x^{r_1}\ln x$$

6: If $r_1 = \overline{r}_2 = \alpha + i\beta$,

$$y = c_1x^{\alpha}\cos(\beta\ln x) + c_2x^{\alpha}\sin(\beta\ln x)$$

2.6 Existence and Uniqueness

Theorem 3 (Existence and Uniqueness Theorem)**.** *The IVP*

$$\begin{aligned} y'' + p(t)y' + q(t) &= g(t) \\ y(t_0) &= y_0 \\ y'(t_0) &= y_0' \end{aligned}$$

has a unique solution in an interval I if and only if the functions $p(t)$, $q(t)$, $g(t)$ are continuous in an interval I, and $t_0 \in I$.

2.7 Reduction of Order : $y'' + p(t)y' + q(t) = 0$, $y_1(t)$

1: Let

$$\begin{aligned} y_2(t) &= y_1(t)\nu(t) \\ \therefore {y_2}' &= {y_1}'(t)\nu(t) + y_1(t)\nu'(t) \\ \therefore {y_2}'' &= {y_1}''(t)\nu(t) + 2{y_1}'(t)\nu'(t) + y_1(t)\nu''(t) \end{aligned}$$

2: Substitute into the equation to get an ODE with $\nu''(t)$ and $\nu'(t)$.

$$\begin{aligned} 0 = \quad & {y_1}''(t)\nu(t) + 2{y_1}'(t)\nu'(t) + y_1(t)\nu''(t) \\ & + \left({y_1}'(t)\nu(t) + y_1(t)\nu'(t)\right) p(t) \\ & + y_1(t)\nu(t)q(t) \end{aligned}$$

3: Let

$$\begin{aligned} k(t) &= \nu'(t) \\ \therefore k'(t) &= \nu''(t) \end{aligned}$$

4: Substitute and solve the first order ODE in k.