ORDINARY DIFFERENTIAL EQUATIONS
ASSIGNMENT 4

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Part 1. Estimation of the Existence Interval

Exercise 1.
Show that the solution $y(t)$ of the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = e^{-t^2} + y^3$$
$$y(0) = 1$$

exists for $|t| \leq \frac{1}{9}$, $0 \leq y \leq 2$.

Solution 1.
Comparing

$$D_1 = \left\{ |t| \leq \frac{1}{9}, |y - 1| \leq 1 \right\}$$

and

$$D_1 = \left\{ |t - t_0| \leq a, |y - y_0| \leq b \right\}$$

$$t_0 = 0$$
$$y_0 = 1$$
$$a = \frac{1}{9}$$
$$b = 1$$

Let $|f(x,y)| < M$.
Therefore,

$$\begin{aligned} M &= \max_{D_1}(e^{-t^2} + y^3) \\ &= (e^{-0^2} + 2^3) \\ &= 9 \end{aligned}$$

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Let $h = \min\left\{a, \frac{b}{M}\right\}$.
Therefore,

$$\begin{aligned} h &= \min\left\{\frac{1}{9}, \frac{1}{9}\right\} \\ &= \frac{1}{9} \end{aligned}$$

Therefore,

$$\begin{aligned} I &= (t_0 - h, t_0 + h) \\ &= \left(-\frac{1}{9}, \frac{1}{9}\right) \end{aligned}$$

Exercise 2.
Calculate the largest interval of existence predicted by the existence and uniqueness theorem for the solution $y(t)$ of the initial-value problem

$$\begin{aligned} y' &= 1 + y^2 \\ y(0) &= 0 \end{aligned}$$

Solution 2.
Let the domain D_1 be $\left\{|x| \leq a, |y| \leq b\right\}$.
Let $|f(x, y)| < M$.
Therefore,

$$\begin{aligned} M &= \max_{D_1}(1 + y^2) \\ &= 1 + b^2 \end{aligned}$$

Let $h = \min\left\{a, \frac{b}{M}\right\}$.
Therefore,

$$h = \min\left\{a, \frac{b}{1 + b^2}\right\}$$

Maximizing $\frac{b}{1+b^2}$, $b = 1$.
Therefore,

$$\max\left\{\frac{b}{1 + b^2}\right\} = \frac{1}{2}$$

Maximizing a, $a = \infty$.
Therefore,

$$\begin{aligned} h &= \min\left\{\infty, \frac{1}{2}\right\} \\ &= \frac{1}{2} \end{aligned}$$

Therefore, the largest interval is

$$\begin{aligned} I &= (x_0 - h, x_0 + h) \\ &= \left(-\frac{1}{2}, \frac{1}{2}\right) \end{aligned}$$

Exercise 3.

Suppose that $|f(t, y)| \leq K$ for $-\infty < t < \infty$, $\infty < y < \infty$. Show that the solution $y(t)$ for the initial-value problem

$$\begin{aligned} y' &= f(t, y) \\ y(t_0) &= y_0 \end{aligned}$$

exists for all t.

Solution 3.

Let the domain D_1 be $\Big\{|t| \leq a, |y| \leq b\Big\}$.
Let $|f(x, y)| < M$.
Therefore,

$$\begin{aligned} M &= \max_{D_1} f(t, y) \\ &\leq K \end{aligned}$$

Let $h = \min\left\{a, \frac{b}{M}\right\}$.
Therefore,

$$h = \min\left\{a, \frac{b}{M}\right\}$$

Maximizing a and $\frac{b}{M}$,

$$h = \infty$$

Therefore,

$$\begin{aligned} I &= (t_0 - h, t_0 + h) \\ &= (-\infty, \infty) \end{aligned}$$

Therefore the solution exists for all t.