ORDINARY DIFFERENTIAL EQUATIONS
ASSIGNMENT 5

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Part 1. Homogeneous High Order Linear ODEs with Constant Coefficients

Exercise 1.
Find the general solution of the following differential equations

(1) $y'' + 2y'3y = 0$
(2) $4y'' - 9y = 0$
(3) $y''2y'3y = 0$
(4) $4y'' + 9y = 0$
(5) $16y'' + 24y' + 9y = 0$
(6) $y'''y''y' + y = 0$
(7) $y^{(6)} + y = 0$

Solution 1.
(1) Let

$$y = e^{\lambda t}$$
$$\therefore y' = \lambda e^{\lambda t}$$
$$\therefore y'' = \lambda^2 e^{\lambda t}$$

Therefore,

$$\lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} - 3e^{\lambda t} = 0$$
$$\therefore \lambda^2 + 2\lambda - 3 = 0$$

Therefore,

$$\lambda = \frac{-2 \pm \sqrt{4 + 12}}{2}$$
$$= \frac{-2 \pm 4}{2}$$
$$= -1 \pm 2$$

Therefore,

$$\lambda = -3 \qquad \text{or} \qquad \lambda = 1$$

Therefore,

$$y = e^{\lambda t}$$

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Therefore,

$$y = e^{-3t} \qquad \text{or} \qquad y = e^{t}$$

(2) Let

$$\begin{aligned} y &= e^{\lambda t} \\ \therefore y' &= \lambda e^{\lambda t} \\ \therefore y'' &= \lambda^2 e^{\lambda t} \end{aligned}$$

Therefore,

$$\begin{aligned} 4\lambda^2 e^{\lambda t} - 9e^{\lambda t} &= 0 \\ \therefore 4\lambda^2 - 9 &= 0 \end{aligned}$$

Therefore,

$$\lambda = \pm\sqrt{\frac{9}{4}}$$

Therefore,

$$\lambda = -\frac{3}{2} \qquad \text{or} \qquad \lambda = \frac{3}{2}$$

Therefore,

$$y = e^{\lambda t}$$

Therefore,

$$y = e^{-\frac{3}{2}t} \qquad \text{or} \qquad y = e^{\frac{3}{2}t}$$

(3) Let

$$\begin{aligned} y &= e^{\lambda t} \\ \therefore y' &= \lambda e^{\lambda t} \\ \therefore y'' &= \lambda^2 e^{\lambda t} \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda^2 e^{\lambda t} - 2\lambda e^{\lambda t} - 3e^{\lambda t} &= 0 \\ \therefore \lambda^2 - 2\lambda - 3 &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda &= \frac{2 \pm \sqrt{4 + 12}}{2} \\ \therefore \lambda &= 1 \pm 2 \end{aligned}$$

Therefore,

$$\lambda = -1 \qquad \text{or} \qquad \lambda = 3$$

Therefore,

$$y = e^{\lambda t}$$

Therefore,

$$y = e^{-t} \qquad \text{or} \qquad y = e^{3t}$$

(4) Let

$$
\begin{aligned}
y &= e^{\lambda t} \\
\therefore y' &= \lambda e^{\lambda t} \\
\therefore y'' &= \lambda^2 e^{\lambda t}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
4\lambda^2 e^{\lambda t} + 9e^{\lambda t} &= 0 \\
\therefore 4\lambda^2 + 9 &= 0
\end{aligned}
$$

Therefore,

$$\lambda = \pm\sqrt{\frac{-9}{4}}$$

$$\lambda = \frac{3i}{2} \qquad \text{or} \qquad \lambda = -\frac{3i}{2}$$

Therefore,

$$y = e^{\lambda t}$$

Therefore,

$$y = e^{-\frac{3i}{2}t} \qquad \text{or} \qquad y = e^{\frac{3i}{2}t}$$

(5) Let

$$
\begin{aligned}
y &= e^{\lambda t} \\
\therefore y' &= \lambda e^{\lambda t} \\
\therefore y'' &= \lambda^2 e^{\lambda t}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
16y'' + 24y' + 9y &= 0 \\
16\lambda^2 e^{\lambda t} + 24\lambda e^{\lambda t} + 9e^{\lambda t} &= 0 \\
\therefore 16\lambda^2 + 24\lambda + 9 &= 0
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lambda &= \frac{-24 \pm \sqrt{24^2 - 4 \cdot 16 \cdot 9}}{2 \cdot 16} \\
\therefore \lambda &= \frac{-24 \pm \sqrt{576 - 576}}{32} \\
\therefore \lambda &= -\frac{3}{4}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
y &= e^{\lambda t} \\
&= e^{-\frac{3}{4}t}
\end{aligned}
$$

(6) Let

$$\begin{aligned} y &= e^{\lambda t} \\ \therefore y' &= \lambda e^{\lambda t} \\ \therefore y'' &= \lambda^2 e^{\lambda t} \\ \therefore y''' &= \lambda^3 e^{\lambda t} \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda^3 e^{\lambda t} - \lambda^2 e^{\lambda t} - \lambda e^{\lambda t} + e^{\lambda t} &= 0 \\ \therefore \lambda^3 - \lambda^2 - \lambda + 1 &= 0 \\ \therefore (\lambda - 1)^2(\lambda - 1) &= 0 \end{aligned}$$

Therefore,

$$\lambda = -1 \qquad \text{or} \qquad \lambda = 1$$

Therefore,

$$y = e^{\lambda t}$$

Therefore,

$$y = e^{-t} \qquad \text{or} \qquad y = e^{t}$$

(7) Let

$$\begin{aligned} y &= e^{\lambda t} \\ \therefore y' &= \lambda e^{\lambda t} \\ \therefore y'' &= \lambda^2 e^{\lambda t} \\ \therefore y''' &= \lambda^3 e^{\lambda t} \\ &\vdots \\ \therefore y^{(6)} &= \lambda^6 e^{\lambda t} \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda^6 e^{\lambda t} + e^{\lambda t} &= 0 \\ \therefore \lambda^6 + 1 &= 0 \end{aligned}$$

Therefore,

$$\lambda = \sqrt[6]{-1}$$

Therefore,

$$\lambda = e^{\frac{i\pi}{6} + \frac{k\pi}{3}}$$

where $k \in \{0, \ldots, 5\}$.
Therefore,

$$\begin{aligned} y &= e^{\lambda t} \\ &= e^{e^{\frac{i\pi}{6} + \frac{k\pi}{3}}} \end{aligned}$$

Exercise 2.
Find the solution for the given initial value problems.

(1)
$$\begin{aligned} y'' + y'2y &= 0 \\ y(0) &= 1 \\ y'(0) &= 1 \end{aligned}$$

(2)
$$\begin{aligned} y'' + 4y &= 0 \\ y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$

(3)
$$\begin{aligned} y'' - 2y' + 5y &= 0 \\ y\left(\frac{\pi}{2}\right) &= 0 \\ y'\left(\frac{\pi}{2}\right) &= 2 \end{aligned}$$

(4)
$$\begin{aligned} 9y'' - 12y' + 4y &= 0 \\ y(0) &= 2 \\ y'(0) &= -1 \end{aligned}$$

(5)
$$\begin{aligned} y''' + y' &= 0 \\ y(0) &= 0 \\ y'(0) &= 1 \\ y''(0) &= 2 \end{aligned}$$

Solution 2.
(1) Let
$$\begin{aligned} y &= e^{\lambda t} \\ \therefore y' &= \lambda e^{\lambda t} \\ \therefore y'' &= \lambda^2 e^{\lambda t} \end{aligned}$$

Therefore,
$$\begin{aligned} \lambda^2 e^{\lambda t} + \lambda e^{\lambda t} - 2e^{\lambda t} &= 0 \\ \therefore \lambda^2 + \lambda - 2 &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned}\lambda &= \frac{-1 \pm \sqrt{1+8}}{2} \\ &= \frac{-1 \pm 3}{2}\end{aligned}$$

Therefore,

$$\lambda = -2 \qquad \text{or} \qquad \lambda = 1$$

Therefore,

$$y = e^{-2t} \qquad \text{or} \qquad y = e^{t}$$

(2) Let

$$\begin{aligned}y &= e^{\lambda t} \\ \therefore y' &= \lambda e^{\lambda t} \\ \therefore y'' &= \lambda^2 e^{\lambda t}\end{aligned}$$

Therefore,

$$\begin{aligned}\lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} &= 0 \\ \therefore \lambda^2 + 4\lambda &= 0 \\ \therefore \lambda(\lambda + 4) &= 0\end{aligned}$$

Therefore,

$$\lambda = -4 \qquad \text{or} \qquad \lambda = 0$$

Therefore,

$$y = e^{-4t} \qquad \text{or} \qquad y = e^{0t}$$

Therefore,

$$y = e^{-4t} \qquad \text{or} \qquad y = 1$$

(3) Let

$$\begin{aligned}y &= e^{\lambda t} \\ \therefore y' &= \lambda e^{\lambda t} \\ \therefore y'' &= \lambda^2 e^{\lambda t}\end{aligned}$$

Therefore,

$$\begin{aligned}\lambda^2 e^{\lambda t} - 2\lambda e^{\lambda t} + 5e^{\lambda t} &= 0 \\ \therefore \lambda^2 - 2\lambda + 5 &= 0\end{aligned}$$

Therefore,

$$\begin{aligned}\lambda &= \frac{2 \pm \sqrt{4-20}}{2}\\ &= \frac{2 \pm 4i}{2}\\ &= 1 \pm 2i\end{aligned}$$

Therefore,

$$\lambda = 1 + 2i \qquad \text{or} \qquad \lambda = 1 - 2i$$

Therefore,

$$y = e^{(1+2i)t} \qquad \text{or} \qquad y = e^{(1-2i)t}$$

(4) Let

$$\begin{aligned}y &= e^{\lambda t}\\ \therefore y' &= \lambda e^{\lambda t}\\ \therefore y'' &= \lambda^2 e^{\lambda t}\end{aligned}$$

Therefore,

$$\begin{aligned}9\lambda^2 e^{\lambda t} - 12\lambda e^{\lambda t} + 4e^{\lambda t} &= 0\\ \therefore 9\lambda^2 - 12\lambda + 4 &= 0\end{aligned}$$

Therefore,

$$\begin{aligned}\lambda &= \frac{12 \pm \sqrt{144-144}}{18}\\ &= \frac{2}{3}\end{aligned}$$

Therefore,

$$y = e^{\frac{2}{3}t}$$

(5) Let

$$\begin{aligned}y &= e^{\lambda t}\\ \therefore y' &= \lambda e^{\lambda t}\\ \therefore y'' &= \lambda^2 e^{\lambda t}\\ \therefore y''' &= \lambda^3 e^{\lambda t}\end{aligned}$$

Therefore,

$$\begin{aligned}y''' + y' &= 0\\ \lambda^3 e^{\lambda t} + \lambda e^{\lambda t} &= 0\\ \therefore \lambda^3 + \lambda &= 0\\ \therefore \lambda(\lambda^2 + 1) &= 0\end{aligned}$$

Therefore,

$$\lambda = 0 \qquad \text{or} \qquad \lambda = i \qquad \text{or} \qquad \lambda = -i$$

Therefore,

$$y = e^{0t} \qquad \text{or} \qquad y = e^{it} \qquad \text{or} \qquad y = e^{-it}$$

Therefore,

$$y = 1 \qquad \text{or} \qquad y = e^{it} \qquad \text{or} \qquad y = e^{-it}$$

Exercise 3.
Consider the ODE $y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$. Determine the values of α, if any, such that all solutions tend to zero as $t \to \infty$. Also determine the values of α, if any, such that all (nonzero) solutions become unbounded as $t \to \infty$.

Solution 3.
Let

$$\begin{aligned} y &= e^{\lambda t} \\ \therefore y' &= \lambda e^{\lambda t} \\ \therefore y'' &= \lambda^2 e^{\lambda t} \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda^2 e^{\lambda t} - (2\alpha - 1)\lambda e^{\lambda t} + \alpha(\alpha - 1)e^{\lambda t} &= 0 \\ \therefore \lambda^2 - (2\alpha - 1)\lambda + \alpha(\alpha - 1) &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda &= \frac{2\alpha - 1 \pm \sqrt{(2\alpha - 1)^2 - 4\alpha(\alpha - 1)}}{2} \\ &= \frac{2\alpha - 1 \pm \sqrt{4\alpha^2 - 4\alpha + 1 - 4\alpha^2 + 4\alpha}}{2} \\ &= \frac{2\alpha - 1 \pm \sqrt{1}}{2} \\ &= \frac{2\alpha - 1 \pm 1}{2} \end{aligned}$$

Therefore,

$$\lambda = \alpha - 1 \qquad \text{or} \qquad \lambda = \alpha$$

Therefore,

$$y = e^{(\alpha - 1)t} \qquad \text{or} \qquad y = e^{\alpha t}$$

$$\lim_{n \to \infty} e^{-n} = 0$$

Therefore, if $y > 1$, $\lim\limits_{t \to \infty} y = 0$

Part 2. Reduction of Order

Exercise 1.
Use the method of reduction of order to find a second solution for the given differential equations.

(1)
$$\begin{aligned} t^2y'' - 4ty' + 6y &= 0 \\ t &> 0 \\ y_1(t) &= t^2 \end{aligned}$$

(2)
$$\begin{aligned} xy'' - y' + 4x^3y &= 0 \\ x &> 0 \\ y_1(x) &= \sin(x^2) \end{aligned}$$

Solution 1.
(1) Let
$$\begin{aligned} y_2(t) &= \nu(t)y_1(t) \\ &= t^2\nu(t) \end{aligned}$$

Therefore,
$$\begin{aligned} {y_2}'(t) &= 2t\nu(t) + t^2\nu'(t) \\ \therefore {y_2}''(t) &= 2\nu(t) + 2t\nu'(t) + 2t\nu'(t) + t^2\nu''(t) \\ &= 2\nu(t) + 4t\nu'(t) + t^2\nu''(t) \end{aligned}$$

Therefore, substituting ${y_2}'$ and ${y_2}''$,
$$\begin{aligned} t^2\left(2\nu(t) + 4t\nu'(t) + t^2\nu''(t)\right) - 4t\left(2t\nu(t) + t^2\nu'(t)\right) + 6\left(t^2\nu(t)\right) &= 0 \\ \therefore 2t^2\nu(t) + 4t^3\nu'(t) + t^4\nu''(t) - 8t^2\nu(t) - 4t^3\nu'(t) + 6t^2\nu(t) &= 0 \\ \therefore t^4\nu''(t) &= 0 \\ \therefore \nu''(t) &= 0 \end{aligned}$$

Therefore, let
$$\nu(t) = k_1t + k_2$$

Therefore,
$$\nu'(t) = k_1$$

Therefore, let $k_1 = 1$, $k_2 = 1$.
Therefore,
$$\begin{aligned} y_2(t) &= t^2t \\ &= t^3 \end{aligned}$$

(2) Let

$$y_2(x) = \nu(x)y_1(x)$$
$$= \sin(x^2)\nu(x)$$

Therefore,

$$y_2{}'(x) = \sin(x^2)\nu'(x) + 2x\nu(x)\cos(x^2)$$
$$y_2{}''(x) = \sin(x^2)\nu''(x) + 4x\cos(x^2)\nu'(x) - 4x^2\nu(x)\sin(x^2) + 2\nu(x)\cos(x^2)$$

Therefore, substituting $y_2{}'$ and $y_2{}''$ and simplifying,

$$\therefore x\sin(x^2)\nu''(x) + \nu'(x)\left(4x^2\cos(x^2) - \sin(x^2)\right) + 4x\nu(x)\cos(x^2) = 0$$

Therefore, the characteristic equation is

$$x\sin(x^2)\lambda^2 + \left(4x^2\cos(x^2) - \sin(x^2)\right)\lambda + 4x\cos(x^2) = 0$$

Therefore,

$$\lambda = \frac{\sin(x^2) - 4x^2\cos(x^2) \pm \sqrt{\sin^2(x^2) - 8x^2\sin(x^2)\cos(x^2) - 16x^2\sin(x^2)\cos(x^2)}}{2x\sin(x^2)}$$
$$= \frac{\sin(x^2) - 4x^2\cos(x^2) \pm \sqrt{\sin^2(x^2) - 24x^2\sin(x^2)\cos(x^2)}}{2x\sin(x^2)}$$

Therefore,

$$\nu(x) = e^{\lambda x}$$

Therefore,

$$y_2(x) = \sin(x^2)\nu(x)$$
$$= \sin(x^2)e^{\frac{\sin(x^2) - 4x^2\cos(x^2) \pm \sqrt{\sin^2(x^2) - 24x^2\sin(x^2)\cos(x^2)}}{2x\sin(x^2)}}$$