

ORDINARY DIFFERENTIAL EQUATIONS : ASSIGNMENT 1

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Part 1. Linear Equations

Exercise 1.

Find the general solution

a) $y' - 2y = t^2 e^{2t}$

b) $y' + \left(\frac{1}{t}\right)y = 3 \cos 2t, t > 0$

c) $ty' + 2y = \sin t, t > 0$

d) $(1 + t^2)y' + 4ty = (1 + t^2)^{-2}$

Solution 1.

Date: Monday 30th March, 2015.

a) Comparing

$$y' - 2y = t^2 e^{2t}$$

and

$$y' + p(t)y = q(t)$$

$$p(t) = -2$$

$$q(t) = t^2 e^{2t}$$

Therefore,

$$\begin{aligned}\mu(x) &= e^{\int p(t) dt} \\ &= e^{\int -2 dt} \\ &= e^{-2t}\end{aligned}$$

Therefore,

$$\begin{aligned}\therefore e^{-2t} (y' - 2y) &= e^{-2t} t^2 e^{2t} \\ \therefore e^{-2t} y' - 2e^{-2t} y &= t^2 \\ \therefore (e^{-2t} y)' &= t^2 \\ \therefore e^{-2t} y &= \frac{t^3}{3} + c \\ \therefore y &= \frac{t^3 e^{2t}}{3} + c e^{2t}\end{aligned}$$

b) Comparing

$$y' - \left(\frac{1}{t}\right)y = 3 \cos 2t$$

and

$$y' + p(t)y = q(t)$$

$$p(t) = -\frac{1}{t}$$

$$q(t) = 3 \cos 2t$$

Therefore,

$$\begin{aligned}\mu(t) &= e^{\int p(t) dt} \\ &= e^{\int \frac{1}{t} dt} \\ &= e^{\ln t} \\ &= t\end{aligned}$$

Therefore,

$$ty' - y = 3t \cos 2t$$

$$\therefore (ty)' = 3t \cos 2t$$

$$\therefore ty = \int 3t \cos 2t dt$$

$$\therefore ty = \frac{3}{4} \cos 2t + \frac{3}{2}t \sin 2t + c$$

$$\therefore y = \frac{3 \cos 2t}{4t} + \frac{3 \sin 2t}{2} + \frac{c}{t}$$

c) Comparing

$$ty' + 2y = \sin t$$

and

$$y' + p(t)y = q(t)$$

$$p(t) = \frac{2}{t}$$

$$q(t) = \frac{\sin t}{t}$$

Therefore,

$$\begin{aligned}\mu(t) &= e^{\int p(t) dt} \\ &= e^{\int \frac{2}{t} dt} \\ &= e^{2 \ln t} \\ &= t^2\end{aligned}$$

Therefore,

$$\begin{aligned}t^2 y' + 2ty &= t \sin t \\ \therefore (t^2 y)' &= t \sin t \\ \therefore t^2 y &= \int t \sin t dt \\ \therefore t^2 y &= -t \cos t + \sin t + c \\ \therefore y &= -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{c}{t^2}\end{aligned}$$

d) Comparing

$$(1 + t^2)y' + 4ty = (1 + t^2)^{-2}$$

and

$$y' + p(t)y = q(t)$$

$$p(t) = \frac{4t}{1 + t^2}$$

$$\begin{aligned} q(t) &= \frac{(1 + t^2)^{-2}}{1 + t^2} \\ &= \frac{1}{(1 + t^2)^3} \end{aligned}$$

Therefore,

$$\begin{aligned} \mu(t) &= e^{\int p(t) dt} \\ &= e^{\int \frac{4t}{1+t^2} dt} \\ &= e^{2 \ln(1+t^2)} \\ &= (1 + t^2)^2 \end{aligned}$$

Therefore,

$$\begin{aligned} y &= \frac{1}{\mu(t)} \int \mu(t)q(t) dt \\ &= (1 + t^2)^{-2} \int (1 + t^2)^2 \cdot (1 + t^2)^{-3} dt \\ &= (1 + t^2)^{-2} \int \frac{1}{1 + t^2} dt \\ &= \frac{\tan^{-1} t + c}{(1 + t^2)^2} \end{aligned}$$

Exercise 2.

In the previous exercise, determine the solution's behaviour for large t .

Solution 2.

a)

$$y = \frac{t^3 e^{2t}}{3} + ce^{2t}$$

$$\therefore \lim_{t \rightarrow \infty} y = \infty$$

b)

$$y = \frac{3 \cos 2t}{4t} + \frac{3 \sin 2t}{2} + \frac{c}{t}$$

If $t \rightarrow \infty$, $\frac{3 \sin 2t}{2} \in \left(-\frac{3}{2}, \frac{3}{2}\right)$, and

$$\lim_{t \rightarrow \infty} \frac{3 \cos 2t}{4t} = 0$$

Therefore,

$$y \in \left[-\frac{3}{2}, \frac{3}{2}\right]$$

c)

$$y = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{c}{t^2}$$

$$\therefore \lim_{t \rightarrow \infty} y = 0$$

d)

$$y = \frac{\tan^{-1} t + c}{(1 + t^2)^2}$$

$$\therefore \lim_{t \rightarrow \infty} y = 0$$

Exercise 3.

Solve the initial value problems

a) $y' + 2y = te^{2t}$, $y(0) = 1$

b) $y' + \left(\frac{2}{t}\right)y = \frac{\cos t}{t^2}$, $y(\pi) = 0$, $t > 0$

c) $ty' + 2y = \sin t$, $y\left(\frac{\pi}{2}\right) = 1$

Solution 3.

a) Comparing

$$y' + 2y = te^{2t}$$

and

$$y' + p(t)y = q(t)$$

$$p(t) = -2$$

$$q(t) = te^{2t}$$

Therefore,

$$\begin{aligned}\mu(t) &= e^{\int p(t) dt} \\ &= e^{\int -2 dt} \\ &= e^{-2t}\end{aligned}$$

Therefore,

$$\begin{aligned}y &= \frac{1}{\mu(t)} \int \mu(t)q(t) dt \\ &= \frac{1}{e^{-2t}} \int e^{-2t} \cdot te^{2t} dt \\ &= \frac{1}{e^{-2t}} \int te^{0} dt \\ &= \frac{e^{2t} \left(\frac{t}{2} - \frac{1}{4} \right) + c}{e^{-2t}} \\ &= e^{2t} \left(\frac{t}{2} - \frac{1}{4} \right) + \frac{c}{e^{-2t}}\end{aligned}$$

Substituting the given condition, $y(0) = 1$,

$$\begin{aligned}1 &= e^{2 \cdot 0} \left(\frac{0}{2} - \frac{1}{4} \right) + \frac{c}{e^{-2 \cdot 0}} \\ \therefore 1 &= c - \frac{1}{4} \\ \therefore c &= \frac{5}{4}\end{aligned}$$

Therefore,

$$y = e^{2t} \left(\frac{t}{2} - \frac{1}{4} \right) + \frac{5}{4e^{-2t}}$$

b) Comparing

$$y' + \left(\frac{2}{t}\right)y = \frac{\cos t}{t^2}$$

and

$$y' + p(t)y = q(t)$$

$$p(t) = \frac{2}{t}$$

$$q(t) = \frac{\cos t}{t^2}$$

Therefore,

$$\begin{aligned}\mu(t) &= e^{\int p(t) dt} \\ &= e^{\int \frac{2}{t} dt} \\ &= e^{2 \ln t} \\ &= t^2\end{aligned}$$

Therefore,

$$\begin{aligned}y &= \frac{1}{\mu(t)} \int \mu(t)q(t) dt \\ &= \frac{1}{t^2} \int t^2 \cdot \frac{\cos t}{t^2} dt \\ &= \frac{1}{t^2} \int \cos t dt \\ &= \frac{\sin t + c}{t^2}\end{aligned}$$

Substituting the given condition, $y(\pi) = 0$,

$$\begin{aligned}0 &= \frac{\sin \pi + c}{\pi^2} \\ \therefore c &= 0\end{aligned}$$

Therefore,

$$y = \frac{\sin t}{t^2}$$

c) Comparing

$$ty' + 2y = \sin t$$

and

$$y' + p(t)y = q(t)$$

$$p(t) = \frac{2}{t}$$

$$q(t) = \frac{\sin t}{t}$$

Therefore,

$$\begin{aligned}\mu(t) &= e^{\int p(t) dt} \\ &= e^{\int \frac{2}{t} dt} \\ &= e^{2 \ln t} \\ &= t^2\end{aligned}$$

Therefore,

$$\begin{aligned}y &= \frac{1}{t^2} \int \mu(x)q(x) dx \\ &= \frac{1}{t^2} \int t^2 \frac{\sin t}{t} dt \\ &= \frac{1}{t^2} \int t \sin t \\ &= \frac{-t \cos t + \sin t + c}{t^2}\end{aligned}$$

Substituting the given condition $y\left(\frac{\pi}{2}\right) = 1$,

$$1 = \frac{-\frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2} + c}{\left(\frac{\pi}{2}\right)^2}$$

$$\therefore \frac{\pi^2}{4} = 1 + c$$

$$\therefore c = \frac{\pi^2}{4} - 1$$

Therefore,

$$y = \frac{-t \cos t + \sin t + \frac{\pi^2}{4} - 1}{t^2}$$

Exercise 4.

Find the initial value y_0 for which the solution of the initial value problem

$$\begin{aligned}y' - y &= 1 + 3 \sin t \\ y(0) &= y_0\end{aligned}$$

remains finite for $t \rightarrow \infty$.

Solution 4.

Comparing

$$y' - y = 1 + 3 \sin t$$

and

$$\begin{aligned}y' + p(t)y &= q(t) \\ p(t) &= -1 \\ q(t) &= 1 + 3 \sin t\end{aligned}$$

Therefore,

$$\begin{aligned}\mu(t) &= e^{\int p(t) dt} \\ &= e^{\int -1 dt} \\ &= e^{-t}\end{aligned}$$

Therefore,

$$\begin{aligned}y &= \frac{1}{\mu(t)} \int \mu(t)q(t) dt \\ &= e^t \int \frac{1 + 3 \sin t}{e^t} dt \\ &= e^t \left(-\frac{e^{-t}}{2} (2 + 3 \cos t + 3 \sin t) + c \right) \\ &= -\frac{2 + 3 \cos t + 3 \sin t}{2} + ce^t\end{aligned}$$

Therefore,

$$\lim_{t \rightarrow \infty} y = \lim_{t \rightarrow \infty} \frac{2 + 3 \cos t + 3 \sin t}{2} + \lim_{t \rightarrow \infty} ce^t$$

Therefore, for $\lim_{t \rightarrow \infty} y$ to be finite, $c = 0$. Substituting the conditions, $y(0) = y_0$ and $c = 0$,

$$\begin{aligned}y_0 &= -\frac{2 + 3 + 0}{2} + 0e^t \\ \therefore y_0 &= -\frac{5}{2}\end{aligned}$$

Exercise 5.

Show that if y_1, y_2, y_3 are private solutions for

$$y' + a(t)y = b(t)$$

then the function $\frac{y_2 - y_3}{y_3 - y_1}$ is constant for all real t .

Solution 5.

$$\begin{aligned}\mu(t) &= e^{\int a(t) dt} \\ \therefore y &= \frac{1}{\mu(t)} \int \mu(t)b(t) dt\end{aligned}$$

Let

$$\int \mu(t)b(t) dt = p(t) + c$$

Therefore,

$$\begin{aligned}y &= \frac{p(t) + c}{\mu(t)} \\ \therefore y_1 &= \frac{p(t) + c_1}{\mu(t)} \\ \therefore y_2 &= \frac{p(t) + c_2}{\mu(t)} \\ \therefore y_3 &= \frac{p(t) + c_3}{\mu(t)}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{y_2 - y_3}{y_3 - y_1} &= \frac{\frac{p(t) + c_2 - p(t) - c_3}{\mu(t)}}{\frac{p(t) + c_3 - p(t) - c_1}{\mu(t)}} \\ &= \frac{c_2 - c_3}{c_3 - c_1}\end{aligned}$$

Therefore, as $\frac{y_2 - y_3}{y_3 - y_1}$ is independent of t , it is constant for all real t .

Exercise 6.

Suppose that there exists $M > 0$ such that for all real x , $|f(x)| \leq M$. Show that for $a > 0$, any solution for the equation $y' + ay = f(x)$ is bounded at $[0, \infty)$.

Solution 6.

$$\begin{aligned}
y' + ay &= f(x) \\
\therefore \mu(x) &= e^{\int a \, dx} \\
&= e^{ax} \\
\therefore y &= \frac{1}{e^{ax}} \int e^{ax} f(x) \, dx
\end{aligned}$$

As $|f(x)| \leq M$,

$$\begin{aligned}
y &\leq \frac{1}{e^{ax}} \int M e^{ax} \, dx \\
\therefore y &\leq \frac{1}{e^{ax}} \left(\frac{M}{a} e^{ax} + c \right) \\
\therefore y &\leq \frac{M}{a} + \frac{c}{e^{ax}}
\end{aligned}$$

or

$$\begin{aligned}
y &\geq \frac{1}{e^{ax}} \cdot - \int M e^{ax} \, dx \\
\therefore y &\geq -\frac{1}{e^{ax}} \left(\frac{M}{a} e^{ax} + c \right) \\
\therefore y &\geq -\frac{M}{a} - \frac{c}{e^{ax}}
\end{aligned}$$

Therefore,

$$y \leq \left| \frac{M}{a} + \frac{c}{e^{ax}} \right|$$

Therefore,

$$0 \leq y < \infty$$

Part 2. Bernoulli Equations**Exercise 1.**

Solve

a) $t^2 y' + 2ty - y^3 = 0, t > 0$

b) $y' = \varepsilon y - \sigma y^3, \varepsilon > 0, \sigma > 0$

Solution 1.

a) Comparing

$$t^2 y' + 2ty - y^3 = 0$$

and

$$y' + p(t)y = q(t)y^n$$

$$p(t) = \frac{2}{t}$$

$$q(t) = \frac{1}{t^2}$$

$$n = 3$$

Therefore, let

$$\nu = y^{1-3}$$

$$= y^{-2}$$

$$\therefore \nu' = (-2)y^{-3}y'$$

Substituting ν and ν' ,

$$\frac{1}{-2}\nu' + \frac{2}{t}\nu = \frac{1}{t^2}$$

$$\therefore \nu' - \frac{4}{t}\nu = -\frac{2}{t^2}$$

Therefore,

$$\begin{aligned}\mu(t) &= e^{\int -\frac{4}{t} dt} \\ &= t^{-4}\end{aligned}$$

Therefore,

$$\begin{aligned}\nu &= t^4 \int t^{-4} t^{-2} dt \\ &= t^4 \left(-\frac{1}{5t^5} + c \right) \\ \therefore y^{-2} &= t^4 \left(\frac{5t^5 c - 1}{5t^5} \right) \\ \therefore y^{-2} &= \frac{5t^5 c - 1}{5t} \\ \therefore y^2 &= \frac{5t}{5t^5 c - 1} \\ \therefore y &= \pm \sqrt{\frac{5t}{5t^5 c - 1}}\end{aligned}$$

b)

$$y' = \varepsilon y - \sigma y^3$$

$$\therefore y' - \varepsilon y = -\sigma y^3$$

Comparing

$$y' - \varepsilon y = -\sigma y^3$$

and

$$y' - p(t)y = q(t)y^n$$

$$p(t) = -\varepsilon$$

$$q(t) = -\sigma$$

$$n = y^{-2}$$

Therefore, let

$$\nu = y^{-2}$$

$$\therefore \nu' = (-2)y^{-3}y'$$

Substituting ν and ν' ,

$$\frac{1}{-2}\nu' - \varepsilon = -\sigma$$

$$\therefore \nu' + 2\varepsilon = 2\sigma$$

Therefore,

$$\begin{aligned}\mu(t) &= e^{\int -\varepsilon dt} \\ &= e^{-\varepsilon t}\end{aligned}$$

Therefore,

$$\begin{aligned}\nu &= e^{\varepsilon t} \int e^{-\varepsilon t} \cdot 2\sigma dt \\ &= e^{\varepsilon t} (-2e^{-\varepsilon t}\sigma t + c) \\ \therefore y^{-2} &= -2\sigma t + e^{\varepsilon t}c \\ \therefore y^2 &= \frac{1}{e^{\varepsilon t}c - 2\sigma t} \\ \therefore y &= \pm \sqrt{\frac{1}{e^{\varepsilon t}c - 2\sigma t}}\end{aligned}$$

Part 3. Separable Equations

Exercise 1.

Find the 'general' solution for the following equations. Keep track of singular solutions, if there are any.

a) $y' + y^2 \sin x = 0$

b) $y' = (\cos^2 x)(\cos^2 2y)$

c) $\frac{dy}{dx} = \frac{x^2}{1+y^2}$

Solution 1.

a)

$$\begin{aligned}\frac{dy}{dx} + y^2 \sin x &= 0 \\ \therefore \frac{dy}{dx} &= -y^2 \sin x\end{aligned}$$

If $y^2 = 0$,
 $y = 0$ is a solution if and only if

$$\frac{dy}{dx} = 0$$

If $y^2 \neq 0$

$$\begin{aligned}\therefore \frac{dy}{y^2} &= -\sin x \, dx \\ \therefore \int \frac{dy}{y^2} &= \int -\sin x \, dx \\ \therefore -\frac{1}{y} &= \cos x + c \\ \therefore y &= -\frac{1}{\cos x + c}\end{aligned}$$

b)

$$\frac{dy}{dx} = (\cos^2 x)(\cos^2 2y)$$

If $\cos^2(2y) = 0$, $y = \frac{\pi}{4} + n\frac{\pi}{2}$ is a solution if and only if

$$\frac{dy}{dx} = 0$$

If $\cos^2(2y) \neq 0$,

$$\therefore \frac{dy}{\cos^2 2y} = \cos^2 x \, dx$$

$$\therefore \int \frac{dy}{\cos^2 2y} = \int \cos^2 x \, dx$$

$$\therefore \frac{1}{2} \tan(2y) = \frac{1}{2} \sin(2x) + c_1$$

$$\therefore \tan(2y) = \sin(2x) + c_2$$

$$\therefore 2y = \tan^{-1}(\sin(2x)) + c_3$$

c)

$$\frac{dy}{dx} = \frac{x^2}{1+y^2}$$

$$\therefore (1+y^2) \, dy = x^2 \, dx$$

$$\therefore \int (1+y^2) \, dy = \int x^2 \, dx$$

$$\therefore y + \frac{y^3}{3} = \frac{x^3}{3} + c_1$$

$$\therefore 3y + y^3 = x^3 + c_2$$

Exercise 2.

Find the solution for the following initial value problems in explicit form, and determine (at least approximately) the interval in which the solution is defined.

a) $y' = (1 - 2x)y^2, y(0) = -\frac{1}{6}$

b) $y' = \frac{2x}{y + x^2y}, y(0) = -2$

c) $y' = \frac{2x}{1+y}, y(0) = -2$

Solution 2.

a)

$$\frac{dy}{dx} = (1 - 2x)y^2$$

If $y^2 = 0$, $y = 0$ is a solution if and only if

$$\frac{dy}{dx} = 0$$

If $y^2 \neq 0$,

$$\frac{dy}{y^2} = (1 - 2x) dx$$

$$\therefore \int \frac{dy}{y^2} = \int (1 - 2x) dx$$

$$\therefore -\frac{1}{y} = x - x^2 + c$$

$$\therefore y = -\frac{1}{x - x^2 + c}$$

Substituting $y(0) = -\frac{1}{6}$,

$$-\frac{1}{6} = -\frac{1}{c}$$

$$\therefore c = 6$$

Therefore, the solution is not defined if and only if

$$x - x^2 + c = 0$$

$$\iff x^2 - x - 6 = 0$$

$$\iff x = \frac{1 \pm \sqrt{1 + 24}}{2}$$

Therefore, the solution is defined on $\mathbb{R} \setminus \{-2, 3\}$

b)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{2x}{y + x^2y} \\
\therefore \frac{dy}{dx} &= \frac{2x}{y(1 + x^2)} \\
\therefore \frac{y}{2} dy &= \frac{x}{1 + x^2} dx \\
\therefore \int \frac{y}{2} dy &= \int \frac{x}{1 + x^2} dx \\
\therefore \frac{y^2}{4} &= \frac{1}{2} \ln(1 + x^2) + c \\
\therefore y &= \sqrt{2 \ln(1 + x^2) + c}
\end{aligned}$$

Substituting $y(0) = -2$,

$$-2 = \sqrt{2 \ln 1 + c}$$

$$\therefore c = 4$$

Therefore, the solution is defined if and only if

$$\begin{aligned}
&\ln(1 + x^2) + 4 \geq 0 \\
\iff \ln(1 + x^2) &\geq -4 \\
\iff 1 + x^2 &\geq e^{-4} \\
\iff x^2 &\geq e^{-4} - 1 \\
\iff x &\in \left(-\infty, \sqrt{e^{-4} - 1}\right] \cup \left[\sqrt{e^{-4} - 1}, \infty\right)
\end{aligned}$$

Therefore, the solution is defined on $\left(-\infty, \sqrt{e^{-4} - 1}\right] \cup \left[\sqrt{e^{-4} - 1}, \infty\right)$.

c)

$$\begin{aligned}\frac{dy}{dx} &= \frac{2x}{1+y} \\ \therefore (1+y) dy &= 2x dx \\ \therefore \int (1+y) dy &= \int 2x dx \\ \therefore y + \frac{y^2}{2} &= x^2 + c\end{aligned}$$

Substituting $y(0) = -2$,

$$\begin{aligned}-2 + \frac{4}{2} &= 0 + c \\ \therefore c &= 0\end{aligned}$$

Therefore,

$$\begin{aligned}y^2 + 2y - 2x^2 &= 0 \\ \therefore y &= \frac{-2 \pm \sqrt{4 + 8x^2}}{2} \\ &= -1 \pm \sqrt{1 + 2x^2}\end{aligned}$$

Therefore, the solution is defined if and only if

$$\begin{aligned}1 + 2x^2 &\geq 0 \\ \iff x^2 &\geq -\frac{1}{2}\end{aligned}$$

Therefore, the solution is defined on \mathbb{R} .**Exercise 3.**

Solve the initial value problem

$$y' = \frac{2 - e^x}{3 + 2y}, y(0) = 0$$

and determine where the solution attains its maximum value.

Solution 3.

$$\begin{aligned}\frac{dy}{dx} &= \frac{2 - e^x}{3 + 2y} \\ \therefore (3 + 2y) dy &= (2 - e^x) dx \\ \therefore \int (3 + 2y) dy &= \int (2 - e^x) dx \\ \therefore 3y + y^2 &= 2x - e^x + c\end{aligned}$$

Substituting $y(0) = 0$,

$$\begin{aligned} 0 &= -1 + c \\ \therefore c &= 1 \end{aligned}$$

Therefore,

$$y^2 + 3y = 2x - e^x + 1$$

If $\frac{dy}{dx} = 0$,

$$\begin{aligned} \frac{2 - e^x}{3 + 2y} &= 0 \\ \therefore 2 &= e^x \\ \therefore x &= \ln 2 \end{aligned}$$

Therefore, the solution attains its maximum value at $x = \ln 2$.

Part 4. Homogeneous Equations

Exercise 1.

Solve

a) $\frac{dy}{dx} = \frac{x + 3y}{x - y}$

b) $\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$

c) $(x^2 + 3xy + y^2) dx - x^2 dy = 0$

d) $xy' - y = (x + y)(\ln(x + y) - \ln(x))$

Solution 1.

a)

$$\begin{aligned}\frac{dy}{dx} &= \frac{x+3y}{x-y} \\ &= \frac{1+\frac{3y}{x}}{1-\frac{y}{x}}\end{aligned}$$

Let

$$\begin{aligned}\frac{y}{x} &= z \\ \therefore y &= xz \\ \therefore \frac{dy}{dx} &= z + x \frac{dz}{dx}\end{aligned}$$

Therefore,

$$\begin{aligned}z + x \frac{dz}{dx} &= \frac{1+3z}{1-z} \\ \therefore x \frac{dz}{dx} &= \frac{1+3z}{1-z} - z \\ \therefore x \frac{dz}{dx} &= \frac{1+3z-z+z^2}{1-z} \\ \therefore \frac{1-z}{1+2z+z^2} &= \frac{dx}{x} \\ \therefore \int \frac{1-z}{1+2z+z^2} &= \int \frac{dx}{x} \\ \therefore -\frac{2}{1+z} - \ln(1+z) &= \ln x + c \\ \therefore -\frac{2}{1+\frac{y}{x}} - \ln\left(1+\frac{y}{x}\right) &= \ln x + c \\ \therefore -\frac{2x}{x+y} - \ln\left(\frac{x+y}{x}\right) &= \ln x + c \\ \therefore -\frac{2x}{x+y} - \ln(x+y) + \ln x &= \ln x + c \\ \therefore -\frac{2x}{x+y} - \ln(x+y) &= c\end{aligned}$$

b)

$$\begin{aligned}\frac{dy}{dx} &= \frac{x^2 + xy + y^2}{x^2} \\ &= \frac{1 + \frac{y}{x} + \frac{y^2}{x^2}}{1}\end{aligned}$$

Let

$$\begin{aligned}\frac{y}{x} &= z \\ \therefore y &= xz \\ \therefore \frac{dy}{dx} &= z + x \frac{dz}{dx}\end{aligned}$$

Therefore,

$$\begin{aligned}z + x \frac{dz}{dx} &= 1 + z + z^2 \\ \therefore x \frac{dz}{dx} &= 1 + z^2 \\ \therefore \frac{dz}{1 + z^2} &= \frac{dx}{x} \\ \therefore \int \frac{dz}{1 + z^2} &= \int \frac{dx}{x} \\ \therefore \tan^{-1} z &= \ln x + c \\ \therefore \tan^{-1} \frac{y}{x} &= \ln x + c \\ \therefore y &= x \tan(\ln x + c)\end{aligned}$$

c)

$$\begin{aligned}
 (x^2 + 3xy + y^2) dx - x^2 dy &= 0 \\
 \therefore \frac{dy}{dx} &= \frac{x^2 + 3xy + y^2}{x^2} \\
 \therefore \frac{dy}{dx} &= \frac{1 + 3\frac{y}{x} + \frac{y^2}{x^2}}{1}
 \end{aligned}$$

Let

$$\begin{aligned}
 \frac{y}{x} &= z \\
 \therefore y &= xz \\
 \therefore \frac{dy}{dx} &= z + x \frac{dz}{dx}
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 z + x \frac{dz}{dx} &= 1 + 3z + z^2 \\
 \therefore x \frac{dz}{dx} &= 1 + 2z + z^2 \\
 \therefore \frac{dz}{(1+z)^2} &= \frac{dx}{x} \\
 \therefore \int \frac{dz}{(1+z)^2} &= \int \frac{dx}{x} \\
 \therefore -\frac{1}{1+z} &= \ln x + c \\
 \therefore -\frac{1}{1+\frac{y}{x}} &= \ln x + c \\
 \therefore -\frac{x}{x+y} &= \ln x + c \\
 \therefore -\frac{x}{\ln x + c} &= x + y \\
 \therefore y &= -x - \frac{x}{\ln x + c}
 \end{aligned}$$

d)

$$\begin{aligned}
 xy' - y &= (x + y) (\ln(x + y) - \ln(x)) \\
 \therefore x \frac{dy}{dx} - y &= (x + y) \left(\ln \left(1 + \frac{y}{x} \right) \right) \\
 \therefore \frac{dy}{dx} - \frac{y}{x} &= \left(1 + \frac{y}{x} \right) \ln \left(1 + \frac{y}{x} \right)
 \end{aligned}$$

Let

$$\begin{aligned}
 \frac{y}{x} &= z \\
 \therefore y &= xz \\
 \therefore \frac{dy}{dx} &= z + x \frac{dz}{dx}
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 z + x \frac{dz}{dx} - z &= (1 + z) \ln(1 + z) \\
 \therefore x \frac{dz}{dx} &= (1 + z) \ln(1 + z) \\
 \therefore \frac{dz}{(1 + z) \ln(1 + z)} &= \frac{dx}{x} \\
 \therefore \int \frac{dz}{(1 + z) \ln(1 + z)} &= \int \frac{dx}{x} \\
 \therefore \ln(\ln(1 + z)) &= \ln x + c \\
 \therefore \ln(\ln(1 + z)) &= \ln x + \ln c \\
 \therefore \ln(\ln(1 + z)) &= \ln xc \\
 \therefore \ln(1 + z) &= xc \\
 \therefore 1 + z &= e^{xc} \\
 \therefore 1 + \frac{y}{x} &= e^{xc} \\
 \therefore y &= xe^{xc} - x
 \end{aligned}$$