# ORDINARY DIFFERENTIAL EQUATIONS : ASSIGNMENT 2

AAKASH JOG
ID : 989323563

## Part 1. Homogeneous Equations

### Exercise 1.

Solve

(1) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x+3y}{x-y}$

(2) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2+xy+y^2}{x^2}$

(3) $(x^2+3xy+y^2)\,\mathrm{d}x - x^2\,\mathrm{d}y = 0$

(4) $xy' - y = (x+y)\big(\ln(x+y) - \ln(x)\big)$

### Solution 1.

(1)

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x+3y}{x-y}\\ &= \frac{1+\frac{3y}{x}}{1-\frac{y}{x}}\end{aligned}$$

Let

$$\begin{aligned}\frac{y}{x} &= z\\ \therefore y &= xz\\ \therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= z + x\frac{\mathrm{d}z}{\mathrm{d}x}\end{aligned}$$

*Date*: Thursday 16th April, 2015.

Therefore,

$$\begin{aligned}
z + x\,\frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{1+3z}{1-z}\\
\therefore x\,\frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{1+3z}{1-z} - z\\
\therefore x\,\frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{1+3z-z+z^2}{1-z}\\
\therefore \frac{1-z}{1+2z+z^2} &= \frac{\mathrm{d}x}{x}\\
\therefore \int \frac{1-z}{1+2z+z^2} &= \int \frac{\mathrm{d}x}{x}\\
\therefore -\frac{2}{1+z} - \ln(1+z) &= \ln x + c\\
\therefore -\frac{2}{1+\frac{y}{x}} - \ln\left(1+\frac{y}{x}\right) &= \ln x + c\\
\therefore -\frac{2x}{x+y} - \ln\left(\frac{x+y}{x}\right) &= \ln x + c\\
\therefore -\frac{2x}{x+y} - \ln(x+y) + \ln x &= \ln x + c\\
\therefore -\frac{2x}{x+y} - \ln(x+y) &= c
\end{aligned}$$

(2)

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^2+xy+y^2}{x^2}\\
&= \frac{1+\frac{y}{x}+\frac{y^2}{x^2}}{1}
\end{aligned}$$

Let

$$\begin{aligned}
\frac{y}{x} &= z\\
\therefore y &= xz\\
\therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= z + x\,\frac{\mathrm{d}z}{\mathrm{d}x}
\end{aligned}$$

Therefore,

$$\begin{aligned}
z + x\,\frac{\mathrm{d}z}{\mathrm{d}x} &= 1 + z + z^2 \\
\therefore x\,\frac{\mathrm{d}z}{\mathrm{d}x} &= 1 + z^2 \\
\therefore \frac{\mathrm{d}z}{1+z^2} &= \frac{\mathrm{d}x}{x} \\
\therefore \int \frac{\mathrm{d}z}{1+z^2} &= \int \frac{\mathrm{d}x}{x} \\
\therefore \tan^{-1} z &= \ln x + c \\
\therefore \tan^{-1} \frac{y}{x} &= \ln x + c \\
\therefore y &= x \tan\left(\ln x + c\right)
\end{aligned}$$

(3)

$$\begin{aligned}
\left(x^2 + 3xy + y^2\right)\mathrm{d}x - x^2\,\mathrm{d}y = 0 \\
\therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^2 + 3xy + y^2}{x^2} \\
\therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1 + 3\frac{y}{x} + \frac{y^2}{x^2}}{1}
\end{aligned}$$

Let

$$\begin{aligned}
\frac{y}{x} &= z \\
\therefore y &= xz \\
\therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= z + x\,\frac{\mathrm{d}z}{\mathrm{d}x}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
z + x\frac{\mathrm{d}z}{\mathrm{d}x} &= 1 + 3z + z^2 \\
\therefore x\frac{\mathrm{d}z}{\mathrm{d}x} &= 1 + 2z + z^2 \\
\therefore \frac{\mathrm{d}z}{(1+z)^2} &= \frac{\mathrm{d}x}{x} \\
\therefore \int \frac{\mathrm{d}z}{(1+z)^2} &= \int \frac{\mathrm{d}x}{x} \\
\therefore -\frac{1}{1+z} &= \ln x + c \\
\therefore -\frac{1}{1+\frac{y}{x}} &= \ln x + c \\
\therefore -\frac{x}{x+y} &= \ln x + c \\
\therefore -\frac{x}{\ln x + c} &= x + y \\
\therefore y &= -x - \frac{x}{\ln x + c}
\end{aligned}
$$

(4)

$$
\begin{aligned}
xy' - y &= (x+y)\big(\ln(x+y) - \ln(x)\big) \\
\therefore x\frac{\mathrm{d}y}{\mathrm{d}x} - y &= (x+y)\left(\ln\left(1+\frac{y}{x}\right)\right) \\
\therefore \frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} &= \left(1+\frac{y}{x}\right)\ln\left(1+\frac{y}{x}\right)
\end{aligned}
$$

Let

$$
\begin{aligned}
\frac{y}{x} &= z \\
\therefore y &= xz \\
\therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= z + x\frac{\mathrm{d}z}{\mathrm{d}x}
\end{aligned}
$$

Therefore,

$$\begin{aligned}
z + x\,\frac{\mathrm{d}z}{\mathrm{d}x} - z &= (1+z)\ln(1+z)\\
\therefore x\,\frac{\mathrm{d}z}{\mathrm{d}x} &= (1+z)\ln(1+z)\\
\therefore \frac{\mathrm{d}z}{(1+z)\ln(1+z)} &= \frac{\mathrm{d}x}{x}\\
\therefore \int \frac{\mathrm{d}z}{(1+z)\ln(1+z)} &= \int \frac{\mathrm{d}x}{x}\\
\therefore \ln\big(\ln(1+z)\big) &= \ln x + c\\
\therefore \ln\big(\ln(1+z)\big) &= \ln x + \ln c\\
\therefore \ln\big(\ln(1+z)\big) &= \ln xc\\
\therefore \ln(1+z) &= xc\\
\therefore 1+z &= e^{xc}\\
\therefore 1+\frac{y}{x} &= e^{xc}\\
\therefore y &= xe^{xc} - x
\end{aligned}$$

## Part 2. Transformations Leading to Separable ODEs

**Exercise 1.**
Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6x+y+4}{6x-y+8}$$

**Solution 1.**
Let

$$\begin{aligned}
6x+y+4 &= 6z+w\\
6x-y+8 &= 6z-w
\end{aligned}$$

Therefore

$$\begin{pmatrix}6 & 1\\ 6 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}4\\ 8\end{pmatrix} = \begin{pmatrix}6 & 1\\ 6 & -1\end{pmatrix}\begin{pmatrix}z\\ w\end{pmatrix}$$

Let

$$\begin{aligned}
A &= \begin{pmatrix}6 & 1\\ 6 & -1\end{pmatrix}\\
\therefore A^{-1} &= \frac{-1}{12}\begin{pmatrix}-1 & -1\\ -6 & 6\end{pmatrix}\\
&= \frac{1}{12}\begin{pmatrix}1 & 1\\ 6 & -6\end{pmatrix}
\end{aligned}$$

Therefore,

$$\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{12}\begin{pmatrix} 1 & 1 \\ 6 & -6 \end{pmatrix}\begin{pmatrix} 4 \\ 8 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} z \\ w \end{pmatrix}$$
$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} z \\ w \end{pmatrix}$$

Therefore,

$$z = x + 1$$
$$w = y - 2$$

Therefore,

$$\mathrm{d}z = \mathrm{d}x$$
$$\mathrm{d}w = \mathrm{d}y$$

Therefore,

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{6z + w}{6z - w} = \frac{6 + \frac{w}{z}}{6 - \frac{w}{z}}$$

Let

$$\frac{w}{z} = t$$
$$\therefore \frac{\mathrm{d}w}{\mathrm{d}z} = t + z\frac{\mathrm{d}t}{\mathrm{d}z}$$

Therefore,

$$t + z\frac{\mathrm{d}t}{\mathrm{d}z} = \frac{6 + t}{6 - t}$$
$$\therefore z\frac{\mathrm{d}t}{\mathrm{d}z} = \frac{t^2 - 5t + 6}{6 - t}$$
$$\therefore \frac{\mathrm{d}z}{z} = \frac{6 - t}{t^2 - 5t + 6}\,\mathrm{d}t$$
$$\therefore \int \frac{\mathrm{d}z}{z} = \int \frac{6 - t}{(t-2)(t-3)}\,\mathrm{d}t$$
$$\therefore \ln z = 3\ln(t - 3) - 4\ln(t - 2) + c_1$$
$$\therefore z = c_2\frac{(t-3)^3}{(t-2)^4}$$
$$\therefore x + 1 = c_2\frac{\big((y - 3x) - 5\big)^3}{\big((y - 2x) - 4\big)^4}(x + 1)$$
$$\therefore (y - 2x - 4)^4 = c_2(y - 3x - 5)^3$$

**Exercise 2.**
Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6x+2y+4}{3x+y+5}$$

**Solution 2.**
Let

$$\begin{aligned} 3x+y &= z \\ \therefore 3+\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}z}{\mathrm{d}x} \\ \therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}z}{\mathrm{d}x} - 3 \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}x} - 3 &= \frac{2z+4}{z+5} \\ \therefore \frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{2z+4+3z+15}{z+5} \\ &= \frac{5z+19}{z+5} \\ \therefore \frac{z+5}{5z+19}\,\mathrm{d}z &= \mathrm{d}x \\ \therefore \int \frac{z+5}{5z+19}\,\mathrm{d}z &= \int \mathrm{d}x \\ \therefore \frac{1}{5}\left(t+\frac{6}{5}\ln\left(t+\frac{19}{5}\right)\right) &= x+c_1 \\ \therefore \frac{1}{5}\left((3x+y)+\frac{6}{5}\ln\left(3x+y+\frac{19}{5}\right)\right) &= x+c_1 \\ \therefore 5(2x-y) &= 6\ln\left(3x+y+\frac{19}{5}\right)+c_2 \end{aligned}$$

$$\therefore 10x = c_2 + 6\ln\left(3x+y+\frac{19}{5}\right) + 5y$$

## Part 3. Exact Equations

**Exercise 1.**
Solve the following exact equations

(1) $(3x^2-2xy+2)\,\mathrm{d}x + (6y^2-x^2+3)\,\mathrm{d}y = 0$
(2) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{ax+by}{bx+cy}$
(3) $(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)\,\mathrm{d}x + (xe^{xy}\cos 2x - 3)\,\mathrm{d}y = 0$

**Solution 1.**

(1) Comparing

$$(3x^2 - 2xy + 2)\,\mathrm{d}x + (6y^2 - x^2 + 3)\,\mathrm{d}y = 0$$

and

$$M(x,y) + N(x,y)y' = 0$$

$$M(x,y) = 3x^2 - 2xy + 2$$
$$N(x,y) = 6y^2 - x^2 + 3$$

Therefore,

$$\begin{aligned}\psi &= \int M(x,y)\,\mathrm{d}x \\ &= \int (3x^2 - 2xy + 2)\,\mathrm{d}x \\ &= x^3 - x^2y + 2x + h(y) \\ \therefore \frac{\mathrm{d}\psi}{\mathrm{d}y} &= -x^2 + h'(y)\end{aligned}$$

Comparing with $N(x,y)$,

$$\begin{aligned}h'(y) &= 6y^2 + 3 \\ \therefore h(y) &= \int (6y^2 + 3)\,\mathrm{d}y \\ &= 2y^3 + 3y + c\end{aligned}$$

Therefore, the solution is

$$\therefore x^3 - x^2y + 2x + 2y^3 + 3y + c = 0$$

(2)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{ax + by}{bx + cy}$$

$$\therefore ax + by + (bx + cy)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

Comparing

$$ax + by + (bx + cy)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and

$$M(x,y) + N(x,y)y' = 0$$

$$M(x,y) = ax + by$$
$$N(x,y) = bx + cy$$

Therefore,

$$\begin{aligned}\psi &= \int M(x,y)\,\mathrm{d}x \\ &= \int (ax+by)\,\mathrm{d}x \\ &= \frac{ax^2}{2} + bxy + h(y) \\ \therefore \frac{\mathrm{d}\psi}{\mathrm{d}y} &= bx + h'(y)\end{aligned}$$

Comparing with $N(x,y)$,

$$\begin{aligned}h'(y) &= cy \\ \therefore h(y) &= \int cy\,\mathrm{d}y \\ &= \frac{cy^2}{2} + c\end{aligned}$$

Therefore, the solution is

$$\frac{ax^2}{2} + bxy + \frac{cy^2}{2} + c = 0$$

(3) Comparing

$$\left(ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x\right)\mathrm{d}x + \left(xe^{xy}\cos(2x) - 3\right)\mathrm{d}y = 0$$

and

$$M(x,y) + N(x,y)y' = 0$$

$$\begin{aligned}M(x,y) &= ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x \\ N(x,y) &= xe^{xy}\cos(2x) - 3\end{aligned}$$

Therefore,

$$\begin{aligned}\psi &= \int \left(ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x\right)\mathrm{d}x \\ &= x^2 + e^{xy}\cos(2x) + h(y) \\ \therefore \frac{\mathrm{d}\psi}{\mathrm{d}y} &= xe^{xy}\cos(2x) + h'(y)\end{aligned}$$

Comparing with $N(x,y)$,

$$\begin{aligned}h'(y) &= -3 \\ \therefore h(y) &= -3y + c\end{aligned}$$

Therefore, the solution is

$$e^{xy}\cos(2x) + x^2 - 3y + c = 0$$

**Exercise 2.**
Solve the exact value problem.

$$\begin{aligned}(9x^2 + y - 1)\,\mathrm{d}x - (4y - x)\,\mathrm{d}y &= 0\\ y(1) &= 3\end{aligned}$$

**Solution 2.**
Comparing

$$(9x^2 + y - 1)\,\mathrm{d}x - (4y - x)\,\mathrm{d}y = 0$$

and

$$M(x, y) + N(x, y)y' = 0$$

$$\begin{aligned}M(x, y) &= 9x^2 + y - 1\\ N(x, y) &= x - 4y\end{aligned}$$

Therefore,

$$\begin{aligned}\psi &= \int (9x^2 + y - 1)\,\mathrm{d}x\\ &= 3x^3 + xy - x + h(y)\\ \therefore \frac{\mathrm{d}\psi}{\mathrm{d}y} &= x + h'(y)\end{aligned}$$

Comparing with $N(x, y)$,

$$\begin{aligned}h'(y) &= -4y\\ \therefore h(y) &= -2y^2 + c\end{aligned}$$

Therefore, the solution is

$$3x^3 + xy - x - 2y^2 + c = 0$$

Substituting the initial condition $y(1) = 3$,

$$\begin{aligned}3(1)^3 + (1)(3) - (1) - 2(3)^2 + c &= 0\\ \therefore 3 + 3 - 1 - 18 + c &= 0\\ \therefore c &= 13\end{aligned}$$

Therefore, the solution is

$$3x^3 + xy - x - 2y^2 + 13 = 0$$

**Exercise 3.**
Find the value of $b$ for which the ODE

$$(xy^2 + bx^2y)\,\mathrm{d}x + (x+y)x^2\,\mathrm{d}y = 0$$

is exact and solve the equation using that value of $b$.

**Solution 3.**
Comparing

$$(xy^2 + bx^2y)\,\mathrm{d}x + (x+y)x^2\,\mathrm{d}y = 0$$

and

$$M(x,y) + N(x,y)y' = 0$$

$$\begin{aligned} M(x,y) &= xy^2 + bx^2y \\ N(x,y) &= (x+y)x^2 \\ &= x^3 + x^2y \end{aligned}$$

Therefore,

$$\begin{aligned} M_y &= 2xy + bx^2 \\ N_x &= 3x^2 + 2xy \end{aligned}$$

For the equation to be exact,

$$\begin{aligned} &M_y = N_x \\ &\therefore b = 3 \end{aligned}$$

Therefore,

$$\begin{aligned} \psi &= \int (xy^2 + 3x^2y)\,\mathrm{d}x \\ &= \frac{x^2y^2}{2} + x^3y + h(y) \\ \therefore \frac{\mathrm{d}\psi}{\mathrm{d}y} &= x^2y + x^3 + h'(y) \end{aligned}$$

Comparing with $N(x,y)$,

$$\begin{aligned} h'(y) &= 0 \\ \therefore h(y) &= c \end{aligned}$$

Therefore, the solution is

$$\frac{x^2y^2}{2} + x^3y + c = 0$$

**Exercise 4.**
Show that any separable ODE $M(x) + N(y)y = 0$ is exact.

**Solution 4.**
As $M$ is a function of $x$ only, and $N$ of $y$ only,

$$\frac{\mathrm{d}M(x)}{\mathrm{d}y} = 0$$
$$\frac{\mathrm{d}N(y)}{\mathrm{d}x} = 0$$

Therefore, the equation is exact.

## Part 4. Integrating Factors

**Exercise 1.**
Show that an ODE $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ has an integrating factor $\mu(y)$ if $\dfrac{M_y - N_x}{M} = -g(y)$ and that $\mu(y) = e^{\int g(y)\,\mathrm{d}y}$.

**Solution 1.**

$$\begin{aligned}
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y &= 0 \\
\therefore \mu(y)M(x,y)\,\mathrm{d}x + \mu(y)N(x,y)\,\mathrm{d}y &= 0
\end{aligned}$$

The equation is exact if and only if

$$\begin{aligned}
\frac{\partial\left(\mu(y)M(x,y)\right)}{\partial y} &= \frac{\partial\left(\mu(y)N(x,y)\right)}{\partial x} \\
\therefore M\,\frac{\partial\mu(y)}{\partial y} + \mu(y)\,\frac{\partial M(x,y)}{\partial y} &= N\,\frac{\partial\mu(y)}{\partial x} + \mu(y)\,\frac{\partial N(x,y)}{\partial x} \\
\therefore M\mu' + \mu M_y &= \mu N_x \\
\therefore \frac{\mathrm{d}\mu}{\mu} &= -\frac{M_y - N_x}{M}\,\mathrm{d}y \\
\therefore \ln\mu &= \int -\frac{M_y - N_x}{M}\,\mathrm{d}y \\
&= \int g(y)\,\mathrm{d}y \\
\therefore \mu &= e^{\int g(y)\,\mathrm{d}y}
\end{aligned}$$

**Exercise 2.**
Show that an ODE $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ has an integrating factor $\mu\left(\dfrac{x}{y}\right)$ if $\dfrac{y^2\left(M_y - N_x\right)}{xM + yN} = h\left(\dfrac{x}{y}\right)$ and the integrating factor.

**Solution 2.**

$$
\begin{aligned}
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y &= 0\\
\therefore \mu\left(\frac{y}{x}\right) M(x,y)\,\mathrm{d}x + \mu\left(\frac{y}{x}\right) N(x,y)\,\mathrm{d}y &= 0
\end{aligned}
$$

The equation is exact if and only if

$$
\begin{aligned}
\frac{\partial}{\partial y}\left(\mu\left(\frac{x}{y}\right) M(x,y)\right) &= \frac{\partial}{\partial x}\left(\mu\left(\frac{x}{y}\right) N(x,y)\right)\\
\therefore \mu \frac{\partial M}{\partial y} + \frac{\partial \mu}{\partial y} M &= \mu \frac{\partial N}{\partial x} + \frac{\partial \mu}{\partial x} N\\
\therefore \mu M_y + \frac{\partial \mu}{\partial y} M &= \mu N_x + \frac{\partial \mu}{\partial x} N\\
\therefore \mu\left(M_y - N_x\right) &= \frac{\partial \mu}{\partial x} N - \frac{\partial \mu}{\partial y} M\\
\therefore \frac{\mathrm{d}\mu}{\mu} &= \frac{y^2\left(M_y - N_x\right)}{xM + yN}\,\mathrm{d}\left(\frac{x}{y}\right)\\
\therefore \ln \mu &= \int \frac{y^2\left(M_y - N_x\right)}{xM + yN}\,\mathrm{d}\left(\frac{y}{x}\right)\\
&= \int h\left(\frac{x}{y}\right)\mathrm{d}\left(\frac{x}{y}\right)\\
\therefore \mu &= e^{\int h\left(\frac{x}{y}\right)\mathrm{d}\left(\frac{x}{y}\right)}
\end{aligned}
$$

**Exercise 3.**
Show that an ODE $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ has an integrating factor $\mu\left(\frac{y}{x}\right)$ if $\frac{x^2\left(N_x - M_y\right)}{xM + yN} = k\left(\frac{y}{x}\right)$ and the integrating factor.

**Solution 3.**

$$
\begin{aligned}
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y &= 0\\
\therefore \mu\left(\frac{y}{x}\right) M(x,y)\,\mathrm{d}x + \mu\left(\frac{y}{x}\right) N(x,y)\,\mathrm{d}y &= 0
\end{aligned}
$$

The equation is exact if and only if

$$
\begin{aligned}
\frac{\partial}{\partial y}\left(\mu\left(\frac{y}{x}\right)M(x,y)\right) &= \frac{\partial}{\partial x}\left(\mu\left(\frac{y}{x}\right)N(x,y)\right)\\
\therefore \mu\,\frac{\partial M}{\partial y} + \frac{\partial \mu}{\partial y}\,M &= \mu\,\frac{\partial N}{\partial x} + \frac{\partial \mu}{\partial x}\,N\\
\therefore \mu M_y + \frac{\partial \mu}{\partial y}\,M &= \mu N_x + \frac{\partial \mu}{\partial x}\,N\\
\therefore \mu\left(M_y - N_x\right) &= \frac{\partial \mu}{\partial x}\,N - \frac{\partial \mu}{\partial y}\,M\\
\therefore \frac{\mathrm{d}\mu}{\mu} &= \frac{x^2\left(N_x - M_y\right)}{xM + yN}\,\mathrm{d}\!\left(\frac{y}{x}\right)\\
\therefore \ln \mu &= \int \frac{y^2\left(M_y - N_x\right)}{xM + yN}\,\mathrm{d}\!\left(\frac{y}{x}\right)\\
&= \int k\left(\frac{y}{x}\right)\mathrm{d}\!\left(\frac{y}{x}\right)\\
\therefore \mu &= e^{\int k\left(\frac{x}{y}\right)\mathrm{d}\left(\frac{y}{x}\right)}
\end{aligned}
$$

**Exercise 4.**

Show that the following ODEs are not exact. Find integrating factors for them and use them to solve the equations.

(1) $(3x^2y + 2xy + y^3)\,\mathrm{d}x + (x^2 + y^2)\,\mathrm{d}y = 0$

(2) $\mathrm{d}x + \left(\frac{x}{y} - \sin y\right)\mathrm{d}y = 0$

(3) $\left(3x + \frac{6}{y}\right) + \left(\frac{x^2}{y} + \frac{3y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

**Solution 4.**

(1) Comparing

$$(3x^2y + 2xy + y^3)\,\mathrm{d}x + (x^2 + y^2)\,\mathrm{d}y = 0$$

and

$$M(x,y) + N(x,y)y' = 0$$

$$
\begin{aligned}
M(x,y) &= 3x^2y + 2xy + y^3\\
N(x,y) &= x^2 + y^2
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
M_y &= 3x^2 + 2x + 3y^2\\
N_x &= 2x
\end{aligned}
$$

Therefore, as $M_y \neq N_x$, the equation is not exact.
Therefore,

$$\begin{aligned} g(x) &= \frac{M_y - N_x}{N} \\ &= \frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2} \\ &= \frac{3x^2 + 3y^2}{x^2 + y^2} \\ &= 3 \end{aligned}$$

$$\therefore \int g(x)\,\mathrm{d}x = 3x$$

Therefore,

$$\begin{aligned} \mu(x) &= e^{\int g(x)\,\mathrm{d}x} \\ &= e^{3x} \end{aligned}$$

Therefore, multiplying the equation by $\mu(x)$,

$$e^{3x}(3x^2y + 2xy + y^3)\,\mathrm{d}x + e^{3x}(x^2 + y^2)\,\mathrm{d}y = 0$$

Comparing

$$e^{3x}(3x^2y + 2xy + y^3)\,\mathrm{d}x + e^{3x}(x^2 + y^2)\,\mathrm{d}y = 0$$

and

$$M'(x, y) + N'(x, y)y' = 0$$

$$\begin{aligned} M'(x, y) &= e^{3x}(3x^2y + 2xy + y^3) \\ N'(x, y) &= e^{3x}(x^2 + y^2) \end{aligned}$$

Therefore,

$$\begin{aligned} \psi &= \int e^{3x}(3x^2y + 2xy + y^3)\,\mathrm{d}x \\ &= e^{3x}\left(x^2y + \frac{y^3}{3}\right) + h(y) \end{aligned}$$

$$\therefore \frac{\mathrm{d}\psi}{\mathrm{d}y} = e^{3x}\left(x^2 + y^2\right)$$

Comparing with $N'(x, y)$,

$$h'(y) = 0$$

$$\therefore h(y) = c$$

Therefore, the solution is

$$e^{3x}\left(x^2y + \frac{y^3}{3}\right) + c = 0$$

(2) Comparing

$$\mathrm{d}x + \left(\frac{x}{y} - \sin y\right) \mathrm{d}y = 0$$

and

$$M(x,y) + N(x,y)y' = 0$$

$$M(x,y) = 1$$
$$N(x,y) = \frac{x}{y} - \sin y$$

Therefore,

$$M_y = 0$$
$$N_x = \frac{1}{y}$$

Therefore, as $M_y \neq N_x$, the equation is not exact.
Therefore,

$$\begin{aligned}\mu(y) &= e^{\int \frac{N_x - M_y}{M}} \\ &= e^{\int \frac{\frac{1}{y} - 0}{1} \mathrm{d}y} \\ &= e^{\ln y} \\ &= y\end{aligned}$$

Therefore, multiplying the equation by $\mu(y)$,

$$y\,\mathrm{d}x + (x - y\sin y)\,\mathrm{d}y = 0$$

Comparing

$$y\,\mathrm{d}x + (x - y\sin y)\,\mathrm{d}y = 0$$

and

$$M'(x,y) + N'(x,y)y' = 0$$

$$M'(x,y) = y$$
$$N'(x,y) = x - y\sin y$$

Therefore,

$$\begin{aligned}\psi &= \int y\,\mathrm{d}x \\ &= xy + h(y)\end{aligned}$$

$$\therefore \frac{\mathrm{d}\psi}{\mathrm{d}y} = x + h'(y)$$

Comapring with $N(x, y)$,

$$h'(y) = -y \sin y$$
$$\therefore h(y) = -\int y \sin y \, \mathrm{d}y$$
$$= y \cos y - \sin y + c$$

Therefore, the solution is

$$x + y \cos y - \sin y + c = 0$$

(3) Comparing

$$\left(3x + \frac{6}{y}\right) + \left(\frac{x^2}{y} + \frac{3y}{x}\right) \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and

$$M(x, y) + N(x, y)y' = 0$$

$$M(x, y) = 3x + \frac{6}{y}$$
$$N(x, y) = \frac{x^2}{y} + \frac{3y}{x}$$

Therefore,

$$M_y = -\frac{6}{y^2}$$
$$N_x = \frac{2x}{y} - \frac{3y}{x^2}$$

Therefore,

$$g(x) = \frac{N_x - M_y}{xM - yN}$$
$$= \frac{\frac{2x}{y} - \frac{3y}{x^2} + \frac{6}{y^2}}{3x^2 + \frac{6x}{y} - x^2 - \frac{3y^2}{x}}$$
$$= \frac{1}{xy}$$

$$\mu(xy) = e^{\int g(xy) \, \mathrm{d}(xy)}$$
$$= e^{\int \frac{1}{xy} \, \mathrm{d}(xy)}$$
$$= e^{\ln(xy)}$$
$$= xy$$

Therefore, multiplying the equation by $\mu(xy)$,

$$\left(3x^2y+6x\right)+\left(x^3+3y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

Comparing

$$\left(3x^2y+6x\right)+\left(x^3+3y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

and

$$M'(x,y)+N'(x,y)y'=0$$

$$M'(x,y)=3x^2y+6x$$
$$N'(x,y)=x^3+3y^2$$

Therefore,

$$\psi=\int(3x^2y+6x)\,\mathrm{d}x$$
$$=x^3y+3x^2+h(y)$$
$$\therefore\frac{\mathrm{d}\psi}{\mathrm{d}y}=x^3+h'(y)$$

Comparing with $N(x,y)$,

$$h'(y)=3y^2$$
$$\therefore h(y)=\int 3y^2\,\mathrm{d}y$$
$$=y^3+c$$

Therefore, the solution is

$$x^3y+3x^2+y^3+c=0$$

## Part 5. Riccati Equations

**Exercise 1.**

(1) Show that $y_1=-x^2$ is a particular solution of $y'=x^3+\frac{2}{x}y-\frac{1}{x}y^2$.
(2) Use $y_1$ to find the general solution for the equation.

**Solution 1.**
(1)

$$y'=x^3+\frac{2}{x}y-\frac{1}{x}y^2$$

Therefore, substituting $y_1=-x^2$,

$$\text{L.H.S.}=(-x^2)'$$
$$=-2x$$

$$\begin{aligned}\text{R.H.S.} &= x^3 + \frac{2y}{x} - \frac{y^2}{x} \\ &= x^3 + \frac{2(-x^2)}{x} - \frac{(-x^2)^2}{x} \\ &= x^3 - 2x + x^3 \\ &= -2x\end{aligned}$$

Therefore, $y_1 = -x^2$ is a solution of the equation.

(2) Comparing

$$y' = x^3 + \frac{2}{x}y - \frac{1}{x}y^2$$

and

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

$$\begin{aligned}f_0(x) &= x^3 \\ f_1(x) &= \frac{2}{x} \\ f_2(x) &= -\frac{1}{x}\end{aligned}$$

Let

$$\begin{aligned}y &= y_1 + \frac{1}{u(x)} \\ &= -x^2 + \frac{1}{u(x)}\end{aligned}$$

$$\therefore y' = -2x + \frac{u'}{u^2}$$

Therefore, substituting $y$ and $y'$ in the original equation,

$$\begin{aligned}-2x + \frac{u'}{u^2} &= x^3 + \frac{2}{x}\left(-x^2 + \frac{1}{u}\right) - \frac{1}{x}\left(-x^2 + \frac{1}{u}\right)^2 \\ \therefore u' &= \big(-f_1(x) - 2f_2(x)y_1\big)\, u - f_2(x) \\ &= \left(-\frac{2}{x} + \frac{2}{x}(-x^2)\right) u + \frac{1}{x} \\ &= \left(-\frac{2}{x} - 2x\right) u + \frac{1}{x}\end{aligned}$$

Therefore,

$$u' = \left(\frac{-2 - 2x^2}{x}\right) u + \frac{1}{x}$$

$$\therefore u' + \left(\frac{2 + 2x^2}{x}\right) u = \frac{1}{x}$$

Comparing with $y' + p(x)y = q(x)$,

$$p(x) = \frac{2+2x^2}{x}$$
$$q(x) = \frac{1}{x}$$

Therefore,

$$\begin{aligned}\mu(x) &= e^{\int \frac{2+2x^2}{x}} \\ &= e^{x^2+2\ln x} \\ &= e^{x^2}x^2\end{aligned}$$

Therefore,

$$\begin{aligned}u &= \frac{1}{e^{x^2}x^2}\int \frac{e^{x^2}x^2}{x}\,\mathrm{d}x \\ &= \frac{1}{x^2e^{x^2}}\int xe^{x^2}\,\mathrm{d}x \\ &= \frac{1}{x^2e^{x^2}}\left(\frac{e^{x^2}}{2}+c\right) \\ &= \frac{1+ce^{-x^2}}{2x^2}\end{aligned}$$

Therefore,

$$y = -x^2 + \frac{2x^2}{1+ce^{-x^2}}$$