ORDINARY DIFFERENTIAL EQUATIONS
ASSIGNMENT 6

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Part 1. Euler's Equations

Exercise 1.
In each of the following sections find the solution for the initial value problem.

(1)
$$
\begin{aligned}
2x^2y'' + xy' - 3y &= 0 \\
y(1) &= 1 \\
y'(1) &= 4
\end{aligned}
$$

(2)
$$
\begin{aligned}
4x^2y'' + 8xy' + 17y &= 0 \\
y(1) &= 2 \\
y'(2) &= -3
\end{aligned}
$$

(3)
$$
\begin{aligned}
x^2y'' - 3xy' + 4y &= 0 \\
y(1) &= 2 \\
y'(1) &= 3
\end{aligned}
$$

Solution 1.
(1)
$$2x^2y'' + xy' - 3y = 0$$

Let
$$y = x^r$$

Therefore,
$$
\begin{aligned}
y' &= rx^{r-1} \\
y'' &= r(r-1)x^{r-2}
\end{aligned}
$$

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Therefore, substituting,

$$\begin{aligned} 2x^2 r(r-1)x^{r-2} + xrx^{r-1} - 3x^r &= 0 \\ \therefore x^r \big(2r(r-1) + r - 3\big) &= 0 \\ \therefore 2r^2 - 2r + r - 3 &= 0 \\ \therefore 2r^2 - r - 3 &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} r &= \frac{1 \pm \sqrt{1+24}}{4} \\ &= \frac{1 \pm 5}{4} \end{aligned}$$

Therefore,

$$\begin{aligned} r_1 &= \frac{3}{2} \\ r_2 &= -1 \end{aligned}$$

Therefore,

$$\begin{aligned} y_1 &= x^{\frac{3}{2}} \\ y_2 &= x^{-1} \end{aligned}$$

Therefore,

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 x^{\frac{3}{2}} + c_2 x^{-1} \end{aligned}$$

Therefore,

$$y' = \frac{3}{2} c_1 x^{\frac{1}{2}} - c_2 x^{-2}$$

Therefore, substituting $y(1) = 1$ and $y'(1) = 4$,

$$\begin{aligned} 1 &= c_1 + c_2 \\ 4 &= \frac{3}{2} c_1 - c_2 \end{aligned}$$

Therefore,

$$\begin{aligned} c_1 &= 2 \\ c_2 &= -1 \end{aligned}$$

Therefore,

$$y = 2x^{\frac{3}{2}} - x^{-1}$$

(2)

$$4x^2 y'' + 8xy' + 17y = 0$$

Let

$$y = x^r$$

Therefore,

$$\begin{aligned} y' &= rx^{r-1} \\ y'' &= r(r-1)x^{r-2} \end{aligned}$$

Therefore, substituting,

$$\begin{aligned} 4x^2r(r-1)x^{r-2} + 8xrx^{r-1} + 17x^r &= 0 \\ \therefore x^r\left(4r(r-1) + 8r + 17\right) &= 0 \\ \therefore 4r^2 - 4r + 8r + 17 &= 0 \\ \therefore 4r^2 + 4r + 17 &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} r &= \frac{-4 \pm \sqrt{16 - 272}}{8} \\ &= \frac{-4 \pm 16i}{8} \end{aligned}$$

Therefore,

$$\begin{aligned} r_1 &= -\frac{1}{2} + 2i \\ r_2 &= -\frac{1}{2} - 2i \end{aligned}$$

Therefore,

$$\begin{aligned} y_1 &= x^{r_1} \\ &= x^{-\frac{1}{2}+2i} \\ &= e^{\ln x^{-\frac{1}{2}+2i}} \\ &= e^{\left(-\frac{1}{2}+2i\right)\ln x} \\ &= e^{-\frac{1}{2}\ln x}e^{2i\ln x} \\ &= e^{\ln x^{-\frac{1}{2}}}e^{2i\ln x} \\ &= x^{-\frac{1}{2}}\left(\cos(2\ln x) + i\sin(2\ln x)\right) \\ &= x^{-\frac{1}{2}}\cos(2\ln x) + ix^{-\frac{1}{2}}\sin(2\ln x) \end{aligned}$$

Therefore,

$$\begin{aligned} y &= c_1\Re(y_1) + c_2\Im(y_1) \\ &= c_1x^{-\frac{1}{2}}\cos(2\ln x) + c_2x^{-\frac{1}{2}}\sin(2\ln x) \end{aligned}$$

Therefore,

$$\begin{aligned} y' = \quad & -\frac{1}{2}c_1x^{-\frac{3}{2}}\cos(2\ln x) - \frac{2}{x}c_1x^{-\frac{1}{2}}\sin(2\ln x) \\ & - \frac{1}{2}c_2x^{-\frac{3}{2}}\sin(2\ln x) + \frac{2}{x}c_2x^{-\frac{1}{2}}\cos(2\ln x) \end{aligned}$$

Therefore, substituting $y(1) = 2$ and $y'(2) = -3$,

$$\begin{aligned}
2 &= c_1 \cdot 1 \cdot \cos(0) + c_2 \cdot 1 \cdot \sin(0) \\
&= c_1 \\
-3 &= -\frac{1}{2}c_1 2^{-\frac{3}{2}} \cos(2\ln 2) - \frac{2}{2}c_2 2^{-\frac{1}{2}} \sin(2\ln 2) \\
&= -\frac{1}{2} \cdot 2 \cdot \frac{1}{2\sqrt{2}} \cos(2\ln 2) - c_2 \frac{1}{\sqrt{2}} \sin(2\ln 2) \\
&= -\frac{\cos(2\ln 2)}{2\sqrt{2}} - c_2 \frac{\sin(2\ln 2)}{\sqrt{2}}
\end{aligned}$$

$$\therefore 3 - \frac{\cos(2\ln 2)}{2\sqrt{2}} = c_2 \frac{\sin(2\ln 2)}{\sqrt{2}}$$

$$\therefore \frac{3\sqrt{2}}{\sin(2\ln 2)} - \frac{1}{2\tan(2\ln 2)} = c_2$$

Therefore,

$$y = 2x^{-\frac{1}{2}} \cos(2\ln x) + \frac{3\sqrt{2}}{\sin(2\ln 2)} - \frac{1}{2\tan(2\ln 2)} x^{-\frac{1}{2}} \sin(2\ln x)$$

(3)

$$x^2 y'' - 3xy' + 4y = 0$$

Let

$$y = x^r$$

Therefore,

$$\begin{aligned}
y' &= r x^{r-1} \\
y'' &= r(r-1)x^{r-2}
\end{aligned}$$

Therefore, substituting,

$$\begin{aligned}
x^2 r(r-1)x^{r-2} - 3xrx^{r-1} + 4x^r &= 0 \\
\therefore x^r \big(r(r-1) - 3r + 4\big) &= 0 \\
\therefore r(r-1) - 3r + 4 &= 0 \\
\therefore r^2 - r - 3r + 4 &= 0 \\
\therefore r^2 - 4r + 4 &= 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
r &= \frac{4 \pm \sqrt{16-16}}{2} \\
&= 2
\end{aligned}$$

Therefore,

$$\begin{aligned}
y_1 &= x^r \\
&= x^2 \\
y_2 &= x^r \ln x \\
&= x^2 \ln x
\end{aligned}$$

Therefore,

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 x^2 + c_2 x^2 \ln x
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
y' &= 2c_1 x + 2c_2 x \ln x + c_2 x^2 \frac{1}{x} \\
&= 2c_1 x + 2c_2 x \ln x + c_2 x
\end{aligned}
$$

Therefore, substituting $y(1) = 2$ and $y'(1) = 3$,

$$
\begin{aligned}
2 &= c_1 + c_2 \ln 1 \\
&= c_1 \\
3 &= 2c_1 + 2c_2 \ln 1 + c_2 \\
&= 2c_1 + c_2
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
c_1 &= 2 \\
c_2 &= -1
\end{aligned}
$$

Therefore,

$$
y = 2x^2 - x^2 \ln x
$$

Exercise 2.
Transformation into a constant coefficients equation:

(1) Substituting $x = e^t$, show that

$$
\begin{aligned}
x \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t} \\
x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}
\end{aligned}
$$

(2) Conclude that

$$
\begin{aligned}
& a x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b x \frac{\mathrm{d}y}{\mathrm{d}x} + c y = h(x) \\
\implies & a \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + (b - a) \frac{\mathrm{d}y}{\mathrm{d}t} + c y = h\left(e^t\right)
\end{aligned}
$$

Solution 2.

(1)

$$
\begin{aligned}
x \frac{\mathrm{d}y}{\mathrm{d}x} &= e^t \frac{\mathrm{d}y}{\mathrm{d}e^t} \\
&= e^t \frac{\mathrm{d}y}{\mathrm{d}t} \frac{\mathrm{d}t}{\mathrm{d}e^t} \\
&= \frac{\mathrm{d}e^t}{\mathrm{d}t} \frac{\mathrm{d}y}{\mathrm{d}t} \frac{\mathrm{d}t}{\mathrm{d}e^t} \\
&= \frac{\mathrm{d}y}{\mathrm{d}t}
\end{aligned}
$$

$$
\begin{aligned}
x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \left(e^t\right)^2 \frac{\mathrm{d}^2 y}{\mathrm{d}(e^t)^2} \\
&= \left(\frac{\mathrm{d}\left(e^t\right)}{\mathrm{d}t}\right)^2 \frac{\mathrm{d}^2 y}{\mathrm{d}(e^t)^2} \\
&= \left(\frac{\mathrm{d}\left(e^t\right)}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}\left(e^t\right)}{\mathrm{d}t}\right) \frac{\mathrm{d}}{\mathrm{d}(e^t)}\left(\frac{\mathrm{d}y}{\mathrm{d}(e^t)}\right) \\
&= \left(\frac{\mathrm{d}\left(e^t\right)}{\mathrm{d}t}\right) \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}(e^t)}\right) \\
&= \quad \left(\frac{\mathrm{d}\left(e^t\right)}{\mathrm{d}t}\right) \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}(e^t)}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\left(e^t\right)}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}(e^t)}\right) \\
&\quad - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\left(e^t\right)}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}(e^t)}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\left(e^t\right)}{\mathrm{d}t} \frac{\mathrm{d}y}{\mathrm{d}(e^t)}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\left(e^t\right)}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}(e^t)}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right) - \left(\frac{\mathrm{d}\left(e^t\right)}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}(e^t)}\right) \\
&= \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}
\end{aligned}
$$

(2)

$$
\begin{aligned}
ax^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + bx \frac{\mathrm{d}y}{\mathrm{d}x} + cy &= h(x) \\
\therefore a \left(e^t\right)^2 \frac{\mathrm{d}^2 y}{\mathrm{d}(e^t)^2} + b \frac{\mathrm{d}y}{\mathrm{d}(e^t)} + cy &= h\left(e^t\right) \\
\therefore a \left(\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}\right) + b \frac{\mathrm{d}y}{\mathrm{d}t} + cy &= h\left(e^t\right) \\
\therefore a \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - a \frac{\mathrm{d}y}{\mathrm{d}t} + b \frac{\mathrm{d}y}{\mathrm{d}t} + cy &= h\left(e^t\right) \\
\therefore a \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + (b-a) \frac{\mathrm{d}y}{\mathrm{d}t} + cy &= h\left(e^t\right)
\end{aligned}
$$

Exercise 3.
Use reduction of order to show that the second solution for a second-order Euler's equation with a double root r is $x^r lnx$.

Solution 3.

$$y_1(x) = x^r$$

is a solution to the differential equation

$$ax^2 y'' + bxy' + cy = 0$$

$$
\begin{aligned}
{y_1}'(x) &= rx^{r-1} \\
{y_1}''(x) &= r(r-1)x^{r-2}
\end{aligned}
$$

Substituting,

$$
\begin{aligned}
ax^2r(r-1)x^{r-2} + bxrx^{r-1} + cx^r &= 0 \\
\therefore x^r\left(ar(r-1) + br + c\right) &= 0 \\
\therefore ar(r-1) + br + c &= 0 \\
\therefore r &= \frac{(a-b) \pm \sqrt{(b-a)^2 - 4ac}}{2a}
\end{aligned}
$$

As the differential equation has a double root,

$$
r = \frac{a-b}{2a}
$$

Let

$$
\begin{aligned}
y_2(x) &= \nu(x)y_1(x) \\
&= \nu(x)x^r \\
&= \nu(x)x^{\frac{a-b}{2a}}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
{y_2}'(x) &= \nu'(x)x^{\frac{a-b}{2a}} + \nu(x)\frac{a-b}{2a}x^{\frac{a-b-2a}{2a}} \\
&= \nu'(x)x^{\frac{a-b}{2a}} + \nu(x)\frac{a-b}{2a}x^{\frac{-a-b}{2a}} \\
{y_2}''(x) &= \quad \nu''(x)x^{\frac{a-b}{2a}} + \nu'(x)\frac{a-b}{2a}x^{\frac{-a-b}{2a}} \\
&\quad + \nu''(x)\frac{a-b}{2a}x^{\frac{-a-b}{2a}} + \nu'(x)\frac{a-b}{2a}\frac{-a-b}{2a}x^{\frac{-3a-b}{2a}}
\end{aligned}
$$

Therefore, solving,

$$
\nu(x) = \ln x
$$

Therefore,

$$
\begin{aligned}
y_2(x) &= \nu(x)x^r \\
x^2 \ln x &
\end{aligned}
$$

$\square$

Part 2. **Existence and Uniqueness for High Order Equations**

Exercise 1.

In each of the following sections determine the largest interval in which the given initial value problem is certain to have a unique solution. Do not attempt to find the solution.

(1)

$$\begin{aligned} ty'' + 3y &= t \\ y(1) &= 1 \\ y'(2) &= 2 \end{aligned}$$

(2)

$$\begin{aligned} t(t-4)y'' - 3ty' + 4y &= \sin t \\ y(-2) &= 2 \\ y'(-2) &= 1 \end{aligned}$$

(3)

$$\begin{aligned} (x-2)y'' + y' + (x-2)\tan x &= 0 \\ y(3) &= 1 \\ y'(3) &= 2 \end{aligned}$$

Solution 1.

(1)

$$\begin{aligned} ty'' + 3y &= 0 \\ \therefore y'' + \frac{3}{t}y &= 0 \end{aligned}$$

$\frac{3}{t}$ is continuous on $\mathbb{R} \setminus \{0\}$.
Therefore, the largest interval in which the above function is continuous, which contains the given points $t = 1$ and $t = 2$, is $(0, \infty)$.
Therefore, by the existence and uniqueness theorem, the initial value problem has a unique solution in $(0, \infty)$.

(2)

$$\begin{aligned} t(t-4)y'' - 3ty' + 4y &= \sin t \\ \therefore y'' - \frac{3t}{t(t-4)} + \frac{4}{t(t-4)} &= \frac{\sin t}{t(t-4)} \end{aligned}$$

$-\frac{3t}{t(t-4)}$ is continuous on $\mathbb{R} \setminus \{4\}$.
$\frac{4}{t(t-4)}$ is continuous on $\mathbb{R} \setminus \{0, 4\}$.
$\sin t$ is continuous on $\mathbb{R}$.
Therefore, the largest interval in which the above functions are continuous, which contains the given point $t = -2$, is $(-\infty, 0)$.
Therefore, by the existence and uniqueness theorem, the initial value problem has a unique solution in $(-\infty, 0)$.

(3)

$$\begin{aligned} (x-2)y'' + y' + (x-2)\tan x &= 0 \\ \therefore y'' + \frac{1}{x-2}y' + \tan x &= 0 \end{aligned}$$

$\frac{1}{x-2}$ is continuous on $\mathbb{R} \setminus \{2\}$.
$\tan x$ is continuous on $\mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi | k \in \mathbb{Z} \right\}$. Therefore, the largest interval in which the above functions are continuous, which contains the given

point $t = 3$, is $\left(2, \frac{3\pi}{2}\right)$.
Therefore, by the existence and uniqueness theorem, the initial value problem has a unique solution in $\left(2, \frac{3\pi}{2}\right)$.

Part 3. The Wronskian

Exercise 1.
In each of the following sections find the Wronskian of the given pair of functions.
(1) e^{2t}, $e^{-\frac{3t}{2}}$
(2) e^{-2t}, te^{-2t}
(3) $e^t \sin t$, $e^t \cos t$
(4) $\cos^2 \theta$, $1 + \cos 2\theta$

Solution 1.
(1)

$$y_1(t) = e^{2t}$$
$$y_2(t) = e^{-\frac{3t}{2}}$$

$$W = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1{}'(t) & y_2{}'(t) \end{vmatrix}$$

Therefore,

$$\begin{aligned} W &= \begin{vmatrix} e^{2t} & e^{-\frac{3t}{2}} \\ 2e^{2t} & \frac{3}{2}e^{-\frac{3t}{2}} \end{vmatrix} \\ &= \frac{3}{2}e^{2t}e^{-\frac{3t}{2}} - 2e^{2t}e^{-\frac{3t}{2}} \\ &= -\frac{e^{\frac{t}{2}}}{2} \end{aligned}$$

(2)

$$y_1(t) = e^{-2t}$$
$$y_2(t) = te^{-2t}$$

$$W = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1{}'(t) & y_2{}'(t) \end{vmatrix}$$

Therefore,

$$\begin{aligned} W &= \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & -2te^{-2t} \end{vmatrix} \\ &= -2te^{-2t}e^{-2t} + 2e^{-2t}e^{-2t} \\ &= 2e^{-4t} - 2te^{-4t} \end{aligned}$$

(3)

$$y_1(t) = e^t \sin t$$
$$y_2(t) = e^t \cos t$$

$$W = \begin{vmatrix} y_1(t) & y_2(t) \\ {y_1}'(t) & {y_2}'(t) \end{vmatrix}$$

Therefore,

$$\begin{aligned} W &= \begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t \sin t + e^t \cos t & e^t \cos t - e^t \sin t \end{vmatrix} \\ &= \left(e^t \sin t\right)\left(e^t \cos t - e^t \sin t\right) - \left(e^t \cos t\right)\left(e^t \sin t + e^t \cos t\right) \\ &= e^{2t} \sin t \cos t - e^{2t} \sin^2 t - e^{2t} \sin t \cos t - e^{2t} \cos^2 t \\ &= -e^{2t} \end{aligned}$$

(4)

$$y_1(\theta) = \cos^2 \theta$$
$$y_2(\theta) = 1 + \cos 2\theta$$

$$W = \begin{vmatrix} y_1(\theta) & y_2(\theta) \\ {y_1}'(\theta) & {y_2}'(\theta) \end{vmatrix}$$

Therefore,

$$\begin{aligned} W &= \begin{vmatrix} \cos^2 \theta & 1 + \cos 2\theta \\ -2 \sin \theta \cos \theta & -2 \sin 2\theta \end{vmatrix} \\ &= -2 \sin 2\theta \cos^2 \theta + 2 \sin \theta \cos \theta (1 + \cos 2\theta) \\ &= -2 \sin 2\theta \cos^2 \theta + 2 \sin \theta \cos \theta + 2 \sin \theta \cos \theta \cos 2\theta \end{aligned}$$

Exercise 2.
If the Wronskian W of f and g is $t^2 e^t$, and if $f(t) = t$, find $g(t)$.

Solution 2.

$$\begin{aligned} W &= \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} \\ &= \begin{vmatrix} t & g(t) \\ 1 & g'(t) \end{vmatrix} \\ &= t g'(t) - g(t) \end{aligned}$$

Therefore,

$$\begin{aligned} t g'(t) - g(t) &= t^2 e^t \\ \therefore g'(t) - \frac{1}{t} g(t) &= t e^t \end{aligned}$$

Therefore,

$$\begin{aligned}\mu(t) &= e^{\int -\frac{1}{t}\,\mathrm{d}t}\\ &= e^{-\ln t}\\ &= t^{-1}\end{aligned}$$

Therefore,

$$\begin{aligned}g(t) &= t\int t^{-1}t^2e^t\,\mathrm{d}t\\ &= t\int te^t\,\mathrm{d}t2e^{-4t} - 2te^{-4t}\\ &= t\left(2e^{-4t} - 2te^{-4t} + c\right)\\ &= 2te^{-4t} - 2t^2e^{-4t} + ct\end{aligned}$$

Exercise 3.
Verify that the functions $y_1(x) = x$ and $y_2(x) = xe^x$ are solutions for the equation

$$\begin{aligned}x^2y'' - x(x+2)y' + (x+2)y = 0\\ x > 0\end{aligned}$$

Do they constitute a fundamental set of solutions?

Solution 3.

$$\begin{aligned}x^2y'' - x(x+2)y' + (x+2)y &= x^2{y_1}'' - x(x+2){y_1}' + (x+2)y_1\\ &= x^2x'' - x(x+2)x' + (x+2)x\\ &= 0 - x(x+2) + x(x+2)\\ &= 0\end{aligned}$$

Therefore, $y_1(x) = x$ is a solution to the equation.

$$\begin{aligned}x^2y'' - x(x+2)y' + (x+2)y &= x^2{y_2}'' - x(x+2){y_2}' + (x+2)y_2\\ &= x^2\left(xe^x\right)'' - x(x+2)\left(xe^x\right)' + (x+2)xe^x\\ &= x^2\left(2e^x + e^xx\right) - x(x+2)\left(e^x + e^xx\right) + (x+2)xe^x\\ &= 0\end{aligned}$$

Therefore, $y_2(x) = xe^x$ is a solution to the equation.

$$\begin{aligned}W &= \begin{vmatrix} x & xe^x \\ 1 & e^x + e^xx \end{vmatrix}\\ &= xe^x + x^2e^x - e^x\end{aligned}$$

As $\forall x \in \mathbb{R}$, $W \neq 0$, the solutions form a fundamental set of solutions.

Part 4. Linear Independence/ Independence of Functions

Exercise 1.

In each of the following sections, determine whether the given pair of functions is linearly independent or linearly dependent using the linear dependence definition and using the Wronskian.

(1) $f(t) = t^2 + 5t$, $g(t) = t^2 - 5t$
(2) $f(\theta) = \cos 3\theta$, $g(\theta) = 4\cos^3\theta - 3\cos\theta$
(3) $f(x) = e^{3x}$, $g(x) = e^{3(x-1)}$

Solution 1.

(1)

$$\begin{aligned} f(t) &= t^2 + 5t \\ g(t) &= t^2 - 5t \end{aligned}$$

Therefore,

$$\begin{aligned} W &= \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} \\ &= \begin{vmatrix} t^2 + 5t & t^2 - 5t \\ 2t + 5 & 2t - 5 \end{vmatrix} \\ &= \left(t^2 + 5t\right)(2t - 5) - \left(t^2 - 5t\right)(2t + 5) \\ &= 10t^2 \end{aligned}$$

If $t = 0$, $W = 0$.
Therefore, as the Wronskian can be 0, $f(t)$ and $g(t)$ are linearly dependent.

(2)

$$\begin{aligned} f(\theta) &= \cos 3\theta \\ g(\theta) &= 4\cos^3\theta - 3\cos\theta \\ &= \cos 3\theta \end{aligned}$$

Therefore,

$$\begin{aligned} W &= \begin{vmatrix} f(\theta) & g(\theta) \\ f'(\theta) & g'(\theta) \end{vmatrix} \\ &= \begin{vmatrix} \cos 3\theta & \cos 3\theta \\ -3\sin 3\theta & -3\sin 3\theta \end{vmatrix} \\ &= 0 \end{aligned}$$

Therefore, as the Wronskian is 0, $f(t)$ and $g(t)$ are linearly dependent.

(3)

$$
\begin{aligned}
f(x) &= e^{3x} \\
g(x) &= e^{3(x-1)} \\
&= e^{3x-3} \\
&= \frac{e^{3x}}{e^3} \\
&= \frac{f(x)}{e^3}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
W &= \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(z) \end{vmatrix} \\
&= \begin{vmatrix} f(x) & \frac{f(x)}{e^3} \\ f'(x) & \frac{f'(x)}{e^3} \end{vmatrix} \\
&= \frac{f(x)f'(x)}{e^3} - \frac{f(x)f'(x)}{e^3} \\
&= 0
\end{aligned}
$$

Therefore, as the Wronskian is 0, $f(t)$ and $g(t)$ are linearly dependent.

Part 5. Abel's Theorem

Exercise 1.
Prove that if y_1 and y_2 are zero at the same point in I, then they cannot be a fundamental set of solutions on that interval.

Solution 1.
Let $t_0 \in I$, such that $y_1(t_0) = y_2(t_0) = 0$.
Therefore,

$$
\begin{aligned}
W(t_0) &= \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ {y_1}'(t_0) & {y_2}'(t_0) \end{vmatrix} \\
&= \begin{vmatrix} 0 & 0 \\ {y_1}'(t_0) & {y_2}'(t_0) \end{vmatrix} \\
&= 0
\end{aligned}
$$

Therefore, as the Wronskian is 0 at t_0, y_1 and y_2 cannot be a fundamental set of solutions in I. $\square$

Exercise 2.
Prove that if y_1 and y_2 have maxima or minima at the same point in I, then they cannot be a fundamental set of solutions on that interval.

Solution 2.
If y_1 and y_2 have maxima or minima at some point t_0 in I, ${y_1}'(t_0) = {y_2}'(t_0) = 0$.
Therefore,

$$\begin{aligned} W(t_0) &= \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ {y_1}'(t_0) & {y_2}'(t_0) \end{vmatrix} \\ &= \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ 0 & 0 \end{vmatrix} \\ &= 0 \end{aligned}$$

Therefore, as the Wronskian is 0 at t_0, y_1 and y_2 cannot be a fundamental set of solutions in I.