ORDINARY DIFFERENTIAL EQUATIONS : ASSIGNMENT 3

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Part 1. Lipschitz Continuous Functions

Exercise 1.
Prove that any Lipschitz function is continuous.

Solution 1.
Let $f(x)$ be a Lipschitz function.
Therefore,

$$|f(x_1) - f(x_2)| \leq k|x_1 - x_2|$$
$$\therefore \left|\frac{f(x_1) - f(x_2)}{x_1 - x_2}\right| \leq k$$
$$\therefore \frac{\mathrm{d}f(x)}{\mathrm{d}x} \leq k$$

Therefore, as the derivative of $f(x)$ exists, $f(x)$ must be continuous.

Exercise 2.
Prove that any continuously differentiable function on an interval $[a, b]$ is Lipschitz there.

Solution 2.
Let the function be $f(x)$.
As $f(x)$ is continuously differentiable, $f'(x)$ exists on $[a, b]$ and is never infinite.
Therefore,

$$\left|\frac{f(x_1) - f(x_2)}{x_1 - x_2}\right| \leq k$$
$$\therefore |f(x_1) - f(x_2)| \leq k|x_1 - x_2|$$

Therefore, $f(x)$ is Lipschitz on $[a, b]$.

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Exercise 3.
Show that $f(x) = |x|$ is a Lipschitz continuous function on the whole real line.

Solution 3.

$$|f(x_1) - f(x_2)| = \big||x_1| - |x_2|\big|$$

By triangle inequality theorem,

$$\begin{aligned} \big||x_1| - |x_2|\big| &\leq |x_1 - x_2| \\ \therefore |f(x_1) - f(x_2) &\leq |x_1 - x_2| \end{aligned}$$

Therefore, $f(x)$ is Lipschitz on $\mathbb{R}$.

Exercise 4.
Show that $g(x) = \sqrt{x}$ is not a Lipschitz continuous function on $[0, 1]$.

Solution 4.
If possible let $g(x) = \sqrt{x}$ be a Lipschitz continuous function on $[0, 1]$.
Therefore, if $x_1 \neq x_2$,

$$\begin{aligned} \left|\sqrt{x_1} - \sqrt{x_2}\right| &\leq k|x_1 - x_2| \\ \therefore \left|\sqrt{x_1} - \sqrt{x_2}\right| - k\left|\sqrt{x_1} - \sqrt{x_2}\right|\left|\sqrt{x_1} + \sqrt{x_2}\right| &\leq 0 \\ \therefore \left|\sqrt{x_1} - \sqrt{x_2}\right|\left(1 - k\left|\sqrt{x_1} + \sqrt{x_2}\right|\right) &\leq 0 \\ \therefore \left|\sqrt{x_1} - \sqrt{x_2}\right| &\leq \frac{1}{1 - k\left|\sqrt{x_1} + \sqrt{x_2}\right|} \end{aligned}$$

Therefore, if $x_1 \neq x_2$, $x_1 \to \infty$, $x_2 \to \infty$, then $|f(x_1) - f(x_2)|$ does not exist.
Therefore, the function is not Lipschitz on $[0, 1]$.

Exercise 5.
Show that $h(x) = x^2$ is Lipschitz continuous function on any closed interval $[a, b]$, but it is not globally Lipschitz continuous on the whole real line.

Solution 5.
$h'(x) = 2x$ is continuous on any closed interval $[a, b]$.
Therefore, $h(x) = x^2$ is Lipschitz on any closed interval $[a, b]$.
For $h(x)$ to be Lipschitz, assuming $x_1 \neq x_2$,

$$\begin{aligned} |{x_1}^2 - {x_2}^2| &\leq k|x_1 - x_2| \\ \therefore |x_1 + x_2||x_1 - x_2| &\leq k|x_1 - x_2| \\ \therefore |x_1 + x_2| &\leq k \end{aligned}$$

If the interval is $\mathbb{R}$, there cannot be one particular k which satisfies the above inequality.
Therefore, $h(x)$ is not Lipschitz on $\mathbb{R}$.

Exercise 6.
Is the function $k(x) = \sqrt{x^2 + 5}$ continuous on the whole real line?

Solution 6.
If possible, let $k(x)$ be Lipschitz on the whole real line.
Therefore,

$$\begin{aligned}
\left|\sqrt{{x_1}^2 + 5} - \sqrt{{x_2}^2 + 5}\right| &\leq k|x_1 - x_2| \\
\therefore \left|{x_1}^2 + 5 - {x_2}^2 - 5\right| &\leq k|x_1 - x_2| \left|\sqrt{{x_1}^2 + 5} + \sqrt{{x_2}^2 + 5}\right| \\
\therefore |x_1 + x_2| &\leq \left|\sqrt{{x_1}^2 + 5} + \sqrt{{x_2}^2 + 5}\right| \\
\therefore \frac{|x_1 + x_2|}{\left|\sqrt{{x_1}^2 + 5} + \sqrt{{x_2}^2 + 5}\right|} &\leq k
\end{aligned}$$

As $|x_1 + x_2| < \left|\sqrt{{x_1}^2 + 5} + \sqrt{{x_2}^2 + 5}\right|$,

$$1 \leq k$$

Therefore, $k(x)$ is Lipschitz continuous on the whole real line.

Part 2. Uniqueness Conditions

Exercise 1.
Find a solution for the initial value problem

$$\begin{aligned}
y' &= \frac{|\sin y|}{y} \\
y(1) &= \pi
\end{aligned}$$

and show it is unique.

Solution 1.
The function $f(x) = \pi$ satisfies the set of equations.

Let y_1, y_2 be two solutions to the initial value problem.

$$
\begin{aligned}
|y_1 - y_2| &\leq \int\limits_0^x \left| \frac{|\sin y_1|}{x'} - \frac{|\sin y_2|}{x_2} \right| \mathrm{d}x' \\
\therefore |y_1 - y_2| &\leq k \int\limits_0^k |y_1 - y_2| \,\mathrm{d}x'
\end{aligned}
$$

Let

$$
\begin{aligned}
a(x) &= \int\limits_0^x |y_1 - y_2| \,\mathrm{d}x' \\
\therefore a'(x) &= |y_1 - y_2|
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
a'(x) &\leq ka(x) \\
\therefore \left(a(x)e^{-kx}\right)' &\leq 0 \\
\therefore a(x) &= 0 \\
\therefore y_1 &= y_2
\end{aligned}
$$

Therefore the solution is unique.

Exercise 2.
Consider the initial value problem

$$
\begin{aligned}
y' &= \frac{-t + (t^2 + 4y)^{\frac{1}{2}}}{2} \\
y(2) &= -1
\end{aligned}
$$

(1) Verify that both $y_1 = 1 - t$ and $y_2 = -\frac{t^2}{4}$ are solutions of the initial value problem. Where are these solutions valid?
(2) Explain why the existence of the two solutions of the given problem does not contradict the uniqueness part of the existence and uniqueness theorem.

Solution 2.
(1)

$$
{y_1}'(t) = -1
$$

Substituting y in y',

$$\begin{aligned} y' &= \frac{-t + (t^2 + 4 - t)^{\frac{1}{2}}}{2} \\ &= \frac{-t + (t-2)^2}{2} \\ &= -1 \end{aligned}$$

$$\begin{aligned} y_1(2) &= 1 - 2 \\ &= -1 \end{aligned}$$

Therefore $y_1 = 1 - t$ is a solution of the initial value problem.
It is valid in $\mathbb{R}$.

$$y_2{}'(t) = -\frac{t}{2}$$

Substituting y in y',

$$\begin{aligned} y' &= \frac{-t + (t^2 - t^2)^{\frac{1}{2}}}{2} \\ &= -\frac{t}{2} \end{aligned}$$

$$\begin{aligned} y_2(2) &= -\frac{2^2}{4} \\ &= 1 \end{aligned}$$

Therefore $y_2 = -\frac{t^2}{4}$ is a solution of the initial value problem.
Due to the square root, it is valid in $[2, \infty)$.

(2) As the function is not Lipschitz in y, the existence and uniqueness theorem is not applicable to it. Therefore, it does not contradict the theorem.

Part 3. Picard Approximations

Exercise 1.

Consider the following initial value problems

(a) $y' = 2(y+1)$, $y(0) = 0$
(b) $y' = y + 1 - t$, $y(0) = 0$

(1) Find the Picard approximations $\varphi_n(t)$ for the solution of the initial value problems for an arbitrary n.
(2) Express $\lim_{n\to\infty} \varphi_n(t)$ in terms of elementary functions.
(3) Solve the initial value problems using order 1 techniques and compare your results.

Solution 1.

(a) (a)

$$\varphi_0(t) = 0$$

$$\therefore \varphi_1(t) = 0 + \int_0^t 2(1+0)\,\mathrm{d}x$$

$$= 2t \therefore \varphi_2(t) \qquad = 0 + \int_0^t 2(1+2x)\,\mathrm{d}x$$

$$= 2t + \frac{4t^2}{2}$$

$$\therefore \varphi_3(t) = 0 + \int_0^t 2\left(1 + 2x + \frac{4x^2}{4}\right)$$

$$= 2t + \frac{4x^2}{2} + \frac{8t^3}{6}$$

$$\vdots$$

$$\therefore \varphi_n(t) = \sum_{r=1}^{n} \frac{(2t)^r}{r!}$$

(b)

$$\lim_{n\to\infty} \varphi_n(t) = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{(2t)^r}{r!}$$

$$= e^{2t} - 1$$

(c)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2(y+1)$$

$$\therefore \frac{\mathrm{d}y}{y+1} = 2\,\mathrm{d}t$$

$$\therefore y + 1 = ce^{2t}$$

Substituting initial conditions, $c = 1$. Therefore,

$$y + 1 = e^{2t}$$

$$\therefore y = e^{2t} - 1$$

(b) (a)

$$
\begin{aligned}
\varphi_0(t) &= 0 \\
\therefore \varphi_1(t) &= \int_0^t (0 + 1 - x)\,\mathrm{d}x \\
&= t - \frac{t^2}{2} \\
\therefore \varphi_2(t) &= \int_0^t \left(0 + 1 - x + x - \frac{x^2}{2}\right) \mathrm{d}x \\
&= t - \frac{t^3}{6} \\
&\vdots \\
\therefore \varphi_n(t) &= t - \frac{t^{n+1}}{(n+1)!}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\lim_{n\to\infty} \varphi_n(t) &= \left(t - \frac{t^{n+1}}{(n+1)!}\right) \\
&= t
\end{aligned}
$$

(c)

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= y + 1 - t \\
\therefore \frac{\mathrm{d}y}{\mathrm{d}t} - y &= 1 - t \\
\therefore y &= t + c
\end{aligned}
$$

Substituting initial conditions, $c = 0$. Therefore,

$$
y = t
$$

Exercise 2.
Calculate the Picard approximations $\varphi_1(t)$, $\varphi_2(t)$, $\varphi_3(t)$ for the initial value problem $y' = t^2 + y^2$, $y(0) = 0$.

Solution 2.

$$\begin{aligned}
\varphi_0(t) &= 0 \\
\therefore \varphi_1(t) &= \int_0^t (x^2 + 0^2)\,\mathrm{d}x \\
&= \frac{t^3}{3} \\
\therefore \varphi_2(t) &= \int_0^t \left(x^2 + \left(\frac{x^3}{3}\right)^2\right) \mathrm{d}x \\
&= \frac{t^3}{3} + \frac{t^7}{3^2 \cdot 7} \\
\therefore \varphi_3(t) &= \int_0^t \left(x^2 + \left(\frac{x^3}{3} + \frac{x^7}{63}\right)^2\right) \mathrm{d}x \\
&= \frac{t^3}{3} + \frac{t^7}{63} + \frac{2}{3^2 \cdot 7} \cdot \frac{t^{11}}{11} + \frac{1}{(3^2 \cdot 7)^2} \cdot \frac{t^{15}}{15}
\end{aligned}$$

Exercise 3.
Use the pattern

$$\begin{aligned}
\varphi_0(t) &= x(0) \qquad & \varphi_{i+1}(t) &= x(0) + \int_0^t f_1\big(s, \varphi_i(x), \psi_i(s)\big)\,\mathrm{d}s \\
\psi_0(t) &= y(0) \qquad & \psi_{i+1}(t) &= y(0) + \int_0^t f_2\big(s, \varphi_i(x), \psi_i(s)\big)\,\mathrm{d}s
\end{aligned}$$

to find the first four Picard approximations for the solution of the initial value problem

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= t + y \qquad & x(0) &= 0 \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= t - x^2 \qquad & y(0) &= 1
\end{aligned}$$

Solution 3.

$$\begin{aligned}
\varphi_0(t) &= 2 \\
\psi_0(t) &= 1
\end{aligned}$$

Therefore,

$$\begin{aligned}
\varphi_1(t) &= 2 + \int_0^t (t' + 1)\,\mathrm{d}t' \\
&= 2 + t + \frac{t^2}{2} \\
\psi_1(t) &= 1 + \int_0^t (t' - 2^2)\,\mathrm{d}t' \\
&= 1 - 4t + \frac{t^2}{2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\varphi_2(t) &= 2 + \int_0^t \left(t' + 1 - 4t' + \frac{t'^2}{2}\right)\mathrm{d}t' \\
&= 2 + t - \frac{3t^2}{2} + \frac{t^3}{6} \\
\psi_2(t) &= 1 + \int_0^t \left(t' - \left(2 + t' + \frac{t'^2}{2}\right)^2\right)\mathrm{d}t' \\
&= 1 + \left(-\frac{t^5}{20} - \frac{t^4}{4} - t^3 - \frac{3t^2}{2} - 4t\right)
\end{aligned}$$