ORDINARY DIFFERENTIAL EQUATIONS : ASSIGNMENT 1

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Part 1. Linear Equations

Exercise 1.
Find the general solution

a) $y' - 2y = t^2 e^{2t}$

b) $y' + \left(\frac{1}{t}\right) y = 3\cos 2t,\ t > 0$

c) $ty' + 2y = \sin t,\ t > 0$

d) $(1 + t^2)y' + 4ty = (1 + t^2)^{-2}$

Solution 1.

Date: Monday 30^{th} March, 2015.

a) Comparing

$$y' - 2y = t^2 e^{2t}$$

and

$$y' + p(t)y = q(t)$$

$$p(t) = -2$$
$$q(t) = t^2 e^{2t}$$

Therefore,

$$\begin{aligned}\mu(x) &= e^{\int p(t)\,\mathrm{d}t} \\ &= e^{\int -2\,\mathrm{d}t} \\ &= e^{-2t}\end{aligned}$$

Therefore,

$$\begin{aligned}\therefore e^{-2t}\left(y' - 2y\right) &= e^{-2t} t^2 e^{2t} \\ \therefore e^{-2t} y' - 2e^{-2t} y &= t^2 \\ \therefore \left(e^{-2t} y\right)' &= t^2 \\ \therefore e^{-2t} y &= \frac{t^3}{3} + c \\ \therefore y &= \frac{t^3 e^{2t}}{3} + ce^{2t}\end{aligned}$$

b) Comparing

$$y' - \left(\frac{1}{t}\right) y = 3\cos 2t$$

and

$$y' + p(t)y = q(t)$$

$$\begin{aligned} p(t) &= -\frac{1}{t} \\ q(t) &= 3\cos 2t \end{aligned}$$

Therefore,

$$\begin{aligned} \mu(t) &= e^{\int p(t)\,\mathrm{d}t} \\ &= e^{\int \frac{1}{t}\,\mathrm{d}t} \\ &= e^{\ln t} \\ &= t \end{aligned}$$

Therefore,

$$\begin{aligned} ty' - y &= 3t\cos 2t \\ \therefore (ty)' &= 3t\cos 2t \\ \therefore ty &= \int 3t\cos 2t\,\mathrm{d}t \\ \therefore ty &= \frac{3}{4}\cos 2t + \frac{3}{2}t\sin 2t + c \\ \therefore y &= \frac{3\cos 2t}{4t} + \frac{3\sin 2t}{2} + \frac{c}{t} \end{aligned}$$

c) Comparing

$$ty' + 2y = \sin t$$

and

$$y' + p(t)y = q(t)$$

$$p(t) = \frac{2}{t}$$

$$q(t) = \frac{\sin t}{t}$$

Therefore,

$$\begin{aligned}\mu(t) &= e^{\int p(t)\,\mathrm{d}t} \\ &= e^{\int \frac{2}{t}\,\mathrm{d}t} \\ &= e^{2\ln t} \\ &= t^2\end{aligned}$$

Therefore,

$$\begin{aligned}t^2y' + 2ty &= t\sin t \\ \therefore (t^2y)' &= t\sin t \\ \therefore t^2y &= \int t\sin t\,\mathrm{d}t \\ \therefore t^2y &= -t\cos t + \sin t + c \\ \therefore y &= -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{c}{t^2}\end{aligned}$$

d) Comparing

$$(1+t^2)y' + 4ty = (1+t^2)^{-2}$$

and

$$y' + p(t)y = q(t)$$

$$\begin{aligned} p(t) &= \frac{4t}{1+t^2} \\ q(t) &= \frac{(1+t^2)^{-2}}{1+t^2} \\ &= \frac{1}{(1+t^2)^3} \end{aligned}$$

Therefore,

$$\begin{aligned} \mu(t) &= e^{\int p(t)\,\mathrm{d}t} \\ &= e^{\int \frac{4t}{1+t^2}\,\mathrm{d}t} \\ &= e^{2\ln(1+t^2)} \\ &= (1+t^2)^2 \end{aligned}$$

Therefore,

$$\begin{aligned} y &= \frac{1}{\mu(t)} \int \mu(t)q(t)\,\mathrm{d}t \\ &= (1+t^2)^{-2} \int (1+t^2)^2 \cdot (1+t^2)^{-3}\,\mathrm{d}t \\ &= (1+t^2)^{-2} \int \frac{1}{1+t^2}\,\mathrm{d}t \\ &= \frac{\tan^{-1} t + c}{(1+t^2)^2} \end{aligned}$$

Exercise 2.
In the previous exercise, determine the solution's behaviour for large t.

Solution 2.

a)

$$\begin{aligned} y &= \frac{t^3e^{2t}}{3} + ce^{2t} \\ \therefore \lim_{t\to\infty} y &= \infty \end{aligned}$$

b)

$$y = \frac{3\cos 2t}{4t} + \frac{3\sin 2t}{2} + \frac{c}{t}$$

If $t \to \infty$, $\dfrac{3\sin 2t}{2} \in \left(\dfrac{-3}{2}, \dfrac{3}{2}\right)$, and

$$\lim_{t\to\infty} \frac{3\cos 2t}{4t} = 0$$

Therefore,

$$y \in \left[\frac{-3}{2}, \frac{3}{2}\right]$$

c)

$$y = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{c}{t^2}$$

$$\therefore \lim_{t\to\infty} y = 0$$

d)

$$y = \frac{\tan^{-1} t + c}{(1+t^2)^2}$$

$$\therefore \lim_{t\to\infty} y = 0$$

Exercise 3.
Solve the initial value problems

a) $y' + 2y = te^{2t}$, $y(0) = 1$

b) $y' + \left(\dfrac{2}{t}\right) y = \dfrac{\cos t}{t^2}$, $y(\pi) = 0$, $t > 0$

c) $ty' + 2y = \sin t$, $y\left(\dfrac{\pi}{2}\right) = 1$

Solution 3.

a) Comparing

$$y' + 2y = te^{2t}$$

and

$$y' + p(t)y = q(t)$$

$$p(t) = -2$$
$$q(t) = te^{2t}$$

Therefore,

$$\begin{aligned}\mu(t) &= e^{\int p(t)\,\mathrm{d}t}\\ &= e^{\int 2\,\mathrm{d}t}\\ &= e^{2t}\end{aligned}$$

Therefore,

$$\begin{aligned}y &= \frac{1}{\mu(t)}\int \mu(t)q(t)\,\mathrm{d}t\\ &= \frac{1}{e^{2t}}\int e^{2t}\cdot te^{2t}\,\mathrm{d}t\\ &= \frac{1}{e^{2t}}\int te^{4t}\,\mathrm{d}t\\ &= \frac{e^{4t}\left(\frac{t}{4}-\frac{1}{16}\right)+c}{e^{2t}}\\ &= e^{2t}\left(\frac{t}{4}-\frac{1}{16}\right)+\frac{c}{e^{2t}}\end{aligned}$$

Substituting the given condition, $y(0) = 1$,

$$1 = e^{2\cdot 0}\left(\frac{0}{4}-\frac{1}{16}\right)+\frac{c}{e^{2\cdot 0}}$$

$$\therefore 1 = c - \frac{1}{16}$$

$$\therefore c = \frac{17}{16}$$

Therefore,

$$y = e^{2t}\left(\frac{t}{4}-\frac{1}{16}\right)+\frac{17}{16e^{2t}}$$

b) Comparing

$$y' + \left(\frac{2}{t}\right) y = \frac{\cos t}{t^2}$$

and

$$y' + p(t)y = q(t)$$

$$p(t) = \frac{2}{t}$$
$$q(t) = \frac{\cos t}{t^2}$$

Therefore,

$$\begin{aligned}\mu(t) &= e^{\int p(t)\,\mathrm{d}t}\\ &= e^{\int \frac{2}{t}\,\mathrm{d}t}\\ &= e^{2\ln t}\\ &= t^2\end{aligned}$$

Therefore,

$$\begin{aligned}y &= \frac{1}{\mu(t)}\int \mu(t)q(t)\,\mathrm{d}t\\ &= \frac{1}{t^2}\int t^2\cdot\frac{\cos t}{t^2}\,\mathrm{d}t\\ &= \frac{1}{t^2}\int \cos t\,\mathrm{d}t\\ &= \frac{\sin t + c}{t^2}\end{aligned}$$

Substituting the given condition, $y(\pi) = 0$,

$$0 = \frac{\sin\pi + c}{\pi^2}$$
$$\therefore c = 0$$

Therefore,

$$y = \frac{\sin t}{t^2}$$

c) Comparing

$$ty' + 2y = \sin t$$

and

$$y' + p(t)y = q(t)$$

$$p(t) = \frac{2}{t}$$
$$q(t) = \frac{\sin t}{t}$$

Therefore,

$$\begin{aligned}\mu(t) &= e^{\int p(t)\,\mathrm{d}t}\\ &= e^{\int \frac{2}{t}\,\mathrm{d}t}\\ &= e^{2\ln t}\\ &= t^2\end{aligned}$$

Therefore,

$$\begin{aligned}y &= \frac{1}{t^2}\int \mu(x)q(x)\,\mathrm{d}x\\ &= \frac{1}{t^2}\int t^2\frac{\sin t}{t}\,\mathrm{d}t\\ &= \frac{1}{t^2}\int t\sin t\\ &= \frac{-t\cos t + \sin t + c}{t^2}\end{aligned}$$

Substituting the given condition $y\left(\frac{\pi}{2}\right) = 1$,

$$1 = \frac{-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + c}{\left(\frac{\pi}{2}\right)^2}$$

$$\therefore \frac{\pi^2}{4} = 1 + c$$
$$\therefore c = \frac{\pi^2}{4} - 1$$

Therefore,

$$y = \frac{-t\cos t + \sin t + \frac{\pi^2}{4} - 1}{t^2}$$

Exercise 4.
Find the initial value y_0 for which the solution of the initial value problem

$$\begin{aligned} y' - y &= 1 + 3\sin t \\ y(0) &= y_0 \end{aligned}$$

remains finite for $t \to \infty$.

Solution 4.
Comparing

$$y' - y = 1 + 3\sin t$$

and

$$y' + p(t)y = q(t)$$

$$\begin{aligned} p(t) &= -1 \\ q(t) &= 1 + 3\sin t \end{aligned}$$

Therefore,

$$\begin{aligned} \mu(t) &= e^{\int p(t)\,\mathrm{d}t} \\ &= e^{\int -\,\mathrm{d}t} \\ &= e^{-t} \end{aligned}$$

Therefore,

$$\begin{aligned} y &= \frac{1}{\mu(t)} \int \mu(t)q(t)\,\mathrm{d}t \\ &= e^t \int \frac{1 + 3\sin t}{e^t}\,\mathrm{d}t \\ &= e^t \left(-\frac{e^{-t}}{2} (2 + 3\cos t + 3\sin t) + c \right) \\ &= -\frac{2 + 3\cos t + 3\sin t}{2} + ce^t \end{aligned}$$

Therefore,

$$\lim_{t\to\infty} y = \lim_{t\to\infty} \frac{2 + 3\cos t + 3\sin t}{2} + \lim_{t\to\infty} ce^t$$

Therefore, for $\lim\limits_{t\to\infty} y$ to be finite, $c = 0$. Substituting the conditions, $y(0) = y_0$ and $c = 0$,

$$\begin{aligned} y_0 &= -\frac{2 + 3 + 0}{2} + 0e^t \\ \therefore y_0 &= -\frac{5}{2} \end{aligned}$$

Exercise 5.
Show that if y_1, y_2, y_3 are private solutions for

$$y' + a(t)y = b(t)$$

then the function $\dfrac{y_2 - y_3}{y_3 - y_1}$ is constant for all real t.

Solution 5.

$$\mu(x) = e^{\int a(t)\,\mathrm{d}t}$$

$$\therefore y = \frac{1}{\mu(t)} \int \mu(t)b(t)\,\mathrm{d}t$$

Let

$$\int \mu(t)b(t)\,\mathrm{d}t = p(t) + c$$

Therefore,

$$y = \frac{p(t) + c}{\mu(t)}$$

$$\therefore y_1 = \frac{p(t) + c_1}{\mu(t)}$$

$$\therefore y_2 = \frac{p(t) + c_2}{\mu(t)}$$

$$\therefore y_3 = \frac{p(t) + c_3}{\mu(t)}$$

Therefore,

$$\frac{y_2 - y_3}{y_3 - y_1} = \frac{\dfrac{p(t) + c_2 - p(t) - c_3}{\mu(t)}}{\dfrac{p(t) + c_3 - p(t) - c_1}{\mu(t)}}$$

$$= \frac{c_2 - c_3}{c_3 - c_1}$$

Therefore, as $\dfrac{y_2 - y_3}{y_3 - y_1}$ is independent of t, it is constant for all real t.

Exercise 6.
Suppose that there exists $M > 0$ such that for all real x, $|f(x)| \leq M$. Show that for $a > 0$, any solution for the equation $y' + ay = f(x)$ is bounded at $[0, \infty)$.

Solution 6.

$$
\begin{aligned}
y' + ay &= f(x) \\
\therefore \mu(x) &= e^{\int a \,\mathrm{d}x} \\
&= e^{ax} \\
\therefore y &= \frac{1}{e^{ax}} \int e^{ax} f(x) \,\mathrm{d}x
\end{aligned}
$$

As $|f(x)| \leq M$,

$$
\begin{aligned}
y &\leq \frac{1}{e^{ax}} \int M e^{ax} \,\mathrm{d}x \\
\therefore y &\leq \frac{1}{e^{ax}} \left(\frac{M}{a} e^{ax} + c \right) \\
\therefore y &\leq \frac{M}{a} + \frac{c}{e^{ax}}
\end{aligned}
$$

or

$$
\begin{aligned}
y &\geq \frac{1}{e^{ax}} \cdot - \int M e^{ax} \,\mathrm{d}x \\
\therefore y &\geq -\frac{1}{e^{ax}} \left(\frac{M}{a} e^{ax} + c \right) \\
\therefore y &\geq -\frac{M}{a} - \frac{c}{e^{ax}}
\end{aligned}
$$

Therefore,

$$
y \leq \left| \frac{M}{a} + \frac{c}{e^{ax}} \right|
$$

Therefore,

$$
0 \leq y < \infty
$$

Part 2. Bernoulli Equations

Exercise 1.
Solve

a) $t^2 y' + 2ty - y^3 = 0$, $t > 0$

b) $y' = \varepsilon y - \sigma y^3$, $\varepsilon > 0$, $\sigma > 0$

Solution 1.

a) Comparing

$$t^2y' + 2ty - y^3 = 0$$

and

$$y' + p(t)y = q(t)y^n$$

$$\begin{aligned} p(t) &= \frac{2}{t} \\ q(t) &= \frac{1}{t^2} \\ n &= 3 \end{aligned}$$

Therefore, let

$$\begin{aligned} \nu &= y^{1-3} \\ &= y^{-2} \\ \therefore \nu' &= (-2)y^{-3}y' \end{aligned}$$

Substituting ν and ν',

$$\begin{aligned} \frac{1}{-2}\nu' + \frac{2}{t}\nu &= \frac{1}{t^2} \\ \therefore \nu' - \frac{4}{t}\nu &= -\frac{2}{t^2} \end{aligned}$$

Therefore,

$$\begin{aligned} \mu(t) &= e^{\int -\frac{4}{t}\,\mathrm{d}t} \\ &= t^{-4} \end{aligned}$$

Therefore,

$$\begin{aligned} \nu &= t^4 \int t^{-4}t^{-2}\,\mathrm{d}t \\ &= t^4\left(-\frac{1}{5t^5} + c\right) \\ \therefore y^{-2} &= t^4\left(\frac{5t^5c - 1}{5t^5}\right) \\ \therefore y^{-2} &= \frac{5t^5c - 1}{5t} \\ \therefore y^2 &= \frac{5t}{5t^5c - 1} \\ \therefore y &= \pm\sqrt{\frac{5t}{5t^5c - 1}} \end{aligned}$$

b)

$$y' = \varepsilon y - \sigma y^3$$
$$\therefore y' - \varepsilon y = -\sigma y^3$$

Comparing

$$y' - \varepsilon y = -\sigma y^3$$

and

$$y' - p(t)y = q(t)y^n$$

$$p(t) = -\varepsilon$$
$$q(t) = -\sigma$$
$$n = y^{-2}$$

Therefore, let

$$\nu = y^{-2}$$
$$\therefore \nu' = (-2)y^{-3}y'$$

Substituting ν and ν',

$$\frac{1}{-2}\nu' - \varepsilon = -\sigma$$
$$\therefore \nu' + 2\varepsilon = 2\sigma$$

Therefore,

$$\mu(t) = e^{\int -\varepsilon \, \mathrm{d}t}$$
$$= e^{-\varepsilon t}$$

Therefore,

$$\nu = e^{\varepsilon t} \int e^{-\varepsilon t} \cdot 2\sigma \, \mathrm{d}t$$
$$= e^{\varepsilon t}\left(-2e^{-\varepsilon t}\sigma t + c\right)$$
$$\therefore y^{-2} = -2\sigma t + e^{\varepsilon t}c$$
$$\therefore y^2 = \frac{1}{e^{\varepsilon t}c - 2\sigma t}$$
$$\therefore y = \pm\sqrt{\frac{1}{e^{\varepsilon t}c - 2\sigma t}}$$

Part 3. Separable Equations

Exercise 1.

Find the ‘general’ solution for the following equations. Keep track of singular solutions, if there are any.

a) $y' + y^2 sinx = 0$

b) $y' = (cos^2 x)(cos^2 2y)$

c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2}{1+y^2}$

Solution 1.

a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 \sin x = 0$$
$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = -y^2 \sin x$$

If $y^2 = 0$,
$y = 0$ is a solution if and only if

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

If $y^2 \neq 0$

$$\therefore \frac{\mathrm{d}y}{y^2} = -\sin x \,\mathrm{d}x$$
$$\therefore \int \frac{\mathrm{d}y}{y^2} = \int -\sin x \,\mathrm{d}x$$
$$\therefore -\frac{1}{y} = \cos x + c$$
$$\therefore y = -\frac{1}{\cos x + c}$$

b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (\cos^2 x)(\cos^2 2y)$$

If $\cos^2(2y) = 0$,
$y = \frac{\pi}{4} + n\frac{\pi}{2}$ is a solution if and only if

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

If $\cos^2(2y) \neq 0$,

$$\begin{aligned}
\therefore \frac{\mathrm{d}y}{\cos^2 2y} &= \cos^2 x \,\mathrm{d}x \\
\therefore \int \frac{\mathrm{d}y}{\cos^2 2y} &= \int \cos^2 x \,\mathrm{d}x \\
\therefore \frac{1}{2}\tan(2y) &= \frac{1}{2}\sin(2x) + c_1 \\
\therefore \tan(2y) &= \sin(2x) + c_2 \\
\therefore 2y &= \tan^{-1}\big(\sin(2x)\big) + c_3
\end{aligned}$$

c)

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^2}{1+y^2} \\
\therefore \left(1+y^2\right)\mathrm{d}y &= x^2 \,\mathrm{d}x \\
\therefore \int \left(1+y^2\right)\mathrm{d}y &= \int x^2 \,\mathrm{d}x \\
\therefore y + \frac{y^3}{3} &= \frac{x^3}{3} + c_1 \\
\therefore 3y + y^3 &= x^3 + c_2
\end{aligned}$$

Exercise 2.
Find the solution for the following initial value problems in explicit form, and determine (at least approximately) the interval in which the solution is defined.

a) $y' = (1-2x)y^2$, $y(0) = -\frac{1}{6}$

b) $y' = \frac{2x}{y + x^2 y}$, $y(0) = -2$

c) $y' = \frac{2x}{1+y}$, $y(0) = -2$

Solution 2.

a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (1-2x)y^2$$

If $y^2 = 0$, $y = 0$ is a solution if and only if

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

If $y^2 \neq 0$,

$$\begin{aligned}\frac{\mathrm{d}y}{y^2} &= (1-2x)\,\mathrm{d}x \\ \therefore \int \frac{\mathrm{d}y}{y^2} &= \int (1-2x)\,\mathrm{d}x \\ \therefore -\frac{1}{y} &= x - x^2 + c \\ \therefore y &= -\frac{1}{x-x^2+c}\end{aligned}$$

Substituting $y(0) = -\frac{1}{6}$,

$$-\frac{1}{6} = -\frac{1}{c}$$
$$\therefore c = 6$$

Therefore, the solution is not defined if and only if

$$\begin{aligned} & x - x^2 + c = 0 \\ \iff \ & x^2 - x - 6 = 0 \\ \iff \ & x = \frac{1 \pm \sqrt{1+24}}{2}\end{aligned}$$

Therefore, the solution is defined on $\mathbb{R} \setminus \{-2, 3\}$

b)

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2}{y + x^2 y} \\
\therefore \frac{y}{2}\,\mathrm{d}y &= \frac{1}{1+x^2}\,\mathrm{d}x \\
\therefore \int \frac{y}{2}\,\mathrm{d}y &= \frac{\mathrm{d}x}{1+x^2} \\
\therefore \frac{y^2}{4} &= \tan^{-1} x + c \\
\therefore y^2 &= 4\tan^{-1} x + 4c \\
\therefore y &= \pm 2\sqrt{\tan^{-1} x + c}
\end{aligned}$$

Substituting $y(0) = -2$,

$$\begin{aligned}
-2 &= \pm 2\sqrt{\tan^{-1} 0 + c} \\
\therefore -1 &= \pm\sqrt{0 + c} \\
\therefore c &= 1
\end{aligned}$$

Therefore, the solution is defined if and only if

$$\begin{aligned}
& \tan^{-1} x + 1 \geq 0 \\
\iff & \tan^{-1} x \geq -1 \\
\iff & x \geq -\frac{\pi}{2}
\end{aligned}$$

Therefore, the solution is defined on $\left[-\frac{\pi}{2}, \infty\right)$.

c)

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x}{1+y}\\ \therefore (1+y)\,\mathrm{d}y &= 2x\,\mathrm{d}x\\ \therefore \int (1+y)\,\mathrm{d}y &= \int 2x\,\mathrm{d}x\\ \therefore y + \frac{y^2}{2} &= x^2 + c\end{aligned}$$

Substituting $y(0) = -2$,

$$\begin{aligned}-2 + \frac{4}{2} &= 0 + c\\ \therefore c &= 0\end{aligned}$$

Therefore,

$$\begin{aligned}y^2 + 2y - 2x^2 &= 0\\ \therefore y &= \frac{-2 \pm \sqrt{4+8x^2}}{2}\\ &= -1 \pm \sqrt{1+2x^2}\end{aligned}$$

Therefore, the solution is defined if and only if

$$\begin{aligned}1 + 2x^2 &\geq 0\\ \iff x^2 &\geq -\frac{1}{2}\end{aligned}$$

Therefore, the solution is defined on $\mathbb{R}$.

Exercise 3.
Solve the initial value problem

$$y' = \frac{2-e^x}{3+2y}, y(0) = 0$$

and determine where the solution attains its maximum value.

Solution 3.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2-e^x}{3+2y}\\ \therefore (3+2y)\,\mathrm{d}y &= (2-e^x)\,\mathrm{d}x\\ \therefore \int (3+2y)\,\mathrm{d}y &= \int (2-e^x)\,\mathrm{d}x\\ \therefore 3y + y^2 &= 2x - e^x + c\end{aligned}$$

Substituting $y(0) = 0$,

$$\begin{aligned} 0 &= -1 + c \\ \therefore c &= 1 \end{aligned}$$

Therefore,

$$y^2 + 3y = 2x - e^x + 1$$

If $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$,

$$\begin{aligned} \frac{2 - e^x}{3 + 2y} &= 0 \\ \therefore 2 &= e^x \\ \therefore x &= \ln 2 \end{aligned}$$

Therefore, the solution attains its maximum value at $x = \ln 2$.

Part 4. Homogeneous Equations

Exercise 1.
Solve

a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x + 3y}{x - y}$

b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2 + xy + y^2}{x^2}$

c) $(x^2 + 3xy + y^2)\,\mathrm{d}x - x^2\,\mathrm{d}y = 0$

d) $xy' - y = (x + y)\big(\ln(x + y) - \ln(x)\big)$

Solution 1.

a)

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x+3y}{x-y}\\ &= \frac{1+\frac{3y}{x}}{1-\frac{y}{x}}\end{aligned}$$

Let

$$\begin{aligned}\frac{y}{x} &= z\\ \therefore y &= xz\\ \therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= z + x\,\frac{\mathrm{d}z}{\mathrm{d}x}\end{aligned}$$

Therefore,

$$\begin{aligned}z + x\,\frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{1+3z}{1-z}\\ \therefore x\,\frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{1+3z}{1-z} - z\\ \therefore x\,\frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{1+3z-z+z^2}{1-z}\\ \therefore \frac{1-z}{1+2z+z^2} &= \frac{\mathrm{d}x}{x}\\ \therefore \int \frac{1-z}{1+2z+z^2} &= \int \frac{\mathrm{d}x}{x}\\ \therefore -\frac{2}{1+z} - \ln(1+z) &= \ln x + c\\ \therefore -\frac{2}{1+\frac{y}{x}} - \ln\left(1+\frac{y}{x}\right) &= \ln x + c\\ \therefore -\frac{2x}{x+y} - \ln\left(\frac{x+y}{x}\right) &= \ln x + c\\ \therefore -\frac{2x}{x+y} - \ln(x+y) + \ln x &= \ln x + c\\ \therefore -\frac{2x}{x+y} - \ln(x+y) &= c\end{aligned}$$

b)

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^2 + xy + y^2}{x^2} \\
&= \frac{1 + \dfrac{y}{x} + \dfrac{y^2}{x^2}}{1}
\end{aligned}$$

Let

$$\begin{aligned}
\frac{y}{x} &= z \\
\therefore y &= xz \\
\therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= z + x\,\frac{\mathrm{d}z}{\mathrm{d}x}
\end{aligned}$$

Therefore,

$$\begin{aligned}
z + x\,\frac{\mathrm{d}z}{\mathrm{d}x} &= 1 + z + z^2 \\
\therefore x\,\frac{\mathrm{d}z}{\mathrm{d}x} &= 1 + z^2 \\
\therefore \frac{\mathrm{d}z}{1+z^2} &= \frac{\mathrm{d}x}{x} \\
\therefore \int \frac{\mathrm{d}z}{1+z^2} &= \int \frac{\mathrm{d}x}{x} \\
\therefore \tan^{-1} z &= \ln x + c \\
\therefore \tan^{-1} \frac{y}{x} &= \ln x + c \\
\therefore y &= x \tan\left(\ln x + c\right)
\end{aligned}$$

c)

$$(x^2 + 3xy + y^2)\,\mathrm{d}x - x^2\,\mathrm{d}y = 0$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + 3xy + y^2}{x^2}$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + 3\frac{y}{x} + \frac{y^2}{x^2}}{1}$$

Let

$$\frac{y}{x} = z$$

$$\therefore y = xz$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = z + x\,\frac{\mathrm{d}z}{\mathrm{d}x}$$

Therefore,

$$z + x\,\frac{\mathrm{d}z}{\mathrm{d}x} = 1 + 3z + z^2$$

$$\therefore x\,\frac{\mathrm{d}z}{\mathrm{d}x} = 1 + 2z + z^2$$

$$\therefore \frac{\mathrm{d}z}{(1+z)^2} = \frac{\mathrm{d}x}{x}$$

$$\therefore \int \frac{\mathrm{d}z}{(1+z)^2} = \int \frac{\mathrm{d}x}{x}$$

$$\therefore -\frac{1}{1+z} = \ln x + c$$

$$\therefore -\frac{1}{1+\frac{y}{x}} = \ln x + c$$

$$\therefore -\frac{x}{x+y} = \ln x + c$$

$$\therefore -\frac{x}{\ln x + c} = x + y$$

$$\therefore y = -x - \frac{x}{\ln x + c}$$

d)

$$\begin{aligned}
xy' - y &= (x+y)\big(\ln(x+y) - \ln(x)\big) \\
\therefore x\,\frac{\mathrm{d}y}{\mathrm{d}x} - y &= (x+y)\left(\ln\left(1+\frac{y}{x}\right)\right) \\
\therefore \frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} &= \left(1+\frac{y}{x}\right)\ln\left(1+\frac{y}{x}\right)
\end{aligned}$$

Let

$$\begin{aligned}
\frac{y}{x} &= z \\
\therefore y &= xz \\
\therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= z + x\,\frac{\mathrm{d}z}{\mathrm{d}x}
\end{aligned}$$

Therefore,

$$\begin{aligned}
z + x\,\frac{\mathrm{d}z}{\mathrm{d}x} - z &= (1+z)\ln(1+z) \\
\therefore x\,\frac{\mathrm{d}z}{\mathrm{d}x} &= (1+z)\ln(1+z) \\
\therefore \frac{\mathrm{d}z}{(1+z)\ln(1+z)} &= \frac{\mathrm{d}x}{x} \\
\therefore \int \frac{\mathrm{d}z}{(1+z)\ln(1+z)} &= \int \frac{\mathrm{d}x}{x} \\
\therefore \ln\big(\ln(1+z)\big) &= \ln x + c \\
\therefore \ln\big(\ln(1+z)\big) &= \ln x + \ln c \\
\therefore \ln\big(\ln(1+z)\big) &= \ln xc \\
\therefore \ln(1+z) &= xc \\
\therefore 1+z &= e^{xc} \\
\therefore 1+\frac{y}{x} &= e^{xc} \\
\therefore y &= xe^{xc} - x
\end{aligned}$$