ORDINARY DIFFERENTIAL EQUATIONS
ASSIGNMENT 5

AAKASH JOG
ID : 989323563

Part 1. Homogeneous High Order Linear ODEs with Constant Coefficients

Exercise 1.
Find the general solution of the following differential equations

(1) $y'' + 2y'3y = 0$
(2) $4y'' - 9y = 0$
(3) $y''2y'3y = 0$
(4) $4y'' + 9y = 0$
(5) $16y'' + 24y' + 9y = 0$
(6) $y'''y''y' + y = 0$
(7) $y^{(6)} + y = 0$

Solution 1.
(1) Let

$$\begin{aligned} y &= e^{\lambda t} \\ \therefore y' &= \lambda e^{\lambda t} \\ \therefore y'' &= \lambda^2 e^{\lambda t} \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} - 3e^{\lambda t} &= 0 \\ \therefore \lambda^2 + 2\lambda - 3 &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda &= \frac{-2 \pm \sqrt{4 + 12}}{2} \\ &= \frac{-2 \pm 4}{2} \\ &= -1 \pm 2 \end{aligned}$$

Therefore,

$$\lambda = -3 \qquad \text{or} \qquad \lambda = 1$$

Therefore,

$$y = e^{\lambda t}$$

Date: Monday 18th May, 2015.

Therefore,

$$y = e^{-3t} \qquad \text{or} \qquad y = e^{t}$$

Therefore,

$$y = c_1 e^{-3t} + c_2 e^{t}$$

(2) Let

$$y = e^{\lambda t}$$
$$\therefore y' = \lambda e^{\lambda t}$$
$$\therefore y'' = \lambda^2 e^{\lambda t}$$

Therefore,

$$4\lambda^2 e^{\lambda t} - 9e^{\lambda t} = 0$$
$$\therefore 4\lambda^2 - 9 = 0$$

Therefore,

$$\lambda = \pm\sqrt{\frac{9}{4}}$$

Therefore,

$$\lambda = -\frac{3}{2} \qquad \text{or} \qquad \lambda = \frac{3}{2}$$

Therefore,

$$y = e^{\lambda t}$$

Therefore,

$$y = e^{-\frac{3}{2}t} \qquad \text{or} \qquad y = e^{\frac{3}{2}t}$$

Therefore,

$$y = c_1 e^{-\frac{3}{2}t} + c_2 e^{\frac{3}{2}}$$

(3) Let

$$y = e^{\lambda t}$$
$$\therefore y' = \lambda e^{\lambda t}$$
$$\therefore y'' = \lambda^2 e^{\lambda t}$$

Therefore,

$$\lambda^2 e^{\lambda t} - 2\lambda e^{\lambda t} - 3e^{\lambda t} = 0$$
$$\therefore \lambda^2 - 2\lambda - 3 = 0$$

Therefore,

$$\lambda = \frac{2 \pm \sqrt{4+12}}{2}$$
$$\therefore \lambda = 1 \pm 2$$

Therefore,

$$\lambda = -1 \qquad \text{or} \qquad \lambda = 3$$

Therefore,

$$y = e^{\lambda t}$$

Therefore,

$$y = e^{-t} \qquad \text{or} \qquad y = e^{3t}$$

Therefore,

$$y = c_1 e^{-t} + c_2 e^{3t}$$

(4) Let

$$\begin{aligned} y &= e^{\lambda t} \\ \therefore y' &= \lambda e^{\lambda t} \\ \therefore y'' &= \lambda^2 e^{\lambda t} \end{aligned}$$

Therefore,

$$\begin{aligned} 4\lambda^2 e^{\lambda t} + 9e^{\lambda t} &= 0 \\ \therefore 4\lambda^2 + 9 &= 0 \end{aligned}$$

Therefore,

$$\lambda = \pm\sqrt{\frac{-9}{4}}$$

$$\lambda = \frac{3i}{2} \qquad \text{or} \qquad \lambda = -\frac{3i}{2}$$

Therefore,

$$y = e^{\lambda t}$$

Therefore,

$$y = e^{-\frac{3i}{2}t} \qquad \text{or} \qquad y = e^{\frac{3i}{2}t}$$

Therefore,

$$y = c_1 e^{-\frac{3i}{2}t} + c_2 e^{\frac{3i}{2}t}$$

(5) Let

$$\begin{aligned} y &= e^{\lambda t} \\ \therefore y' &= \lambda e^{\lambda t} \\ \therefore y'' &= \lambda^2 e^{\lambda t} \end{aligned}$$

Therefore,

$$\begin{aligned} 16y'' + 24y' + 9y &= 0 \\ 16\lambda^2 e^{\lambda t} + 24\lambda e^{\lambda t} + 9e^{\lambda t} &= 0 \\ \therefore 16\lambda^2 + 24\lambda + 9 &= 0 \end{aligned}$$

Therefore,

$$\lambda = \frac{-24 \pm \sqrt{24^2 - 4 \cdot 16 \cdot 9}}{2 \cdot 16}$$
$$\therefore \lambda = \frac{-24 \pm \sqrt{576 - 576}}{32}$$
$$\therefore \lambda = -\frac{3}{4}$$

Therefore,

$$y = e^{\lambda t}$$
$$= e^{-\frac{3}{4}t}$$

Therefore,

$$y = (c_1 + c_2)e^{-\frac{3}{4}t}$$

(6) Let

$$y = e^{\lambda t}$$
$$\therefore y' = \lambda e^{\lambda t}$$
$$\therefore y'' = \lambda^2 e^{\lambda t}$$
$$\therefore y''' = \lambda^3 e^{\lambda t}$$

Therefore,

$$\lambda^3 e^{\lambda t} - \lambda^2 e^{\lambda t} - \lambda e^{\lambda t} + e^{\lambda t} = 0$$
$$\therefore \lambda^3 - \lambda^2 - \lambda + 1 = 0$$
$$\therefore (\lambda - 1)^2(\lambda - 1) = 0$$

Therefore,

$$\lambda = -1 \qquad \text{or} \qquad \lambda = 1$$

Therefore,

$$y = e^{\lambda t}$$

Therefore,

$$y = e^{-t} \qquad \text{or} \qquad y = e^{t}$$

Therefore,

$$y = c_1 e^{-t} + c_2 e^{t}$$

(7) Let

$$y = e^{\lambda t}$$
$$\therefore y' = \lambda e^{\lambda t}$$
$$\therefore y'' = \lambda^2 e^{\lambda t}$$
$$\therefore y''' = \lambda^3 e^{\lambda t}$$
$$\vdots$$
$$\therefore y^{(6)} = \lambda^6 e^{\lambda t}$$

Therefore,

$$\lambda^6 e^{\lambda t} + e^{\lambda t} = 0$$
$$\therefore \lambda^6 + 1 = 0$$

Therefore,

$$\lambda = \sqrt[6]{-1}$$

Therefore,

$$\lambda = e^{\frac{i\pi}{6} + \frac{k\pi}{3}}$$

where $k \in \{0, \dots, 5\}$.
Therefore,

$$\begin{aligned} y &= e^{\lambda t} \\ &= e^{e^{\frac{i\pi}{6} + \frac{k\pi}{3}}} \end{aligned}$$

Therefore,

$$y = c_1 e^{e^{\frac{i\pi}{6}}} + c_2 e^{e^{\frac{i\pi}{6} + \frac{\pi}{3}}} + c_3 e^{e^{\frac{i\pi}{6} + \frac{2\pi}{3}}} + c_4 e^{e^{\frac{i\pi}{6} + \frac{3\pi}{3}}} + c_5 e^{e^{\frac{i\pi}{6} + \frac{4\pi}{3}}} + c_6 e^{e^{\frac{i\pi}{6} + \frac{5\pi}{3}}}$$

Exercise 2.
Find the solution for the given initial value problems.

(1)

$$\begin{aligned} y'' + y'2y &= 0 \\ y(0) &= 1 \\ y'(0) &= 1 \end{aligned}$$

(2)

$$\begin{aligned} y'' + 4y &= 0 \\ y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$

(3)

$$\begin{aligned} y'' - 2y' + 5y &= 0 \\ y\left(\frac{\pi}{2}\right) &= 0 \\ y'\left(\frac{\pi}{2}\right) &= 2 \end{aligned}$$

(4)

$$\begin{aligned} 9y'' - 12y' + 4y &= 0 \\ y(0) &= 2 \\ y'(0) &= -1 \end{aligned}$$

(5)

$$
\begin{aligned}
y''' + y' &= 0 \\
y(0) &= 0 \\
y'(0) &= 1 \\
y''(0) &= 2
\end{aligned}
$$

Solution 2.

(1) Let

$$
\begin{aligned}
y &= e^{\lambda t} \\
\therefore y' &= \lambda e^{\lambda t} \\
\therefore y'' &= \lambda^2 e^{\lambda t}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lambda^2 e^{\lambda t} + \lambda e^{\lambda t} - 2e^{\lambda t} &= 0 \\
\therefore \lambda^2 + \lambda - 2 &= 0
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lambda &= \frac{-1 \pm \sqrt{1+8}}{2} \\
&= \frac{-1 \pm 3}{2}
\end{aligned}
$$

Therefore,

$$\lambda = -2 \qquad \text{or} \qquad \lambda = 1$$

Therefore,

$$y = e^{-2t} \qquad \text{or} \qquad y = e^{t}$$

Therefore,

$$
\begin{aligned}
y &= c_1 e^{-2t} + c_2 e^{t} \\
\therefore y' &= -2c_1 e^{-2t} + c_2 e^{t}
\end{aligned}
$$

Therefore, substituting $y(0) = 1$ and $y'(0) = 1$,

$$
\begin{aligned}
1 &= c_1 + c_2 \\
1 &= -2c_1 + c_2
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= 1
\end{aligned}
$$

Therefore,

$$y = e^{t}$$

(2) Let

$$y = e^{\lambda t}$$
$$\therefore y' = \lambda e^{\lambda t}$$
$$\therefore y'' = \lambda^2 e^{\lambda t}$$

Therefore,

$$\lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} = 0$$
$$\therefore \lambda^2 + 4\lambda = 0$$
$$\therefore \lambda(\lambda + 4) = 0$$

Therefore,

$$\lambda = -4 \quad \text{or} \quad \lambda = 0$$

Therefore,

$$y = e^{-4t} \quad \text{or} \quad y = e^{0t}$$

Therefore,

$$y = e^{-4t} \quad \text{or} \quad y = 1$$

Therefore,

$$y = c_1 e^{-4t} + c_2$$
$$\therefore y' = -4c_1 e^{-4t}$$

Therefore, substituting $y(0) = 1$ and $y'(0) = 1$,

$$1 = c_1 + c_2$$
$$1 = -4c_1$$

Therefore,

$$c_1 = -\frac{1}{4}$$
$$c_2 = \frac{1}{4}$$

Therefore,

$$y = -\frac{1}{4}e^{-4t} + \frac{1}{4}$$

(3) Let

$$y = e^{\lambda t}$$
$$\therefore y' = \lambda e^{\lambda t}$$
$$\therefore y'' = \lambda^2 e^{\lambda t}$$

Therefore,

$$\lambda^2 e^{\lambda t} - 2\lambda e^{\lambda t} + 5e^{\lambda t} = 0$$
$$\therefore \lambda^2 - 2\lambda + 5 = 0$$

Therefore,

$$\begin{aligned}\lambda &= \frac{2 \pm \sqrt{4-20}}{2}\\ &= \frac{2 \pm 4i}{2}\\ &= 1 \pm 2i\end{aligned}$$

Therefore,

$$\lambda = 1 + 2i \qquad \text{or} \qquad \lambda = 1 - 2i$$

Therefore,

$$y = e^{(1+2i)t} \qquad \text{or} \qquad y = e^{(1-2i)t}$$

Therefore,

$$\begin{aligned} y &= c_1 e^{(1+2i)t} + c_2 e^{(1-2i)t}\\ \therefore y' &= (1+2i)c_1 e^{(1+2i)t} + (1-2i)c_2 e^{(1-2i)t}\end{aligned}$$

(4) Let

$$\begin{aligned} y &= e^{\lambda t}\\ \therefore y' &= \lambda e^{\lambda t}\\ \therefore y'' &= \lambda^2 e^{\lambda t}\end{aligned}$$

Therefore,

$$\begin{aligned} 9\lambda^2 e^{\lambda t} - 12\lambda e^{\lambda t} + 4e^{\lambda t} &= 0\\ \therefore 9\lambda^2 - 12\lambda + 4 &= 0\end{aligned}$$

Therefore,

$$\begin{aligned}\lambda &= \frac{12 \pm \sqrt{144-144}}{18}\\ &= \frac{2}{3}\end{aligned}$$

Therefore,

$$y = e^{\frac{2}{3}t}$$

Therefore,

$$\begin{aligned} y &= (c_1 + c_2)e^{\frac{2}{3}t}\\ \therefore y' &= \frac{2}{3}(c_1 + c_2)e^{\frac{2}{3}t}\end{aligned}$$

Therefore, substituting $y(0) = 2$ and $y'(0) = -1$,

$$\begin{aligned} 2 &= c_1 + c_2\\ -1 &= \frac{2}{3}(c_1 + c_2)\end{aligned}$$

Therefore, the system is inconsistent.
Hence a solution does not exist.

(5) Let

$$\begin{aligned} y &= e^{\lambda t} \\ \therefore y' &= \lambda e^{\lambda t} \\ \therefore y'' &= \lambda^2 e^{\lambda t} \\ \therefore y''' &= \lambda^3 e^{\lambda t} \end{aligned}$$

Therefore,

$$\begin{aligned} y''' + y' &= 0 \\ \lambda^3 e^{\lambda t} + \lambda e^{\lambda t} &= 0 \\ \therefore \lambda^3 + \lambda &= 0 \\ \therefore \lambda(\lambda^2 + 1) &= 0 \end{aligned}$$

Therefore,

$$\lambda = 0 \qquad \text{or} \qquad \lambda = i \qquad \text{or} \qquad \lambda = -i$$

Therefore,

$$y = e^{0t} \qquad \text{or} \qquad y = e^{it} \qquad \text{or} \qquad y = e^{-it}$$

Therefore,

$$y = 1 \qquad \text{or} \qquad y = e^{it} \qquad \text{or} \qquad y = e^{-it}$$

Therefore,

$$\begin{aligned} y &= c_1 + c_2 e^{it} + c_3 e^{-it} \\ \therefore y' &= ic_2 e^{it} - ic_3 e^{-it} \\ \therefore y'' &= -c_2 e^{it} + c_3 e^{-it} \end{aligned}$$

Therefore, substituting $y(0) = 0$, $y'(0) = 1$, $y''(0) = 2$,

$$\begin{aligned} 0 &= c_1 + c_2 + c_3 \\ 1 &= c_2 - c_3 \\ 2 &= -c_2 + c_3 \end{aligned}$$

Therefore, the system is inconsistent.
Hence a solution does not exist.

Exercise 3.
Consider the ODE $y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$. Determine the values of α, if any, such that all solutions tend to zero as $t \to \infty$. Also determine the values of α, if any, such that all (nonzero) solutions become unbounded as $t \to \infty$.

Solution 3.
Let

$$\begin{aligned} y &= e^{\lambda t} \\ \therefore y' &= \lambda e^{\lambda t} \\ \therefore y'' &= \lambda^2 e^{\lambda t} \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda^2 e^{\lambda t} - (2\alpha - 1)\lambda e^{\lambda t} + \alpha(\alpha - 1)e^{\lambda t} &= 0 \\ \therefore \lambda^2 - (2\alpha - 1)\lambda + \alpha(\alpha - 1) &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda &= \frac{2\alpha - 1 \pm \sqrt{(2\alpha - 1)^2 - 4\alpha(\alpha - 1)}}{2} \\ &= \frac{2\alpha - 1 \pm \sqrt{4\alpha^2 - 4\alpha + 1 - 4\alpha^2 + 4\alpha}}{2} \\ &= \frac{2\alpha - 1 \pm \sqrt{1}}{2} \\ &= \frac{2\alpha - 1 \pm 1}{2} \end{aligned}$$

Therefore,

$$\lambda = \alpha - 1 \qquad \text{or} \qquad \lambda = \alpha$$

Therefore,

$$y = e^{(\alpha-1)t} \qquad \text{or} \qquad y = e^{\alpha t}$$

$$\lim_{n \to \infty} e^{-n} = 0$$

Therefore, if $y > 1$, $\lim\limits_{t \to \infty} y = 0$

Part 2. **Reduction of Order**

Exercise 1.
Use the method of reduction of order to find a second solution for the given differential equations.

(1)

$$\begin{aligned} t^2 y'' - 4ty' + 6y &= 0 \\ t &> 0 \\ y_1(t) &= t^2 \end{aligned}$$

(2)

$$\begin{aligned} xy'' - y' + 4x^3 y &= 0 \\ x &> 0 \\ y_1(x) &= \sin(x^2) \end{aligned}$$

Solution 1.

(1) Let

$$\begin{aligned} y_2(t) &= \nu(t)y_1(t) \\ &= t^2\nu(t) \end{aligned}$$

Therefore,

$$\begin{aligned} {y_2}'(t) &= 2t\nu(t) + t^2\nu'(t) \\ \therefore {y_2}''(t) &= 2\nu(t) + 2t\nu'(t) + 2t\nu'(t) + t^2\nu''(t) \\ &= 2\nu(t) + 4t\nu'(t) + t^2\nu''(t) \end{aligned}$$

Therefore, substituting ${y_2}'$ and ${y_2}''$,

$$\begin{aligned} t^2\left(2\nu(t) + 4t\nu'(t) + t^2\nu''(t)\right) - 4t\left(2t\nu(t) + t^2\nu'(t)\right) + 6\left(t^2\nu(t)\right) &= 0 \\ \therefore 2t^2\nu(t) + 4t^3\nu'(t) + t^4\nu''(t) - 8t^2\nu(t) - 4t^3\nu'(t) + 6t^2\nu(t) &= 0 \\ \therefore t^4\nu''(t) &= 0 \\ \therefore \nu''(t) &= 0 \end{aligned}$$

Therefore, let

$$\nu(t) = k_1 t + k_2$$

Therefore,

$$\nu'(t) = k_1$$

Therefore, let $k_1 = 1$, $k_2 = 1$.
Therefore,

$$\begin{aligned} y_2(t) &= t^2 t \\ &= t^3 \end{aligned}$$

(2) Let

$$\begin{aligned} y_2(x) &= \nu(x)y_1(x) \\ &= \sin(x^2)\nu(x) \end{aligned}$$

Therefore,

$$\begin{aligned} {y_2}'(x) &= \sin(x^2)\nu'(x) + 2x\nu(x)\cos(x^2) \\ {y_2}''(x) &= \sin(x^2)\nu''(x) + 4x\cos(x^2)\nu'(x) - 4x^2\nu(x)\sin(x^2) + 2\nu(x)\cos(x^2) \end{aligned}$$

Therefore, substituting ${y_2}'$ and ${y_2}''$ and simplifying,

$$\therefore x\sin(x^2)\nu''(x) + \nu'(x)\left(4x^2\cos(x^2) - \sin(x^2)\right) + 4x\nu(x)\cos(x^2) = 0$$

Therefore, the characteristic equation is

$$x\sin(x^2)\lambda^2 + \left(4x^2\cos(x^2) - \sin(x^2)\right)\lambda + 4x\cos(x^2) = 0$$

Therefore,

$$\begin{aligned}\lambda &= \frac{\sin(x^2) - 4x^2\cos(x^2) \pm \sqrt{\sin^2(x^2) - 8x^2\sin(x^2)\cos(x^2) - 16x^2\sin(x^2)\cos(x^2)}}{2x\sin(x^2)} \\ &= \frac{\sin(x^2) - 4x^2\cos(x^2) \pm \sqrt{\sin^2(x^2) - 24x^2\sin(x^2)\cos(x^2)}}{2x\sin(x^2)}\end{aligned}$$

Therefore,

$$\nu(x) = e^{\lambda x}$$

Therefore,

$$\begin{aligned}y_2(x) &= \sin(x^2)\nu(x) \\ &= \sin(x^2)e^{\frac{\sin(x^2)-4x^2\cos(x^2)\pm\sqrt{\sin^2(x^2)-24x^2\sin(x^2)\cos(x^2)}}{2x\sin(x^2)}}\end{aligned}$$