PARTIAL DIFFERENTIAL EQUATIONS
ASSIGNMENT 3

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Exercise 1.

(1) Using the d'Alembert formula, calculate $u\left(\frac{3}{4}, \frac{1}{2}\right)$ for

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(x,0) &= x(x-1) \\ u_t(x,0) &= 0 \\ u(0,t) &= 0 \\ u(1,t) &= 0 \end{aligned}$$

where $0 \le x \le 1$, and $t \ge 0$.

(2) Using the d'Alembert formula, calculate $u\left(\frac{5}{8}, \frac{9}{8}\right)$ for

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(x,0) &= \sin(\pi x) \\ u_t(x,0) &= x\left(1-x^2\right) \\ u(0,t) &= 0 \\ u(1,t) &= 0 \end{aligned}$$

where $0 \le x \le 1$, and $t \ge 0$.

Solution 1.

(1) Comparing to the standard form,

$$\begin{aligned} a &= 1 \\ l &= 1 \\ f(x) &= x(x-1) \\ g(x) &= 0 \end{aligned}$$

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As the boundary is fixed, let $\widetilde{f}(x)$ and $\widetilde{g}(x)$ be the odd $2l$ periodic extensions of $f(x)$ and $g(x)$. Therefore,

$$\begin{aligned}
\widetilde{u}(x,t) &= \frac{\widetilde{f}(x-at)+\widetilde{f}(x+at)}{2} + \frac{1}{2a}\int\limits_{x-at}^{x+at} \widetilde{g}(s)\,\mathrm{d}s \\
&= \frac{\widetilde{f}(x-t)+\widetilde{f}(x+t)}{2} + \frac{1}{2}\int\limits_{x-t}^{x+t} 0\,\mathrm{d}t \\
&= \frac{\widetilde{f}(x-t)+\widetilde{f}(x+t)}{2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\widetilde{u}\left(\frac{3}{4},\frac{1}{2}\right) &= \frac{\widetilde{f}\left(\frac{3}{4}-\frac{1}{2}\right)+\widetilde{f}\left(\frac{3}{4}+\frac{1}{2}\right)}{2} \\
&= \frac{\widetilde{f}\left(\frac{1}{4}\right)+\widetilde{f}\left(\frac{5}{4}\right)}{2} \\
&= \frac{\widetilde{f}\left(\frac{1}{4}\right)+\widetilde{f}\left(\frac{5}{4}-2\right)}{2} \\
&= \frac{\widetilde{f}\left(\frac{1}{4}\right)+\widetilde{f}\left(-\frac{3}{4}\right)}{2} \\
&= \frac{f\left(\frac{1}{4}\right)-f\left(\frac{3}{4}\right)}{2} \\
&= \frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)-\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)}{2} \\
&= 0
\end{aligned}$$

(2) Comparing to the standard form,

$$\begin{aligned}
a &= 1 \\
l &= 1 \\
f(x) &= \sin(\pi x) \\
g(x) &= x\left(1-x^2\right)
\end{aligned}$$

As the boundary is fixed, let $\widetilde{f}(x)$ and $\widetilde{g}(x)$ be the odd $2l$ periodic extensions of $f(x)$ and $g(x)$. Therefore,

$$\begin{aligned}
\widetilde{u}(x,t) &= \frac{\widetilde{f}(x-at)+\widetilde{f}(x+at)}{2} + \frac{1}{2a}\int\limits_{x-at}^{x+at} \widetilde{g}(s)\,\mathrm{d}s \\
&= \frac{\widetilde{f}(x-t)+\widetilde{f}(x+t)}{2} + \frac{1}{2}\int\limits_{x-t}^{x+t} \widetilde{g}(s)\,\mathrm{d}s
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\widetilde{u}\left(\frac{5}{8},\frac{9}{8}\right) &= \frac{\widetilde{f}\left(-\frac{4}{8}\right)+\widetilde{f}\left(\frac{14}{8}\right)}{2} + \frac{1}{2}\int\limits_{-\frac{4}{8}}^{\frac{14}{8}} \widetilde{g}(s)\,\mathrm{d}s \\
&= \frac{\widetilde{f}\left(-\frac{4}{8}\right)+\widetilde{f}\left(-\frac{2}{8}\right)}{2} + \frac{1}{2}\left(\int\limits_{-\frac{4}{8}}^{\frac{12}{8}} \widetilde{g}(s)\,\mathrm{d}s + \int\limits_{\frac{12}{8}}^{\frac{14}{8}} \widetilde{g}(s)\,\mathrm{d}s\right) \\
&= \frac{-f\left(\frac{4}{8}\right)-f\left(\frac{2}{8}\right)}{2} + \frac{1}{2}\left(0 + \int\limits_{-\frac{4}{8}}^{-\frac{2}{8}} \widetilde{g}(s)\,\mathrm{d}s\right) \\
&= \frac{-f\left(\frac{4}{8}\right)-f\left(\frac{2}{8}\right)}{2} + \frac{1}{2}\left(-\int\limits_{\frac{4}{8}}^{\frac{2}{8}} g(s)\,\mathrm{d}s\right) \\
&= \frac{-\sin\left(\frac{4}{8}\pi\right)-\sin\left(\frac{2}{8}\pi\right)}{2} - \frac{1}{2}\int\limits_{\frac{4}{8}}^{\frac{2}{8}} s\left(1-s^2\right)\mathrm{d}s \\
&= \frac{-1-\frac{1}{\sqrt{2}}}{2} - \frac{1}{2}\left(\frac{s^2}{2}-\frac{s^4}{4}\Bigg|_{\frac{4}{8}}^{\frac{2}{8}}\right) \\
&= -\frac{1}{2}-\frac{1}{2\sqrt{2}}-\frac{1}{2}\left(\frac{\frac{1}{16}}{2}-\frac{\frac{1}{256}}{2}-\frac{\frac{1}{4}}{2}+\frac{\frac{1}{16}}{2}\right) \\
&= -\frac{1}{2}-\frac{1}{2\sqrt{2}}-\frac{1}{2}\left(\frac{1}{32}-\frac{1}{512}-\frac{1}{8}+\frac{1}{32}\right) \\
&= -\frac{1}{2}-\frac{1}{2\sqrt{2}}-\frac{1}{32}+\frac{1}{1024}-\frac{1}{16}
\end{aligned}
$$

Exercise 2.
Solve using the separation of variables method.

$$
\begin{aligned}
u_{tt} &= u_{xx} \\
u(x,0) &= 0 \\
u_t(x,0) &= v_0 \\
u(0,t) &= 0 \\
u(l,t) &= 0
\end{aligned}
$$

where $0 \le x \le l$, and $t \ge 0$.

Exercise 3.
Solve using the separation of variables method.

$$
\begin{aligned}
u_{tt} &= a^2 u_{xx} \\
u(x,0) &= 0 \\
u_t(x,0) &= \sin\left(\frac{\pi}{l}x\right) \\
u(0,t) &= 0 \\
u_x(l,t) &= 0
\end{aligned}
$$

where $0 \leq x \leq l$, and $t \geq 0$.

Exercise 4.
Solve using the separation of variables method.

$$
\begin{aligned}
u_{tt} &= a^2 u_{xx} \\
u(x,0) &= \sin\left(\frac{\pi}{l}x\right) \\
u_t(x,0) &= \frac{x(l-x)}{l^2} \\
u_x(0,t) &= 0 \\
u(l,t) &= 0
\end{aligned}
$$

where $0 \leq x \leq l$, and $t \geq 0$.