PARTIAL DIFFERENTIAL EQUATIONS
ASSIGNMENT 4

AAKASH JOG
ID : 989323563

Exercise 1.
Using the energy method, prove uniqueness of the solution to the Dirichlet problem of the heat equation

$$\begin{aligned} u_t - ku_{xx} &= F(x,t) \\ u(0,t) &= a(t) \\ u(L,t) &= b(t) \\ u(x,0) &= f(x) \end{aligned}$$

where $0 \leq x \leq L$, $t \geq 0$, and k is a positive constant.
Hint: Show that the energy

$$E(t) = \frac{1}{2} \int\limits_0^L w^2 \, \mathrm{d}x$$

satisfies

$$E'(t) \leq 0$$

where w is a solution of the homogeneous problem.

Solution 1.
If possible, let u_1 and u_2 be two distinct solutions of the problem.
Let

$$v(x,t) = u_1(x,t) - u_2(x,t)$$

Therefore,

$$\begin{aligned} v_t - kv_{xx} &= 0 \\ v(0,t) &= 0 \\ v(L,t) &= 0 \\ v(x,0) &= 0 \end{aligned}$$

Date: Thursday 31^{st} March, 2016.

Therefore,

$$
\begin{aligned}
E(t) &= \frac{1}{2}\int_0^L v^2 \,\mathrm{d}x \\
\therefore E'(t) &= \frac{1}{2}\int_0^L 2vv_t \,\mathrm{d}x \\
&= \int_0^L kvv_{xx} \,\mathrm{d}x \\
&= k\int_0^L vv_{xx} \,\mathrm{d}x \\
&= kv\int v_{xx} \,\mathrm{d}x - k\int v_x \int v_{xx} \,\mathrm{d}x \Big|_0^L \\
&= kvv_x|_0^L - \int_0^L k{v_x}^2 \,\mathrm{d}x \\
&= -\int_0^L {v_x}^2 \,\mathrm{d}x \\
&\leq 0
\end{aligned}
$$

Also,

$$
\begin{aligned}
E(t) &= \frac{1}{2}\int_0^L v^2 \,\mathrm{d}x \\
&\geq 0
\end{aligned}
$$

Therefore,

$$E(t) \equiv 0$$

Therefore,

$$
\begin{aligned}
\frac{1}{2}\int_0^L v^2 \,\mathrm{d}x &\equiv 0 \\
\therefore v &\equiv 0
\end{aligned}
$$

Therefore,

$$u_1 \equiv u_2$$

This contradicts the assumption that u_1 and u_2 are distinct. Therefore, the problem has a unique solution.

Exercise 2.
Using the energy method, prove uniqueness of the solution to the following problem of the string equation

$$\begin{aligned} u_{tt} - u_{xx} &= xt \\ u_x(0,t) &= g(t) \\ u(1,t) &= h(t) \\ u(x,0) &= x^2 - 1 \\ u_t(x,0) &= x^{2016} - 1 \end{aligned}$$

where $0 \leq x \leq 1$, $t \geq 0$.

Solution 2.
If possible, let u_1 and u_2 be two distinct solutions of the problem.
Let

$$v(x,t) = u_1(x,t) - u_2(x,t)$$

Therefore,

$$\begin{aligned} v_{tt} - v_{xx} &= 0 \\ v_x(0,t) &= 0 \\ v(1,t) &= 0 \\ v(x,0) &= 0 \\ v_t(x,0) &= 0 \end{aligned}$$

Therefore, comparing to the standard form,

$$\begin{aligned} \rho(t) &= 1 \\ k(x) &= 1 \end{aligned}$$

Therefore,

$$\begin{aligned} E(t) &= \frac{1}{2}\int\limits_0^l \left(k{v_x}^2 + \rho{v_t}^2\right) \mathrm{d}x \\ &= \frac{1}{2}\int\limits_0^1 {v_x}^2 + {v_t}^2 \,\mathrm{d}x \\ \therefore E'(t) &= \frac{1}{2}\int\limits_0^1 2v_x v_{xt} + 2v_t v_{tt} \,\mathrm{d}x \\ &= \int\limits_0^1 v_x v_{xt} \,\mathrm{d}t + \int\limits_0^1 v_t v_{tt} \,\mathrm{d}x \end{aligned}$$

Assuming the mixed derivatives exist and are continuous,

$$v_{xt} = v_{tx}$$

Therefore,

$$\begin{aligned}
E'(t) &= \int_0^1 v_x v_{tx}\,\mathrm{d}x + \int_0^1 v_t v_{tt}\,\mathrm{d}x \\
&= \int_0^1 v_x v_{tt}\,\mathrm{d}x + v_x v_t\Big|_0^1 - \int_0^1 v_t v_{xx}\,\mathrm{d}x \\
&= v_x v_t\Big|_0^1 \\
&\equiv 0
\end{aligned}$$

Therefore,

$$E(t) \equiv c$$

Therefore, as $v(x,0) = 0$,

$$v_x(x,0) = 0$$

Also,

$$v_t(x,0) = 0$$

Therefore,

$$E(t) \equiv 0$$

Therefore,

$$\begin{aligned}
v_x &\equiv v_t \\
&\equiv 0
\end{aligned}$$

Therefore,

$$u_1 \equiv u_2$$

This contradicts the assumption that u_1 and u_2 are distinct. Therefore, the problem has a unique solution.

Exercise 3.
Using the energy method, prove uniqueness of the solution to the Neumann problem of the string equation

$$\begin{aligned}
u_{tt} - 4u_{xx} &= xt \\
u(x,0) &= \cos^2(\pi x) \\
u_t(x,0) &= \sin^2(\pi x)\cos(\pi x) \\
u_x(0,t) &= 0 \\
u_x(1,t) &= 0
\end{aligned}$$

where $0 \leq x \leq 1$, $t \geq 0$.

Solution 3.
If possible, let u_1 and u_2 be two distinct solutions of the problem.
Let

$$v(x,t) = u_1(x,t) - u_2(x,t)$$

Therefore,

$$\begin{aligned} v_{tt} - 4v_{xx} &= 0 \\ v(x,0) &= 0 \\ v_t(x,0) &= 0 \\ v_x(0,t) &= 0 \\ v_x(1,t) &= 0 \end{aligned}$$

Therefore, comparing to the standard form,

$$\begin{aligned} \rho(t) &= 1 \\ k(t) &= 4 \end{aligned}$$

Therefore,

$$\begin{aligned} E(t) &= \frac{1}{2}\int\limits_0^l \left(k{v_x}^2 + \rho{v_t}^2\right)\mathrm{d}x \\ &= \frac{1}{2}\int\limits_0^1 4{v_x}^2 + {v_t}^2\,\mathrm{d}x \\ \therefore E'(t) &= \frac{1}{2}\int\limits_0^1 8v_xv_{xt} + 2v_tv_{tt}\,\mathrm{d}x \\ &= \int\limits_0^1 4v_xv_{xt}\,\mathrm{d}x + \int\limits_0^1 v_tv_{tt}\,\mathrm{d}x \end{aligned}$$

Assuming the mixed derivatives exist and are continuous,

$$v_{xt} = v_{tx}$$

Therefore,

$$\begin{aligned} E'(t) &= \int\limits_0^1 4v_xv_{tx}\,\mathrm{d}x + \int\limits_0^1 v_tv_{tt}\,\mathrm{d}x \\ &= \int\limits_0^1 v_tv_{tt}\,\mathrm{d}x + 4v_xv_t\big|_0^1 - \int\limits_0^1 4v_tv_{xx}\,\mathrm{d}x \\ &= 4v_xv_t\big|_0^1 \\ &\equiv 0 \end{aligned}$$

Therefore,

$$E(t) \equiv c$$

Therefore, as $v(x,0) = 0$,

$$v_x(x,0) = 0$$

Also,

$$v_t(x,0) = 0$$

Therefore,

$$E(t) \equiv 0$$

Therefore,

$$\begin{aligned} v_x &\equiv v_t \\ &\equiv 0 \end{aligned}$$

Therefore,

$$u_1 \equiv u_2$$

This contradicts the assumption that u_1 and u_2 are distinct. Therefore, the problem has a unique solution.