PARTIAL DIFFERENTIAL EQUATIONS
ASSIGNMENT 5

AAKASH JOG
ID : 989323563

Exercise 1.
Bring to a canonical form and solve.

$$u_{xx} + 2u_{xy} + \cos^2 x u_{yy} - \cot x(u_x + u_y) = 0$$

Solution 1.

$$u_{xx} + 2u_{xy} + \cos^2 x u_{yy} - \cot x(u_x + u_y) = 0$$

Therefore, the characteristic equation is

$$(y')^2 + 2y' + \cos^2 x = 0$$

Therefore,

$$\begin{aligned} y' &= \frac{1 \pm \sqrt{1 - \cos^2 x}}{1} \\ &= 1 \pm \sin x \end{aligned}$$

Therefore,

$$y = x \pm \cos x + c$$

Therefore, let

$$\begin{aligned} \xi &= y - x + \cos x \\ \eta &= y - x - \cos x \end{aligned}$$

Therefore,

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x \\ &= u_\xi \left(-1 - \sin x\right) + u_\eta \left(-1 + \sin x\right) \\ u_y &= u_\xi \xi_y + u_\eta \eta_y \\ &= u_\xi + u_\eta \\ u_{xx} &= u_{\xi\xi} \left(1 - 2\sin x + \sin^2 x\right) + 2u_{\xi\eta} \cos^2 x + u_{\eta\eta} \left(1 - 2\sin x + \sin^2 x\right) + (u_\eta - u_\xi) \cos x \\ u_{xy} &= -u_{\xi\xi}(1 + \sin x) - 2u_{\xi\eta} - u_{\eta\eta}(1 - \sin x) \\ u_{yy} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \end{aligned}$$

Therefore, substituting,

$$-4\left(\sin^2 x\right) u_{\xi\eta} = 0$$

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Therefore, a canonical form is

$$u_{\xi\eta} = 0$$

Therefore, the general solution is

$$u(\xi, \eta) = F(\xi) + G(\eta)$$
$$\therefore u(x, y) = F(y - x + \cos x) + G(y - x - \cos x)$$

Exercise 2.

(1) Bring to a canonical form and solve.

$$u_{xx} + 6u_{xy} - 16u_{yy} = 0$$

(2) Find a solution to the above which satisfies

$$u(x, 8x) = \cos x$$
$$u(x, -2x) = e^x$$

Solution 2.

(1)

$$u_{xx} + 6u_{xy} - 16u_{yy} = 0$$

Therefore, the characteristic equation is

$$(y')^2 + 6y' - 16 = 0$$

Therefore,

$$y' = \frac{3 \pm \sqrt{9 + 16}}{1}$$
$$= 3 \pm 5$$

Therefore,

$$y_1 = 8x + c$$
$$y_2 = -2x + c$$

Therefore, let

$$\xi = y - 8x$$
$$\eta = y + 2x$$

Therefore,

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$
$$= 2u_\xi - 8u_\eta$$
$$u_y = u_\xi \xi_y + u_\eta \eta_y$$
$$= u_\xi + u_\eta$$
$$u_{xx} = 4u_{\xi\xi} - 32u_{\xi\eta} + 64u_{\eta\eta}$$
$$u_{xy} = 2u_{\xi\xi} - 6u_{\xi\eta} - 8u_{\eta\eta}$$
$$u_{yy} = 2_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

Therefore, substituting,

$$-100u_{\xi\eta} = 0$$

Therefore, a canonical form is

$$u_{\xi\eta} = 0$$

Therefore, the general solution is

$$u(\xi, \eta) = F(\xi) + G(\eta)$$

$$\therefore u(x, y) = F(y - 8x) + G(y + 2x)$$