PARTIAL DIFFERENTIAL EQUATIONS
ASSIGNMENT 1

AAKASH JOG
ID : 989323563

Exercise 1.
Solve

$$
\begin{aligned}
u_{tt} &= u_{xx} \\
u(x,0) &= \frac{x}{1+x^2} \\
u_t(x,0) &= \sin x
\end{aligned}
$$

where $-\infty < x < \infty$, and $t > 0$.

Solution 1.
Comparing to the standard form,

$$
\begin{aligned}
a &= 1 \\
f(x) &= \frac{x}{1+x^2} \\
g(x) &= \sin x
\end{aligned}
$$

Therefore, by the d'Alembert formula,

$$
\begin{aligned}
u(x,t) &= \frac{f(x-at)+f(x+at)}{2} + \frac{1}{2a} \int\limits_{x-at}^{x+at} g(s) \,\mathrm{d}s \\
&= \frac{f(x-t)+f(x+t)}{2} + \frac{1}{2} \int\limits_{x-t}^{x+t} g(s) \,\mathrm{d}s \\
&= \frac{\frac{x-t}{1+(x-t)^2} + \frac{x+t}{1+(x+t)^2}}{2} + \frac{1}{2} \int\limits_{x-t}^{x+t} \sin s \,\mathrm{d}s \\
&= \frac{\frac{x-t}{1+(x-t)^2} + \frac{x+t}{1+(x+t)^2}}{2} + \frac{1}{2} \left(-\cos s \,\mathrm{d}s\right)\Big|_{x-t}^{x+t} \\
&= \frac{\frac{x-t}{1+(x-t)^2} + \frac{x+t}{1+(x+t)^2}}{2} + \frac{1}{2} \big(-\cos(x+t) + \cos(x-t)\big) \\
&= \frac{1}{2}\left(\frac{x-t}{1+(x-t)^2} + \frac{x+t}{1+(x+t)^2}\right) + \sin x \sin t
\end{aligned}
$$

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Exercise 2.
Solve the problem and sketch the solution for different time segments.

$$u_{tt} = 3u_{xx}$$
$$u(x,0) = \begin{cases} 1 & ; \quad 1 \leq x \leq 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$
$$u_t(x,0) = 0$$

Solution 2.
Comparing to the standard form,

$$a = 2$$
$$f(x) = \begin{cases} 1 & ; \quad 1 \leq x \leq 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$
$$g(x) = 0$$

Therefore, by the d'Alembert formula,

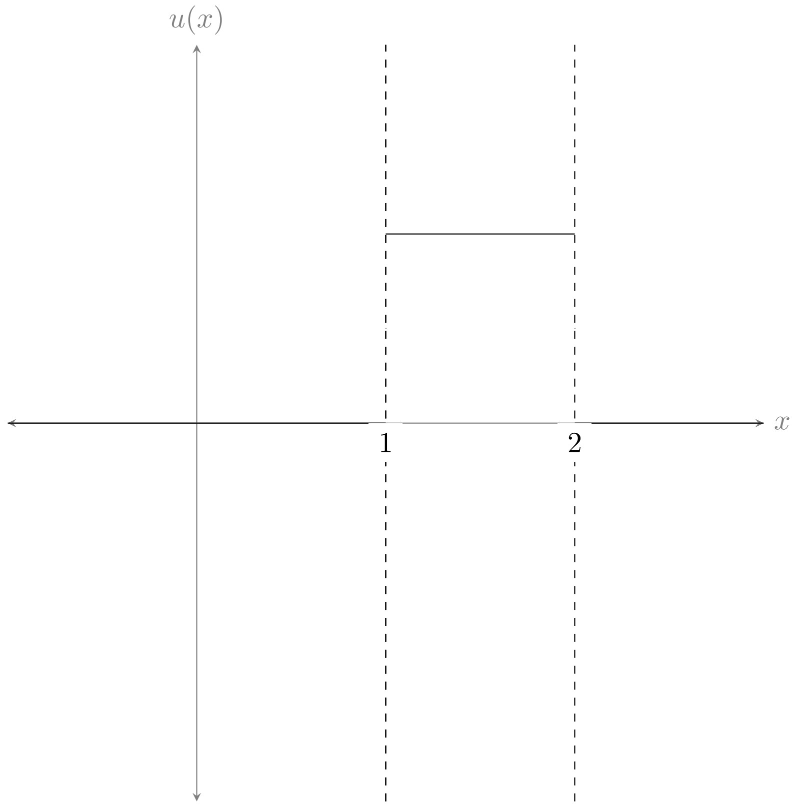
$$\begin{aligned} u(x,t) &= \frac{f(x-at) + f(x+at)}{2} + \frac{1}{2a} \int\limits_{x-at}^{x+at} g(s) \, \mathrm{d}s \\ &= \frac{f(x-2t) + f(x+2t)}{2} + \frac{1}{4} \int\limits_{x-2t}^{x+2t} 0 \, \mathrm{d}s \\ &= \frac{1}{2} \big(f(x-2t) + f(x+2t) \big) \end{aligned}$$

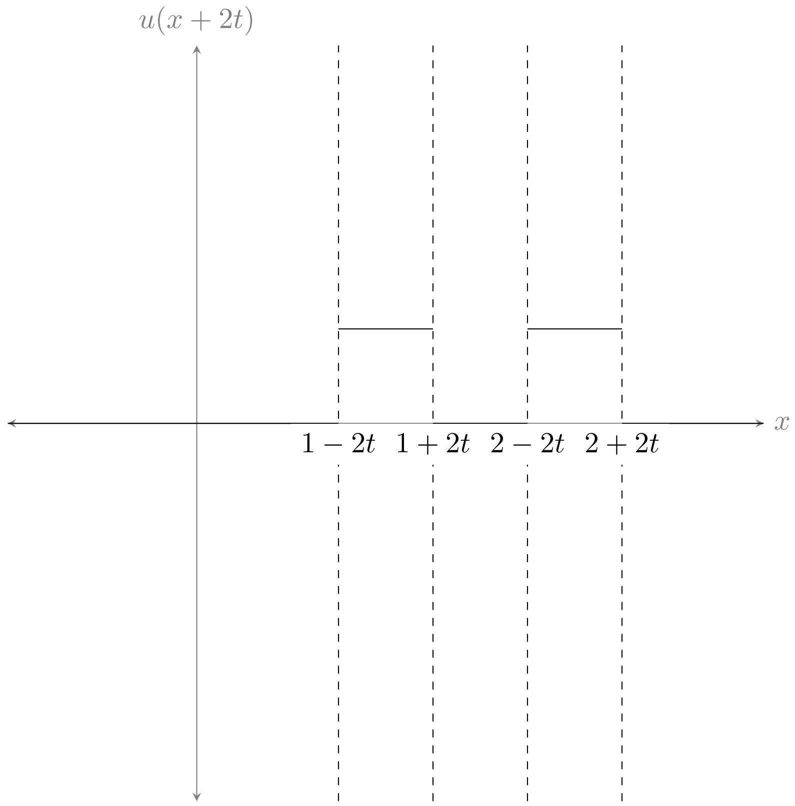
Therefore,

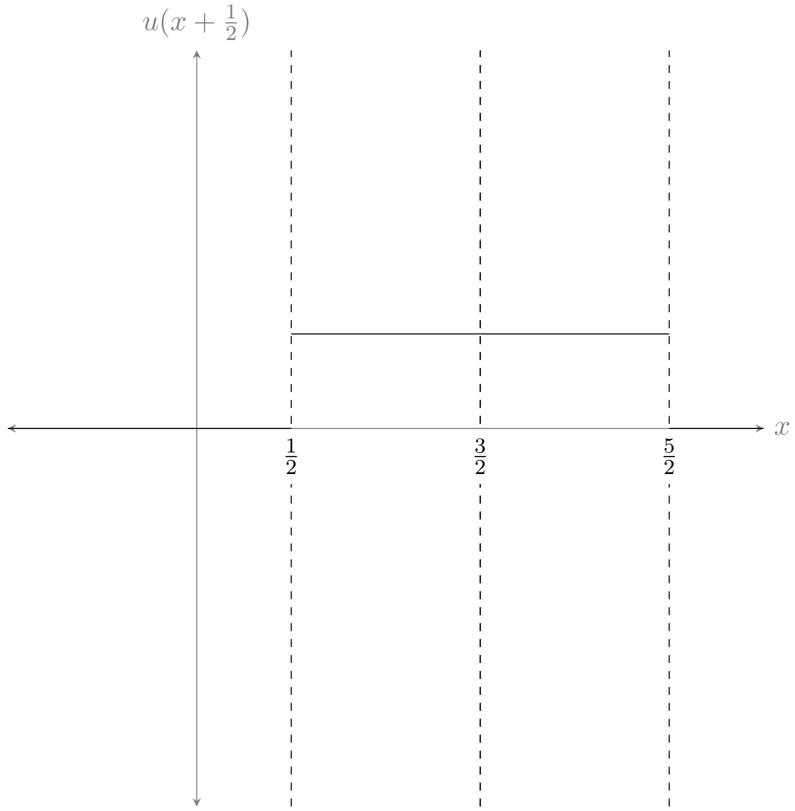
$$\begin{aligned} f(x-2t) &= \begin{cases} 1 & ; \quad 1 \leq x - 2t \leq 2 \\ 0 & ; \quad \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & ; \quad 1 + 2t \leq x \leq 2 + 2t \\ 0 & ; \quad \text{otherwise} \end{cases} \\ f(x+2t) &= \begin{cases} 1 & ; \quad 1 \leq x + 2t \leq 2 \\ 0 & ; \quad \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & ; \quad 1 - 2t \leq x \leq 2 - 2t \\ 0 & ; \quad \text{otherwise} \end{cases} \end{aligned}$$

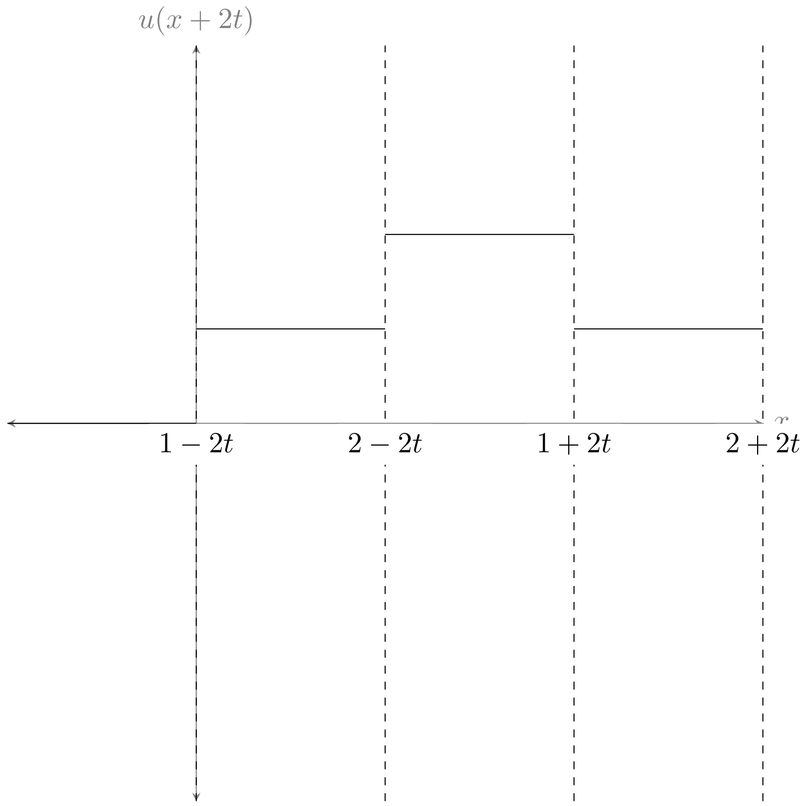
For $t = 0$,

$$\begin{aligned} u(x,t) &= u(x,0) \\ &= \frac{1}{2} \big(f(x) + f(x) \big) \\ &= f(x) \\ &= \begin{cases} 1 & ; \quad 1 \leq x \leq 2 \\ 0 & ; \quad \text{otherwise} \end{cases} \end{aligned}$$


$u(x)$
x
1
2


$u(x+2t)$
x
$1-2t$
$1+2t$
$2-2t$
$2+2t$


$u(x + \frac{1}{2})$
x
$\frac{1}{2}$
$\frac{3}{2}$
$\frac{5}{2}$


$u(x + 2t)$
$1 - 2t$
$2 - 2t$
$1 + 2t$
$2 + 2t$

For $0 < t < \frac{1}{4}$,

$$u(x,t) = \frac{1}{2}\big(f(x-2t) + f(x+2t)\big)$$
$$= \begin{cases} 0 & ; \quad x < 1-2t \\ \frac{1}{2} & ; \quad 1-2t < x < 1+2t \\ 1 & ; \quad 1+2t \le x \le 2-2t \\ \frac{1}{2} & ; \quad 2-2t < x < 2+2t \\ 0 & ; \quad 2+2t < x \end{cases}$$

For $t = \frac{1}{4}$,

$$u(x,t) = u\left(x, \frac{1}{4}\right)$$
$$= \begin{cases} 0 & ; \quad x < \frac{1}{2} \\ \frac{1}{2} & ; \quad \frac{1}{2} \le x < \frac{3}{2} \\ 1 & ; \quad x = \frac{3}{2} \\ \frac{1}{2} & ; \quad \frac{3}{2} < x \le \frac{5}{2} \\ 0 & ; \quad \frac{5}{2} < x \end{cases}$$

For $\frac{1}{4} < t$,

$$u(x,t) = \frac{1}{2}\big(f(x-2t) + f(x+2t)\big)$$
$$= \begin{cases} 0 & ; \quad x < 1-2t \\ \frac{1}{2} & ; \quad 1-2t \le x \le 1+2t \\ 0 & ; \quad 1+2t < x < 2-2t \\ \frac{1}{2} & ; \quad 2-2t \le x \le 2+2t \\ 0 & ; \quad 2+2t < x \end{cases}$$

Therefore, the graphs of the solution are

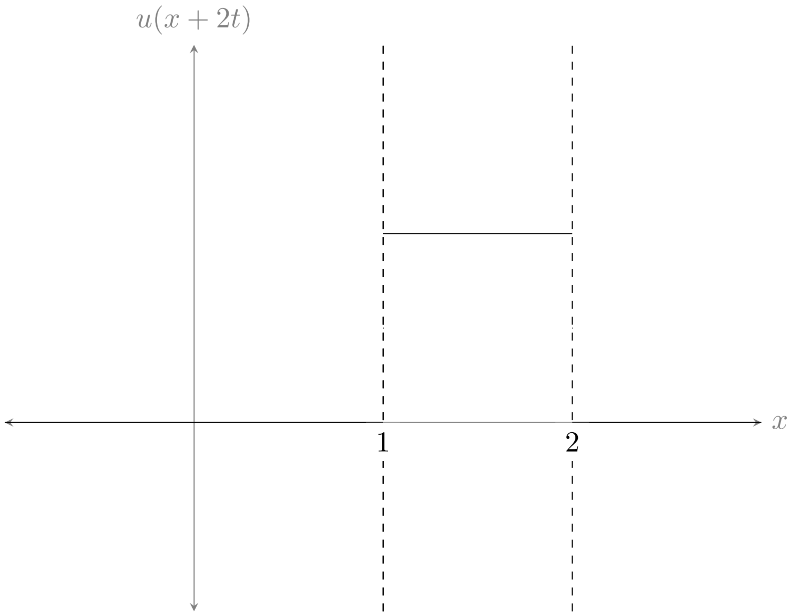


FIGURE 1. $t = 0$

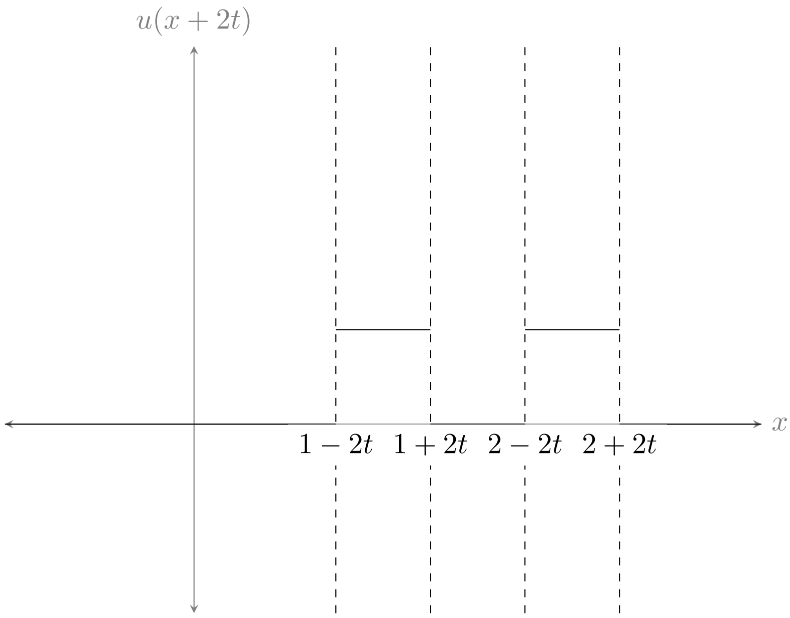


FIGURE 2. $0 < t < \frac{1}{4}$

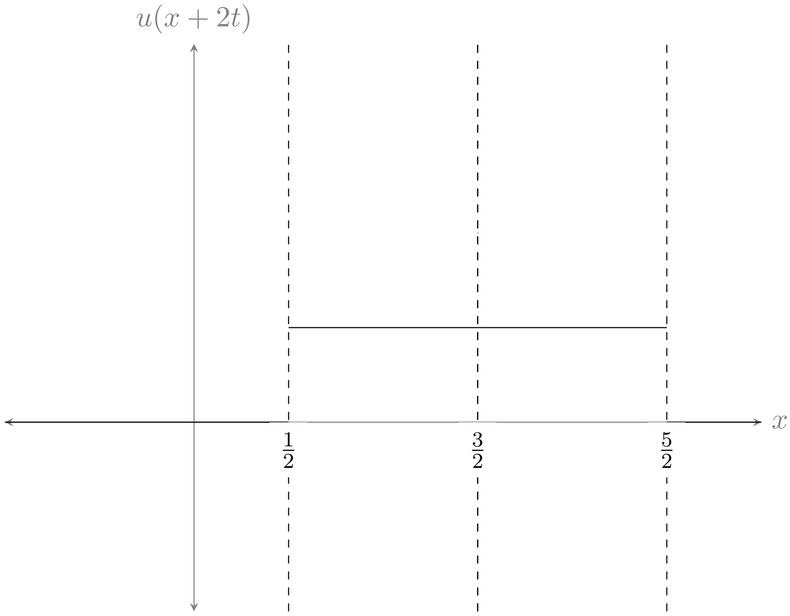


FIGURE 3. $t = \frac{1}{4}$

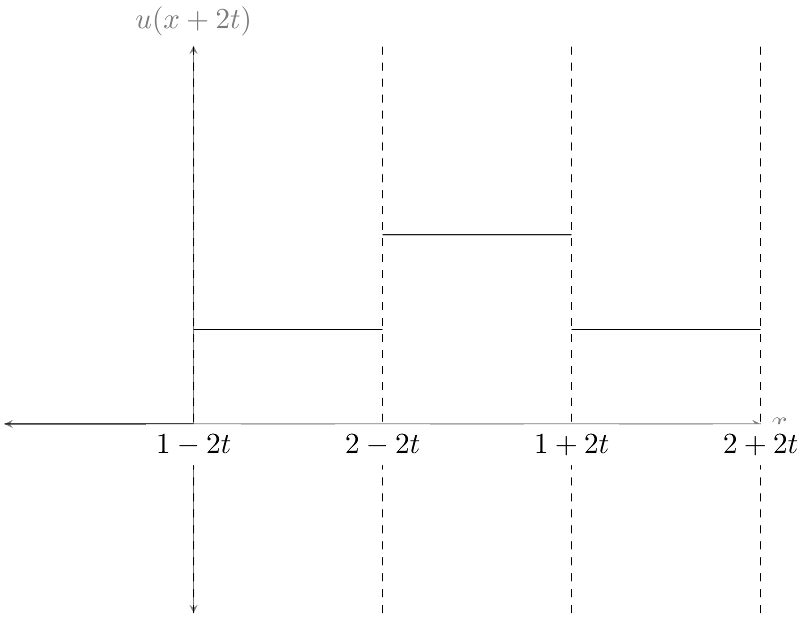


FIGURE 4. $\frac{1}{4} < t$

Exercise 3.
Solve

(1)

$$
\begin{aligned}
u_{tt} &= u_{xx} \\
u(x,0) &= \begin{cases} \sin x & ; \quad x \geq 0 \\ -\sin x & ; \quad x < 0 \end{cases} \\
u_t(x,0) &= -\cos x
\end{aligned}
$$

where $-\infty < x < \infty$, and $t > 0$.

(2)

$$
\begin{aligned}
u_{tt} &= u_{xx} \\
u(x,0) &= -\cos x \\
u_t(x,0) &= \begin{cases} \sin x & ; \quad x \geq 0 \\ -\sin x & ; \quad x < 0 \end{cases}
\end{aligned}
$$

where $-\infty < x < \infty$, and $t > 0$.

Solution 3.

(1) Comparing to the standard form,

$$
\begin{aligned}
a &= 1 \\
f(x) &= \begin{cases} \sin x & ; \quad x \geq 0 \\ -\sin x & ; \quad x < 0 \end{cases} \\
g(x) &= -\cos x
\end{aligned}
$$

Therefore, by the d'Alembert formula,

$$
\begin{aligned}
u(x,t) &= \frac{f(x-at)+f(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,\mathrm{d}s \\
&= \frac{f(x-t)+f(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} (-\cos s)\,\mathrm{d}s \\
&= \frac{f(x-t)+f(x+t)}{2} + \frac{1}{2}\big(\sin(x-t)-\sin(x+t)\big) \\
&= \begin{cases} \frac{1}{2}\big(\sin(x-t)+\sin(x+t)\big) + \frac{1}{2}\big(\sin(x-t)-\sin(x+t)\big) & ; \quad x \geq 0 \\ \frac{1}{2}\big(-\sin(x-t)-\sin(x+t)\big) + \frac{1}{2}\big(\sin(x-t)-\sin(x+t)\big) & ; \quad x < 0 \end{cases} \\
&= \begin{cases} \sin(x-t) & ; \quad x \geq 0 \\ -\sin(x+t) & ; \quad x < 0 \end{cases}
\end{aligned}
$$

(2) Comparing to the standard form,

$$
\begin{aligned}
a &= 1 \\
f(x) &= -\cos x \\
g(x) &= \begin{cases} \sin x & ; \quad x \geq 0 \\ -\sin x & ; \quad x < 0 \end{cases}
\end{aligned}
$$

Therefore, by the d'Alembert formula,

$$
\begin{aligned}
u(x,t) &= \frac{f(x-at)+f(x+at)}{2} + \frac{1}{2a}\int\limits_{x-at}^{x+at} g(s)\,\mathrm{d}s \\
&= \frac{-\cos(x-t)-\cos(x+t)}{2} + \frac{1}{2}\int\limits_{x-t}^{x+t} g(s)\,\mathrm{d}s \\
&= \quad \frac{-\cos(x-t)-\cos(x+t)}{2} \\
&\quad + \begin{cases} \frac{1}{2}\int\limits_{x-t}^{x+t} \sin(s)\,\mathrm{d}s & ; \quad x \geq t \\ \frac{1}{2}\left(\int\limits_{x-t}^{0} -\sin(s)\,\mathrm{d}s + \int\limits_{0}^{x+t} \sin(s)\,\mathrm{d}s\right) & ; \quad x < t \end{cases} \\
&= \quad \frac{-\cos(x-t)-\cos(x+t)}{2} \\
&\quad + \begin{cases} \frac{1}{2}\big(-\cos(x+t)+\cos(x-t)\big) & ; \quad x \geq t \\ \frac{1}{2}\Big(\big(\cos(0)-\cos(x-t)\big)+\big(-\cos(x+t)+\cos(0)\big)\Big) & ; \quad x < t \end{cases} \\
&= \quad \frac{-\cos(x-t)-\cos(x+t)}{2} \\
&\quad + \begin{cases} -\cos(x-t) & ; \quad x \geq t \\ 1-\cos(x-t)-\cos(x+t) & ; \quad x < t \end{cases}
\end{aligned}
$$