Partial Differential Equations

Aakash Jog

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1 Lecturer Information

Prof. Yakov Yakubov

Office: Schreiber 233
E-mail: yakubov@post.tau.ac.il

2 Recommended Reading

1. Tikhonov, A.N. and Samarskii, N.A: Equations of Mathematical Physics, Pergamon Press, Oxfort, 1963.

2. Weinberger, H.F, A first Course in Partial Differential Equations, Dover, NY, 1995.

Part I
String Equations

1 Solution using d'Alembert Formula

Definition 1 (Partial differential equation)**.** An equation

$$F(x_1, x_2, \dots, u, u_{x_1}, u_{x_2}, \dots, u_{x_1 x_2}, \dots) = 0$$

where all x_i are independent variables, and $u(x_1, .., x_n)$ is an unknown function, is called a partial differential equation.
A partial differential equation describes a connection between an unknown function of several variables and its partial derivatives.

Definition 2 (Order of a PDE)**.** The order of a PDE is defined to be the highest order of partial derivatives in the equation.

Definition 3 (Linear PDE)**.** A PDE is said o be linear if and only if it is a linear function of u and its partial derivatives.

Definition 4 (String equation/1D Wave Equation)**.** Consider an ideal string on the x-axis. Let the string oscillate in the direction normal to the x-axis. Let u be the position function of a point on the string. Therefore, u depends on the position of the point on the string and on the time, i.e. it is a function of x and t.
Therefore, solving using Newton's Laws,

$$\rho(x) u_{tt}(x,t) = T u_{xx}(x,t)$$

where ρ is the mass density of the string, and T is the tension in the string.
If

$$\rho(x_0) = \rho_0$$

then,

$$u_{tt}(x,t) = a^2 u_{xx}(x,t)$$

where

$$a = \sqrt{\frac{T}{\rho_0}}$$

If there is an external force applied to the string,

$$\rho(x) u_{tt}(x,t) = a^2 u_{xx}(x,t) + F(x,t)$$

Definition 5 (Cauchy problem)**.** Consider an infinite string, i.e. $x \in (-\infty, \infty)$. If the initial position and the initial velocity of the string are given to be $f(x)$ and $g(x)$ respectively, then,

$$\begin{aligned} u(x,0) &= f(x) \\ u_x(x,0) &= g(x) \end{aligned}$$

The problem

$$\begin{aligned} u_{tt}(x,t) &= a^2 u_{xx}(x,t) \\ u(x,0) &= f(x) \\ u_x(x,0) &= g(x) \end{aligned}$$

is called the Cauchy problem.

Definition 6 (Dirichlet's boundary conditions)**.** Consider a finite string, such that $x \in [0, l]$. If the ends of the string are fixed, the boundary conditions

$$\begin{aligned} u(0,t) &= 0 \\ u(l,t) &= 0 \end{aligned}$$

are called Dirichlet's boundary conditions.

Definition 7 (General string equation)**.** Consider a PDE

$$u_{tt}(x,t) = a^2 u_{xx}(x,t)$$

Let

$$\begin{aligned} \zeta &= x - at \\ \eta &= x + at \end{aligned}$$

Therefore,

$$\begin{aligned} u(x,t) &= F(\zeta) + G(\eta) \\ &= F(x - at) + G(x + at) \end{aligned}$$

where F and G are functions of a single variable, and are differentiable twice.

1.1 Infinite Strings

Theorem 1 (Solution to Cauchy Problem (Infinite String))**.** *The solution to the Cauchy problem*

$$\begin{aligned}
u_{tt}(x,t) &= a^2 u_{xx}(x,t) \\
u(x,0) &= f(x) \\
u_x(x,0) &= g(x)
\end{aligned}$$

where $-\infty < x < \infty$*,* $t \geq 0$ *is given by the d'Alembert formula, i.e.*

$$u(x,t) = \frac{f(x-at) + f(x+at)}{2} + \frac{1}{2a} \int\limits_{x-at}^{x+at} g(s) \, \mathrm{d}s$$

where f *is twice differentiable and* g *is differentiable.*

Proof.

$$\begin{aligned}
u_{tt}(x,t) &= a^2 u_{xx}(x,t) \\
u(x,0) &= f(x) \\
u_x(x,0) &= g(x)
\end{aligned}$$

Let the solution be

$$u(x,t) = F(x-at) + G(x+at)$$

Therefore,

$$\begin{aligned}
u_t(x,t) &= \frac{\mathrm{d}u(x,t)}{\mathrm{d}(x-at)} \frac{\mathrm{d}(x-at)}{\mathrm{d}t} \\
&= F'(x-at)(-a) + G'(x+at)(a) \\
&= -aF'(x-at) + aG'(x+at)
\end{aligned}$$

Substituting the initial conditions,

$$\begin{aligned}
u(x,0) &= f(x) \\
&= F(x) + G(x) \\
u_t(x,0) &= g(x) \\
&= -aF'(x) + aG'(x)
\end{aligned}$$

Therefore,

$$a \int_0^x \big(-F'(s) + G'(s)\big)\,\mathrm{d}s = \int_0^x g(s)\,\mathrm{d}s$$

$$\therefore -F(x) + G(x) = \frac{1}{a}\int_0^x g(s)\,\mathrm{d}s + c$$

Therefore, solving with the initial conditions corresponding to $u(x,0)$,

$$
\begin{aligned}
2G(x) &= f(x) + \frac{1}{a}\int_0^x g(s)\,\mathrm{d}s + c \\
\therefore G(x) &= \frac{f(x)}{2} + \frac{1}{2a}\int_0^x g(s)\,\mathrm{d}s + \frac{c}{2} \\
2F(x) &= f(x) - \frac{1}{a}\int_0^x g(s)\,\mathrm{d}s - c \\
\therefore F(x) &= \frac{f(x)}{2} - \frac{1}{2a}\int_0^x g(s)\,\mathrm{d}s - \frac{c}{2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
u(x,t) &= F(x-at) + G(x+at) \\
&= \frac{f(x-at)}{2} - \frac{1}{2a}\int_0^{x-at} g(s)\,\mathrm{d}s - \frac{c}{2} \\
&\quad + \frac{f(x+at)}{2} + \frac{1}{2a}\int_0^{x+at} g(s)\,\mathrm{d}s + \frac{x}{2} \\
&= \frac{f(x-at)+f(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,\mathrm{d}s
\end{aligned}
$$

$\square$

1.2 Half-infinite Strings

Theorem 2 (Solution to initial boundary value problem for half-infinite string with fixed boundary)**.** *The solution to the initial boundary value problem*

$$\begin{aligned} u_{tt}(x,t) &= a^2 u_{xx}(x,t) \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \\ u(0,t) &= 0 \end{aligned}$$

where $0 \leq x < \infty$*,* $t \geq 0$ *is*

$$\widetilde{u}(x,t) = \frac{\widetilde{f}(x-at) + \widetilde{f}(x+at)}{2} + \frac{1}{2a} \int\limits_{x-at}^{x+at} \widetilde{g}(s) \, \mathrm{d}s$$

where

$$\widetilde{f} = \begin{cases} f(x) & ; \quad x \geq 0 \\ -f(-x) & ; \quad x < 0 \end{cases}$$

$$\widetilde{g} = \begin{cases} g(x) & ; \quad x \geq 0 \\ -g(-x) & ; \quad x < 0 \end{cases}$$

where f *is twice differentiable,* $f(0) = 0$*,* g *is differentiable, and* $g(0) = 0$*.*

Proof. By the initial and boundary conditions,

$$\begin{aligned} u(x,0) &= f(x) \\ \therefore u(0,0) &= f(0) \\ u(0,t) &= 0 \\ \therefore u(0,0) &= 0 \end{aligned}$$

Therefore,

$$\therefore f(0) = 0$$

Similarly,

$$\begin{aligned} u_t(x,0) &= g(x) \\ \therefore u_t(0,0) &= g(0) \\ u(0,t) &= 0 \\ \therefore u_t(0,t) &= 0 \\ \therefore u_t(0,0) &= 0 \end{aligned}$$

Therefore,

$$g(0) = 0$$

These conditions are called compatibility conditions.
Let

$$\widetilde{f} = \begin{cases} f(x) & ; \quad x \geq 0 \\ -f(-x) & ; \quad x < 0 \end{cases}$$

$$\widetilde{g} = \begin{cases} g(x) & ; \quad x \geq 0 \\ -g(-x) & ; \quad x < 0 \end{cases}$$

Therefore, due to the compatibility conditions, the odd extensions are continuous. Therefore, by the Solution to Cauchy Problem (Infinite String),

$$\widetilde{u}(x,t) = \frac{\widetilde{f}(x-at) + \widetilde{f}(x+at)}{2} + \frac{1}{2a} \int\limits_{x-at}^{x+at} \widetilde{g}(s)\,\mathrm{d}s$$

□

Theorem 3 (Solution to initial boundary value problem for half-infinite string with free boundary)**.** *The solution to the initial boundary value problem*

$$\begin{aligned} u_{tt}(x,t) &= a^2 u_{xx}(x,t) \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \\ u_x(0,t) &= 0 \end{aligned}$$

where $0 \leq x < \infty$, $t \geq 0$ *is*

$$\widetilde{u}(x,t) = \frac{\widetilde{f}(x-at) + \widetilde{f}(x+at)}{2} + \frac{1}{2a} \int\limits_{x-at}^{x+at} \widetilde{g}(s)\,\mathrm{d}s$$

where

$$\widetilde{f} = \begin{cases} f(x) & ; \quad x \geq 0 \\ -f(-x) & ; \quad x < 0 \end{cases}$$

$$\widetilde{g} = \begin{cases} g(x) & ; \quad x \geq 0 \\ -g(-x) & ; \quad x < 0 \end{cases}$$

where f *is twice differentiable,* $f'(0) = 0$, g *is differentiable, and* $g'(0) = 0$.

Proof. By the initial and boundary conditions,

$$\begin{aligned}
u(x,0) &= f(x) \\
\therefore u_x(x,0) &= f'(x) \\
\therefore u_x(0,0) &= f'(0) \\
u_x(0,t) &= 0 \\
\therefore u_x(0,0) &= 0
\end{aligned}$$

Therefore,

$$\therefore f'(0) = 0$$

Similarly,

$$\begin{aligned}
u_t(x,0) &= g(x) \\
\therefore u_{tx}(x,0) &= g'(x) \\
\therefore u_{tx}(0,0) &= g'(0) \\
u_x(0,t) &= 0 \\
\therefore u_{xt}(0,t) &= 0 \\
\therefore u_{xt}(0,0) &= 0
\end{aligned}$$

Therefore,

$$g'(0) = 0$$

These conditions are called compatibility conditions.
Let

$$\widetilde{f} = \begin{cases} f(x) & ; \quad x \geq 0 \\ f(-x) & ; \quad x < 0 \end{cases}$$

$$\widetilde{g} = \begin{cases} g(x) & ; \quad x \geq 0 \\ g(-x) & ; \quad x < 0 \end{cases}$$

Therefore, by the Solution to Cauchy Problem (Infinite String),

$$\widetilde{u}(x,t) = \frac{\widetilde{f}(x-at) + \widetilde{f}(x+at)}{2} + \frac{1}{2a} \int\limits_{x-at}^{x+at} \widetilde{g}(s) \, \mathrm{d}s$$

$\square$

Exercise 1.

Solve

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(x,0) &= x(2-x) \\ u_t(x,0) &= 0 \\ u(0,t) &= 0 \end{aligned}$$

where $x > 0$, $t > 0$.

Solution 1.

Comparing to the standard form,

$$\begin{aligned} a &= 1 \\ f(x) &= x(2-x) \\ g(x) &= 0 \end{aligned}$$

Therefore, as the boundary is fixed, let

$$\begin{aligned} \widetilde{f} &= \begin{cases} f(x) & ; \quad x \geq 0 \\ -f(-x) & ; \quad x < 0 \end{cases} \\ &= \begin{cases} x(2-x) & ; \quad x \geq 0 \\ -\big(-x(2+x)\big) & ; \quad x < 0 \end{cases} \\ \widetilde{g} &= \begin{cases} g(x) & ; \quad x \geq 0 \\ -g(-x) & ; \quad x < 0 \end{cases} \\ &= 0 \end{aligned}$$

Therefore,

$$
\begin{aligned}
\widetilde{u}(x,t) &= \frac{\widetilde{f}(x-at)+\widetilde{f}(x+at)}{2} + \frac{1}{2a}\int\limits_{x-at}^{x+at} \widetilde{g}(s)\,\mathrm{d}s \\
&= \frac{\widetilde{f}(x-t)+\widetilde{f}(x+t)}{2} \\
&= \begin{cases} \frac{1}{2}(x-t)\left(2-(x-t)\right) & ; \quad x-t \geq 0 \\ \frac{1}{2}(x-t)\left(2+(x-t)\right) & ; \quad x-t < 0 \end{cases} \\
&\quad + \begin{cases} \frac{1}{2}(x+t)\left(2-(x+t)\right) & ; \quad x-t \geq 0 \\ \frac{1}{2}(x+t)\left(2+(x+t)\right) & ; \quad x-t < 0 \end{cases} \\
&= \begin{cases} \frac{1}{2}(x-t)\left(2-(x-t)\right) & ; \quad x \geq t \\ \frac{1}{2}(x-t)\left(2+(x-t)\right) & ; \quad x < t \end{cases} \\
&\quad + \begin{cases} \frac{1}{2}(x+t)\left(2-(x+t)\right) & ; \quad x \geq -t \\ \frac{1}{2}(x+t)\left(2+(x+t)\right) & ; \quad x < -t \end{cases}
\end{aligned}
$$

Therefore, the restricted solution, i.e. the solution on the given domain $x > 0$, $t > 0$ is

$$
u(x,t) = \begin{cases} \frac{1}{2}\left((x+t)(2-x-t)+(x-t)(2+x-t)\right) & ; \quad 0 < x < t \\ \frac{1}{2}\left((x+t)(2-x-t)+(x-t)(2-x+t)\right) & ; \quad t \leq x \end{cases}
$$

Exercise 2.
Solve

$$
\begin{aligned}
u_{tt} &= 2u_{xx} \\
u(x,0) &= x^2 \\
u_t(x,0) &= \sin x \\
u_x(0,t) &= 0
\end{aligned}
$$

where $x > 0$, $t > 0$.

Solution 2.
Comparing to the standard form,

$$
\begin{aligned}
a &= 2 \\
f(x) &= x^2 \\
g(x) &= \sin x
\end{aligned}
$$

Therefore, as the boundary is free, let

$$\begin{aligned}
\widetilde{f} &= \begin{cases} f(x) & ; \quad x \geq 0 \\ f(-x) & ; \quad x < 0 \end{cases} \\
&= \begin{cases} x^2 & ; \quad x \geq 0 \\ (-x)^2 & ; \quad x < 0 \end{cases} \\
&= x^2 \\
\widetilde{g} &= \begin{cases} g(x) & ; \quad x \geq 0 \\ g(-x) & ; \quad x < 0 \end{cases} \\
&= \begin{cases} \sin x & ; \quad x \geq 0 \\ \sin(-x) & ; \quad x < 0 \end{cases} \\
&= \sin |x|
\end{aligned}$$

Therefore,

$$\begin{aligned}
\widetilde{u}(x,t) &= \frac{\widetilde{f}(x-at) + \widetilde{f}(x+at)}{2} + \frac{1}{2a} \int\limits_{x-at}^{x+at} \widetilde{g}(s)\,\mathrm{d}s \\
&= \frac{(x-2t)^2 + (x+2t)^2}{2} + \frac{1}{4} \int\limits_{x-2t}^{x+2t} \sin|s|\,\mathrm{d}s \\
&= \frac{2x^2 + 8t^2}{2} + \frac{1}{4} \int\limits_{x-2t}^{x+2t} \sin|s|\,\mathrm{d}s \\
&= x^2 + 4t^2 + \frac{1}{4} \int\limits_{x-2t}^{x+2t} \sin|s|\,\mathrm{d}s
\end{aligned}$$

Therefore, the restricted solution, i.e. the solution on the given domain $x > 0$, $t > 0$ is

$$\begin{aligned}
u(x,t) &= x^2 + 4t^2 + \begin{cases} \frac{1}{4} \int\limits_{x-2t}^{x+2t} \sin s \, \mathrm{d}s & ; \quad x - 2t \geq 0 \\ \frac{1}{4} \int\limits_{x-2t}^{0} \sin(-s) \, \mathrm{d}s + \frac{1}{4} \int\limits_{0}^{x+2t} \sin s \, \mathrm{d}s & ; \quad x - 2t < 0 \end{cases} \\
&= x^2 + 4t^2 + \begin{cases} \frac{1}{4} \big(\cos(x-2t) - \cos(x+2t)\big) & ; \quad x \geq 2t \\ \frac{1}{4} \big(2\cos(0) - \cos(x-2t) - \cos(x+2t)\big) & ; \quad 0 < x < 2t \end{cases} \\
&= x^2 + 4t^2 + \begin{cases} \frac{1}{4} \big(2 \sin x \sin(2t)\big) & ; \quad x \geq 2t \\ \frac{1}{4} \big(2 - 2\cos x \cos(2t)\big) & ; \quad 0 < x < 2t \end{cases} \\
&= \begin{cases} x^2 + 4t^2 + \frac{1}{2} \sin x \sin(2t) & ; \quad x \geq 2t \\ x^2 + 4t^2 + \frac{1}{2} \big(1 - \cos x \cos(2t)\big) & ; \quad 0 < x < 2t \end{cases}
\end{aligned}$$

In this case, even though $g'(0) \neq 0$, the calculated solution is a valid solution for the problem.

1.3 Finite Strings

Theorem 4 (Solution to boundary value problem for finite string with fixed boundary)**.** *The solution to the initial boundary value problem*

$$\begin{aligned}
u_{tt}(x,t) &= a^2 u_{xx}(x,t) \\
u(x,0) &= f(x) \\
u_t(x,0) &= g(x) \\
u(0,t) &= 0 \\
u(l,t) &= 0
\end{aligned}$$

where $0 \leq x \leq l$, $t \geq 0$ *is*

$$\widetilde{u}(x,t) = \frac{\widetilde{f}(x-at) + \widetilde{f}(x+at)}{2} + \frac{1}{2a} \int\limits_{x-at}^{x+at} \widetilde{g}(s) \, \mathrm{d}s$$

where $\widetilde{f}$ *and* $\widetilde{g}$ *are the* $2l$ *periodic extensions of the odd extensions of* f *and* g *respectively, where* f *is twice differentiable,* $f'(0) = 0$*,* g *is differentiable, and* $g'(0) = 0$*.*

2 A Particular Case of Sturm-Liouville Problem

Consider the problem

$$\begin{aligned}X''(x) + \lambda X(x) &= 0\\ X(0) &= 0\\ X(l) &= 0\end{aligned}$$

on $[0, l]$.
Let $\lambda > 0$. Therefore, let

$$\lambda = \omega^2$$

where $\omega > 0$.
Therefore, the characteristic equation is

$$r^2 + \omega^2 = 0$$

Therefore, solving,

$$r = \pm i\omega$$

Therefore the solution of the ODE is

$$X(s) = A\cos(\omega x) + B\sin(\omega x)$$

Therefore, substituting the given boundary conditions,

$$\begin{aligned}X(0) &= 0\\ \therefore A &= 0\\ X(l) &= 0\\ \therefore B\sin(\omega l) &= 0\\ \therefore \sin(\omega l) &= 0\\ \therefore \omega l &= n\pi\end{aligned}$$

If $n \in \mathbb{Z}$, then $\omega \leq 0$. This contradicts the assumption $\omega > 0$.

where $n \in \mathbb{N}$.

$$\begin{aligned}\lambda_n &= {\omega_n}^2\\ &= \left(\frac{n\pi}{l}\right)^2\end{aligned}$$

where $n \in \mathbb{N}$, is called an eigenvalue of the problem.
The corresponding solution to the problem is

$$X_n = B_n \sin\left(\frac{n\pi}{l}x\right)$$

The function

$$X_n = \sin\left(\frac{n\pi}{l}x\right)$$

is called an eigenfunction of the problem, corresponding to the eigenvalue λ_n.

3 Method of Separation of Variables (Fourier Method)

Theorem 5 (Fourier Method solution to boundary value problem for finite string with fixed boundary)**.** *The solution to the initial boundary value problem*

$$\begin{aligned}
u_{tt}(x,t) &= a^2 u_{xx}(x,t) \\
u(0,t) &= 0 \\
u(l,t) &= 0 \\
u(x,0) &= f(x) \\
u_t(x,0) &= g(x)
\end{aligned}$$

where $0 \leq x \leq l$, $t \geq 0$ *is*

$$u(x,t) = \left(A_n \cos\left(\frac{n\pi a}{l}t\right) + B_n \sin\left(\frac{n\pi a}{l}t\right)\right) \sin\left(\frac{n\pi}{l}x\right)$$

where

$$\begin{aligned}
A_n &= \frac{2}{l}\int_0^l f(x) \sin\left(\frac{k\pi}{l}x\right) \mathrm{d}x \\
B_n &= \frac{2}{n\pi a}\int_0^l g(x) \sin\left(\frac{n\pi}{l}x\right) \mathrm{d}x
\end{aligned}$$

Proof. Let

$$u(x,t) = X(x)T(t)$$

Therefore, substituting into the problem,

$$\begin{aligned}
u_{tt}(x,t) &= a^2 u_{xx}(x,t) \\
\therefore X(x)T''(t) &= a^2 X''(x)T(t) \\
\therefore \frac{T''(t)}{a^2T(t)} &= \frac{X''(x)}{X(x)}
\end{aligned}$$

Therefore, the LHS is dependent only on t, and the RHS is dependent only on x. Therefore, for them to be equal, both sides must be constant.
Therefore, let

$$\begin{aligned}
\frac{X''(x)}{X(x)} &= -\lambda \\
\therefore X''(x) + \lambda X(x) &= 0 \\
\frac{T''(t)}{a^2T(t)} &= -\lambda \\
\therefore T'' + a^2\lambda T(t) &= 0
\end{aligned}$$

Therefore, substituting into the boundary conditions,

$$\begin{aligned}
u(0,t) &= 0 \\
\therefore X(0)T(t) &= 0 \\
u(l,t) &= 0 \\
\therefore X(l)T(t) &= 0
\end{aligned}$$

Therefore, as $T(t) \not\equiv 0$,

$$\begin{aligned}
X(0) &= 0 \\
X(l) &= 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
X''(x) + \lambda X(x) &= 0 \\
X(0) &= 0 \\
X(l) &= 0
\end{aligned}$$

This is a particular case of the Strum-Liouville problem. Therefore, the eigenvalues are

$$\begin{aligned}
\lambda_n &= {\omega_n}^2 \\
&= \left(\frac{n\pi}{l}\right)^2
\end{aligned}$$

where $n \in \mathbb{N}$, and the corresponding eigenfunctions are

$$X_n(x) = \sin\left(\frac{n\pi}{l}x\right)$$

Similarly,

$$\begin{aligned}
T''(x) + a^2\lambda T(t) &= 0 \\
\therefore T''(x) + a^2\left(\frac{n\pi}{l}\right)^2 T(t) &= 0
\end{aligned}$$

Therefore, the characteristic equation is

$$\begin{aligned}
r^2 + a^2\left(\frac{n\pi}{l}\right)^2 = 0 \\
\therefore r = \pm ia\frac{n\pi}{l}
\end{aligned}$$

Therefore,

$$T_n(t) = A_n \cos\left(\frac{n\pi a}{l}t\right) + B_n \sin\left(\frac{n\pi a}{l}t\right)$$

where $n \in \mathbb{N}$.
Therefore,

$$\begin{aligned}
u_n(x,t) &= X_n(x)T_n(t) \\
&= \left(A_n \cos\left(\frac{n\pi a}{l}t\right) + B_n \sin\left(\frac{n\pi a}{l}t\right)\right)\sin\left(\frac{n\pi}{l}x\right)
\end{aligned}$$

Therefore, taking the infinite summation,

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) \\
&= \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi a}{l}t\right) + B_n \sin\left(\frac{n\pi a}{l}t\right)\right)\sin\left(\frac{n\pi}{l}x\right)
\end{aligned}$$

Substituting the first initial condition,

$$
\begin{aligned}
f(x) &= u(x,0) \\
&= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right) \\
\therefore f(x)\sin\left(\frac{k\pi}{l}x\right) &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{k\pi}{l}x\right) \\
\therefore \int\limits_0^l f(x)\sin\left(\frac{k\pi}{l}x\right)\mathrm{d}x &= \sum_{n=1}^{\infty} A_n \int\limits_0^l \sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{k\pi}{l}x\right) \\
&= \sum_{n=1}^{\infty} A_n \frac{l}{2}\delta_{nk} \\
&= A_n \frac{l}{2} \\
\therefore A_n &= \frac{2}{l}\int\limits_0^l f(x)\sin\left(\frac{k\pi}{l}x\right)\mathrm{d}x
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{n\pi a}{l}t\right) + B_n\sin\left(\frac{n\pi a}{l}t\right)\right)\sin\left(\frac{n\pi}{l}x\right) \\
\therefore u_t(x,t) &= \sum_{n=1}^{\infty}\left(-A_n\frac{n\pi a}{l}\sin\left(\frac{n\pi a}{l}t\right) + B_n\frac{n\pi a}{l}\cos\left(\frac{n\pi a}{l}t\right)\right)\sin\left(\frac{n\pi}{l}x\right)
\end{aligned}
$$

Therefore, substituting the second initial condition,

$$\begin{aligned}
\therefore g(x) &= u_t(x,0) \\
&= \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \sin\left(\frac{n\pi x}{l}\right) \\
\therefore g(x) \sin\left(\frac{k\pi}{l}x\right) &= \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \\
\therefore \int\limits_0^l g(x) \sin\left(\frac{k\pi}{l}x\right) \mathrm{d}x &= \sum_{n=1}^{\infty} B_n \int\limits_0^l \frac{n\pi a}{l} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x \\
&= \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \frac{l}{2} \delta_{nk} \\
&= B_n \frac{n\pi a}{2} \\
\therefore B_n &= \frac{2}{n\pi a} \int\limits_0^l g(x) \sin\left(\frac{n\pi}{l}x\right) \mathrm{d}x
\end{aligned}$$

Therefore, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi a}{l}t\right) + B_n \sin\left(\frac{n\pi a}{l}t\right)\right) \sin\left(\frac{n\pi}{l}x\right)$$

where

$$\begin{aligned}
A_n &= \frac{2}{l} \int\limits_0^l f(x) \sin\left(\frac{k\pi}{l}x\right) \mathrm{d}x \\
B_n &= \frac{2}{n\pi a} \int\limits_0^l g(x) \sin\left(\frac{n\pi}{l}x\right) \mathrm{d}x
\end{aligned}$$

□

Definition 8 (Standing wave)**.** Let

$$u_n(x,t) = \left(A_n \cos\left(\frac{n\pi a}{l}t\right) + B_n \sin\left(\frac{n\pi a}{l}t\right)\right) \sin\left(\frac{n\pi}{l}x\right)$$

where $n \in \mathbb{N}$.
Therefore,

$$u_n(x,t) = F_n \sin\left(\frac{n\pi a}{l}t + \varphi_n\right) \sin\left(\frac{n\pi}{l}x\right)$$

where

$$\begin{aligned} F_n &= \sqrt{{A_n}^2 + {B_n}^2} \\ \tan\varphi_n &= \frac{A_n}{B_n} \end{aligned}$$

Therefore, every point on the string at distance x_0 oscillates with amplitude $F_n \sin\left(\frac{n\pi}{l}x_0\right)$ and phase φ_n. Such oscillations are called standing waves.

Definition 9 (Node)**.** The points on a standing wave, for which the solution is zero are called nodes.

At the nodes,

$$\begin{aligned} \sin\left(\frac{n\pi}{l}x\right) &= 0 \\ \frac{n\pi}{l}x &= k\pi \\ \therefore x &= \frac{kl}{n} \end{aligned}$$

where $k \in \mathbb{N}$.
Exercise 3.
Solve

$$\begin{aligned} u_{tt} &= a^2 u_{xx} \\ u(0,t) &= 0 \\ u(l,t) &= 0 \\ u(x,0) &= \begin{cases} \frac{hx}{c} & ; \quad 0 \le x \le c \\ \frac{h(l-x)}{l-c} & ; \quad c \le x \le l \end{cases} \\ u_t(x,0) &= 0 \end{aligned}$$

where $0 \le x \le l$, $t \ge 0$.

Solution 3.
Comparing to the standard form,

$$f(x) = \begin{cases} \frac{hx}{c} & ; \quad 0 \le x \le c \\ \frac{h(l-x)}{l-c} & ; \quad c \le x \le l \end{cases}$$
$$g(x) = 0$$

Therefore,

$$\begin{aligned} A_n &= \frac{2}{l} \int_0^l f(x) \sin\left(\frac{k\pi}{l} x\right) \mathrm{d}x \\ &= \frac{2}{l} \int_0^c \frac{hx}{c} \sin\left(\frac{k\pi}{l} x\right) \mathrm{d}x + \frac{2}{l} \int_c^l \frac{h(l-x)}{l-c} \sin\left(\frac{k\pi}{l} x\right) \mathrm{d}x \\ &= \frac{2hl^2}{n^2\pi^2 c(l-c)} \sin\left(\frac{n\pi c}{l}\right) \\ B_n &= \frac{2}{n\pi a} \int_0^l g(x) \sin\left(\frac{n\pi}{l} x\right) \mathrm{d}x \\ &= \frac{2}{n\pi a} \int_0^l 0 \, \mathrm{d}x \\ &= 0 \end{aligned}$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2hl^2}{n^2\pi^2 c(l-c)} \sin\left(\frac{n\pi c}{l}\right) \cos\left(\frac{n\pi a}{l} t\right) \right) \sin\left(\frac{n\pi}{l} x\right)$$

Exercise 4.
Given

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(0,t) &= 0 \\ u(2,t) &= 0 \\ u(x,0) &= 0 u_t(x,0) & &= \begin{cases} x & ; \quad 0 \le x \le 1 \\ 2-x & ; \quad 1 \le x \le 2 \end{cases} \end{aligned}$$

where $0 \leq x \leq 2$, $t \geq 0$, find $u(1.5, 5.3)$.

Solution 4.

Comparing to the standard form,

$$\begin{aligned}
a &= 1 \\
l &= 2 \\
f(x) &= 0 \\
g(x) &= \begin{cases} x & ; \quad 0 \leq x \leq 1 \\ 2 - x & ; \quad 1 \leq x \leq 2 \end{cases}
\end{aligned}$$

As the boundary is fixed, let $\widetilde{f}(x)$ and $\widetilde{g}(x)$ be the $2l$ periodic extensions of $f(x)$ and $g(x)$. Therefore,

$$\begin{aligned}
\widetilde{u}(x, t) &= \frac{\widetilde{f}(x - at) + \widetilde{f}(x + at)}{2} + \frac{1}{2a} \int\limits_{x-at}^{x+at} \widetilde{g}(s) \, \mathrm{d}s \\
&= \frac{1}{2} \int\limits_{x-t}^{x+t} \widetilde{g}(s) \, \mathrm{d}s
\end{aligned}$$

Therefore,

$$\begin{aligned}
u(1.5, 5.3) &= \widetilde{u}(1.5, 5.3) \\
&= \frac{1}{2} \int\limits_{-3.8}^{6.8} \widetilde{g}(s) \, \mathrm{d}s \\
&= \frac{1}{2} \left(\int\limits_{-3.8}^{0.2} \widetilde{g}(s) \, \mathrm{d}s + \int\limits_{0.2}^{4.2} \widetilde{g}(s) \, \mathrm{d}s + \int\limits_{4.2}^{6.8} \widetilde{g}(s) \, \mathrm{d}s \right) \\
&= \frac{1}{2} \left(0 + 0 + \frac{4.2}{6.8} \widetilde{g}(s) \, \mathrm{d}s \right) \\
&= \frac{1}{2} \int\limits_{0.2}^{2.8} \widetilde{g}(s) \, \mathrm{d}s \\
&= \frac{1}{2} \left(\int\limits_{0.2}^{1} s \, \mathrm{d}s + \int\limits_{1}^{2.8} (2 - s) \, \mathrm{d}s \right) \\
&= 0.33
\end{aligned}$$

Exercise 5.
Solve

$$\begin{aligned} u_{tt} &= 4u_{xx} \\ u(x,0) &= \cos^2(\pi x) \\ u_t(x,0) &= \sin^2(\pi x)\cos(\pi x) \\ u_x(0,t) &= 0 \\ u_x(1,t) &= 0 \end{aligned}$$

where $0 \le x \le 1$, $t \ge 0$.

4 Impulse Response

Theorem 6. *The solution to the initial boundary value probelm*

$$\begin{aligned} u_{tt} &= a^2 u_{xx} \\ u(0,t) &= 0 \\ u(l,t) &= 0 \\ u(x,0) &= 0 \\ u_t(x,0) &= \begin{cases} v_0 & ; \quad \alpha \le x \le \beta \\ 0 & ; \quad otherwise \end{cases} \end{aligned}$$

where $0 \le x \le l$, $t \ge 0$ *is*

$$u(x,t) = \frac{2v_0 l}{\pi^2 a} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\cos\left(\frac{n\pi\alpha}{l}\right) - \cos\left(\frac{n\pi\beta}{l}\right) \right) \sin\left(\frac{n\pi a}{l} t\right) \sin\left(\frac{n\pi}{l} x\right)$$

Proof. Comparing to the standard form,

$$\begin{aligned} f(x) &= 0 \\ g(x) &= \begin{cases} v_0 & ; \quad \alpha \le x \le \beta \\ 0 & ; \quad \text{otherwise} \end{cases} \end{aligned}$$

Therefore, by Fourier Method solution to boundary value problem for finite string with fixed boundary,

$$\begin{aligned}
A_n &= 0 \\
B_n &= \frac{2}{n\pi a} \int\limits_0^l v_0 \sin\left(\frac{n\pi}{l} x\right) \mathrm{d}x \\
&= \frac{2v_0}{n\pi a} \int\limits_\alpha^\beta \sin\left(\frac{n\pi}{l} x\right) \mathrm{d}x \\
&= \frac{2v_0 l}{n^2\pi^2 a} \left(\cos\left(\frac{n\pi\alpha}{a}\right) - \cos\left(\frac{n\pi\beta}{l}\right)\right)
\end{aligned}$$

Therefore,

$$u(x,t) = \frac{2v_0 l}{\pi^2 a} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\cos\left(\frac{n\pi\alpha}{l}\right) - \cos\left(\frac{n\pi\beta}{l}\right)\right) \sin\left(\frac{n\pi a}{l} t\right) \sin\left(\frac{n\pi}{l} x\right)$$

□

The solution to the initial boundary value problem

$$\begin{aligned}
u_{tt} &= a^2 u_{xx} \\
u(0,t) &= 0 \\
u(l,t) &= 0 \\
u(x,0) &= 0 \\
u_t(x,0) &= \delta(x-c)
\end{aligned}$$

where $0 \le x \le l$, $t \ge 0$, $0 \le c \le l$ is

$$u(x,t) = \lim_{\varepsilon \to 0} \frac{Il}{\varepsilon\rho\pi^2 a} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\cos\left(\frac{n\pi(c-\varepsilon)}{l}\right) - \cos\left(\frac{n\pi(c+\varepsilon)}{l}\right)\right) \sin\left(\frac{n\pi a}{l} t\right) \sin\left(\frac{n\pi}{l} x\right)$$

Proof. Let the impulse be I.
Let the impulse act on the interval $(c-\varepsilon, c+\varepsilon)$.
Therefore,

$$I = \Delta p$$

where p is the momentum. Therefore,

$$\begin{aligned}
\Delta p &= \Delta m v_0 \\
&= \rho \Delta x
\end{aligned}$$

Therefore,

$$
\begin{aligned}
I &= 2\varepsilon\rho v_0 \\
\therefore v_0 &= \frac{I}{2\varepsilon\rho}
\end{aligned}
$$

Let the solution to the problem be $u_\varepsilon(x,t)$. Therefore,

$$
u_\varepsilon(x,t) = \frac{2v_0 l}{\pi^2 a}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\cos\left(\frac{n\pi\alpha}{l}\right) - \cos\left(\frac{n\pi\beta}{l}\right)\right)\sin\left(\frac{n\pi a}{l}t\right)\sin\left(\frac{n\pi}{l}x\right)
$$

Therefore, the solution to the impulse response problem is

$$
\lim_{\varepsilon\to 0} u_\varepsilon(x,t) = \lim_{\varepsilon\to 0}\frac{Il}{\varepsilon\rho\pi^2 a}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\cos\left(\frac{n\pi(c-\varepsilon)}{l}\right) - \cos\left(\frac{n\pi(c+\varepsilon)}{l}\right)\right)\sin\left(\frac{n\pi a}{l}t\right)\sin\left(\frac{n\pi}{l}x\right)
$$

□

5 Uniqueness of Solution using Energy Method

Theorem 7 (Uniqueness Theorem)**.** *If there exists a solution to the problem*

$$
\begin{aligned}
\rho(x)u_{tt} &= \big(k(x)u_x\big)_x + F(x,t) \\
u(x,0) &= f(x) \\
u_t(x,0) &= g(x) \\
u(0,t) &= h_1(t) \\
u(l,t) &= h_2(t)
\end{aligned}
$$

where $0 \le x \le l$, $t \ge 0$, $k(x) > 0$, $\rho(x) > 0$, *then the solution is unique.*

Proof. If possible, let there be two distinct solutions $u_1(x,t)$ and $u_2(x,t)$.

Therefore,

$$
\begin{aligned}
\rho(x)(u_1)_{tt} &= \big(k(x)(u_1)_x\big)_x + F(x,t) \\
u_1(x,0) &= f(x) \\
(u_1)_t(x,0) &= g(x) \\
u_1(0,t) &= h_1(t) \\
u_1(l,t) &= h_2(t) \\
\rho(x)(u_2)_{tt} &= \big(k(x)(u_2)_x\big)_x + F(x,t) \\
u_2(x,0) &= f(x) \\
(u_2)_t(x,0) &= g(x) \\
u_2(0,t) &= h_1(t) \\
u_2(l,t) &= h_2(t)
\end{aligned}
$$

Let

$$v(x,t) = u_1(x,t) - u_2(x,t)$$

Therefore,

$$
\begin{aligned}
\rho(x)v_{tt}(x,t) &= \big(k(x)v_x(x,t)\big) \\
v(x,0) &= 0 \\
v_t(x,0) &= 0 \\
v(0,t) &= 0 \\
v(l,t) &= 0
\end{aligned}
$$

Therefore, the total energy of the string at time t is

$$E(t) = \frac{1}{2}\int\limits_0^l \left(k{v_x}^2 + \varepsilon {v_t}^2\right) \mathrm{d}x$$

Therefore,

$$
\begin{aligned}
E'(t) &= \frac{1}{2}\int\limits_0^l (2kv_xv_{xt} + 2\rho v_tv_{tt})\,\mathrm{d}x \\
&= \int\limits_0^l (kv_xv_{xt} + \rho v_tv_{tt})\,\mathrm{d}x
\end{aligned}
$$

Assuming the mixed derivatives exist and are continuous,

$$v_{xt} = v_{tx}$$

Therefore,

$$E'(t) = \int\limits_0^l k v_x v_{tx} \, \mathrm{d}x + \int\limits_0^l \rho v_t v_{tt} \, \mathrm{d}x$$

Substituting the initial conditions,

$$\int\limits_0^l k v_x v_{tx} \, \mathrm{d}x = -\int\limits_0^l v_t (k v_x)_x \, \mathrm{d}x$$

Therefore,

$$\begin{aligned} E'(x) &= \int\limits_0^l \rho v_t v_{tt} \, \mathrm{d}x - \int\limits_0^l v_t (k v_x)_x \, \mathrm{d}x \\ &= \int\limits_0^l v_t \left(\rho v_{tt} - (k v_x)_x \right) \mathrm{d}x \end{aligned}$$

Therefore, as $\rho(x) v_{tt}(x,t) = \big(k(x) v_x(x,t)\big)$,

$$E'(x) = 0$$

Therefore, $E(t)$ is constant.
As $v(x,0) = 0$,

$$v_x(x,0) = 0$$

Also, $v_t(x,0) = 0$. Therefore,

$$\begin{aligned} E(0) &= \frac{1}{2} \int\limits_0^l \left(k v_x(x,0)^2 + \rho v_t(x,0)^2 \right) \mathrm{d}x \\ &= 0 \end{aligned}$$

Therefore, as k and ρ are positive,

$$\begin{aligned} v_x(x,t) &= 0 \\ v_t(x,t) &= 0 \end{aligned}$$

Therefore, $v(x,t)$ must be constant.
Therefore, let

$$v(x,t) = c$$

Therefore,

$$\begin{aligned} v(x,0) &= c \\ \therefore 0 &= c \end{aligned}$$

Therefore,

$$\begin{aligned} v(x,t) &= 0 \\ \therefore u_1(x,t) &= u_2(x,t) \end{aligned}$$

This contradicts the assumption that u_1 and u_2 are distinct. Therefore, the solution to the problem is unique. □

6 Well-posedness

Definition 10 (Well-posed problem)**.** A problem is said to be well-posed if it has a unique solution, continuously dependent on the conditions of the problem.

Definition 11 (Continuous dependence)**.** If small changes in the conditions of a problem imply small changes in the solution, it is called continuous dependence.

Theorem 8. *The Cauchy problem*

$$\begin{aligned} u_{tt}(x,t) &= a^2 u_{xx}(x,t) \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

where $-\infty < x < \infty$*,* $t \geq 0$ *is well-posed.*

Proof. Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of the Cauchy problem. Therefore,

$$\begin{aligned} (u_1)_{tt}(x,t) &= a^2 (u_2)_{xx}(x,t) \\ u_1(x,0) &= f_1(x) \\ (u_1)_t(x,0) &= g_1(x) \\ u_2(x,0) &= f_2(x) \\ (u_2)_t(x,0) &= g_2(x) \end{aligned}$$

$\forall x$, let

$$\begin{aligned}
&\big|f_1(x) - f_2(x)\big| < \varepsilon \\
&\big|g_1(x) - g_2(x)\big| < \varepsilon
\end{aligned}$$

Therefore, by Solution to Cauchy Problem (Infinite String),

$$\begin{aligned}
u_1(x,t) &= \frac{f_1(x-at) + f_1(x+at)}{2} + \frac{1}{2a} \int\limits_{x-at}^{x+at} g_1(s)\,\mathrm{d}s \\
u_2(x,t) &= \frac{f_2(x-at) + f_2(x+at)}{2} + \frac{1}{2a} \int\limits_{x-at}^{x+at} g_2(s)\,\mathrm{d}s
\end{aligned}$$

Therefore, for $0 \leq t \leq t_0$,

$$\begin{aligned}
\big|u_1(x,t) - u_2(x,t)\big| &\leq \frac{\big|f_1(x-at) - f_2(x-at)\big| + \big|f_1(x+at) - f_2(x+at)\big|}{2} + \frac{1}{2a} \int\limits_{x-at}^{x+at} \big|g_1(s) - g_2 \\
&\leq \frac{\varepsilon + \varepsilon}{2} + \frac{1}{2a} \int\limits_{x-at}^{x+at} \varepsilon\,\mathrm{d}x \\
&= \varepsilon + \varepsilon t \\
&\leq \varepsilon + \varepsilon t_0 \\
\therefore \big|u_1(x,t) - u_2(x,t)\big| &\leq \varepsilon(1 + t_0)
\end{aligned}$$

Therefore, as a small change in ε implies a small change in $u(x,t)$, the problem is well-posed. $\square$

Exercise 6.

The telegraph problem describes the voltage inside a piece of wire with some specific electrical properties. Prove the uniqueness of the solution of the following particular case of the telegraph problem.

$$\begin{aligned}
u_{tt} + c^2 u_t - u_{xx} &= 0 \\
u(x,0) &= f(x) \\
u_t(x,0) &= g(x) \\
u(a,t) &= 0 \\
u_x(b,t) &= 0
\end{aligned}$$

where $a \leq x \leq b$, $t \geq 0$.
Hint: Use the energy integral

$$E(t) = \frac{1}{2} \int_a^b \left({v_t}^2 + {v_x}^2\right) \mathrm{d}x$$

Solution 6.

If possible, let u_1 and u_2 be two distinct solutions of the problem.
Let

$$v(x,t) = u_1(x,t) - u_2(x,t)$$

Therefore,

$$\begin{aligned} v_{tt} + c^2 v_t - v_{xx} &= 0 \\ v(x,0) &= 0 \\ v(x,0) &= 0 \\ v(a,t) &= 0 \\ v_x(b,t) &= 0 \end{aligned}$$

Let

$$E(t) = \frac{1}{2} \int_a^b \left({v_t}^2 + {v_x}^2\right) \mathrm{d}x$$

Therefore,

$$E'(t) = \int_a^b (v_t v_{tt} + v_x v_{xt}) \, \mathrm{d}x$$

Assuming the mixed derivatives exist and are continuous,

$$v_{xt} = v_{tx}$$

Solving using integration by parts and substituting the initial conditions,

$$\begin{aligned}
\int_a^b v_x v_{xt} \, \mathrm{d}x &= \int_a^b v_x v_{tx} \, \mathrm{d}x \\
&= v_x(b,t)v_t(b,t) - v_x(a,t)v_t(a,t) - \int_a^b v_t v_{xx} \, \mathrm{d}t \\
&= -\int_a^b v_t v_{xx} \, \mathrm{d}x
\end{aligned}$$

Therefore, substituting $v_{tt} + c^2 v_t - v_{xx} = 0$,

$$\begin{aligned}
E'(t) &= \int_a^b (v_t v_{tt} - v_t v_{xx}) \, \mathrm{d}x \\
&= \int_a^b v_t (v_{tt} - v_{xx}) \, \mathrm{d}x \\
&= -c^2 \int_a^b (v_t)^2 \, \mathrm{d}x \\
&\leq 0
\end{aligned}$$

Therefore, $E(t)$ is a decreasing function, i.e.,

$$E(t) \leq E(0)$$

Also,

$$E(0) = \frac{1}{2} \int_a^b \left({v_t}^2(x,0) + {v_x}^2(x,0) \right) \mathrm{d}x$$

Therefore, substituting the given conditions,

$$E(0) = 0$$

Therefore, $\forall t \geq 0$,

$$0 \leq E(t) \leq 0$$

Therefore,

$$E(t) \equiv 0$$

Therefore,

$$\begin{aligned} v_t(x,t) &= 0 \\ v_x(x,t) &= 0 \\ \therefore v(x,t) &= c(x) \\ \therefore v_x(x,t) &= c'(x) \\ \therefore c'(x) &= 0 \end{aligned}$$

Therefore, let

$$c(x) = c$$

Therefore,

$$\begin{aligned} v(x,t) &= c \\ \therefore v(x,0) &= c \\ \therefore 0 &= c \end{aligned}$$

Therefore,

$$\begin{aligned} v(x,t) &= 0 \\ \therefore u_1(x,t) &= u_2(x,t) \end{aligned}$$

This contradicts the assumption that u_1 and u_2 are distinct. Therefore, the solution to the problem is unique.

Part II

General Second Order Partial Differential Equations

1 Classification

Consider a PDE

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y + cu + f = 0$$

where the coefficients are functions of x and y.
Therefore, the PDE can be written as

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + F(x, y, u, u_x, u_y) = 0$$

If

$$a_{11} = 0$$
$$a_{22} = 0$$

the equation is said to be of a simple form.
If

$$a_{11} \neq 0$$

or

$$a_{22} \neq 0$$

then, let

$$\xi = \varphi(x, y)$$
$$\eta = \psi(x, t)$$

such that the Jacobian

$$J = \begin{vmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{vmatrix}$$
$$\neq 0$$

Therefore,

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x \\
u_y &= u_\xi \xi_y + u_\eta \eta_y \\
u_{xx} &= (u_\xi)_x \xi_x + u_\xi \xi_{xx} + (u_\eta)_x \eta_x + u_\eta \eta_{xx} \\
&= (u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x)\xi_x + (u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x)\eta_x + u_\xi \xi_{xx} + u_\eta \eta_{xx}
\end{aligned}$$

Therefore,

$$\begin{aligned}
u_{xx} &= (u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x)\xi_x + (u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x)\eta_x + u_\xi \xi_{xx} + u_\eta \eta_{xx} \\
u_{xy} &= (u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)\xi_x + (u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y)\eta_x + u_\xi \xi_{xy} + u_\eta \eta_{xy} \\
u_{yy} &= (u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)\xi_y + (u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y)\eta_y + u_\xi \xi_{yy} + u_\eta \eta_{yy}
\end{aligned}$$

Therefore, substituting into the shorter form of the original PDE,

$$\widetilde{a_{11}} u_{\xi\xi} + 2\widetilde{a_{12}} u_{\xi\eta} + \widetilde{a_{22}} u_{\eta\eta} + \widetilde{F} = 0$$

where

$$\begin{aligned}
\widetilde{a_{11}} &= a_{11}{\xi_x}^2 + 2a_{12}\xi_x\xi_y + a_{22}{\xi_y}^2 \\
\widetilde{a_{12}} &= a_{11}\xi_x\eta_x + a_{12}(\xi_x\eta_y + \xi_y\eta_x) + a_{22}\xi_y\eta_y \\
\widetilde{a_{22}} &= a_{11}{\eta_x}^2 + 2a_{12}\eta_x\eta_y + a_{22}{\eta_y}^2
\end{aligned}$$

Let ξ and η be chosen such that

$$\begin{aligned}
0 &= \widetilde{a_{11}} \\
&= a_{11}{\varphi_x}^2 + 2a_{12}\varphi_x\varphi_y + a_{22}{\varphi_y}^2
\end{aligned}$$

Let

$$a_{11} \neq 0$$

If φ_y is zero, φ_x must also be zero. Therefore,

$$J = 0$$

Therefore, as the Jacobian must be non-zero, φ_y also must be non-zero. Therefore,

$$\begin{aligned}
a_{11}{\varphi_x}^2 + 2a_{12}\varphi_x\varphi_y + a_{22}{\varphi_y}^2 &= 0 \\
\therefore a_{11}\left(-\frac{\varphi_x}{\varphi_y}\right)^2 - 2a_{12}\left(-\frac{\varphi_x}{\varphi_y}\right) + a_{22} &= 0
\end{aligned}$$

Also, $-\frac{\varphi_x}{\varphi_y}$ is the derivative of the implicit function of $\varphi(x,y) = c$. Therefore, substituting,

$$a_{11}{y'}^2 - 2a_{12}y' + a_{22} = 0$$

Therefore, $\varphi(x,y)$ satisfies the equation if and only if

$$\varphi(x,y) = c$$

is the general solution of the ODE.
The equation

$$a_{11}{y'}^2 - 2a_{12}y' + a_{22} = 0$$

is called the characteristic equation of the original PDE. Its solutions are called characteristic curves of the original PDE.
Therefore, solving,

$$y' = \frac{a_{12} \pm \sqrt{{a_{12}}^2 - a_{11}a_{22}}}{a_{11}}$$

Therefore, the two roots are

$$k_1 = \frac{a_{12} + \sqrt{{a_{12}}^2 - a_{11}a_{22}}}{a_{11}}$$
$$k_2 = \frac{a_{12} - \sqrt{{a_{12}}^2 - a_{11}a_{22}}}{a_{11}}$$

As a_{11}, a_{12}, and a_{22} depend on x and y, k_1 and k_2 are also dependent on x and y.
If $k_1(x,y)$ and $k_2(x,y)$ are real and distinct, the PDE is said to be hyperbolic at (x,y).
If $k_1(x,y)$ and $k_2(x,y)$ are real and equal, the PDE is said to be parabolic at (x,y).
If $k_1(x,y)$ and $k_2(x,y)$ are complex, the PDE is said to be elliptical at (x,y).

$k_1(x,y)$ and $k_2(x,y)$ are real and distinct, if and only if

$${a_{12}}^2 - a_{11}a_{22} > 0$$

Therefore, the two solutions of the PDE correspond to

$$y' = k_1(x,y)$$
$$y' = k_2(x,y)$$

Therefore, let the two solutions of the PDE be

$$\varphi(x, y) = c$$
$$\psi(x, y) = c$$

Therefore, let

$$\xi = \varphi(x, y)$$
$$\eta = \psi(x, y)$$

Therefore,

$$\widetilde{a_{11}} = 0$$
$$\widetilde{a_{22}} = 0$$

Therefore, substituting,

$$2\widetilde{a_{12}}u_{\xi\eta} + \widetilde{F} = 0$$

Therefore,

$$u_{\xi\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta)$$

is the canonical form of the PDE.

$k_1(x, y)$ and $k_2(x, y)$ are real and equal, if and only if If the equation is parabolic, i.e.

$${a_{12}}^2 - a_{11}a_{22} = 0$$

Therefore, the solution of the PDE corresponds to

$$\begin{aligned} y' &= k_1(x, y) \\ &= k_2(x, y) \end{aligned}$$

Therefore, let the solution of the PDE be

$$\varphi(x, y) = c$$

Therefore, let

$$\xi = \varphi(x, y)$$
$$\eta = \psi(x, y)$$

where $\psi(x, y)$ is any function such that the Jacobian is non zero.
Therefore,

$$
\begin{aligned}
{a_{12}}^2 - a_{11}a_{22} &= 0 \\
\therefore a_{12} &= \pm\sqrt{a_{11}a_{22}}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\widetilde{a_{11}} a_{11}{\varphi_x}^2 + 2\sqrt{a_{11}a_{22}}\varphi_x\varphi_y + a_{22}{\varphi_y}^2 &= 0 \\
\therefore \left(\sqrt{a_{11}}\varphi_x + \sqrt{a_{22}}\varphi_y\right)^2 &= 0
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\widetilde{a_{12}} &= a_{11}\varphi_x\psi_x + a_{12}\left(\varphi_x\psi_y + \varphi_y\psi_x\right) + a_{22}\varphi_y\psi_y \\
&= \left(\sqrt{a_{11}}\varphi_x + \sqrt{a_{22}}\varphi_y\right)\left(\sqrt{a_{11}}\psi_x + \sqrt{a_{22}}\psi_y\right) \\
&= 0
\end{aligned}
$$

Therefore, substituting,

$$\therefore \widetilde{a_{22}} u_{\eta\eta} + \widetilde{F} = 0$$

Therefore,

$$\therefore u_{\eta\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta)$$

is the canonical form of the PDE.

$k_1(x, y)$ and $k_2(x, y)$ are complex, if and only if

$${a_{12}}^2 - a_{11}a_{22} < 0$$

Therefore, the two solutions of the PDE correspond to

$$
\begin{aligned}
y' &= k(x, y) \\
y' &= \overline{k(x, y)}
\end{aligned}
$$

Therefore, let the two solutions of the PDE be

$$
\begin{aligned}
\varphi(x, y) &= c \\
\overline{\varphi(x, y)} &= c
\end{aligned}
$$

Therefore, let

$$
\begin{aligned}
\xi &= \varphi(x, y) \\
\eta &= \overline{\varphi(x, y)}
\end{aligned}
$$

Therefore,

$$\widetilde{a_{11}} = 0$$
$$\widetilde{a_{22}} = 0$$

Therefore, substituting,

$$2\widetilde{a_{12}}u_{\xi\eta} + \widetilde{F} = 0$$

where the coefficients are complex.
Therefore, let

$$\begin{aligned}\alpha &= \frac{\xi + \eta}{2} \\ &= \Re\{\xi\} \\ \beta &= \frac{\xi - \eta}{2} \\ &= \Im\{\xi\}\end{aligned}$$

Therefore, with respect to α and β,

$$\widetilde{a_{12}} = 0$$
$$\widetilde{a_{11}} = \widetilde{a_{22}}$$

Therefore,

$$u_{\alpha\alpha} + u_{\beta\beta} = \Phi(\alpha, \beta, u, u_\alpha, u_\beta)$$

is the canonical form of the PDE.
Exercise 7.
Reduce the equation

$$u_{xx} + 2u_{xy} - 3u_{yy} = 0$$

to a canonical form and solve the equation.

Solution 7.
Comparing to the standard form,

$$a_{11} = 1$$
$$a_{12} = 1$$
$$a_{22} = -3$$

Therefore, the characteristic equation is

$$y'^2 - 2y' - 3 = 0$$

Therefore,

$$\begin{aligned} k_1 &= -1 \\ k_2 &= 3 \end{aligned}$$

Therefore, the PDE is hyperbolic.
Therefore,

$$\begin{aligned} y_1 &= -x + c \\ y_2 &= -3x + c \end{aligned}$$

Therefore,

$$\begin{aligned} \xi &= x + y \\ \eta &= 3x - y \end{aligned}$$

Therefore, substituting,

$$\begin{aligned} \widetilde{a_{11}} &= a_{11}{\xi_x}^2 + 2a_{12}\xi_x\xi_y + a_{22}{\xi_y}^2 \\ &= 0 \\ \widetilde{a_{22}} &= a_{11}{\eta_x}^2 + 2a_{12}\eta_x\eta_y + a_{22}{\eta_y}^2 \\ &= 0 \end{aligned}$$

Therefore,

$$2\widetilde{a_{12}}u_{\xi\eta} + \widetilde{F} = 0$$

Therefore,

$$u_{\xi\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta)$$

is a canonical form of the PDE.
Therefore,

$$
\begin{aligned}
u_x &= x_\xi \xi_x + u_\eta \eta_x \\
&= u_\xi + 3u_\eta \\
u_y &= u_\xi \xi_y + u_\eta \eta_y \\
&= u_\xi - u_\eta \\
u_{xx} &= (u_\xi)_x + 3(u_\eta)_x \\
&= u_{\xi\xi} + 3u_{\xi\eta} + 3(u_{\eta\xi} + 3u_{\eta\eta}) \\
&= u_{\xi\xi} + 6u_{\xi\eta} + 9u_{\eta\eta} \\
u_{xy} &= (u_\xi)_y + 3(u_\eta)_y \\
&= u_{\xi\xi} - u_{\xi\eta} + 3(u_{\eta\xi} - u_{\eta\eta}) \\
&= u_{\xi\xi} + 2u_{\xi\eta} - 3u_{\eta\eta} \\
u_{yy} &= (u_\xi)_y - (u_\eta)_y \\
&= u_{\xi\xi} - u_{\xi\eta} - (u_{\eta\xi} - u_{\eta\eta}) \\
&= u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}
\end{aligned}
$$

Therefore, substituting into the original equation and solving,

$$u_{\xi\eta} = 0$$

This is a canonical form of the PDE.
Therefore, integrating with respect to η,

$$u_\xi = f(\xi)$$

Therefore, integrating with respect to ξ,

$$
\begin{aligned}
u &= \int f(\xi)\,\mathrm{d}\xi + G(\eta) \\
&= F(\xi) + G(\eta) \\
&= F(x + y) + G(3x - y)
\end{aligned}
$$

Therefore,

$$u(x, y) = F(x + y) + G(3x - y)$$

where F and G are twice differentiable.

Exercise 8.
Consider the equation

$$u_{xx} + 2u_{xy} - 3u_{yy} = -2(y-3x)^2 + \sin\big(2(x+y)\big)$$

1. Assume that the canonical form of the corresponding homogeneous PDE is

   $$16u_{\xi\eta} = 0$$

   and that the homogeneous PDE is hyperbolic.
   Classify the given equation and bring it to a canonical form.

2. Find a general solution.

3. Find a solution which satisfies

   $$\begin{aligned} u(x,3x) &= x^2 \\ u(x,-x) &= -\frac{1}{8}x \end{aligned}$$

Exercise 9.
Consider the equation

$$x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} + xu_x + yu_y = 0$$

where $x > 0$.

1. Classify the equation.
2. Find the canonical form of the equation.
3. Find the general solution.

Solution 9.
1. Comparing to the standard form

$$\begin{aligned} a_{11} &= x^2 \\ a_{12} &= -xy \\ a_{22} &= y^2 \end{aligned}$$

Therefore,

$$\begin{aligned} {a_{12}}^2 - a_{11}a_{22} &= x^2y^2 - x^2y^2 \\ &= 0 \end{aligned}$$

Therefore, the equation is parabolic.

2. The characteristic equation is

$$x^2 {y'}^2 + 2xyy' + y^2 = 0$$
$$\therefore \left(xy' + y\right)^2 = 0$$

Therefore,

$$y' = -\frac{y}{x}$$

Therefore,

$$\frac{\mathrm{d}y}{y} = -\frac{\mathrm{d}x}{x}$$
$$\therefore \ln|y| = -\ln x + c_1$$
$$\therefore |y| = \frac{1}{x} + c_2$$
$$= \frac{c_2}{x}$$
$$\therefore xy = c_2$$

Therefore, let

$$\xi = xy$$
$$\eta = x$$

Therefore,

$$J = \begin{vmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{vmatrix}$$
$$= \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix}$$
$$= -x$$

Therefore, as the Jacobian is non-zero, the choice of η is valid.
Therefore,

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$
$$= y u_\xi + u_\eta$$
$$u_y = u_\xi \xi_y + u_\eta \eta_y$$
$$= x u_\xi$$
$$u_{xx} = y^2 u_{\xi\xi} + 2y U_{\xi\eta} + u_{\eta\eta}$$
$$u_{xy} = u_\xi + xy u_{\xi\xi} + x u_{\xi\eta}$$
$$u_{yy} = x^2 u_{\xi\xi}$$

Therefore, substituting,

$$x^2 u_{\eta\eta} + x u_\eta = 0$$
$$\therefore x u_{\eta\eta} + u_\eta = 0$$
$$\therefore \eta u_{\eta\eta} + u_\eta = 0$$

Therefore, a canonical form is

$$u_{\eta\eta} = -\frac{1}{\eta} u_\eta$$

3. Let

$$w = u_\eta$$

Therefore,

$$w_\eta = u_{\eta\eta}$$

Therefore, substituting in the canonical form,

$$w = w_\eta$$

Therefore, solving,

$$\ln |w| = -\ln \eta + f(\xi)$$
$$\therefore |w| = \frac{1}{\eta} e^{f(\xi)}$$

Also,

$$\begin{aligned} u_\eta &= w \\ &= \frac{1}{\eta} F(\xi) \\ \therefore u &= \int \frac{1}{\eta} \, \mathrm{d}\eta F(\xi) \\ &= \ln \eta F(\xi) + G(\xi) \end{aligned}$$

Therefore,

$$u(x, y) = F(xy) \ln x + G(xy)$$

2 Laplace and Poisson Equations

Definition 12 (Laplacian)**.**

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is called the Laplacian.

Definition 13 (Laplace equation)**.** The equation

$$\Delta u = 0$$

is called the Laplace equation.

Definition 14 (Harmonic function)**.** A function which satisfies the Laplace equation is called a harmonic function.

Definition 15. The equation

$$\Delta u = F(x, y)$$

is called the Poisson equation.

Definition 16 (Dirichlet problem)**.** Let D be an open and bounded domain in $\mathbb{R}^2$. Let ∂D be the boundary of the domain D. Then, the problem

$$\begin{aligned} \Delta u &= F(x, y) \\ u(x, y) &= f(x, y) \end{aligned}$$

for all $(x, y) \in D$ is called the Dirichlet problem.
The boundary condition

$$u(x, y) = f(x, y)$$

is called the Dirichlet boundary condition.

2.1 Maximum Principle

Theorem 9 (Maximum Principle)**.** *Let u be continuous on a bounded and closed domain $D \cup \partial D$, twice differentiable on the open domain D and satisfying the Poisson equation*

$$\Delta u(x, y) = F(x, y)$$

Let

$$F \geq 0$$

on D.
Then, the maximum value of u in $D \cup \partial D$ is on the boundary ∂D.

Proof. Let

$$F > 0$$

Therefore,

$$\Delta u > 0$$

If there is a point of maximum inside the domain, then at this point,

$$\begin{aligned} u_x &= 0 \\ u_y &= 0 \\ u_{xx} &\leq 0 \\ u_{yy} &\leq 0 \end{aligned}$$

Therefore,

$$\begin{aligned} u_{xx} + u_{yy} &\leq 0 \\ \therefore \Delta u &\leq 0 \end{aligned}$$

This contradicts the assumption that

$$\Delta u > 0$$

Therefore, there cannot be a point of maximum inside the domain.
Let

$$F \geq 0$$

Let

$$M = \max_{(x,y) \in \partial D} u(x, y)$$

Let

$$w(x, y) = x^2 + y^2$$

Therefore,

$$\Delta w = 4$$

Let

$$v(x, y) = u(x, y) + \varepsilon w(x, y)$$

for all $\varepsilon > 0$.
Therefore,

$$\begin{aligned}\Delta v &= \Delta u + e\Delta w\\ &= F + 4\varepsilon\\ &> 0\end{aligned}$$

Therefore, as there cannot be a point of maximum inside the domain, $\forall (x, y) \in D$,

$$\begin{aligned}v(x, y) &\leq \max_{(x,y)\in\partial D} v(x, y)\\ &\leq \max_{(x,y)\in\partial D} u(x, y) + \max_{(x,y)\in\partial D} v(x, y)\\ &\leq \max_{(x,y)\in\partial D} u(x, y) + \max_{(x,y)\in\partial D} x^2 + y^2\end{aligned}$$

Let

$$x^2 + y^2 \leq {R_0}^2$$

in the domain D.
Therefore,

$$v(x, y) \leq M + \varepsilon {R_0}^2$$

Therefore,

$$\begin{aligned}u(x, y) &\leq v(x, y)\\ &\leq M + \varepsilon {R_0}^2\end{aligned}$$

Therefore, as $\varepsilon \to 0$, $\forall (x, y) \in D$,

$$u(x, y) \leq M$$

$\square$

Theorem 10. *Let u be continuous on a bounded and closed domain $D \cup \partial D$, twice differentiable on the open domain D and satisfying the Poisson equation*

$$\Delta u(x, y) = F(x, y)$$

Let

$$F \leq 0$$

on D.
Then, the minimum value of u in $D \cup \partial D$ is on the boundary ∂D.

Proof.

$$\Delta u(x, y) = F(x, y)$$

Therefore,

$$-\Delta u(x, y) = -F(x, y)$$

Therefore, $-u$ gets its maximum value on the boundary. Therefore, u gets its minimum value on the boundary. □

Theorem 11. *Let u be continuous on a bounded and closed domain $D \cup \partial D$, twice differentiable on the open domain D and satisfying the Poisson equation*

$$\Delta u(x, y) = F(x, y)$$

Let

$$F = 0$$

on D.
Then, the minimum value and the maximum value of u in $D \cup \partial D$ are on the boundary ∂D, i.e. if

$$\max_{(x,y)\in\partial D} u = M$$

$$\min_{(x,y)\in\partial D} u = m$$

then

$$m \leq u(x, y) \leq M$$

in D.

Theorem 12. *If there exists a solution to the Dirichlet problem*

$$\begin{aligned} \Delta u &= F(x, y) \\ u(x, y) &= f(x, y) \end{aligned}$$

where F is defined on D and f is defined on ∂D, then the problem is well-posed.

Proof. If possible let u_1 and u_2 be two solutions to

$$\begin{aligned} \Delta u &= F(x, y) \\ u(x, y) &= f(x, y) \end{aligned}$$

Therefore,

$$\begin{aligned}
\Delta u_1 &= F(x,y)\\
u_1(x,y) &= f(x,y)\\
\Delta u_2 &= F(x,y)\\
u_2(x,y) &= f(x,y)
\end{aligned}$$

Let

$$v(x,y) = u_1(x,y) - u_2(x,y)$$

Therefore,

$$\begin{aligned}
\Delta v_1 &= 0\\
v(x,y) &= 0
\end{aligned}$$

Therefore, u gets it minimum and maximum values on the boundary ∂D, but on ∂D,

$$v(x,y) = 0$$

Therefore,

$$0 \leq v(x,y) \leq 0$$

Therefore,

$$v(x,y) \equiv 0$$

$\square$

Exercise 10.
Let

$$D = (-1,1) \times (-1,1)$$

Let $u \in \mathcal{C}^2(D) \cap \mathcal{C}\left(\overline{D}\right)$ be a solution to the Dirichlet problem

$$\Delta u = -1$$

where $(x,y) \in D$, and

$$u(x,y) = 0$$

where $(x,y) \in \partial D$.
Prove

$$\frac{1}{4} \leq u(0,0) \leq \frac{1}{2}$$

Hint: Define the function

$$v(x,y) = u(x,y) + \frac{1}{4}\left(x^2 + y^2\right)$$

Solution 10.

$$\begin{aligned}
\Delta v &= \Delta u + \frac{1}{4}\Delta\left(x^2 + y^2\right) \\
&= -1 + 1 \\
&= 0
\end{aligned}$$

Therefore, by the Maximum Principle, the maximum and minimum of $v(x,y)$ are on the boundary of D.
Therefore, as $u(x,y)$ is zero on the boundary of D,

$$\begin{aligned}
\max_{(x,y)\in\partial D} v(x,y) &= \max_{(x,y)\in\partial D} \frac{1}{4}\left(x^2 + y^2\right) \\
&= \frac{1}{2} \\
\min_{(x,y)\in\partial D} v(x,y) &= \min_{(x,y)\in\partial D} \frac{1}{4}\left(x^2 + y^2\right) \\
&= \frac{1}{4}
\end{aligned}$$

Therefore, $\forall (x,y) \in D$,

$$\frac{1}{4} \leq v(x,y) \leq \frac{1}{2}$$

Also,

$$v(0,0) = u(0,0)$$

Therefore,

$$\frac{1}{4} \leq u(0,0) \leq \frac{1}{2}$$

2.2 Solution of the Laplace Equation in a Rectangular Domain

Theorem 13. *A solution to*

$$\Delta u = 0$$

for all $(x,y) \in D$*, where* $D = [0,a] \times [0,b]$*, with boundary conditions*

$$\begin{aligned} u(x,0) &= \varphi_0(x) \\ u(x,b) &= \varphi_1(x) \\ u(0,y) &= \psi_0(y) \\ u(a,y) &= \psi_1(y) \end{aligned}$$

is

$$u(x,y) = v(x,y) + w(x,y)$$

where

$$\begin{aligned} v(x,y) &= \sum_{n=1}^{\infty} \left(A_n \sinh\left(\frac{n\pi}{a} y\right) + B_n \sinh\left(\frac{n\pi}{a}(b-y)\right)\right) \sin\left(\frac{n\pi}{a} x\right) \\ w(x,y) &= \sum_{n=1}^{\infty} \left(C_n \sinh\left(\frac{n\pi}{b} x\right) + D_n \sinh\left(\frac{n\pi}{b}(a-x)\right)\right) \sin\left(\frac{n\pi}{a} y\right) \end{aligned}$$

where

$$\begin{aligned} A_n &= \frac{2}{a \sinh\left(\frac{n\pi}{a} b\right)} \int_0^a \varphi_1(x) \sin\left(\frac{n\pi}{a} x\right) \mathrm{d}x \\ B_n &= \frac{2}{a \sinh\left(\frac{n\pi}{a} b\right)} \int_0^a \varphi_0(x) \sin\left(\frac{n\pi}{a} x\right) \mathrm{d}x \\ C_n &= \frac{2}{b \sinh\left(\frac{n\pi}{b} a\right)} \int_0^b \psi_1 \sin\left(\frac{n\pi}{b} y\right) \mathrm{d}y \\ D_n &= \frac{2}{b \sinh\left(\frac{n\pi}{b} a\right)} \int_0^b \psi_0 \sin\left(\frac{n\pi}{b} y\right) \mathrm{d}y \end{aligned}$$

Proof. Let

$$\begin{aligned}
\Delta v &= 0 \\
v(x,0) &= \varphi_0(x) \\
v(x,b) &= \varphi_1(x) \\
v(0,y) &= 0 \\
v(a,y) &= 0 \\
\Delta w &= 0 \\
w(x,0) &= 0 \\
w(x,b) &= 0 \\
w(0,y) &= \psi_0(y) \\
w(a,y) &= \psi_1(y)
\end{aligned}$$

for all $(x,y) \in D$.
Therefore,

$$u(x,y) = v(x,y) + w(x,y)$$

is a solution to the problem.
Let

$$V(x,y) = X(x)Y(y)$$

Therefore, substituting, let

$$\begin{aligned}
-\frac{X''(x)}{X(x)} &= \lambda \\
\frac{Y''(y)}{Y(y)} &= \lambda
\end{aligned}$$

Therefore,

$$\begin{aligned}
X''(x) + \lambda X(x) &= 0 \\
X(0) &= 0 \\
X(a) &= 0
\end{aligned}$$

Therefore, the eigenvalues of the Strum-Liouville problem are

$$\begin{aligned}
\lambda_n &= {\omega_n}^2 \\
&= \left(\frac{n\pi}{a}\right)^2
\end{aligned}$$

Therefore,

$$X_n(x) = \sin\left(\frac{n\pi}{a}x\right)$$

Similarly,

$$\begin{aligned} Y''(y) - \lambda Y(y) &= 0 \\ \therefore Y''(y) - \left(\frac{n\pi}{a}\right)^2 Y(y) &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} Y_n(y) &= \widetilde{A}_n e^{\frac{n\pi}{a}y} + \widetilde{B}_n e^{-\frac{n\pi}{a}y} \\ &= A_n \sinh\left(\frac{n\pi}{a}y\right) + B_n \sinh\left(\frac{n\pi}{a}(b-h)\right) \end{aligned}$$

Therefore,

$$v(x,y) = \sum_{n=1}^{\infty} \left(A_n \sinh\left(\frac{n\pi}{a}y\right) + B_n \sinh\left(\frac{n\pi}{a}(b-y)\right)\right) \sin\left(\frac{n\pi}{a}x\right)$$

Therefore, substituting the boundary conditions,

$$\begin{aligned} v(x,0) &= \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi}{a}b\right) \sin\left(\frac{n\pi}{a}x\right) \\ \therefore \varphi_0(x) &= \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi}{a}b\right) \sin\left(\frac{n\pi}{a}x\right) \\ v(x,b) &= \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a}b\right) \sin\left(\frac{n\pi}{a}x\right) \\ \varphi_1(x) &= \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a}b\right) \sin\left(\frac{n\pi}{a}x\right) \end{aligned}$$

Therefore, multiplying both sides by $\sin\left(\frac{k\pi}{a}x\right)$ and integrating from 0 to a,

$$\begin{aligned} \int_0^a \varphi_0(x) \sin\left(\frac{k\pi}{a}x\right) \mathrm{d}x &= B_k \sinh\left(\frac{k\pi}{a}b\right) \frac{a}{2} \\ \int_0^a \varphi_1(x) \sin\left(\frac{k\pi}{a}x\right) \mathrm{d}x &= A_k \sinh\left(\frac{k\pi}{a}b\right) \frac{a}{2} \end{aligned}$$

Therefore,

$$A_n = \frac{2}{a \sinh\left(\frac{n\pi}{a}b\right)} \int\limits_0^a \varphi_1(x) \sin\left(\frac{n\pi}{a}x\right) \mathrm{d}x$$

$$B_n = \frac{2}{a \sinh\left(\frac{n\pi}{a}b\right)} \int\limits_0^a \varphi_0(x) \sin\left(\frac{n\pi}{a}x\right) \mathrm{d}x$$

Similarly,

$$w(x, y) = \sum_{n=1}^{\infty} \left(C_n \sinh\left(\frac{n\pi}{b}x\right) + D_n \sinh\left(\frac{n\pi}{b}(a - x)\right)\right) \sin\left(\frac{n\pi}{a}y\right)$$

where

$$C_n = \frac{2}{b \sinh\left(\frac{n\pi}{b}a\right)} \int\limits_0^b \psi_1 \sin\left(\frac{n\pi}{b}y\right) \mathrm{d}y$$

$$D_n = \frac{2}{b \sinh\left(\frac{n\pi}{b}a\right)} \int\limits_0^b \psi_0 \sin\left(\frac{n\pi}{b}y\right) \mathrm{d}y$$

□

Exercise 11.
Solve

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \\ u(x, 0) &= \sin^3 x \\ u(x, 1) &= \sin^3 x \\ u(0, y) &= 0 \\ u(\pi, y) &= 0 \end{aligned}$$

where $0 \leq x \leq \pi$ and $0 \leq y \leq 1$.

Solution 11.
Let

$$u(x, y) = \psi(x)\varphi(y)$$

Therefore,

$$\psi''(x)\varphi(y) + \psi(x)\varphi''(y) = 0$$

Therefore,

$$\frac{\varphi''(y)}{\varphi(y)} = \lambda$$
$$\frac{\psi''(x)}{\psi(x)} = \lambda$$

Therefore,

$$\begin{aligned} \psi''(x) + \lambda\psi(x) &= 0 \\ \psi(0) &= 0 \\ \psi(\pi) &= 0 \end{aligned}$$

Therefore, for $\lambda \leq 0$, there exists a trivial solution.
If $\lambda > 0$,

$$\psi(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c^2 \sin\left(\sqrt{\lambda}x\right)$$

Substituting the boundary conditions and solving,

$$c_1 = 0$$

Therefore,

$$\begin{aligned} \lambda_n &= n^2 \\ \psi_n(x) &= \sin(nx) \end{aligned}$$

Similarly,

$$\varphi_n''(y) + n^2\varphi_n = 0$$

Therefore,

$$\varphi_n(y) = A_n \sinh\big(n(1-y)\big) + B_n \sinh(ny)$$

Therefore,

$$u(x,y) = \sum_{n=1}^{\infty} \Big(A_n \sinh\big(n(1-y)\big) + B_n \sinh(ny)\Big) \sin(nx)$$

Therefore, substituting the boundary conditions,

$$A_n = \begin{cases} \frac{3}{4\sinh(1)} & ; \quad n = 1 \\ -\frac{1}{4\sinh(3)} & ; \quad n = 3 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$B_n = \begin{cases} \frac{3}{4\sinh(1)} & ; \quad n = 1 \\ -\frac{1}{4\sinh(3)} & ; \quad n = 3 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} u(x,y) = & \left(\frac{3}{4\sinh(1)}\sinh(1-y) + \frac{3}{4\sinh(1)}\sinh(y)\right)\sin(x) \\ & - \left(\frac{1}{4\sinh(3)}\sinh(3-3y) + \frac{1}{4\sinh(3)}\sinh(3y)\right)\sin(3x) \end{aligned}$$

2.3 Solution of the Laplace Equation in a Disk

Theorem 14. *A solution to*

$$\Delta u = 0$$

for all $(x,y) \in D$, *where* $D = \left\{(x,y) | x^2 + y^2 < {R_0}^2\right\}$, *with boundary conditions*

$$u(x,y) = f(x,y)$$

where $(x,y) \in \partial D$, *is*

$$u(r,\theta) = C_0 + D_0 \ln(r) + \sum_{n=1}^{\infty} \left(C_n r^n + D_n r^{-n}\right)\left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)$$

where

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\,\mathrm{d}\theta$$

$$A_n = \frac{1}{\pi {R_0}^n}\int_0^{2\pi} h(\theta)\cos(n\theta)\,\mathrm{d}\theta$$

$$B_n = \frac{1}{\pi {R_0}^n}\int_0^{2\pi} h(\theta)\sin(n\theta)\,\mathrm{d}\theta$$

Proof.

$$\Delta u = u_{xx} + u_{yy}$$

Therefore, substituting the polar form of the coordinates,

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

Let

$$u(r, \theta) = R(r)\Theta(\theta)$$

Therefore, substituting,

$$\begin{aligned} 0 &= \Delta u \\ &= R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) \end{aligned}$$

Therefore, multiplying by r^2 and dividing by $R(r)\Theta(\theta)$,

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$

Therefore,

$$\begin{aligned} r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} &= \lambda \\ -\frac{\Theta''(\theta)}{\Theta(\theta)} &= \lambda \end{aligned}$$

Therefore,

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0$$

Therefore,

$$\lambda_n = n^2$$

Therefore,

$$\Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)$$

where $\Theta(\theta)$ is 2π periodic.
Therefore,

$$r^2R''(r) + rR'(r) - n^2R(r) = 0$$

Therefore, the characteristic equation for this Euler's ODE is

$$r^2\alpha(\alpha-1)r^{\alpha-2} + r\alpha r^{\alpha-1} + \left(-n^2\right) r^{\alpha} = 0$$

where

$$\alpha = \pm n$$

If $n = 0$,

$$rR''(r) + rR'(r) = 0$$

Therefore,

$$R_0(r) = C_0 + D_0 \ln(r)$$

If $n \neq 0$,

$$R_n(r) = C_n r^n + D_n r^{-n}$$

Therefore,

$$\begin{aligned} u(r,\theta) &= \sum_{n=0}^{\infty} R_n(r)\Theta_n(\theta) \\ &= C_0 + D_0 \ln(r) + \sum_{n=1}^{\infty} \left(C_n r^n + D_n r^{-n}\right)\left(A_n \cos(n\theta) + B_n \sin(n\theta)\right) \end{aligned}$$

As $\ln(r)$ and r^{-n} are undefined at $r = 0$, the solution to the equation for a region inside a disk is

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right) r^n$$

As $\ln(r)$ and r^n are unbounded as $r \to \infty$, the solution to the equation for a region outside a disk is

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right) r^{-n}$$

For an annular region,

$$u(r,\theta) = C_0 + D_0 \ln(r) + \sum_{n=1}^{\infty} \left(C_n r^n + D_n r^{-n}\right)\left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)$$

Let the boundary condition in polar coordinates be

$$u(R_0, \theta) = h(\theta)$$

Therefore, solving using the Fourier Series method,

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) \, \mathrm{d}\theta$$

$$A_n {R_0}^n = \frac{1}{\pi} \int_0^{2\pi} h(\theta) \cos(n\theta) \, \mathrm{d}\theta$$

$$B_n {R_0}^n = \frac{1}{\pi} \int_0^{2\pi} h(\theta) \sin(n\theta) \, \mathrm{d}\theta$$

Therefore,

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) \, \mathrm{d}\theta$$

$$A_n = \frac{1}{\pi {R_0}^n} \int_0^{2\pi} h(\theta) \cos(n\theta) \, \mathrm{d}\theta$$

$$B_n = \frac{1}{\pi {R_0}^n} \int_0^{2\pi} h(\theta) \sin(n\theta) \, \mathrm{d}\theta$$

□

Definition 17 (Poisson kernel)**.** The expression

$$I = \frac{{R_0}^2 - r^2}{{R_0}^2 - 2R_0 r \cos(\theta - \psi) + r^2}$$

is called the Poisson kernel.

Definition 18.

$$G(r, \theta - \psi) = \frac{1}{2\pi} I$$

where I is the Poisson kernel, is called the Green function for the Dirichlet problem of the Laplace equation in a disk.

Theorem 15. *The solution to*

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$
$$u(R_0, \theta) = h(\theta)$$

where $0 \leq r \leq R_0$, $0 \leq \theta \leq 2\pi$, *is*

$$u(r, \theta) = \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{{R_0}^2 - r^2}{{R_0}^2 2R_0 r\cos(\theta - \psi) + r^2} h(\psi)\,\mathrm{d}\psi$$

and the boundary condition is

$$h(\theta_0) = \lim_{(r,\theta)\to(R_0,\theta_0)} u(r, \theta)$$

Exercise 12.
Solve

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$
$$u(r, 0) = 0$$
$$u_\theta\left(r, \frac{\pi}{2}\right) = 0$$
$$u(1, \theta) = \theta$$

where $0 \leq r \leq 1$, $0 \leq \theta \leq \frac{\pi}{2}$.

Solution 12.
Let

$$u(r, \theta) = \Psi(r)\Phi(\theta)$$

Therefore,

$$\Psi''(r)\Phi(\theta) + \frac{1}{r}\Psi'(r)\Phi(\theta) + \frac{1}{r^2}\Psi(r)\Phi''(\theta) = 0$$

Therefore,

$$\frac{r^2\Psi''(r) + r\Psi'(r)}{\Psi(r)} = \lambda$$
$$-\frac{\Phi''(\theta)}{\Phi(\theta)} = \lambda$$

Therefore,

$$\Phi''(\theta) + \lambda\Phi(\theta) = 0$$
$$\Phi(0) = 0$$
$$\Phi'\left(\frac{\pi}{2}\right) = 0$$

For $\lambda \leq 0$, there exists a trivial solution. If $\lambda > 0$,

$$\Phi(\theta) = c_1 \cos\left(\sqrt{\lambda}\theta\right) + c_2 \sin\left(\sqrt{\lambda}\theta\right)$$

Substituting the boundary condition $\Phi(0) = 0$,

$$c_1 = 0$$

Therefore,

$$\Phi(\theta) = c_2 \sin\left(\sqrt{\lambda}\theta\right)$$

Therefore,

$$\Phi'(\theta) = c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\theta\right)$$

Therefore, substituting the boundary condition $\Phi'\left(\frac{\pi}{2}\right) = 0$,

$$\sqrt{\lambda}\frac{\pi}{2} = \frac{\pi}{2} + k\pi$$

Therefore,

$$\sqrt{\lambda_k} = 1 + 2k$$

Therefore, the eigenvalues are

$$\lambda_k = (1 + 2k)^2$$

and the corresponding eigenfunctions are

$$\Phi_k(\theta) = \sin\big((1 + 2k)\varphi\big)$$

Therefore,

$$\Psi_k(r) = A_k r^{1+2k} + B_k e^{-(1+2k)}$$

As the solution is continuous at $r = 0$,

$$B_k = 0$$

Therefore,

$$\Psi_k(r) = B_k e^{-(1+2k)}$$

Therefore,

$$\begin{aligned} u(r,\theta) &= \sum_{k=0}^{\infty} \Psi_k(r)\Phi_k(\theta) \\ &= \sum_{k=0}^{\infty} A_k e^{2k+1} \sin\big((2k+1)\theta\big) \end{aligned}$$

Therefore, substituting the boundary condition $u(1,\theta) = \theta$,

$$A_n = \frac{4}{\pi}\frac{(-1)^n}{(2n+1)^2}$$

Therefore,

$$u(r,\theta) = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} r^{2k+1} \sin\big((2k+1)\theta\big)$$

2.4 Ill-posedness of the Cauchy Problem for the Laplace-Hadamard Example

Theorem 16. *The problem*

$$\begin{aligned} \Delta u &= 0 \\ u(x,0) &= f(x) \\ u_y(x,0) &= g(x) \end{aligned}$$

is ill-posed, i.e. small changes in the initial conditions may result in large changes in the solution.

Proof.

$$\begin{aligned} \Delta u &= 0 \\ u(x,0) &= f(x) \\ u_y(x,0) &= g(x) \end{aligned}$$

Therefore, let

$$\begin{aligned}
f_n(x) &= 0 \\
g_n(x) &= \frac{\cos(nx)}{n}
\end{aligned}$$

Therefore,

$$u_n(x, y) = \frac{1}{n^2} \cos(nx) \sinh(ny)$$

is a solution to the problem.
Therefore,

$$\begin{aligned}
\lim_{n\to\infty} f_n(x) &= 0 \\
\lim_{n\to\infty} g_n(x) &= 0
\end{aligned}$$

However,

$$\lim_{n\to\infty} u_n(x, y) \neq 0$$

Therefore, the problem is ill-posed. $\square$

3 Green's Formula

Theorem 17. *Let $\frac{\partial u}{\partial n}$ be the directional derivative of $u(x, y)$ with respect to the unit outward normal $\hat{n}$ to ∂D. Then,*

$$\iint\limits_D \Delta u \, \mathrm{d}x \, \mathrm{d}y = \int\limits_{\partial D} \frac{\partial u}{\partial n} \, \mathrm{d}s$$

Proof.

$$\begin{aligned}
\operatorname{div}(\nabla u) &= \operatorname{div}\big((u_x, u_y)\big) \\
&= (u_x)_x + (u_y)_y \\
&= u_{xx} + u_{yy} \\
&= \Delta u
\end{aligned}$$

Therefore,

$$\iint\limits_D \Delta u \, \mathrm{d}x \, \mathrm{d}y = \iint\limits_D \operatorname{div}(\nabla u) \, \mathrm{d}x \, \mathrm{d}y$$

Therefore, by Green's Theorem,

$$\iint\limits_D \operatorname{div}(\nabla u)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{\partial D} \nabla u \cdot \hat{n}\,\mathrm{d}s$$
$$= \int\limits_{\partial D} \frac{\partial u}{\partial n}\,\mathrm{d}s$$

□

Theorem 18 (First Green's Formula)**.** *Let* $\dfrac{\partial u}{\partial n}$ *be the directional derivative of* $u(x,y)$ *with respect to the unit outward normal* $\hat{n}$ *to* ∂D*. Then,*

$$\iint\limits_D u\Delta v\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{\partial D} u\,\frac{\partial v}{\partial n}\,\mathrm{d}s - \iint\limits_D \nabla u\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}y$$

Proof.

$$\begin{aligned}\operatorname{div}(u\nabla v) &= \operatorname{div}\big((uv_x, uv_y)\big)\\ &= (uv_x)_x + (uv_y)_y\\ &= u_xv_x + uv_{xx} + u_yv_y + uv_{yy}\\ &= u\Delta v + \nabla u\cdot\nabla v\end{aligned}$$

Therefore,

$$\begin{aligned}\iint\limits_D u\Delta v\,\mathrm{d}x\,\mathrm{d}y &= \iint\limits_D \operatorname{div}(u\nabla)\,\mathrm{d}x\,\mathrm{d}y - \iint\limits_D \nabla u\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}y\\ &= \iint\limits_D u\nabla v\cdot\hat{n}\,\mathrm{d}s - \iint\limits_D \nabla u\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}y\\ &= \int\limits_{\partial D} \frac{\partial v}{\partial n}\,\mathrm{d}s - \iint\limits_D \nabla u\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

□

Theorem 19 (Second Green's Formula)**.** *Let* $\dfrac{\partial u}{\partial n}$ *be the directional derivative of* $u(x,y)$ *with respect to the unit outward normal* $\hat{n}$ *to* ∂D*. Then,*

$$\iint\limits_D (u\Delta v - v\Delta u)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{\partial D} \left(u\,\frac{\partial v}{\partial n} - v\,\frac{\partial u}{\partial n}\right)\mathrm{d}s$$

Proof. By First Green's Formula,

$$\iint\limits_D u\Delta v \, \mathrm{d}x \, \mathrm{d}y = \int\limits_{\partial D} u \frac{\partial v}{\partial n} \, \mathrm{d}s - \iint\limits_D \nabla u \cdot \nabla v \, \mathrm{d}x \, \mathrm{d}y$$

Therefore, replacing u by v, and v by u,

$$\iint\limits_D v\Delta u \, \mathrm{d}x \, \mathrm{d}y = \int\limits_{\partial D} v \frac{\partial u}{\partial n} \, \mathrm{d}s - \iint\limits_D \nabla v \cdot \nabla u \, \mathrm{d}x \, \mathrm{d}y$$

Therefore, subtracting,

$$\iint\limits_D (u\Delta v - v\Delta u) \, \mathrm{d}x \, \mathrm{d}y = \int\limits_{\partial D} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) \mathrm{d}s$$

□

4 Neumann Problem of the Poisson Equation

Theorem 20. *A necessary condition for existence of a solution to the Neumann problem of the Poisson equation*

$$\Delta u = F(x, y)$$

for $(x, y) \in D$, *with boundary condition*

$$\frac{\partial u(x, y)}{\partial n} = g(x, y)$$

for $(x, y) \in \partial D$ *is*

$$\iint\limits_D F(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int\limits_{\partial D} g(x, y) \, \mathrm{d}s$$

Proof. By First Green's Formula,

$$\iint\limits_D \Delta u \, \mathrm{d}x \, \mathrm{d}y = \int\limits_{\partial D} \frac{\partial u}{\partial n} \, \mathrm{d}s$$

Therefore, substituting the functions of the Neumann problem,

$$\iint\limits_D F(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int\limits_{\partial D} g(x, y) \, \mathrm{d}s$$

□

Theorem 21. *A sufficient condition for existence of a solution to the Neumann problem of the Poisson equation*

$$\Delta u = F(x, y)$$

for $(x, y) \in D$*, with boundary condition*

$$\frac{\partial u(x, y)}{\partial n} = g(x, y)$$

for $(x, y) \in \partial D$ *is*

$$\iint\limits_{D} F(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{\partial D} g(x, y)\,\mathrm{d}s$$

Theorem 22. *Let* $u_1(x, y)$ *and* $u_2(x, y)$ *be solutions to the Neumann problem of the Poisson equation*

$$\Delta u = F(x, y)$$

for $(x, y) \in D$*, with boundary condition*

$$\frac{\partial u(x, y)}{\partial n} = g(x, y)$$

for $(x, y) \in \partial D$*.*
Then,

$$u_1(x, y) - u_2(x, y) = c$$

where c *is a constant.*

Proof. Let

$$v(x, y) = u_1(x, y) - u_2(x, y)$$

Therefore,

$$\Delta v = 0$$

for $(x, y) \in D$, and

$$\frac{\partial v}{\partial n} = 0$$

for $(x, y) \in \partial D$.
Therefore, substituting v in First Green's Formula, for both u and v,

$$\iint_D v\Delta v \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial D} v\,\frac{\partial v}{\partial n}\,\mathrm{d}s - \iint_D |\nabla v|^2 \,\mathrm{d}x\,\mathrm{d}y$$
$$\therefore 0 = -\iint_D |\nabla v|^2 \,\mathrm{d}x\,\mathrm{d}y$$

Therefore,

$$\nabla v = 0$$

Therefore,

$$v_x = 0$$
$$v_y = 0$$

Therefore,

$$v = c(y)$$

Therefore,

$$v_y = c'(y)$$
$$\therefore 0 = c'(y)$$

Therefore,

$$c(y) = c$$

Therefore,

$$v(x, y) = c$$

Therefore,

$$u_1(x, y) - u_2(x, y) = c$$

□

Theorem 23. *The solution to the Dirichlet problem of the Poisson equation*

$$\Delta u = F(x, y)$$

for $(x, y) \in D$, *and*

$$u(x, y) = f(x, y)$$

for $(x, y) \in \partial D$ *is unique.*

Proof. Let $u_1(x, y)$ and $u_2(x, y)$ be solutions to the Dirichlet problem of the Poisson equation. Let

$$v(x, y) = u_1(x, y) + u_2(x, y)$$

Therefore, substituting v in First Green's Formula, for both u and v,

$$\iint\limits_D v\Delta v \,\mathrm{d}x \,\mathrm{d}y = \int\limits_{\partial D} v \frac{\partial v}{\partial n} \,\mathrm{d}s - \iint\limits_D |\nabla v|^2 \,\mathrm{d}x \,\mathrm{d}y$$
$$\therefore 0 = -\iint\limits_D |\nabla v|^2 \,\mathrm{d}x \,\mathrm{d}y$$

Therefore,

$$\nabla v = 0$$

for $(x, y) \in D$, and

$$v(x, y) = 0$$

for $(x, y) \in \partial D$.
Therefore,

$$v_x = 0$$
$$v_y = 0$$

Therefore, integrating,

$$v(x, y) = c(y)$$

Therefore,

$$v_y = c'(y)$$
$$\therefore 0 = c'(y)$$

Therefore,

$$v(x, y) = c$$

Also, $\forall (x, y) \in \partial D$,

$$v(x, y) = 0$$

Therefore,

$$c = 0$$

Therefore, $\forall (x, y)$,

$$v(x, y) = 0$$

$\square$

Exercise 13.
Prove that there is no solution to the problem

$$\Delta u = 10$$

for $x^2 + y^2 < 4$, and

$$\frac{\partial u}{\partial n} = 7$$

for $x^2 + y^2 = 4$.

Solution 13.
Comparing to the standard form,

$$\begin{aligned} F(x, y) &= 10 \\ g(x, y) &= 7 \end{aligned}$$

If there exists a solution to the problem,

$$\begin{aligned} \iint\limits_{D} F(x, y) \, \mathrm{d}x \, \mathrm{d}y &= \int\limits_{\partial D} g(x, y) \, \mathrm{d}s \\ \iff \iint\limits_{x^2+y^2<4} 10 \, \mathrm{d}x \, \mathrm{d}y &= \int\limits_{x^2+y^2=4} 7 \, \mathrm{d}s \\ \iff 10 \left(\pi 2^2\right) &= 7 \left((2\pi)(2)\right) \\ \iff 40\pi &= 28\pi \end{aligned}$$

Therefore, there is no solution to the problem.

Part III
Heat Equation

1 Maximum and Minimum Principles

Definition 19 (Heat equation)**.** An equation

$$u_t = a^2 u_{xx} + F(x,t)$$

where $0 \leq x \leq l$, $0 \leq t \leq t_1$, with initial condition

$$u(x,0) = f(x)$$

for $0 \leq x \leq l$, and boundary condition

$$\begin{aligned} u(0,t) &= \mu(t) \\ u(l,t) &= \nu(t) \end{aligned}$$

for $0 \leq t \leq t_1$, is said to be a non-homogeneous heat equation.
The function $u(x,t)$ represents the temperature of a wire from 0 to l, at position x at time t.

Theorem 24 (Maximum Principle for Heat Equation)**.** *Let $u(x,t)$ and $F(x,t)$ be continuous on the domain $0 \leq x \leq l$, $0 \leq t \leq t_1$.*
$\forall (x,y)$, let

$$F(x,t) \leq 0$$

Let $u(x,t)$ satisfy

$$u_t = a^2 u_{xx} + F(x,t)$$

Let

$$u(x,t) \leq M$$

for $t = 0$, $x = 0$, or $x = l$.
Then, for $0 \leq x \leq l$, $0 \leq t \leq t_1$,

$$u(x,t) \leq M$$

that is, the maximum of $u(x,t)$ is on the boundary of the domain, excluding the top boundary.

Proof. Let

$$F < 0$$

If possible, let the maximum of $u(x,t)$ be at a point (x,t) such that $0 < x < l$, $0 < t < t_1$. Therefore,

$$\begin{aligned} u_x(x,t) &= 0 \\ u_t(x,t) &= 0 \\ u_{xx} &\leq 0 \end{aligned}$$

Therefore,

$$\begin{aligned} u_t - a^2 u_{xx} &\geq 0 \\ \therefore F &\geq 0 \end{aligned}$$

This contradicts the assumption that $F < 0$. Therefore, the maximum of $u(x,t)$ cannot be at a point (x,t) such that $0 < x < l$, $0 < t < t_1$.
If possible, let the maximum of $u(x,t)$ be at a point (x,t_1) such that $0 < x < l$. Therefore,

$$\begin{aligned} u_x(x,t_1) &= 0 \\ {u_t}^-(x,t_1) &\geq 0 \\ u_{xx}(x,t_1) &\leq 0 \end{aligned}$$

Therefore,

$$\begin{aligned} u_t - a^2 u_{xx} &\geq 0 \\ \therefore F &\geq 0 \end{aligned}$$

This contradicts the assumption that $F < 0$. Therefore, the maximum of $u(x,t)$ cannot be at a point (x,t_1) such that $0 < x < l$.

Let

$$F \leq 0$$

Let

$$v(x,t) = u(x,t) + \varepsilon x^2$$

where $\varepsilon > 0$.
Therefore,

$$\begin{aligned} v_t - a^2 v_{xx} &= u_t - a^2(u_{xx} + 2\varepsilon) \\ &= F - 2a^2\varepsilon \end{aligned}$$

Let

$$F_1 = F - 2a^2\varepsilon$$

Therefore,

$$F_1 < 0$$

Therefore, for $v(x,t)$ and $F_1 < 0$,

$$\begin{aligned} u(x,t) &\le v(x,t) \\ &\le \max_{\{t=0\}\cup\{x=0\}\cup\{x=l\}} v(x,t) \\ &\le \max_{\{t=0\}\cup\{x=0\}\cup\{x=l\}} u(x,t) + \varepsilon l^2 \\ &\le M + \varepsilon l^2 \end{aligned}$$

Therefore, for $\varepsilon \to 0$,

$$u(x,t) \le M$$

for (x,t) in the entire domain. □

Theorem 25 (Minimum Principle for Heat Equation)**.** *Let $u(x,t)$ and $F(x,t)$ be continuous on the domain $0 \le x \le l$, $0 \le t \le t_1$.*
$\forall (x,y)$, let

$$F(x,t) \ge 0$$

Let $u(x,t)$ satisfy

$$u_t = a^2 u_{xx} + F(x,t)$$

Let

$$u(x,t) \ge m$$

for $t = 0$, $x = 0$, or $x = l$.
Then, for $0 \le x \le l$, $0 \le t \le t_1$,

$$u(x,t) \ge m$$

that is, the minimum of $u(x,t)$ is on the boundary of the domain, excluding the top boundary.

Theorem 26 (Maximum-minimum Principle for Heat Equation)**.** *Let $u(x,t)$ be continuous on the domain $0 \leq x \leq l$, $0 \leq t \leq t_1$.*
Let $u(x,t)$ satisfy

$$u_t = a^2 u_{xx} + F(x,t)$$

If

$$m \leq u(x,t) \leq M$$

for $t = 0$, $x = 0$, or $x = l$.
Then, for $0 \leq x \leq l$, $0 \leq t \leq t_1$,

$$m \leq u(x,t) \leq M$$

Exercise 14.
Find $\max\limits_{0 \leq x \leq 1, 0 \leq t \leq 1} u(x,t)$ for the solution of

$$\begin{aligned} u_t - t_{xx} &= -x \\ u(0,t) &= 0 \\ u(1,t) &= t \\ u(x,0) &= \frac{1}{2}\sin(\pi x) \end{aligned}$$

where $0 \leq x \leq 1$, $0 \leq t \leq 1$.

Solution 14.
Comparing to the standard form,

$$F(x,t) = -x$$

Therefore, for $0 \leq x \leq 1$, and $0 \leq t \leq 1$,

$$F(x,t) \leq 0$$

Therefore, by Maximum Principle for Heat Equation, $\max\limits_{0 \leq x \leq 1, 0 \leq t \leq 1} u(x,t)$ is on the boundary of the domain, excluding $t = 1$.
Therefore,

$$\begin{aligned} \max_{0 \leq x \leq 1, 0 \leq t \leq 1} u(x,t) &= \max\left\{\max_{0 \leq x \leq 1} \frac{1}{2}\sin(\pi x), \max_{0 \leq t \leq 1}(-t), \max_{0 \leq t \leq 1} 0\right\} \\ &= \max\left\{\frac{1}{2}, 0, 0\right\} \\ &= \frac{1}{2} \end{aligned}$$

Exercise 15.
Prove that

$$u(x,t) \leq 2$$

in the domain $0 \leq x \leq 1$, $t \geq 0$, for the solution of the problem

$$\begin{aligned} u_t - u_{xx} &= -e^{-t}\sin(\pi x) \\ u(0,t) &= 0 \\ u(1,t) &= 0 \\ u(x,0) &= 1 - x + \sin(2\pi x) \end{aligned}$$

Solution 15.
Comparing to the standard form,

$$F(x,t) = -e^{-t}\sin(\pi x)$$

Therefore, for $0 \leq x \leq 1$, and $t \geq 0$,

$$F(x,t) \leq 0$$

Therefore, by Maximum Principle for Heat Equation,

$$\begin{aligned} u(x,t) &\leq \max\left\{0, 0, \max_{0\leq x\leq 1}\big(1 - x + \sin(2\pi x)\big)\right\} \\ &\leq 2 \end{aligned}$$

2 Well-posedness

Theorem 27. *If there exists a solution to the problem*

$$\begin{aligned} u_t &= a^2 u_{xx} + F(x,t) \\ u(x,0) &= f(x) \\ u(0,t) &= \mu(t) \\ u(l,t) &= \nu(t) \end{aligned}$$

where $0 \leq x \leq l$, $0 \leq t \leq t_1$, *then the problem is well-posed.*

3 Separation of Variables

Theorem 28. *The solution to the problem*

$$\begin{aligned} u_t &= a^2 u_{xx} \\ u(x,0) &= f(x) \\ u(0,t) &= 0 \\ u(l,t) &= 0 \end{aligned}$$

where $0 \leq x \leq l$*,* $t \geq 0$*, is*

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{l} \int\limits_0^l f(x) \sin\left(\frac{n\pi}{l}x\right) \mathrm{d}x \right) e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin\left(\frac{n\pi}{l}x\right)$$

Proof. Let

$$u(x,t) = X(x)T(t)$$

Therefore, substituting into the equation,

$$X(x)T'(t) = a^2 X''(x)T(t)$$

Therefore, let

$$\begin{aligned} \frac{T'(t)}{a^2 T(t)} &= -\lambda \\ \frac{X''(x)}{X(x)} &= -\lambda \end{aligned}$$

Therefore,

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \\ X(0) &= 0 \\ X(l) &= 0 \end{aligned}$$

Therefore, the eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2$$

and the corresponding eigenfunctions are

$$X_n(x) = \sin\left(\frac{n\pi}{l}x\right)$$

where $n \in \mathbb{N}$.
Similarly,

$$T_n{}'(t) + \left(\frac{n\pi}{l}\right)^2 a^2 T_n(t) = 0$$

Therefore, solving,

$$T_n = A_n e^{-\left(\frac{n\pi a}{l}\right)^2 t}$$

where $n \in \mathbb{N}$.
Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin\left(\frac{n\pi}{l} x\right)$$

Therefore, substituting the initial condition and solving,

$$A_n = \frac{2}{l} \int\limits_0^l f(x) \sin\left(\frac{n\pi}{l} x\right) \mathrm{d}x$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{l} \int\limits_0^l f(x) \sin\left(\frac{n\pi}{l} x\right) \mathrm{d}x\right) e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin\left(\frac{n\pi}{l} x\right)$$

$\square$

4 Cauchy Problem for the Heat Equation

Theorem 29 (Solution to the Cauchy Problem for the Heat Equation)**.**

$$\begin{aligned} u_t &= a^2 u_{xx} \\ u(x,0) &= f(x) \end{aligned}$$

is the Poisson formula, i.e.

$$u(x,t) = \int\limits_{-\infty}^{\infty} G(x,y,t) f(y) \, \mathrm{d}y$$

where

$$G(x,y,t) = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4a^2 t}}$$

is the Green function for the Cauchy problem of the heat equation on an infinite interval.

Proof. The Fourier transform of $g(x)$ is

$$\hat{g}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-i\omega x} \, \mathrm{d}x$$

and the inverse Fourier transform of $\hat{g}(\omega)$ is

$$g(x) = \hat{g}(\omega) e^{i\omega x} \, \mathrm{d}\omega$$

Therefore,

$$\begin{aligned}
\hat{u_t}(\omega,t) &= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} u(x,t) e^{i\omega x} \, \mathrm{d}x \right)_t \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} u_t(x,t) e^{-i\omega x} \, \mathrm{d}x \\
&= \hat{u_t}(\omega,t)
\end{aligned}$$

Also,

$$\hat{g^{(n)}}(\omega) = (i\omega)^n \hat{g}(\omega)$$

Therefore, taking the Fourier transform of both sides of the PDE,

$$\begin{aligned}
\hat{u_t}(\omega,t) &= a^2 \omega^2 \hat{u}(\omega,t) \\
\therefore \hat{u_t}(\omega,t) &= a^2 \omega^2 \hat{u}(\omega,t)
\end{aligned}$$

Similarly, taking the Fourier transform of both sides of the initial condition,

$$\hat{u}(\omega,t) = \hat{f}(\omega)$$

Therefore, the problem is an ODE of $\hat{u}(\omega,t)$ in t.
Therefore, solving,

$$\hat{u}(\omega,t) = \hat{f}(\omega) e^{-a^2 \omega^2 t}$$

Therefore, taking the inverse Fourier transform,

$$\begin{aligned}
u(x,t) &= \int_{-\infty}^{\infty} \hat{u}(\omega,t)e^{i\omega x}\,\mathrm{d}\omega \\
&= \int_{-\infty}^{\infty} \hat{f}(\omega)e^{-a^2\omega^2 t}e^{i\omega x}\,\mathrm{d}\omega \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} f(y)e^{-i\omega y}\,\mathrm{d}y\right)e^{-a^2\omega^2 t}e^{i\omega x}\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\int_{-\infty}^{\infty} e^{a^2\omega^2 t}e^{i\omega(x-y)}\,\mathrm{d}\omega\,\mathrm{d}y
\end{aligned}$$

Therefore, for $t \neq 0$,

$$-a^2\omega^2 t + i\omega(x-y) = -\left(a\omega\sqrt{t} - \frac{i(x-y)}{2a\sqrt{t}}\right)^2 - \frac{(x-y)^2}{4a^2 t}$$

Therefore,

$$\int_{-\infty}^{\infty} e^{-a^2\omega^2 t}e^{i\omega(x-y)}\,\mathrm{d}\omega = e^{-\frac{(x-y)^2}{4a^2 t}}\int_{-\infty}^{\infty} e^{-\left(a\omega\sqrt{t}-\frac{i(x-y)}{2a\sqrt{t}}\right)}\,\mathrm{d}\omega$$

Let

$$s = a\omega\sqrt{t} - \frac{i(x-y)}{2a\sqrt{t}}$$

Therefore,

$$\mathrm{d}s = a\sqrt{t}\,\mathrm{d}\omega$$

Therefore,

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-a^2\omega^2 t}e^{i\omega(x-y)}\,\mathrm{d}\omega &= e^{-\frac{(x-y)^2}{4a^2 t}}\frac{1}{a\sqrt{t}}\int_{-\infty}^{\infty} e^{-s^2}\,\mathrm{d}s \\
&= e^{-\frac{(x-y)^2}{4a^2 t}}\frac{\sqrt{\pi}}{a\sqrt{t}}
\end{aligned}$$

Therefore, substituting,

$$\begin{aligned}
u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\int_{-\infty}^{\infty} e^{a^2\omega^2 t} e^{i\omega(x-y)}\,\mathrm{d}\omega\,\mathrm{d}y \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y) e^{-\frac{(x-y)^2}{4a^2t}} \frac{\sqrt{\pi}}{a\sqrt{t}}\,\mathrm{d}y \\
&= \frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4a^2t}} f(y)\,\mathrm{d}y
\end{aligned}$$

Let

$$G(x,y,t) = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4a^2t}}$$

Therefore,

$$u(x,t) = \int_{-\infty}^{\infty} G(x,y,t) f(y)\,\mathrm{d}y$$

□

Definition 20 (Gaussian error function)**.** The function

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}\,\mathrm{d}s$$

is called the Gaussian error function.

Theorem 30.

$$\operatorname{erf}(\infty) = 1$$

Exercise 16.
Solve

$$\begin{aligned}
u_t &= a^2 u_{xx} \\
u(x,0) &= \begin{cases} x & ; \quad x \geq 0 \\ 0 & ; \quad x < 0 \end{cases}
\end{aligned}$$

for $-\infty < x < \infty$, $t > 0$.

Solution 16.
By Solution to the Cauchy Problem for the Heat Equation,

$$\begin{aligned}
u(x,t) &= \frac{1}{2a\sqrt{\pi t}} \int\limits_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4a^2 t}} f(y)\,\mathrm{d}y \\
&= \frac{1}{2a\sqrt{\pi t}} \int\limits_{0}^{\infty} e^{-\frac{(x-y)^2}{4a^2 t}} y\,\mathrm{d}y
\end{aligned}$$

Let

$$z = \frac{x-y}{2a\sqrt{t}}$$

Therefore,

$$\mathrm{d}y = -2a\sqrt{t}\,\mathrm{d}z$$

Therefore,

$$\begin{aligned}
u(x,t) &= \frac{1}{2a\sqrt{\pi t}} \int\limits_{\frac{x}{2a\sqrt{t}}}^{-\infty} e^{-z^2} \left(x - 2az\sqrt{t}\right)\left(-2a\sqrt{t}\right) \mathrm{d}z \\
&= -\frac{1}{\sqrt{\pi}} \int\limits_{\frac{x}{2a\sqrt{t}}}^{-\infty} e^{-z^2} \left(x - 2az\sqrt{t}\right) \mathrm{d}z \\
&= \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\frac{x}{2a\sqrt{t}}} e^{-z^2} \left(z - 2az\sqrt{t}\right) \mathrm{d}z \\
&= \frac{x}{\sqrt{\pi}} \int\limits_{-\infty}^{\frac{x}{2a\sqrt{t}}} e^{-z^2}\,\mathrm{d}z + \frac{a\sqrt{t}}{\sqrt{\pi}} \int\limits_{-\infty}^{\frac{x}{2a\sqrt{t}}} e^{-z^2}(-2z)\,\mathrm{d}z \\
&= \frac{x}{\sqrt{\pi}} \left(\int\limits_{-\infty}^{0} e^{-z^2}\,\mathrm{d}z + \int\limits_{0}^{\frac{x}{2a\sqrt{t}}} e^{-z^2}\,\mathrm{d}z \right) + \frac{a\sqrt{t}}{\sqrt{\pi}} e^{-z^2} \Bigg|_{-\infty}^{\frac{x}{2a\sqrt{t}}} \\
&= \frac{x}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{x}{2a\sqrt{t}}\right)\right) + \frac{a\sqrt{t}}{\sqrt{\pi}} e^{-\frac{x^2}{4a^2 t}} \\
&= \frac{x}{2} \left(1 + \operatorname{erf}\left(\frac{x}{2a\sqrt{t}}\right)\right) + \frac{a\sqrt{t}}{\sqrt{\pi}} e^{-\frac{x^2}{4a^2 t}}
\end{aligned}$$