**PARTIAL DIFFERENTIAL EQUATIONS**
**ASSIGNMENT 2**

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**Exercise 1.**
Solve

$$\begin{aligned}
u_{tt} &= a^2 u_{xx}\\
u(x,0) &= \frac{x^2}{1+x^2}\\
u_t(x,0) &= 0\\
u(0,t) &= 0
\end{aligned}$$

where $0 \le x \le \infty$, and $t \ge 0$.

**Solution 1.**
Comparing to the standard form,

$$\begin{aligned}
f(x) &= \frac{x^2}{1+x^2}\\
g(x) &= 0
\end{aligned}$$

Therefore, as the boundary is fixed, let

$$\begin{aligned}
\widetilde{f}(x) &= \begin{cases} f(x) & ; \quad x \ge 0 \\ -f(-x) & ; \quad x < 0 \end{cases}\\
&= \begin{cases} \frac{x^2}{1+x^2} & ; \quad x \ge 0 \\ -\frac{(-x)^2}{1+(-x)^2} & ; \quad x < 0 \end{cases}\\
&= \begin{cases} \frac{x^2}{1+x^2} & ; \quad x \ge 0 \\ -\frac{x^2}{1+x^2} & ; \quad x < 0 \end{cases}\\
\widetilde{g}(x) &= \begin{cases} g(x) & ; \quad x \ge 0 \\ -g(-x) & ; \quad x < 0 \end{cases}\\
&= 0
\end{aligned}$$

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Therefore, by the d'Alembert formula,

$$\begin{aligned}
\widetilde{u}(x,t) &= \frac{\widetilde{f}(x-at)+\widetilde{f}(x+at)}{2} + \frac{1}{2a}\int\limits_{x-at}^{x+at} \widetilde{g}(s)\,\mathrm{d}s \\
&= \frac{\widetilde{f}(x-at)+\widetilde{f}(x+at)}{2} + \frac{1}{2a}\int\limits_{x-at}^{x+at} 0\,\mathrm{d}s \\
&= \begin{cases} \frac{(x-at)^2}{1+(x-at)^2} & ; \quad x-at \geq 0 \\ -\frac{(x-at)^2}{1+(x-at)^2} & ; \quad x-at < 0 \end{cases} \\
&\quad + \begin{cases} \frac{(x+at)^2}{1+(x+at)^2} & ; \quad x+at \geq 0 \\ -\frac{(x+at)^2}{1+(x+at)^2} & ; \quad x+at < 0 \end{cases} \\
&= \begin{cases} \frac{(x-at)^2}{1+(x-at)^2} & ; \quad x \geq at \\ -\frac{(x-at)^2}{1+(x-at)^2} & ; \quad x < at \end{cases} \\
&\quad + \begin{cases} \frac{(x+at)^2}{1+(x+at)^2} & ; \quad x \geq -at \\ -\frac{(x+at)^2}{1+(x+at)^2} & ; \quad x < -at \end{cases}
\end{aligned}$$

Therefore, the restricted solution, i.e. the solution on the given domain $x > 0$, $t > 0$ is

$$\begin{aligned}
u(x,t) &= \begin{cases} \frac{(x-at)^2}{1+(x-at)^2} & ; \quad x \geq at \\ -\frac{(x-at)^2}{1+(x-at)^2} & ; \quad 0 < x < at \end{cases} \\
&\quad + \frac{(x+at)^2}{1+(x+at)^2} \\
&= \begin{cases} \frac{(x+at)^2}{1+(x+at)^2} + \frac{(x-at)^2}{1+(x-at)^2} & ; \quad x \geq at \\ \frac{(x+at)^2}{1+(x+at)^2} - \frac{(x-at)^2}{1+(x-at)^2} & ; \quad 0 < x < at \end{cases}
\end{aligned}$$

**Exercise 2.**
Solve

$$\begin{aligned}
u_{tt} &= a^2 u_{xx} \\
u(x,0) &= 0 \\
u_t(x,0) &= \sin(x) \\
u_x(0,t) &= 0
\end{aligned}$$

where $0 \leq x \leq \infty$, and $t \geq 0$.

**Solution 2.**

Comparing to the standard form,

$$f(x) = 0$$
$$g(x) = \sin(x)$$

Therefore, as the boundary is fixed, let

$$\begin{aligned}\widetilde{f}(x) &= \begin{cases} f(x) & ; \quad x \geq 0 \\ f(-x) & ; \quad x < 0 \end{cases} \\ &= 0 \\ \widetilde{g}(x) &= \begin{cases} g(x) & ; \quad x \geq 0 \\ g(-x) & ; \quad x < 0 \end{cases} \\ &= \begin{cases} \sin(x) & ; \quad x \geq 0 \\ \sin(-x) & ; \quad x < 0 \end{cases} \\ &= \sin|x|\end{aligned}$$

Therefore, by the d'Alembert formula,

$$\begin{aligned}\widetilde{u}(x,t) &= \frac{\widetilde{f}(x-at) + \widetilde{f}(x+at)}{2} + \frac{1}{2a}\int\limits_{x-at}^{x+at} \widetilde{g}(s)\,\mathrm{d}s \\ &= \frac{1}{2a}\int\limits_{x-at}^{x+at} \sin|s|\,\mathrm{d}s\end{aligned}$$

Therefore, the restricted solution, i.e. the solution on the given domain $x > 0$, $t > 0$ is

$$\begin{aligned}&= \begin{cases} \frac{1}{2a}\int\limits_{x-at}^{x+at} \sin s\,\mathrm{d}s & ; \quad x - at \geq 0 \\ \frac{1}{2a}\int\limits_{x-at}^{0} \sin(-s)\,\mathrm{d}s + \frac{1}{2a}\int\limits_{0}^{x+2t} & ; \quad x - at < 0 \end{cases} \\ &= \begin{cases} \frac{1}{2a}\big(\cos(x-at) - \cos(x+at)\big) & ; \quad x \geq 2t \\ \frac{1}{2a}\big(2\cos(0) - \cos(x-at) - \cos(x+at)\big) & ; \quad 0 < x < 2t \end{cases} \\ &= \begin{cases} \frac{1}{2a}\big(2\sin(x)\sin(2t)\big) & ; \quad x \geq 2t \\ \frac{1}{2a}\big(2 - 2\cos(x)\cos(2t)\big) & ; \quad 0 < x < 2t \end{cases}\end{aligned}$$