Lecture 14

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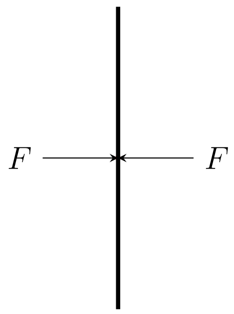
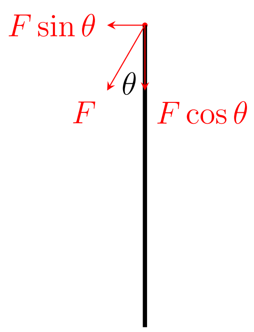
Contents

1 Rigid Body Mechanics

1.1 Definitions



$$
\begin{aligned}
M_{\text{total}}\overrightarrow{a_{\text{COM}}} &= \sum \overrightarrow{F_{\text{ext}}} \\
\therefore v_{\text{COM}} &= \text{ constant}
\end{aligned}
$$



Definition 1 (Torque)**.**

$$
\begin{aligned}
\tau &\doteq rF\sin\theta \\
\overrightarrow{\tau} &\doteq \overrightarrow{r} \times \overrightarrow{F}
\end{aligned}
$$

Definition 2 (Angular momentum)**.**

$$
\overrightarrow{L} \doteq \overrightarrow{r} \times \overrightarrow{p}
$$

$$
\begin{aligned}
\overrightarrow{\tau_A} &= 0 \\
\therefore \frac{\mathrm{d}\overrightarrow{L_A}}{\mathrm{d}t} &= 0 \\
\therefore \overrightarrow{L_A} &= \text{ constant} \\
&= rmv\hat{z} \\
&= mr^2\omega\hat{z} \\
&= (mr^2)\overrightarrow{\omega}
\end{aligned}
$$

Definition 3 (Moment of Inertia)**.**

$$
I \doteq mr^2
$$

$$
\begin{aligned}
E_k &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}mr^2\omega^2 \\
&= \frac{1}{2}I\omega^2
\end{aligned}
$$

Example 1. A particle attached to a string is on a horizontal table, moving in a circle of radius r_0 with v_0. The other end on the string goes through the table, through a hole in the centre of the circle. It is pulled down by a force F. Find the velocity of the particle as a function of the radius.

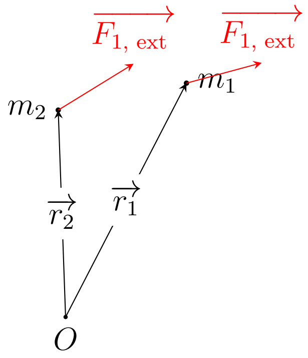
Solution.

$$
\begin{aligned}
\overrightarrow{\tau} &= 0 \\
\therefore \overrightarrow{L_A} &= r_0mv_0\hat{z} \\
\therefore \overrightarrow{L} &= rmv\hat{z}
\end{aligned}
$$

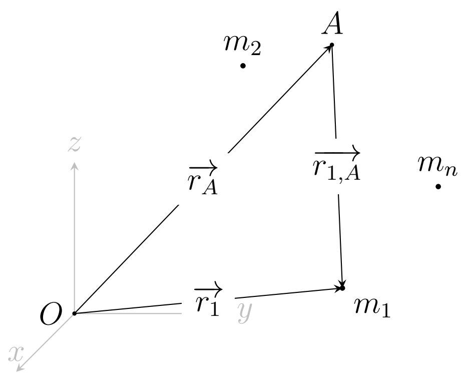
Example 2.

Solution.

$$
\begin{aligned}
\overrightarrow{F_{1,2}} &= -\overrightarrow{F_{2,1}} \\
\overrightarrow{\tau_{\text{ total },0}} &= \overrightarrow{\tau_{1,0}} + \overrightarrow{\tau_{2,0}} \\
&= \overrightarrow{r_1} \times \overrightarrow{F_{1,\text{ ext}}} + \overrightarrow{r_1} \times \overrightarrow{F_{1,2}} + \overrightarrow{r_2} \times \overrightarrow{F_{2,\text{ ext}}} + \overrightarrow{r_2} \times \overrightarrow{F_{2,1}} \\
&= \overrightarrow{r_1} \times \overrightarrow{F_{1,\text{ ext}}} + \overrightarrow{r_2} \times \overrightarrow{F_{2,\text{ ext}}} + \cancelto{0}{(\overrightarrow{r_1} - \overrightarrow{r_2}) \times \overrightarrow{F_{1,2}}} \\
&= \overrightarrow{\tau_{\text{ total},O,\text{ ext}}}
\end{aligned}
$$



1.2 Conditions for Equilibrium

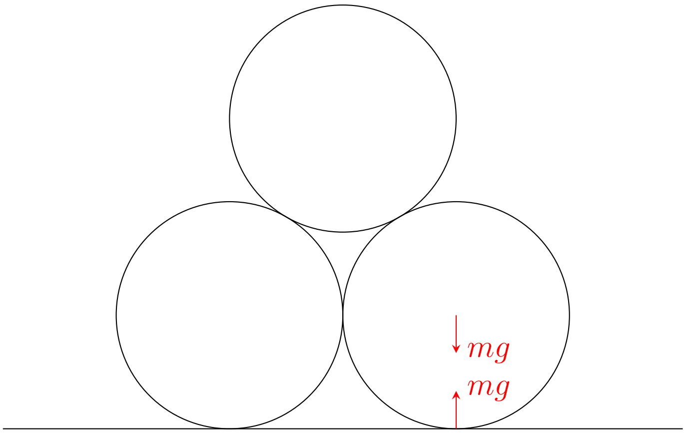


$$\sum \overrightarrow{F_{i,\text{ ext}}} = 0$$
$$\sum \overrightarrow{\tau_{i,O,\text{ ext}}} = 0$$

$$\begin{aligned}\sum \overrightarrow{\tau_{i,A,\text{ ext}}} &= \sum \overrightarrow{r_{i,A}} \times \overrightarrow{F_{i,\text{ ext}}} \\ &= \sum \overrightarrow{r_{i,O}} \times \overrightarrow{F_{i,\text{ ext}}} - \sum \overrightarrow{r_A} \times \overrightarrow{F_{i,\text{ ext}}} \\ &= 0\end{aligned}$$

Therefore, equilibrium is independent of the point of reference.
Hence the conditions for equilibrium are

$$\sum \overrightarrow{F_{i,\text{ ext}}} = 0$$
$$\sum \overrightarrow{\tau_{i,\text{ ext}}} = 0$$



Example 3. Find the minimum μ for which the bodies remain at rest.

Solution.

$$\begin{aligned} \sum \overrightarrow{F_{i,\text{ ext}}} &= 0 \\ \therefore 2N_f &= 3mg \\ \therefore N_f &= \frac{3}{2}mg \end{aligned}$$

With respect to an axis passing through A,

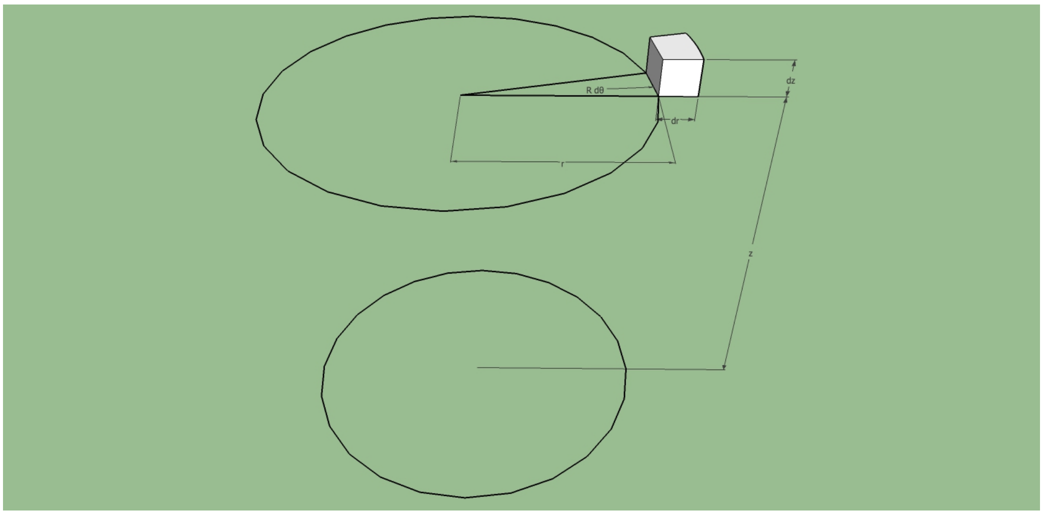
$$\begin{aligned} \sum \overrightarrow{\tau_{i,\text{ ext}}} &= 0 \\ \therefore (-\hat{z})dmg + \hat{z}d \cdot \frac{3}{2}mg + (-\hat{z})Hf_f &= 0 \\ \therefore f_f &= \frac{{}^1\!/_2 \cdot mgd}{H} \\ &= \frac{{}^1\!/_2 \cdot mgR\cos 60}{R + R\sin 60} \\ &\leq \mu N_f \\ \therefore \frac{{}^1\!/_2 \cdot mg \cdot {}^1\!/_2}{1 + {}^{\sqrt{3}}\!/_2} &= \mu \cdot {}^3\!/_2 \cdot mg \\ \therefore \mu &\geq \frac{{}^1\!/_6}{1 + {}^{\sqrt{3}}\!/_2} \end{aligned}$$

2 Coordinate Systems

2.1 Cylindrical Coordinates

$$(x, y, z) \to (r, \theta, z)$$

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z
\end{aligned}
$$

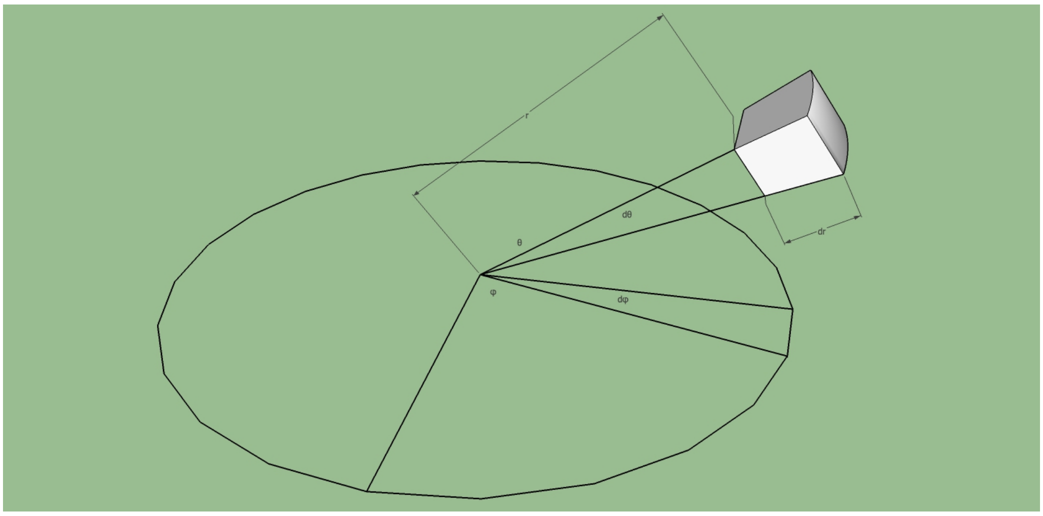


$$\mathrm{d}V = r\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}z$$

2.2 Spherical Coordinates

$$(x, y, z) \to (r, \theta, \varphi)$$

$$
\begin{aligned}
\mathrm{d}V &= r\,\mathrm{d}\theta \cdot r\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}r \\
&= r^2\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}r\,\mathrm{d}\theta
\end{aligned}
$$


r
dθ
θ
dr
φ
dφ