Lecture 25

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Example 1. A disk with mass m_1 and radius R is rotating with ω. A rod is fixed at its centre and two masses of mass m_2 are fixed to the rod with springs of coefficient k. The natural length of the springs is r_0.
At some $t > 0$, and a small enough ω, the masses are in radial equilibrium. Find the r in such a case.
At some Ω, $r = R$. Find Ω.
In the external frame of reference, find the total mechanical energy.

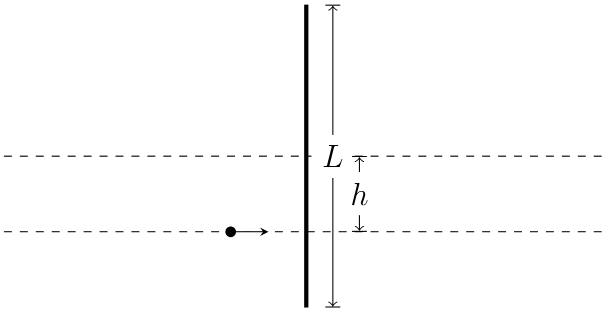
Solution. In the wedge frame of reference,

$$\begin{aligned} k(r - r_0) &= m_2\omega^2 r \\ \therefore kr - kr_0 &= m_2\omega^2 r \\ \therefore r(k - m_2\omega^2) &= kr_0 \\ \therefore r &= \frac{kr_0}{k - m_2\omega^2} \end{aligned}$$

$$\begin{aligned} \therefore R &= \frac{kr_0}{k - m_2\Omega^2} \\ \therefore k - m_2\Omega^2 &= \frac{kr_0}{R} \\ \therefore m_2\Omega^2 &= k - \frac{kr_0}{R} \\ \therefore \Omega^2 &= \frac{k}{m_2}\left(1 - \frac{r_0}{R}\right) \end{aligned}$$

$$\begin{aligned} E &= \frac{1}{2}I\omega^2 + 2 \cdot \frac{1}{2}k(r - r_0)^2 \\ &= \frac{1}{2}\left(\frac{1}{2}m_1R^2 + 2m_2r^2\right)\omega^2 + k(r - r_0)^2 \\ &= \frac{1}{2}\left(\frac{1}{2}m_1R^2 + 2m_2r^2\right)\omega^2 + k(r - r_0)^2 \end{aligned}$$

Example 2. A particle of mass m_1 hits a rod of length L and mass m_2, at height h from the rod's centre of mass.



Assuming that the collision is elastic, find h so that m_1 stops after the collision. Find the range of $\dfrac{m_2}{m_1}$ such that m_1 stops after the collision.

Solution. As the collision is elastic,

$$\begin{aligned}
m_1 v_0 &= m_2 v \\
m_1 v_0 h &= \frac{m_2 L^2}{12}\omega \\
\frac{1}{2} m_1 {v_0}^2 &= \frac{1}{2}\frac{m_2 L^2}{12}\omega^2 + \frac{1}{2} m_2 v^2
\end{aligned}$$

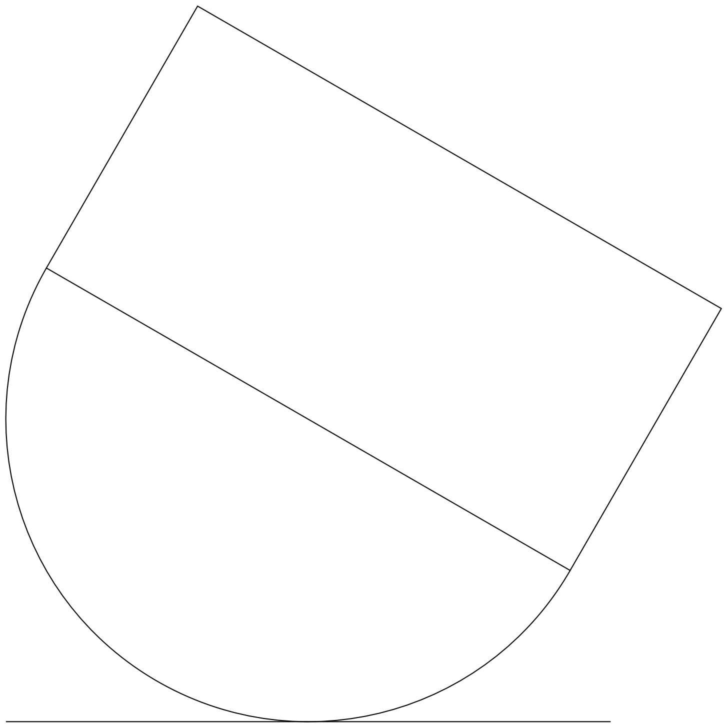
Therefore,

$$\begin{aligned}
m_2 v h &= \frac{m_2 L^2}{12}\omega \\
\therefore vh &= \frac{L^2}{12}\omega \\
\therefore \omega &= \frac{12vh}{L^2} \\
\therefore \frac{1}{2} m_1 {v_0}^2 &= \frac{m_2 L^2}{24}\cdot\frac{144 v^2 h^2}{L^4} + \frac{1}{2} m_2 v^2 \\
\therefore h &= \sqrt{\frac{1}{2} L^2 \left(\frac{m_2}{m_1} - 1\right)}
\end{aligned}$$

This condition holds if and only if $h \leq \dfrac{L}{2}$.

$$\begin{aligned}
0 \leq \frac{L}{\sqrt{12}}\sqrt{\frac{m_2}{m_1} - 1} \leq \frac{L}{2} \\
1 \leq \frac{m_2}{m_1} \leq 4
\end{aligned}$$

Example 3. A body made of a hemisphere of radius a and a cylinder of radius a and height h has mass m. θ is the angle between the axis of the vertical. Let the gravitational potential energy be zero at the ground level. Given $z_{\mathrm{COM}} = \left(a + \dfrac{\dfrac{h^2}{2} - \dfrac{a^2}{4}}{h + \dfrac{2a}{3}} \cos\theta \right)$ with respect to the centre of the hemisphere, on the cylinder side of the body, find the gravitational potential energy as a function of θ. Find the condition for the equilibrium at $\theta = 0$ to be stable.



Solution.

$$
\begin{aligned}
y_{\text{COM}} &= a + z_{\text{COM}} \cos\theta \\
\therefore U &= mg \left(a + \frac{\dfrac{h^2}{2} - \dfrac{a^2}{4}}{h + \dfrac{2a}{3}} \cos\theta \right) \\
\therefore U' &= mg \left(a + \frac{\dfrac{h^2}{2} - \dfrac{a^2}{4}}{h + \dfrac{2a}{3}} \cos\theta \right) (-\sin\theta) \\
\therefore U'' &= -mg \left(a + \frac{\dfrac{h^2}{2} - \dfrac{a^2}{4}}{h + \dfrac{2a}{3}} \cos\theta \right)
\end{aligned}
$$

For the equilibrium at $\theta = 0$ to be stable,

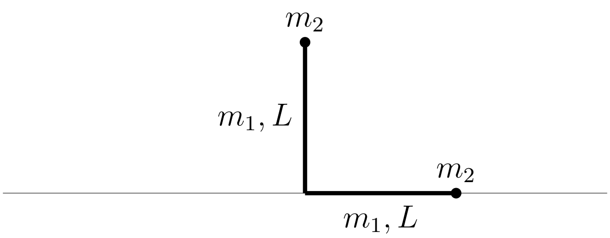
$$
\begin{aligned}
U'' &> 0 \\
\therefore \left(a + \frac{\dfrac{h^2}{2} - \dfrac{a^2}{4}}{h + \dfrac{2a}{3}} \cos\theta \right) &< 0
\end{aligned}
$$

Example 4. A solid hemisphere of mass m and radius a is pivoted at its centre, and is used as a pendulum. Find the angular frequency of its oscillations.

Solution.

$$
\begin{aligned}
\omega &= \sqrt{\frac{dmg}{I_0}} \\
&= \sqrt{\frac{\left({}^3\!/\!{}_8\right) amg}{\left({}^2\!/\!{}_5\right) ma^2}} \\
&= \sqrt{\frac{15g}{16a}}
\end{aligned}
$$

Example 5. A system is arranged on the ground as shown.



Solution.

$$I = \frac{2m_1L^2}{3} + 2m_2L^2$$

$$x_{\text{COM}} = \frac{m_1 \cdot 0 + m_2 \cdot 0 + m_1 \cdot {}^{L}\!/\!{}_{2} + m_2 \cdot L}{2m_1 + 2m_2}$$

$$\therefore x_{\text{COM}} = \frac{m_1L + 2m_2L}{4(m_1 + m_2)}$$

$$y_{\text{COM}} = \frac{m_1L + 2m_2L}{4(m_1 + m_2)}$$

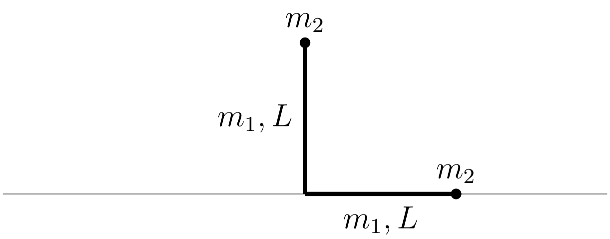


Figure 1: Stable equilibrium condition

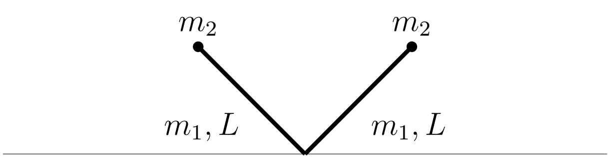


Figure 2: Unstable equilibrium condition

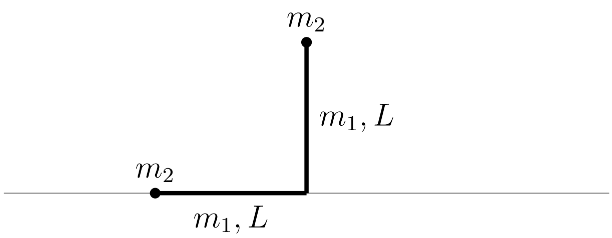


Figure 3: Stable equilibrium condition

For the body to topple,

$$\frac{1}{2}I{\omega_{\text{min}}}^2 = mg\left(\sqrt{{x_{\text{COM}}}^2 + {y_{\text{COM}}}^2} - y_{\text{COM}}\right)$$