Lecture 16

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Contents

1 Rigid Body Collisions

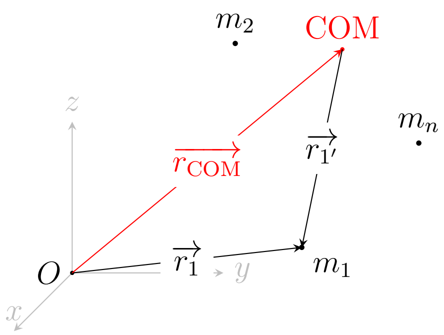
1.1 Conservation of Angular Momentum

If

$$\sum \overrightarrow{F_{\text{ext}}} = 0$$

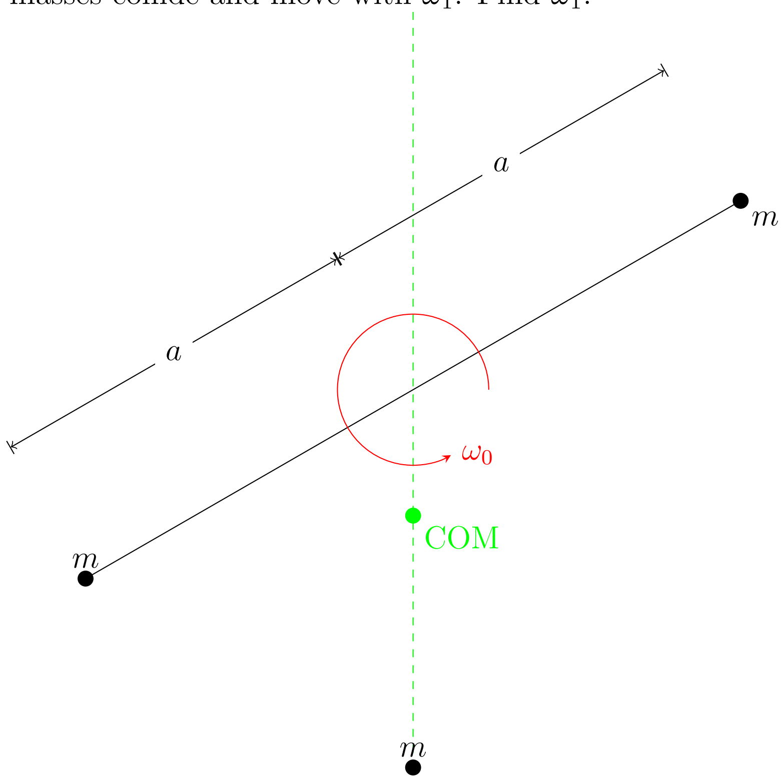
then

$$\begin{aligned}
\overrightarrow{\tau_A} &= \sum \overrightarrow{r} \times \overrightarrow{F_{\text{ext}}} \\
&= 0 \\
\therefore \frac{\mathrm{d}}{\mathrm{d}t} \overrightarrow{L_A} &= 0 \\
\therefore \overrightarrow{L_A} &= \text{ constant}
\end{aligned}$$



$$\begin{aligned}
\overrightarrow{L_O} &= \sum \overrightarrow{r_i} \times \left(m_i \overrightarrow{v_i}\right) \\
&= \sum \left(\overrightarrow{r_i'} + \overrightarrow{r_{\text{COM}}}\right) \times \left(m_i \left(\overrightarrow{v_{\text{COM}}} + \overrightarrow{v_i'}\right)\right) \\
&= \sum \overrightarrow{r_i'} \times \left(m_i \overrightarrow{v_{\text{COM}}}\right) + \sum \overrightarrow{r_i'} \times \left(m_i \overrightarrow{v_i'}\right) \\
&\quad + \sum \overrightarrow{r_{\text{COM}}} \times \left(m_i \overrightarrow{v_{\text{COM}}}\right) + \sum \overrightarrow{r_{\text{COM}}} \times \left(m_i \overrightarrow{v_i'}\right) \\
&= \sum \cancelto{0}{\left(m_i \overrightarrow{r_i'}\right)} \times \overrightarrow{v_{\text{COM}}} + \overrightarrow{L_{COM}} \\
&\quad + \overrightarrow{r_{\text{COM}}} \times \left(\sum m_i\right) \overrightarrow{v_{\text{COM}}} + \overrightarrow{r_{\text{COM}}} \cancelto{0}{\sum m_i \overrightarrow{v_i'}} \\
&= \overrightarrow{L_{\text{COM}}} + \overrightarrow{r_{\text{COM}}} \times \left(\sum m_i\right) \overrightarrow{v_{\text{COM}}}
\end{aligned}$$

Example 1. Consider a system of two point masses rotating about their centre of mass, with ω_0. Another point mass is positioned as shown. The masses collide and move with ω_1. Find ω_1.



Solution. Before the collision, wrt the COM of all 3 bodies,

$$\begin{aligned}\overrightarrow{L_{\text{COM}}} &= 0 + \frac{4a}{3}ma\omega_0\hat{z} + \frac{2a}{3}ma\omega_0\hat{z}\\ &= 2a^2m\omega_0\hat{z}\end{aligned}$$

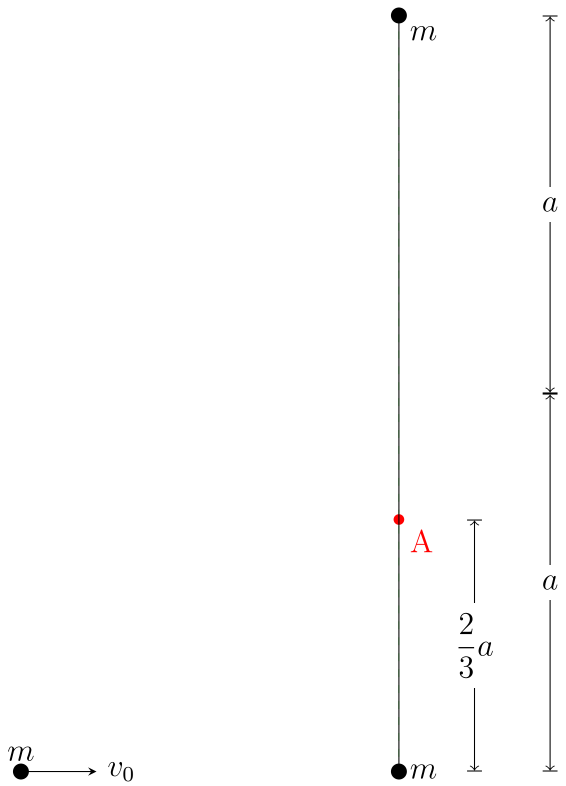
After the collision, wrt the COM of all 3 bodies,

$$\frac{4}{3}am\frac{4}{3}a\omega_1\hat{z} + \frac{2}{3}a(2m)\frac{2}{3}a\omega_1\hat{z}$$

Applying COAM,

$$\omega_1 = \frac{3}{4}\omega_0$$

Example 2. A particle of mass m hits a system of 2 masses, as shown.



Solution. Before the collision,

$$\overrightarrow{L} = \frac{2a}{3} m v_0 \hat{z}$$

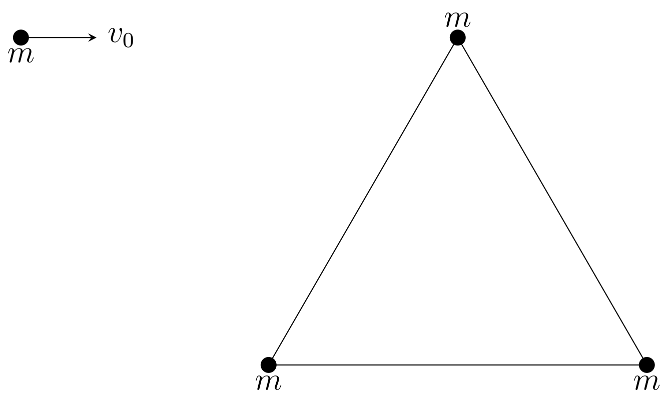
After the collision,

$$\begin{aligned}\overrightarrow{L} &= \overrightarrow{L'_{\text{COM}}} + \overrightarrow{r_{\text{COM}}} \times 3m\frac{v_0}{3}\hat{x} \\ &= \frac{4a}{3}m\frac{4a}{3}\omega\hat{z} + \frac{2a}{3}(2m)\frac{2a}{3}\omega\hat{z}\end{aligned}$$

Applying COAM,

$$\omega = \frac{v_0}{4a}$$

Example 3. 3 masses form an equilateral triangle of side length L. A particle of mass m is moving towards the apex of the triangle as shown. Given that the particle moves with a velocity of $-\frac{v_0}{5}\hat{x}$ after the collision, and that the collision is inelastic, find



Solution. Before the collision,

$$\overrightarrow{L_A} = \frac{2}{3}L\frac{\sqrt{3}}{2}mv_0(-\hat{z})$$

After the collision,

$$\begin{aligned}
\overrightarrow{L_A} &= \frac{2}{3}\cdot L\cdot\frac{\sqrt{3}}{2}\cdot m\frac{v_0}{5}\,(+\hat{z}) + \overrightarrow{L_{\triangle,A}} \\
&= \frac{2}{3}\cdot L\frac{\sqrt{3}}{2}\cdot m\frac{v_0}{5}\,(+\hat{z}) + 3d^2m\omega\,(-\hat{z}) + \overrightarrow{r_{\text{COM}}}\times\left(3m\cdot\frac{2}{5}\cdot v_0\hat{x}\right) \\
\therefore \omega &= \frac{2\sqrt{3}}{5}\cdot\frac{v_0}{L}
\end{aligned}$$

2 Rigid Body Mechanics

Theorem 1 (Steiner Theorem (Theorem of Parallel Axes))**.**

$$I = I_{COM} + Mr_{COM}^2$$

Theorem 2 (Theorem of Perpendicular Axes)**.** *For a body in the xy-plane,*

$$I_z = I_x + I_y$$