Lecture 19

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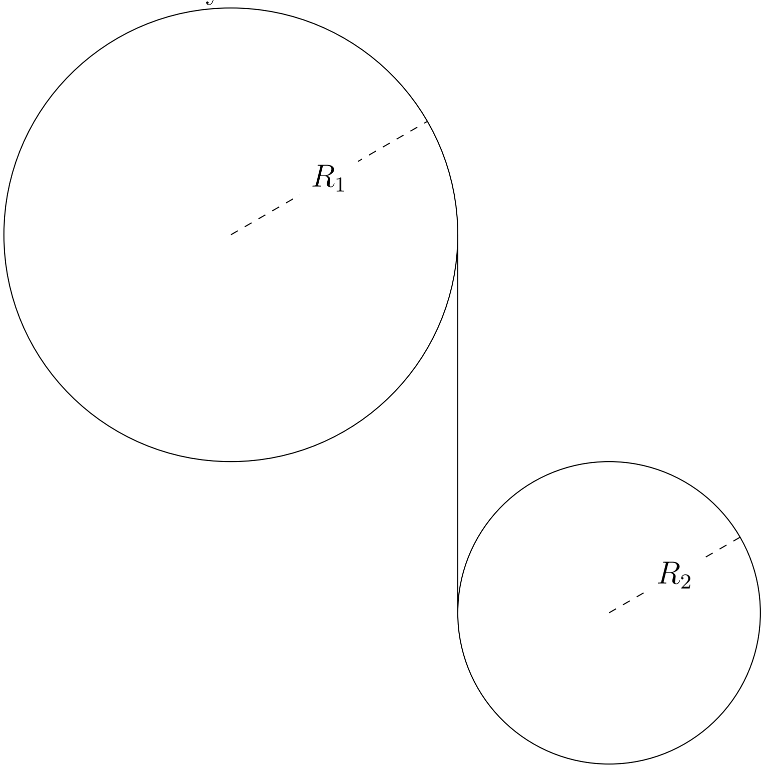
Contents

1 Rigid Body Mechanics

Example 1. A string is wound around a pulley of mass m_1 and radius R_1, which is fixed at its centre. The other end of the string is wound onto another pulley of mass m_2 and radius R_2. The string unwinds and the second pulley moves vertically downwards. Find the acceleration of the second pulley.



Solution. For m_1, about its centre,

$$\begin{aligned} TR_1 &= I_{O,1}\alpha_1 \\ &= \frac{1}{2}m_2{R_1}^2\alpha_1 \\ \therefore T &= \frac{1}{2}m_1R_1\alpha_1 \end{aligned}$$

For m_1,

$$m_2g - T = m_2a_2$$

For m_2, about its centre,

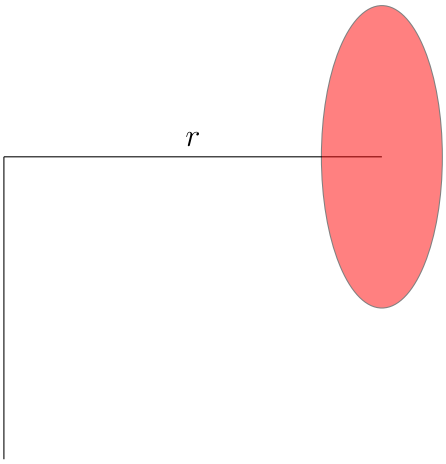
$$\begin{aligned} TR_2 &= \frac{1}{2}m_2{R_2}^2\alpha_2 \\ \therefore T &= \frac{1}{2}m_2R_2\alpha_2 \end{aligned}$$

As the second pulley goes down vertically, the string is being unwound from both pulleys. Therefore,

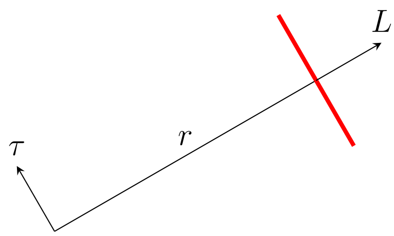
$$a_2 = \alpha_1 R_1 + \alpha_2 R_2$$

1.1 Gyroscope

A disk is attached to rods as shown, and is rotating about itself with ω.



The torque is directed $\otimes$.
Seen from the top,



$$\overrightarrow{\tau} = \tau \hat{\theta}$$

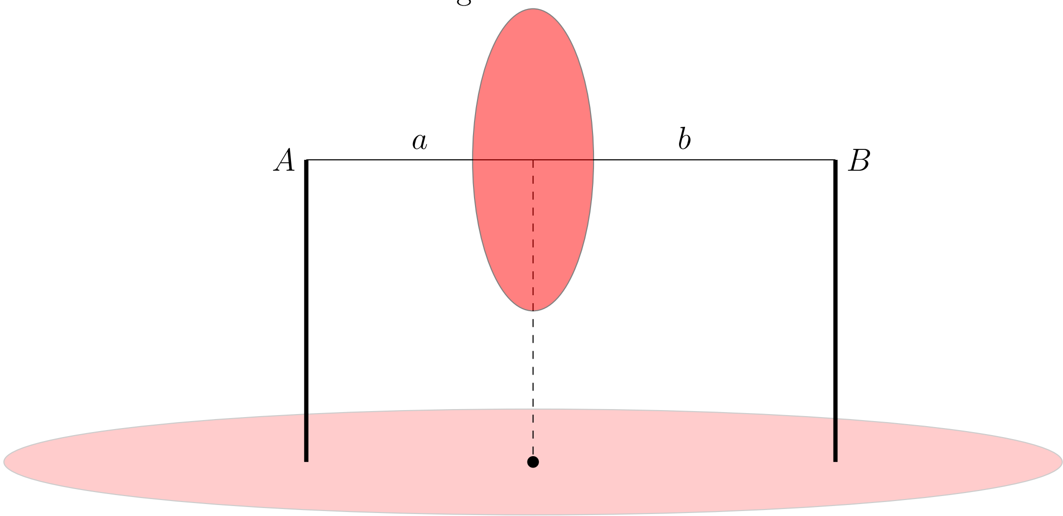
with respect to the joint,

$$\begin{aligned} \overrightarrow{L} &= \frac{1}{2} m R^2 \omega \hat{r} \\ \therefore \frac{\mathrm{d}\overrightarrow{L}}{\mathrm{d}t} &= \frac{1}{2} m R^2 \omega \cdot \frac{\mathrm{d}\hat{r}}{\mathrm{d}t} \\ \therefore \tau &= \frac{1}{2} m R^2 \omega \dot{\theta} \\ \therefore mgr &= \frac{1}{2} m R^2 \omega \dot{\theta} \end{aligned}$$

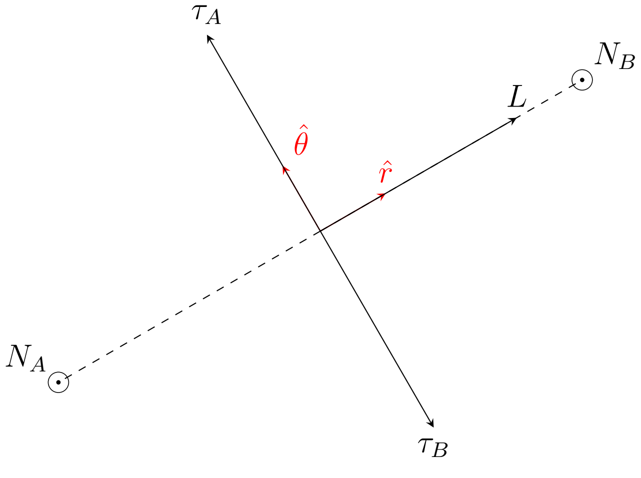
Therefore,

$$\dot{\theta} = \frac{2gr}{R^2\omega}$$

Example 2. A disk of mass m and radius R is fixed on a rod and is kept on two stands fixed on another disk which is rotating with Ω. Find the normal forces that the stands are exerting on the rod.



Solution. As viewed from the top,



About the point O,

$$\begin{aligned}
\overrightarrow{L} &= \frac{1}{4}mR^2\Omega\hat{z} + \frac{1}{2}mR^2\omega\hat{r} \\
\therefore \overrightarrow{\tau} &= \frac{\mathrm{d}\overrightarrow{L}}{\mathrm{d}t} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{4}mR^2\Omega\hat{z} + \frac{1}{2}mR^2\omega\hat{r}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mR^2\omega\hat{r}\right)
\end{aligned}$$

The net torque about point O is only due to the normal forces.
Therefore, the net torque is in the $\hat{\theta}$ direction. Hence, it cannot change the magnitude of ω, but only the direction.
Therefore,

$$\begin{aligned}
\overrightarrow{\tau} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{4}mR^2\Omega\hat{z} + \frac{1}{2}mR^2\omega\hat{r}\right) \\
&= \frac{1}{2}mR^2\omega\,\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} \\
&= \frac{1}{2}mR^2\omega\dot{\theta}\hat{\theta} \\
&= \frac{1}{2}mR^2\omega\Omega\hat{\theta} \\
\therefore aN_A\hat{\theta} + bN_b(-\hat{\theta}) &= \frac{1}{2}mR^2\omega\Omega\hat{\theta}
\end{aligned}$$

Therefore,

$$aN_A - bN_B = \frac{1}{2}mR^2\omega\Omega$$

Also,

$$N_A + N_B = mg$$

2 Second Order Linear Differential Equations with Constant Coefficients

2.1 Homogeneous Differential Equations

$$ay'' + by' + c = 0$$

Let $y = e^{\lambda x}$

$$\therefore a\lambda^2 + b\lambda + c = 0$$

$$\therefore \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $\Delta > 0$,

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

is the general solution to the differential equation.
If $\Delta < 0$,

$$y = e^{\alpha x}\big(C\cos(\beta x) + D\sin(\beta x)\big)$$

is the general solution to the differential equation.
If $\Delta = 0$,

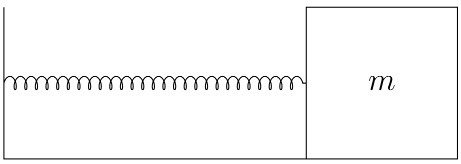
$$y = Ae^{\lambda x} + Bxe^{\lambda x}$$

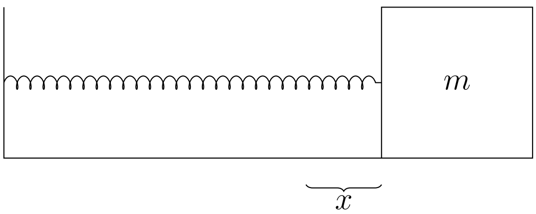
is the general solution to the differential equation.

3 Simple Harmonic Oscillator

A mass m is attached to a spring of coefficient k, which is at its natural length. The spring is stretched by x and released.

$$x(t=0) = A$$
$$\dot{x}(t=0) = 0$$





$$
\begin{aligned}
m\ddot{x} &= -kx \\
\therefore \ddot{x} + \frac{k}{m}x &= 0
\end{aligned}
$$

Therefore, the characteristic equation is

$$
\begin{aligned}
\lambda^2 + \frac{k}{m} &= 0 \\
\therefore \lambda &= \pm i\sqrt{\frac{k}{m}}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
x &= C\cos\left(\sqrt{\frac{k}{m}}t\right) + D\sin\left(\sqrt{\frac{k}{m}}t\right) \\
\therefore \dot{x} &= \sqrt{\frac{k}{m}}\left(-C\sin\left(\sqrt{\frac{k}{m}}t\right) + D\cos\left(\sqrt{\frac{k}{m}}t\right)\right)
\end{aligned}
$$

Solving with initial conditions,

$$
\begin{aligned}
x &= A\cos\left(\sqrt{\frac{k}{m}}t\right) \\
\dot{x} &= -A\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}t\right)
\end{aligned}
$$

Alternatively, as the mechanical energy is constant throughout,

$$
\begin{aligned}
E &= \frac{1}{2}m(\dot{x})^2 + \frac{1}{2}kx^2 \\
\therefore \dot{E} &= 0 \\
\therefore 0 &= \frac{1}{2}m(2\dot{x})\ddot{x} + \frac{1}{2}k(2x)\dot{x} \\
\therefore 0 &= \dot{x}\left(m\ddot{x} + kx\right) \\
\therefore \ddot{x} + \frac{k}{m}x &= 0
\end{aligned}
$$

Definition 1 (Angular frequency)**.** The angular frequency is defined as

$$
\omega_0 = \sqrt{\frac{k}{m}}
$$