PHYSICS 1 : COMPENDIUM

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1. Non-Linear Motion

$$\overrightarrow{r} = r\hat{r}$$
$$\overrightarrow{v} = \dot{\overrightarrow{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$
$$\overrightarrow{a} = \ddot{\overrightarrow{r}} = \left(\ddot{r} - r(\dot{\theta})^2\right)\hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta}$$

2. Conservative Forces

$$\operatorname{curl}\overrightarrow{F} \doteq \overrightarrow{\nabla} \times \overrightarrow{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

If $\overrightarrow{F}$ is conservative, $\operatorname{curl}\overrightarrow{F} = 0$.

$$F = -\frac{\mathrm{d}U}{\mathrm{d}r}$$

3. Conservation of Momenta

Example 1. On one side of a boat of mass M, a box of mass m is placed and the boat's engine is pulling the box via a massless rope. At $t = 0$ the entire system is at rest. Then the engine is turned on and it starts pulling the box with force $\overrightarrow{F}(t) = \alpha t$ where α is a positive constant. The friction coefficients between the box and the boat are μ_s and μ_k. The engine is working for time $\tau = \dfrac{mg}{\alpha}$ and then stops. Assuming that the box does not reach the end of the boat nor collide with any other object, what will be the boat's velocity after a very long time $t >> \tau$? Find the box's velocity w.r.t. the boat from the moment the engine is turned on.

Solution.

$$
\begin{aligned}
F &= \alpha t && ; \quad 0 \leq t \leq \tau = \frac{mg}{\alpha} \\
N_s &= \mu_s mg \\
N_k &= \mu_k mg
\end{aligned}
$$

Let t_0 be the time when the box starts moving.

$$
\begin{aligned}
F(t) &= \mu_s mg \\
&= \alpha t \\
\therefore t_0 &= \frac{\mu_s mg}{\alpha}
\end{aligned}
$$

For $t_0 < t < \tau$,

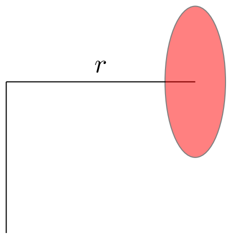
$$
\begin{aligned}
\alpha t - \mu_k mg &= ma \\
\therefore a &= \frac{\alpha t}{m} - \mu_k g \\
\therefore v &= \int\limits_{t_0}^{\tau} a(t) \, \mathrm{d}t \\
&= \frac{\alpha}{2m}(\tau^2 - t_0^2) - \mu_k g(\tau - t_0)
\end{aligned}
$$

For $t > \tau$,

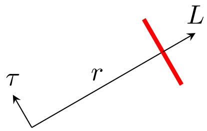
$$
\begin{aligned}
a &= \mu_k g \\
\therefore v &= v_\tau + \int\limits_{\tau}^{t} g \, \mathrm{d}t \\
&= v_\tau - \mu_k g(t - \tau)
\end{aligned}
$$

4. Rigid Body Motion

4.1. **Gyroscope.** A disk is attached to rods as shown, and is rotating about itself with ω.



The torque is directed $\otimes$.
Seen from the top,



$$\overrightarrow{\tau} = \tau\hat{\theta}$$

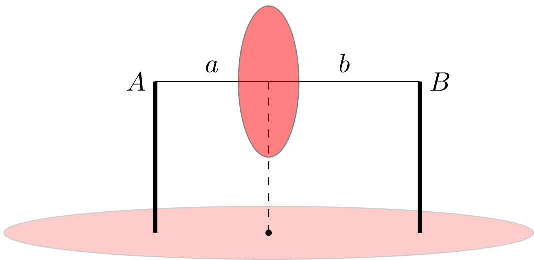
with respect to the joint,

$$\begin{aligned}
\overrightarrow{L} &= \frac{1}{2}mR^2\omega\hat{r} \\
\therefore \frac{\mathrm{d}\overrightarrow{L}}{\mathrm{d}t} &= \frac{1}{2}mR^2\omega \cdot \frac{\mathrm{d}\hat{r}}{\mathrm{d}t} \\
\therefore \tau &= \frac{1}{2}mR^2\omega\dot{\theta} \\
\therefore mgr &= \frac{1}{2}mR^2\omega\dot{\theta}
\end{aligned}$$

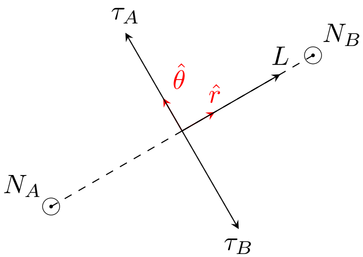
Therefore,

$$\dot{\theta} = \frac{2gr}{R^2\omega}$$

Example 2. A disk of mass m and radius R is fixed on a rod and is kept on two stands fixed on another disk which is rotating with Ω. Find the normal forces that the stands are exerting on the rod.



Solution. As viewed from the top,



About the point O,

$$
\begin{aligned}
\overrightarrow{L} &= \frac{1}{4}mR^2\Omega\hat{z} + \frac{1}{2}mR^2\omega\hat{r} \\
\therefore \overrightarrow{\tau} &= \frac{\mathrm{d}\overrightarrow{L}}{\mathrm{d}t} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{4}mR^2\Omega\hat{z} + \frac{1}{2}mR^2\omega\hat{r}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mR^2\omega\hat{r}\right)
\end{aligned}
$$

The net torque about point O is only due to the normal forces. Therefore, the net torque is in the $\hat{\theta}$ direction. Hence, it cannot change the magnitude of ω, but only the direction.

Therefore,

$$\begin{aligned}
\vec{\tau} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{4}mR^2\Omega\hat{z} + \frac{1}{2}mR^2\omega\hat{r}\right) \\
&= \frac{1}{2}mR^2\omega\,\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} \\
&= \frac{1}{2}mR^2\omega\dot{\theta}\hat{\theta} \\
&= \frac{1}{2}mR^2\omega\Omega\hat{\theta} \\
\therefore aN_A\hat{\theta} + bN_b(-\hat{\theta}) &= \frac{1}{2}mR^2\omega\Omega\hat{\theta}
\end{aligned}$$

Therefore,

$$aN_A - bN_B = \frac{1}{2}mR^2\omega\Omega$$

Also,

$$N_A + N_B = mg$$

5. Variable Mass Systems

If a rocket is releasing gasses with velocity u with respect to it,

$$\frac{\mathrm{d}p}{\mathrm{d}t} = m\,\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\mathrm{d}m}{\mathrm{d}t}\,u$$

6. Centres of Mass

Solid hemisphere	$(3/8)\ R$
Hollow hemisphere	$(1/2)\ R$
Solid cone (from vertex)	$(3/4)\ h$
Hollow cone (from vertex)	$(2/3)\ h$

Example 3. Find the centre of mass of an eighth of a solid sphere.

Solution. Consider an elemental mass $\mathrm{d}m$ at (r, θ, φ).

$$\begin{aligned}
x_{\text{COM}} &= \frac{\int\limits_{r=0}^{R}\int\limits_{\theta=0}^{\frac{\pi}{2}}\int\limits_{\varphi=0}^{\frac{\pi}{2}} r\sin\theta\cos\varphi\,\mathrm{d}V}{\iiint \mathrm{d}V} \\
&= \frac{\int\limits_{r=0}^{R}\int\limits_{\theta=0}^{\frac{\pi}{2}}\int\limits_{\varphi=0}^{\frac{\pi}{2}} r\sin\theta\cos\varphi(r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi)}{\frac{1}{8}\cdot\frac{4}{3}\pi R^3}
\end{aligned}$$

Therefore,

$$\therefore x_{\text{COM}} = y_{\text{COM}} = z_{\text{COM}} = \frac{3}{8}R$$

7. Moments of Inertia

$$I = \int r^2 \, \mathrm{d}m$$

Ring ($\perp$ to plane)	mR^2
Disk ($\perp$ to plane)	$\left(1/2\right) mR^2$
Solid sphere	$\left(2/5\right) mR^2$
Hollow sphere	$\left(2/3\right) mR^2$
Rod (centre)	$\left(1/12\right) ml^2$
Rod (end)	$\left(1/3\right) ml^2$
Cone (axis of symmetry)	$\left(3/10\right) mR^2$

8. Accelerating Systems

$$F_{\text{centrifugal}} = -m\overrightarrow{\omega} \times \left(\overrightarrow{\omega} \times \overrightarrow{r}\right)$$

$$F_{\text{coriolis}} = -2m\overrightarrow{\omega} \times \overrightarrow{v}$$

9. Oscillations

9.1. Simple Oscillations.

$$\ddot{x} = -\omega^2 x$$

$$\omega_{\text{physical pendulum}} = \sqrt{\frac{d_{\text{axis,COM}} mg}{I_{\text{axis}}}}$$

9.2. Damped Oscillations.

$$\ddot{x} + \frac{\beta}{m}\dot{x} + {\omega_0}^2 x = 0$$

$$\begin{array}{cl} \text{Strong damping} & \dfrac{\beta}{2m} > \omega_0 \\ \text{Critical damping} & \dfrac{\beta}{2m} = \omega_0 \\ \text{Weak damping} & \dfrac{\beta}{2m} < \omega_0 \end{array}$$

Oscillations occur in case of weak damping.

$$\text{Let } \omega_1 = \sqrt{{\omega_0}^2 - \left(\frac{\beta}{2m}\right)^2}.$$

For weak damping,

$$x = e^{-\beta/2m \cdot t}\left(\widetilde{A}\cos\omega_1 t + \widetilde{B}\sin\omega_1 t\right)$$

For critical damping,

$$x = e^{-\beta/2m \cdot t}\left(\widetilde{A} + \widetilde{B}t\right)$$

In case of strong damping,

$$x = \widetilde{A}e^{\left(-\beta/2m + \sqrt{-{\omega_1}^2}\right)} + \widetilde{B}e^{\left(-\beta/2m - \sqrt{-{\omega_1}^2}\right)t}$$

9.3. **Forced Oscillations.**

$$\begin{aligned} m\ddot{x} + \frac{k}{m}x &= \frac{F_0}{m}\cos\omega t \\ \therefore \ddot{x} + \frac{k}{m}x &= \frac{F_0}{m}\cos\omega t \end{aligned}$$

Therefore, solving

$$\begin{aligned} x &= A\cos\omega_0 t + B\sin\omega_0 t + \frac{F_0}{k - m\omega^2}\cos\omega t \\ \therefore \dot{x} &= \omega_0(-A\sin\omega_0 t + B\cos\omega_0 t) - \frac{F_0}{k - m\omega^2}\omega\sin\omega t \end{aligned}$$

Substituting initial conditions,

$$x = \frac{\dfrac{F_0}{m}}{\dfrac{k}{m} - \omega^2}(-\cos\omega_0 t + \cos\omega t)$$

Let $\frac{F_0}{m} = f_0$

$$\therefore x = -\frac{2f_0}{{\omega_0}^2 - \omega^2} \sin\left(\frac{\omega t - \omega_0 t}{2}\right) \sin\left(\frac{\omega t + \omega_0 t}{2}\right)$$

$\omega - \omega_0 = \Delta\omega$ and $\omega + \omega_0 \approx 2\omega_0$

$$\therefore x \approx \frac{2f_0}{\Delta\omega \cdot 2\omega_0} \sin\left(\frac{\Delta\omega}{2} t\right) \cdot \sin(\omega_0 t)$$