Lecture 11

Aakash Jog

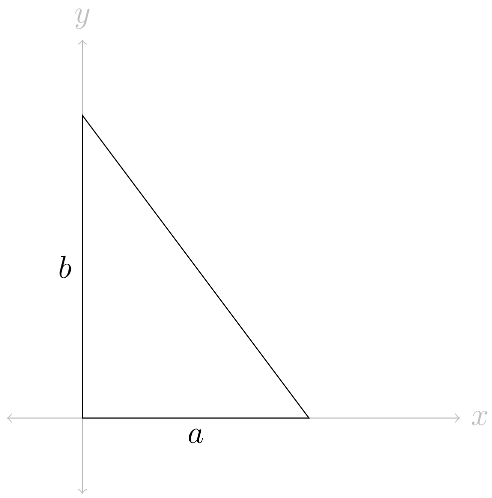
Tuesday 2nd December, 2014

Contents

1 Centre of Mass

Example 1.

$$\sigma = \frac{M_0}{\frac{1}{2}ab}$$



Find the centre of mass of the triangle.

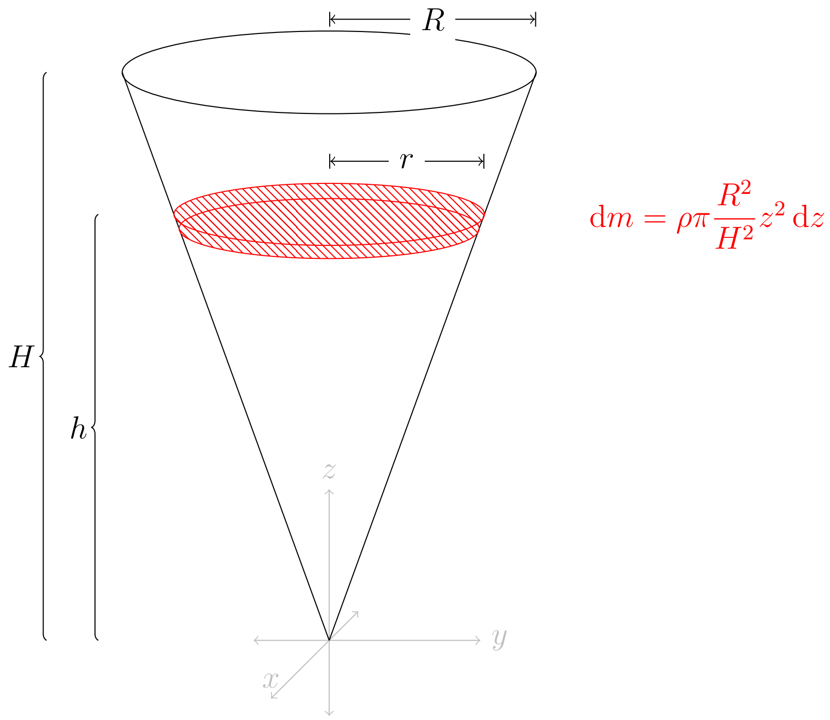
Solution.

$$\begin{aligned}
\mathrm{d}m &= \left(-\frac{b}{a}x + b\right)\mathrm{d}x\sigma \\
\therefore x_{\mathrm{COM}} &= \frac{\int x\,\mathrm{d}m}{M_0} \\
&= \frac{\int\limits_0^a x\left(-\frac{b}{a}x + b\right)\mathrm{d}x\sigma}{M_0} \\
&= \frac{\sigma\int\limits_0^a \left(-\frac{b}{a}x^2 + bx\right)\mathrm{d}x}{M_0} \\
&= \frac{M_0}{\frac{1}{2}abM_0}\left(-\frac{ba^3}{3a} + \frac{ba^2}{2}\right) \\
&= \frac{1}{3}a
\end{aligned}$$

Example 2. Find the COM of a solid cone.

Solution.

$$\rho = \frac{M_0}{\frac{1}{3}\pi R^2 H}$$



$$\frac{r}{R} = \frac{z}{H}$$

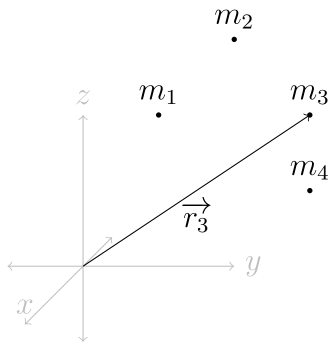
$$\therefore r = \frac{R}{H} z$$

$$x_{\text{COM}} = y_{\text{COM}} = 0$$

$$z_{\text{COM}} = \frac{\int_0^{M_0} z \,\mathrm{d}m}{M_0}$$

$$= \frac{\int_0^{H} z\rho\pi \frac{R^2}{H^2} z^2 \,\mathrm{d}z}{M_0}$$

2 Energy and Centre of Mass



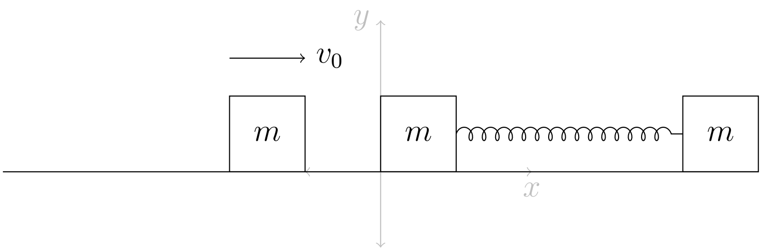
Example 3.

Solution.

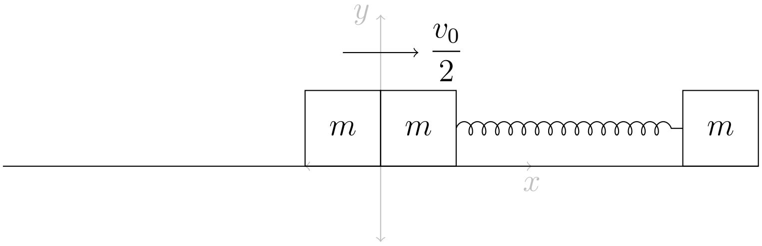
$$
\begin{aligned}
\overrightarrow{r_k} &= \overrightarrow{r_{\text{COM}}} + \overrightarrow{r'_k} \\
\therefore \overrightarrow{v_k} &= \overrightarrow{v_{\text{COM}}} + \overrightarrow{v'_k}
\end{aligned}
$$

$$
\begin{aligned}
E_{\text{kinetic}} &= \sum \frac{1}{2} m_i (\dot{\overrightarrow{r_i}})^2 \\
&= \sum \frac{1}{2} m_i (\overrightarrow{v_i})^2 \\
&= \sum \frac{1}{2} m_i (\overrightarrow{v_{\text{COM}}} + \overrightarrow{v'_k})^2 \\
&= \sum \frac{1}{2} m_i (\overrightarrow{v_{\text{COM}}} + \overrightarrow{v'_k}) \cdot (\overrightarrow{v_{\text{COM}}} + \overrightarrow{v'_k}) \\
&= \sum \frac{1}{2} m_i (v_{\text{COM}})^2 + \sum \frac{1}{2} m_i \overrightarrow{v_{\text{COM}}} \cdot \overrightarrow{v'_k} + \sum \frac{1}{2} m_i (v_k)^2 \\
&= \frac{1}{2} \left(\sum m_i\right) (v_{\text{COM}})^2 + \overrightarrow{v_{\text{COM}}} \left(\sum m_i \overrightarrow{v'_k}\right) + E'_{\text{kinetic}} \\
&= \frac{1}{2} \left(\sum m_i\right) (v_{\text{COM}})^2 + 0 + E'_{\text{kinetic}} \\
&= \frac{1}{2} M_{\text{total}} (v_{\text{COM}})^2 + E'_{\text{kinetic}}
\end{aligned}
$$

Example 4. Find the maximum compression of the spring.



Solution. As during the collision, mechanical energy is not conserved, COME cannot be applied to the system. However COLM can be applied in the x direction, as the net force is zero.

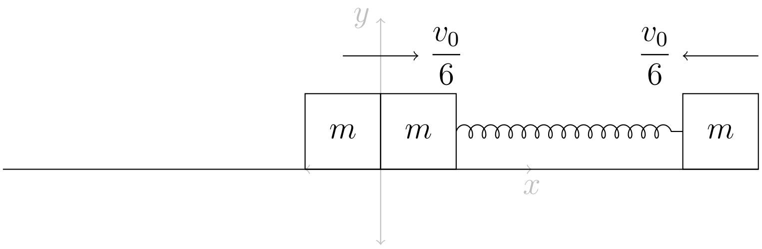


$$
\begin{aligned}
v_{\text{COM}} &= \frac{mv_0 + m(0) + m(0)}{3m} \\
&= \frac{v_0}{3} \\
p_{\text{total}} &= mv_0
\end{aligned}
$$

Consider the time of collision of the bodies to be $t = 0$.
Therefore,

$$
x_{\text{COM}}(t = 0) = \frac{l}{3}
$$

Therefore, in the COM frame of reference,



Therefore,

$$
\begin{aligned}
E' &= \frac{1}{2}(2m)\left(\frac{v_0}{6}\right)^2 + \frac{1}{2}m\left(\frac{1}{3}v_0\right)^2 \\
&= \frac{3mv_0^2}{36} \\
&= \frac{1}{12}mv_0^2
\end{aligned}
$$

After the collision, we can apply COME to the system.

Therefore,

$$\frac{1}{2}kx_{\max}^2 = \frac{1}{12}mv_0^2$$
$$\therefore x_{\max} = \sqrt{\frac{mv_0^2}{6k}}$$