Lecture 12

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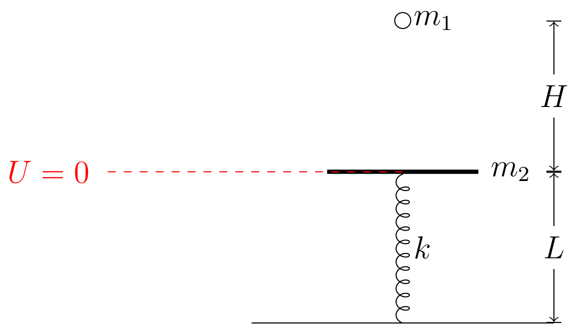
Thursday 4th December, 2014

Contents

1 Impulse and Momentum

Example 1. Assuming a plastic collision, what is the maximum compression of the string?



Solution. Applying COME before the collision,

$$\begin{aligned} m_1 g H &= \frac{1}{2} m_1 v_1^2 \\ \therefore v_1 &= \sqrt{2gH} \end{aligned}$$

The spring is already compressed due to m_2. Therefore

$$\begin{aligned} k x_i &= m_2 g \\ \therefore x_i &= \frac{m_2 g}{k} \end{aligned}$$

Therefore, the natural length of the spring is

$$x_i + L = \frac{m_2 g}{k} + L$$

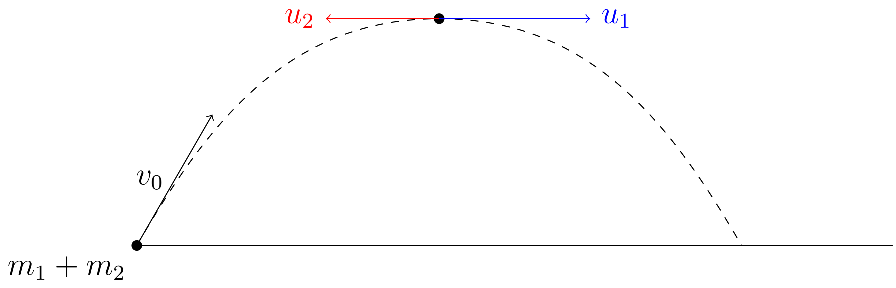
Applying COLM during the collision,

$$\begin{aligned} m_1 v_1 &= (m_1 + m_2) u_{12} \\ \therefore u_{12} &= \frac{m_1}{m_1 + m_2} v_1 \end{aligned}$$

Applying COME after the collision,

$$\frac{1}{2}(m_1 + m_2) u_{12}^2 + 0 + \frac{1}{2} k x_i^2 = 0 + (m_1 + m_2) g (-x_{\max}) + \frac{1}{2} k (x_i + x_{\max})^2$$

Example 2. Find x_2.



Solution. Applying COLM,

$$(m_1 + m_2)v_0 \cos\alpha = m_1 u_1 + m_2 u_2$$

$$\begin{aligned} v_y &= v_0 \sin\alpha - gt_1 \\ \therefore 0 &= v_0 \sin\alpha - gt_1 \\ \therefore t_1 &= \frac{v_0 \sin\alpha}{g} \end{aligned}$$

$$\begin{aligned} u_1 t &= x_1 \\ \therefore u_1 \frac{v_0 \sin\alpha}{g} &= x_1 \\ \therefore u_1 &= \frac{x_1 g}{v_0 \sin\alpha} \end{aligned}$$

2 Variable Mass Systems

Example 3. For a body, given $\dfrac{\mathrm{d}M}{\mathrm{d}t} = \rho A v$, find v_t.

Solution. Let the velocity and mass of the body at time t be v and m respectively. Therefore, applying COLM,

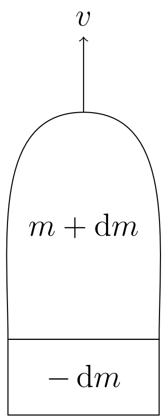
$$\begin{aligned}
m_0 v_0 &= mv \\
\therefore m &= \frac{m_0 v_0}{v} \\
\therefore \frac{\mathrm{d}M}{\mathrm{d}t} &= -\frac{m_0 v_0}{v^2} \cdot \frac{\mathrm{d}v}{\mathrm{d}t} \\
\therefore \rho A v &= -\frac{m_0 v_0}{v^2} \cdot \frac{\mathrm{d}v}{\mathrm{d}t} \\
\therefore \int_{v_0}^{v} \frac{\mathrm{d}v}{v^3} &= \int_0^t \frac{\rho A}{m_0 v_0} \, \mathrm{d}t \\
\therefore \frac{1}{2v^2} - \frac{1}{2v_0^2} &= \frac{\rho A}{m_0 v_0} t
\end{aligned}$$

Example 4. Given m_0 is the initial mass of the rocket, $\dfrac{\mathrm{d}m}{\mathrm{d}t} = -k$, and u is the velocity of the gas emitted from the rocket relative to the rocket, find the rocket's velocity as a function of time.

Solution.

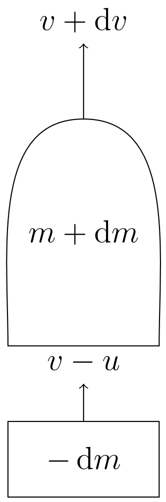
$$\mathrm{d}p = p_{t+\mathrm{d}t} - p_t$$

At t,



$$P_t = mv$$

At $t + \mathrm{d}t$,



$$p_{t+\mathrm{d}t} = (m + \mathrm{d}f)(v + \mathrm{d}v) + (-\,\mathrm{d}m)(v - u)$$

$$\begin{aligned}
\mathrm{d}p &= p_{t+\mathrm{d}t} - P_t \\
&= m\,\mathrm{d}v + \cancel{dmv} + \mathrm{d}m\,\mathrm{d}v - \cancel{dmv} + \mathrm{d}mu \\
\therefore \frac{\mathrm{d}p}{\mathrm{d}t} &= M\,\frac{\mathrm{d}v}{\mathrm{d}t} + \cancelto{0}{\frac{\mathrm{d}m}{\mathrm{d}t}\,\mathrm{d}v} + \frac{\mathrm{d}m}{\mathrm{d}t}\,u \\
&= M\,\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\mathrm{d}m}{\mathrm{d}t}\,u
\end{aligned}$$

$$\begin{aligned}
-mg &= \frac{\mathrm{d}p}{\mathrm{d}t} \\
&= M\,\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\mathrm{d}m}{\mathrm{d}t}\,u \\
&= m\,\frac{\mathrm{d}v}{\mathrm{d}t} - ku \\
\therefore m\,\frac{\mathrm{d}v}{\mathrm{d}t} &= ku - mg
\end{aligned}$$

Therefore the rocket accelerates upwards iff

$$\begin{aligned}
&ku - mg > 0 \\
\iff\ &ku > mg
\end{aligned}$$

Therefore, the condition for take-off is

$$ku > m_0 g$$

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -k$$
$$\therefore m = m_0 - kt$$

$$
\begin{aligned}
m\,\frac{\mathrm{d}v}{\mathrm{d}t} &= ku - mg \\
\therefore (m_0 - kt)\,\frac{\mathrm{d}v}{\mathrm{d}t} &= ku - (m_0 - kt)g \\
\therefore \int\limits_0^v \mathrm{d}v &= \int\limits_0^t \left(\frac{ku}{m_0 - kt} - g\right) \mathrm{d}t \\
&= -u\ln(m_0 - kt) + u\ln m_0 - gt \\
&= u\ln\left(\frac{m_0}{m_0 - kt}\right) - gt \\
\therefore v(t) &= u\ln\left(\frac{m_0}{m_0 - kt}\right) - gt
\end{aligned}
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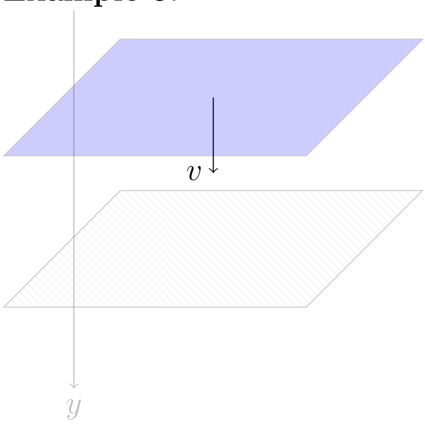
Example 5.



Solution.

$$
\begin{aligned}
p_t &= m \cdot v + \rho A v\,\mathrm{d}t \cdot 0 \\
p_{t+\mathrm{d}t} &= (m + \rho A v\,\mathrm{d}t)(v + \mathrm{d}v) \\
\therefore \mathrm{d}p &= m\,\mathrm{d}v + \rho A v^2\,\mathrm{d}t + \rho A v\,\mathrm{d}v\,\mathrm{d}t \\
\therefore \frac{\mathrm{d}p}{\mathrm{d}t} &= m\,\frac{\mathrm{d}v}{\mathrm{d}t} + \rho A v^2 + \rho A v\,\mathrm{d}v
\end{aligned}
$$