Lecture 9

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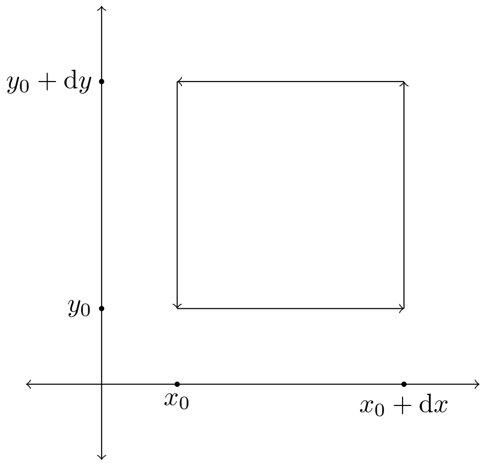
Contents

1 Curl



$$
\begin{aligned}
\oint \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} &= \int \overrightarrow{F}_x(x_0 + \frac{\mathrm{d}x}{2}, y_0, z_0)\,\mathrm{d}x + \int \overrightarrow{F}_y(x_0 + \mathrm{d}x, y_0 + \frac{\mathrm{d}y}{2}, z_0)\,\mathrm{d}y \\
&- \int \overrightarrow{F}_x(x_0 + \frac{\mathrm{d}x}{2}, y_0 + \mathrm{d}y, z_0)\,\mathrm{d}x - \int \overrightarrow{F}_y(x_0, y_0 + \frac{\mathrm{d}y}{2}, z_0)\,\mathrm{d}y \\
&= \Big(\int \overrightarrow{F}_x(x_0 + \frac{\mathrm{d}x}{2}, y_0, z_0) - \int \overrightarrow{F}_x(x_0 + \frac{\mathrm{d}x}{2}, y_0 + \mathrm{d}y, z_0)\Big)\,\mathrm{d}x \\
&+ \Big(\int \overrightarrow{F}_y(x_0 + \mathrm{d}x, y_0 + \frac{\mathrm{d}y}{2}, z_0) - \int \overrightarrow{F}_y(x_0, y_0 + \frac{\mathrm{d}y}{2}, z_0)\Big)\,\mathrm{d}y \\
&= \frac{\int \overrightarrow{F}_x(x_0 + \frac{\mathrm{d}x}{2}, y_0, z_0) - \int \overrightarrow{F}_x(x_0 + \frac{\mathrm{d}x}{2}, y_0 + \mathrm{d}y, z_0)}{\mathrm{d}y}\,\mathrm{d}x\,\mathrm{d}y \\
&+ \frac{\int \overrightarrow{F}_y(x_0 + \mathrm{d}x, y_0 + \frac{\mathrm{d}y}{2}, z_0) - \int \overrightarrow{F}_y(x_0, y_0 + \frac{\mathrm{d}y}{2}, z_0)}{\mathrm{d}x}\,\mathrm{d}x\,\mathrm{d}y \\
&= -\frac{\partial F(x)}{\partial y}\,\mathrm{d}x\,\mathrm{d}y + \frac{\partial F_y}{\partial x}\,\mathrm{d}x\,\mathrm{d}y \\
&= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y
\end{aligned}
$$

As $\overrightarrow{F}$ is conservative,

$$
\begin{aligned}
\oint \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} &= 0 \\
\therefore \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} &= 0 \\
\therefore \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} &= 0 \\
\therefore \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} &= 0
\end{aligned}
$$

Definition 1.

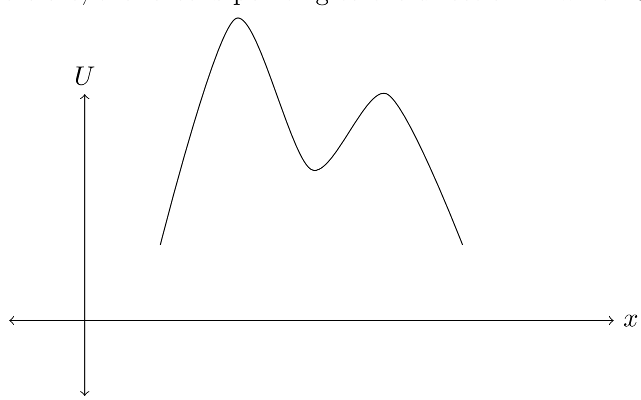
$$
\begin{aligned}
\operatorname{curl} \overrightarrow{F} &\doteq \overrightarrow{\nabla} \times \overrightarrow{F} \\
&= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}
\end{aligned}
$$

If $\overrightarrow{F}$ is conservative, $\operatorname{curl} \overrightarrow{F} = 0$.

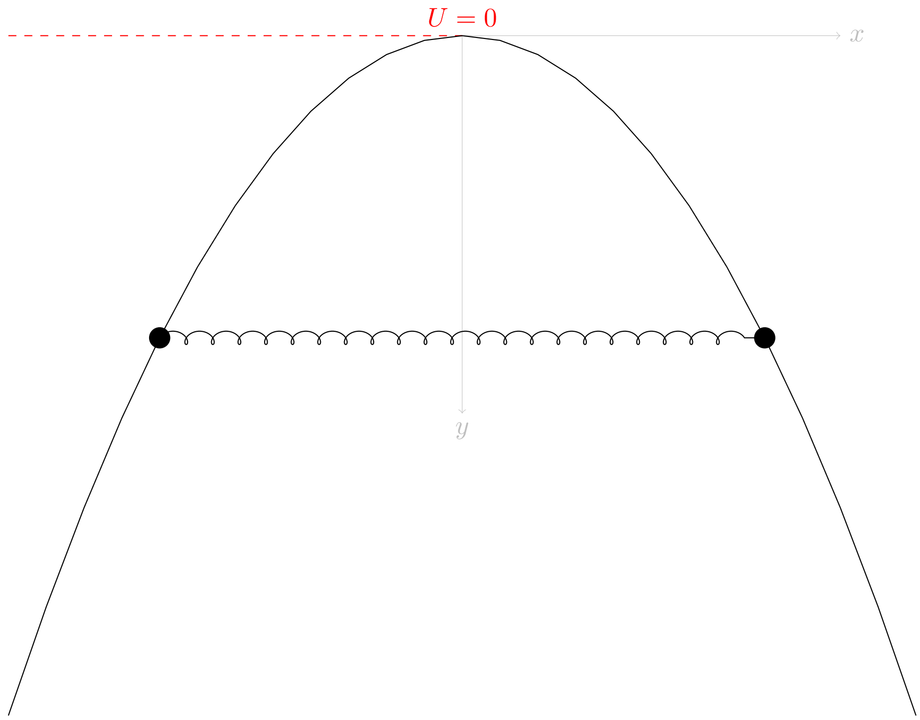
2 Equilibria

$$F = -\frac{\mathrm{d}U}{\mathrm{d}x}$$

Therefore, the force is pointing to the direction in which U decreases.



At the local extrema of U, the force is always zero.
In the neighbourhood of the point with the local minima, all forces are pointing towards the point, and at the local maxima, they are pointing away from it.
Therefore at the point where U is minimum, there is stable equilibrium. At the point where U is maximum, there is unstable equilibrium.



Example 1.

$$y = bx^2$$

Find x_{eq}.

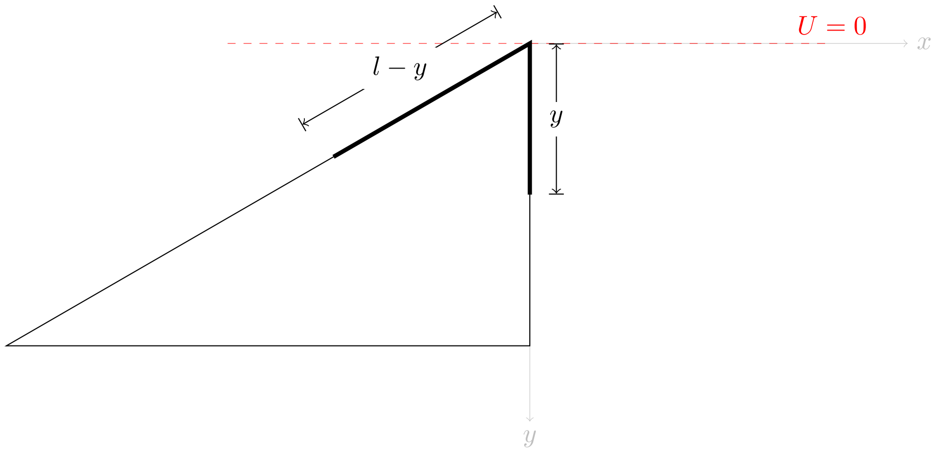
Solution. [Solution using energy]

$$\begin{aligned} U &= -2mgy + \frac{1}{2}k(2x - l)^2 \\ &= -2mgbx^2 + \frac{1}{2}k(2x - l)^2 \end{aligned}$$

Differentiating w.r.t. x,

$$0 = U' = -4mgbx_{\text{eq}} + 2k(2x-)$$

$$\therefore x_{\text{eq}} = \frac{kl}{2(k - mgb)}$$



Example 2.

Solution.

$$
\begin{aligned}
U_{\text{right side}} &= \int \mathrm{d}m g(-\tilde{y}) \\
&= \int_0^y -\left(\frac{m_0}{l}\,\mathrm{d}\tilde{y}\right) g\tilde{y} \\
&= -\frac{m_0 g}{l}\int_0^y \tilde{y}\,\mathrm{d}\tilde{y} \\
&= -\frac{m_0 g}{l}\frac{y^2}{2} \\
&= -\left(\frac{m_0}{l}y\right) g \left(\frac{y}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
U_{\text{left side}} &= \int \mathrm{d}m g(-\tilde{y}\sin\theta) \\
&= \int_0^{l-y} -\left(\frac{m_0}{l}\,\mathrm{d}\tilde{y}\right) g\tilde{y}\sin\theta \\
&= -\frac{m_0 g}{l}\int_0^{l-y} \tilde{y}\,\mathrm{d}\tilde{y}\sin\theta \\
&= -\frac{m_0 g\sin\theta}{l}\frac{(l-y)^2}{2} \\
&= -\left(\frac{m_0}{l}(l-y)\right) g \left(\frac{(l-y)\sin\theta}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\therefore U &= U_{\text{right side}} + U_{\text{left side}} \\
&= -\frac{m_0 g}{2l}y^2 - \frac{m_0 g\sin\theta}{2l}(l-y)^2
\end{aligned}
$$

Minimizing U,

$$y_{\text{eq}} = \sin\theta(l-y)$$

3 Principle of Conservation of Momentum

$$\begin{aligned}
m\ddot{\overrightarrow{r_1}} &= \overrightarrow{F_{1,\text{ext}}} + \overrightarrow{F_{1,2}} \\
m\ddot{\overrightarrow{r_2}} &= \overrightarrow{F_{2,\text{ext}}} + \overrightarrow{F_{2,1}} \\
\therefore m\ddot{\overrightarrow{r_1}} + m\ddot{\overrightarrow{r_2}} &= \overrightarrow{F_{1,\text{ext}}} + \overrightarrow{F_{2,\text{ext}}} \\
\therefore \sum m_i \ddot{\overrightarrow{r_i}} &= \sum \overrightarrow{F_{i,\text{ext}}}
\end{aligned}$$

If $\sum \overrightarrow{F_{\text{ext}}} = 0$,

$$\begin{aligned}
\sum m_i \ddot{\overrightarrow{r_i}} &= 0 \\
\therefore \sum m_i \frac{\mathrm{d}\overrightarrow{v_i}}{\mathrm{d}t} &= 0 \\
\therefore \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum m_i \overrightarrow{v_i}\right) &= 0 \\
\therefore \overrightarrow{p_{\text{total}}} = \sum m_i \overrightarrow{v_i} &= \text{ constant}
\end{aligned}$$

4 Centre of Mass

$$\begin{aligned}
\sum m_i a_i &= \frac{\mathrm{d}}{\mathrm{d}t}\overrightarrow{p_{\text{total}}} = \sum \overrightarrow{F_{\text{ext}}} \\
\therefore \left(\sum m_i\right) a_{\text{total}} &= \sum m_i \ddot{\overrightarrow{r_i}} \\
\therefore a_{\text{total}} &= \frac{\sum m_i \ddot{\overrightarrow{r_i}}}{\sum m_i} \\
&= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\sum m_i \overrightarrow{r_i}}{\sum m_i}\right) \\
&\doteq \ddot{\overrightarrow{r_{\text{COM}}}} \\
\overrightarrow{r_{\text{COM}}} &\doteq \frac{\sum m_i \overrightarrow{r_i}}{\sum m_i}
\end{aligned}$$