Lecture 6

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1 Circular Motion

1.1 Time Period

Time period is defined as the time required to complete a full circle.

$$T \doteq \frac{2\pi}{\omega}$$
$$f \doteq \frac{1}{T} = \frac{\omega}{2\pi}$$

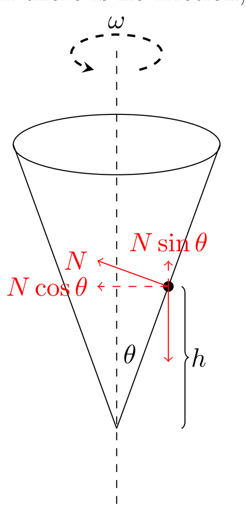
1.2 Angular Velocity

$$\overrightarrow{v} = \overrightarrow{\omega} \times \overrightarrow{r}$$

1.3 Examples

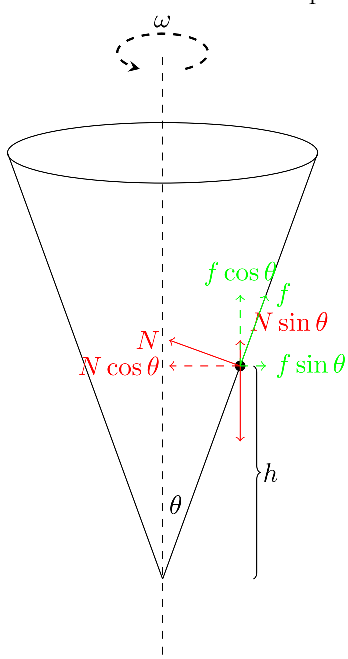
1.3.1 Example 1

If there is no friction,



$$N \sin\theta_0 = mg$$
$$N \cos\theta_0 = m\omega^2 R$$
$$\therefore \cot\theta_0 = \frac{\omega^2 h \tan\theta_0}{g}$$
$$\therefore T_0 = 2\pi\sqrt{\frac{h}{g}}\tan\theta_0$$

If friction is directed upwards,

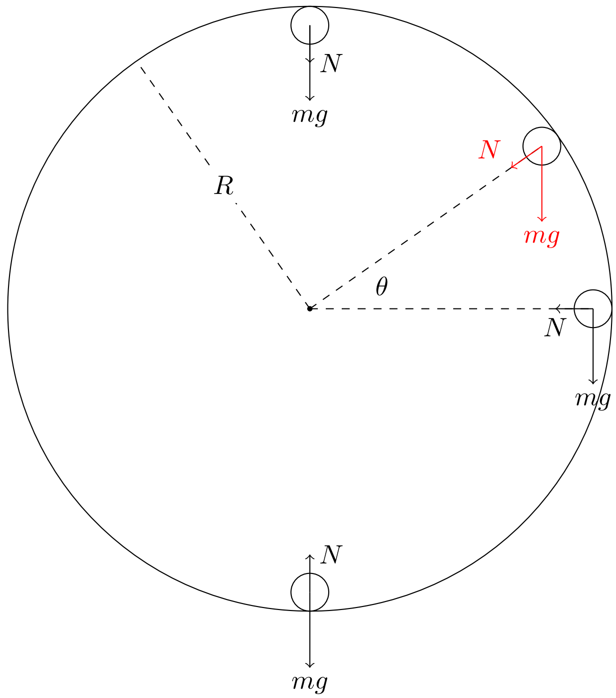


$$
\begin{aligned}
f\cos\theta_0 + N\sin\theta_0 &= mg \\
N\cos\theta_0 - f\sin\theta_0 &= m\omega^2 R \\
&= m\omega^2(h\tan\theta_0) \\
\therefore T_{\max} &= T_0\sqrt{\frac{\cos\theta_0}{\sin\theta_0}\cdot\frac{\mu_s\cos\theta_0+\sin\theta_0}{\cos\theta_0-\mu\sin\theta_0}}
\end{aligned}
$$

If friction is directed downwards,

$$
\begin{aligned}
-f\cos\theta_0 + N\sin\theta_0 &= mg \\
N\cos\theta_0 + f\sin\theta_0 &= m\omega^2 R \\
&= m\omega^2(h\tan\theta_0) \\
\therefore T_{\min} &= T_0\sqrt{\frac{\cos\theta_0}{\sin\theta_0}\cdot\frac{-\mu_s\cos\theta_0+\sin\theta_0}{\cos\theta_0+\mu\sin\theta_0}}
\end{aligned}
$$

1.3.2 Example 2



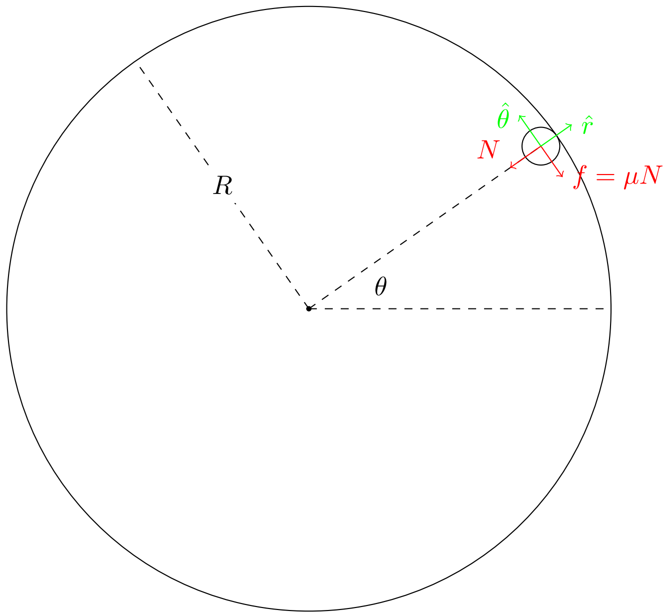
At the topmost point, if the ball just completes the circular motion

$$mg = m\frac{v^2}{R}$$
$$\therefore v = \sqrt{gR}$$

At a point at angle θ from the horizontal,

$$(-N - mg\sin\theta)\hat{r} - mg\cos\theta\hat{\theta} = -mR(\dot{\theta})^2 + mR\ddot{\theta}\hat{\theta}$$
$$\therefore -N - mg\sin\theta = -mR(\dot{\theta})^2$$
$$\therefore -mg\cos\theta = mR\ddot{\theta}$$

1.3.3 Example 3



$$
\begin{aligned}
-N\hat{r} - \mu N\hat{\theta} &= -mR(\dot{\theta})^2\hat{r} + mR\ddot{\theta}\hat{\theta} \\
\mu mR(\dot{\theta})^2 &= -mR\ddot{\theta} \\
\therefore \mu\omega^2 &= -\dot{\omega} \\
&= -\frac{\mathrm{d}\omega}{\mathrm{d}t} \\
\therefore \int \mu \,\mathrm{d}t &= \int -\frac{\mathrm{d}\omega}{\omega^2} \therefore \mu t + c = \frac{1}{\omega}
\end{aligned}
$$

At $t = 0$, $\omega = \omega_0$

$$
\begin{aligned}
\therefore c &= \frac{1}{\omega_0} \\
\therefore \mu t + \frac{1}{\omega_0} &= \frac{1}{\omega} \\
\therefore \frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega(t) &= \frac{\omega_0}{\mu\omega_0 t + 1} \\
\therefore \theta(t) &= \frac{1}{\mu}\ln\left(\mu\omega_0 t + 1\right) + c_1
\end{aligned}
$$

1.3.4 Example 4

If,

$$\begin{aligned}
r &= 4 \\
\dot{r} = \ddot{r} &= 0 \\
\theta &= \frac{\pi}{2}t \\
\dot{\theta} &= \frac{\pi}{2} \\
\ddot{\theta} &= 0
\end{aligned}$$

Find $\overrightarrow{v}$

Solution

$$\begin{aligned}
\overrightarrow{r} &= r\hat{r} \\
\overrightarrow{v} = \dot{\overrightarrow{r}} &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \\
\overrightarrow{a} = \ddot{\overrightarrow{r}} &= \left(\ddot{r} - r(\dot{\theta})^2\right)\hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\overrightarrow{r} &= 4\hat{r} \\
\overrightarrow{v} &= 4\frac{\pi}{2}\hat{\theta} = 2\pi\hat{\theta}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\overrightarrow{r} &= 4\cos\left(\frac{\pi}{2}t\right)\hat{x} + 4\sin\left(\frac{\pi}{2}t\right)\hat{y} \\
\overrightarrow{v} &= -\frac{\pi}{2}\cdot 4\sin\left(\frac{\pi}{2}t\right)\hat{x} + \frac{\pi}{2}\cdot 4\cos\left(\frac{\pi}{2}t\right)\hat{y} \\
&= -2\pi\sin\left(\frac{\pi}{2}t\right)\hat{x} + 2\pi\cos\left(\frac{\pi}{2}t\right)\hat{y}
\end{aligned}$$

1.3.5 Example 5

If,

$$\begin{aligned}
r &= v_0 t & \dot{r} &= v_0 & \ddot{r} &= 0 \\
\theta &= \omega_0 t & \dot{\theta} &= \omega_0 & \ddot{\theta} &= 0
\end{aligned}$$

Solution

$$
\begin{aligned}
\overrightarrow{r} &= r\hat{r} \\
\overrightarrow{v} = \dot{\overrightarrow{r}} &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \\
\overrightarrow{a} = \ddot{\overrightarrow{r}} &= \left(\ddot{r} - r(\dot{\theta})^2\right)\hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\overrightarrow{r} &= v_0 t\hat{r} \\
\overrightarrow{v} &= v_0\hat{r} + v_0 t\omega_0\hat{\theta} \\
\overrightarrow{a} &= -v_0 t\omega_0^2\hat{r} + 2v_0\omega_0\hat{\theta}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\overrightarrow{r} &= v_0 t\cos\omega_0 t\hat{x} + v_0 t\sin\omega_0 t\hat{y} \\
\overrightarrow{v} &= \big(v_0\cos\omega_0 t + v_0 t(-\omega_0\sin\omega_0 t)\big)\,\hat{x} + (v_0\sin\omega_0 t + v_0 t\omega_0\cos\omega_0 t)\,\hat{y}
\end{aligned}
$$