# Lecture 20

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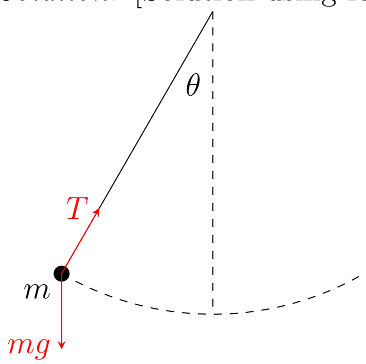
## Contents

# 1 Simple Harmonic Oscillators

$$
\begin{aligned}
\ddot{x} + {\omega_0}^2 x = 0 \\
\therefore x &= \widetilde{A}\cos\omega_0 t + \widetilde{B}\sin\omega_0 t \\
&= A\sin(\omega_0 t + \varphi)
\end{aligned}
$$

**Example 1.** A simple pendulum with with length $l$ and mass $m$ attached at its end is oscillating under gravity. Find its angular frequency.

*Solution.* [Solution using force dynamics]

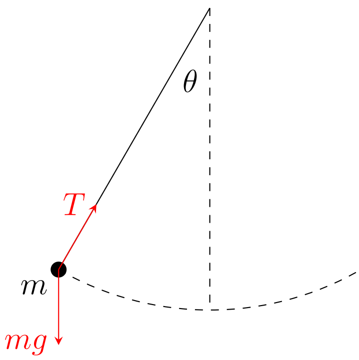



$$
\begin{aligned}
ml\ddot{\theta} &= -mg\sin\theta \\
\therefore \ddot{\theta} + \frac{g}{l}\sin\theta &= 0
\end{aligned}
$$

If $\theta$ is very small, $\sin\theta \approx \theta$

$$
\begin{aligned}
\therefore \ddot{\theta} + \frac{g}{l}\theta &= 0 \\
\therefore \omega &= \sqrt{\frac{g}{l}}
\end{aligned}
$$

*Solution.* [Solution using conservation of energy]




$$
\begin{aligned}
E &= \frac{1}{2}m(l\dot{\theta})^2 + mgl(1-\cos\theta) \\
\therefore \dot{E} &= ml^2\dot{\theta}\ddot{\theta} + mgl\dot{\theta}\sin\theta
\end{aligned}
$$

If $\theta$ is very small, $\sin\theta \approx \theta$

$$
\begin{aligned}
\therefore &= ml^2\dot{\theta}\ddot{\theta} + mgl\dot{\theta}\theta \\
\therefore &= l\ddot{\theta} + g\theta \\
\therefore &= \ddot{\theta} + \frac{g}{l}\theta \\
\therefore \omega &= \sqrt{\frac{g}{l}}
\end{aligned}
$$

## 1.1 Analysis of Potential Energy

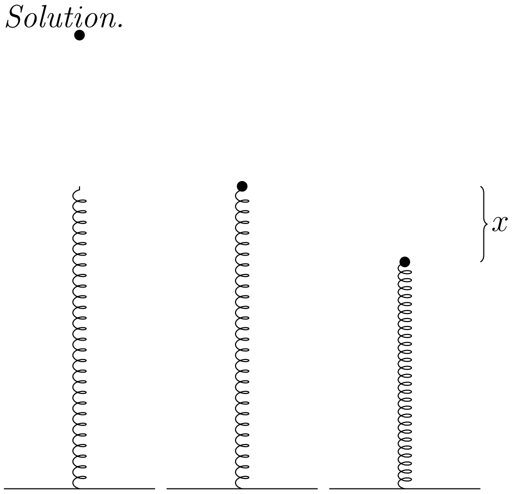
Let $\xi = x_0 - x$.
By Taylor's expansion of $U(x)$,

$$
\begin{aligned}
U(x) &= U(x_0) + \cancelto{0}{U'(x_0)}(x - x_0) + \frac{U''(x_0)}{2}(x-x_0)^2 \\
\therefore U(\xi) &= U(x_0) + \frac{1}{2}U''(x_0)\xi^2 \\
\therefore \omega &= \sqrt{\frac{U''(x_0)}{\text{“}m\text{”}}}
\end{aligned}
$$

$$
\begin{aligned}
E &= \frac{1}{2}\text{“}m\text{”}(\dot{x})^2 + U(x) \\
&= \frac{1}{2}\text{“}m\text{”}(\dot{x})^2 + U(x_0) + \frac{1}{2}U''(x_0)\xi^2 \\
&= \frac{1}{2}\text{“}m\text{”}(\dot{\xi})^2 + U(x_0) + \frac{1}{2}U''(x_0)\xi^2 \\
\therefore \dot{E} &= \dot{\xi}\,\text{“}m\text{”}\left(\ddot{\xi} + \frac{U''(x_0)}{\text{“}m\text{”}}\xi\right) \\
\therefore 0 &= \ddot{\xi} + \frac{U''(x_0)}{\text{“}m\text{”}}\xi \\
\therefore \omega &= \sqrt{\frac{U''(x_0)}{\text{“}m\text{”}}}
\end{aligned}
$$

**Example 2.** A particle of mass $m$ is dropped from a height $h$ above a spring of natural length $l$. Find angular frequency of the oscillations of the system.

*Solution.*



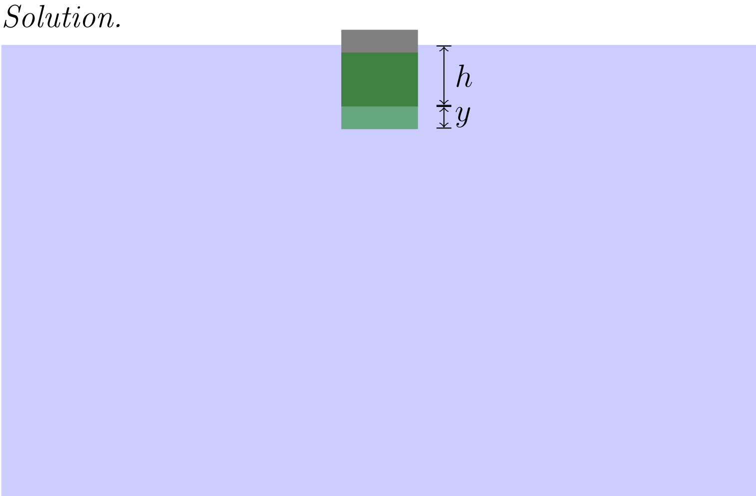
$$
\begin{aligned}
E &= \frac{1}{2}m(\dot{x})^2 + \frac{1}{2}kx^2 - mgx \\
U(x) &= \frac{1}{2}kx^2 - mgx \\
\therefore U'(x_0) &= kx_0 - mg \\
\therefore 0 &= kx_0 - mg \\
\therefore kx_0 &= mg \\
\therefore x_0 &= \frac{mg}{k} \\
\therefore U''(x_0) &= k \\
&> 0 \\
\therefore U(x) &= U(x_0) + \frac{1}{2}k\left(x - \frac{mg}{k}\right)^2
\end{aligned}
$$

Let $\xi = x - \dfrac{mg}{k}$

$$
\begin{aligned}
\therefore E &= \frac{1}{2}m(\dot{\xi})^2 + \frac{1}{2}k\xi^2 + U(x_0) \\
\therefore \xi^2 + \frac{k}{m}\xi &= 0 \\
\therefore \omega &= \sqrt{\frac{k}{m}}
\end{aligned}
$$

**Example 3.** A body of volume $V$ and volume density $\rho$ is immersed in a fluid of volume density $\rho_0$. It floats with some portion above the surface. It is perturbed from its equilibrium position.

*Solution.*



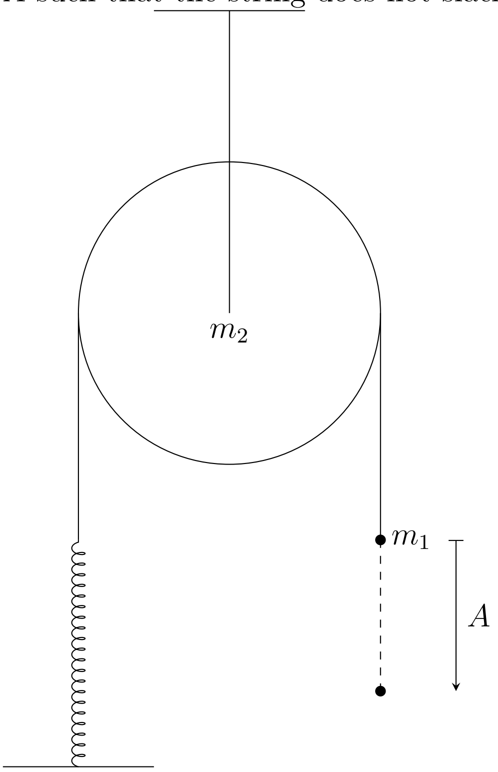
When the body is in equilibrium, let the height of the submerged portion be $h$. Therefore,

$$\begin{aligned}\rho_0 a^2 h g &= \rho a^3 g\\ \therefore h &= \frac{\rho}{\rho_0} a\end{aligned}$$

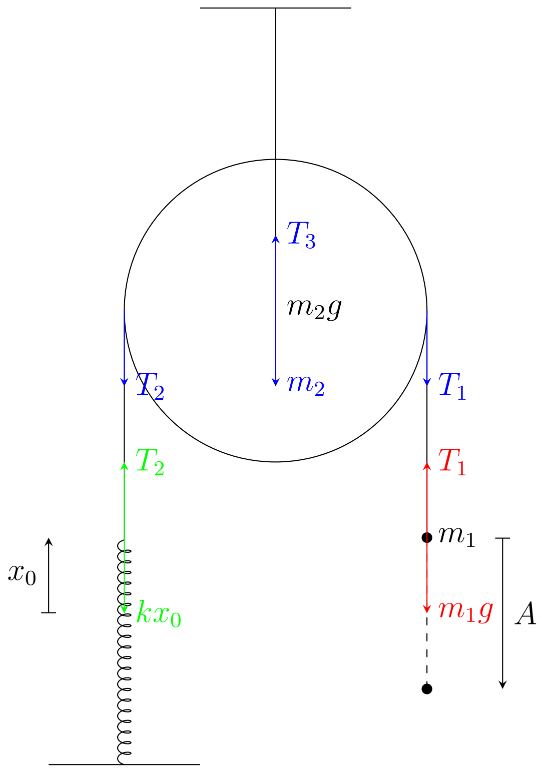
After the body is pushed down by a small distance $y$,

$$\begin{aligned}\rho a^3 \ddot{y} &= \rho a^3 g - \rho_0 (h+y) a^2 g\\ &= \cancel{\rho a^3 g} - \cancel{\rho_0 h a^2 g} - \rho_0 y a^2 g\\ \therefore \rho a^3 \ddot{y} &= -\rho_0 a^2 g y\\ \therefore \ddot{y} + \frac{\rho_0}{\rho} \cdot \frac{g}{a} \cdot y &= 0\\ \therefore {\omega_0}^2 &= \frac{\rho_0}{\rho} \cdot \frac{g}{a}\end{aligned}$$

**Example 4.** Find the angular frequency of the oscillations and the maximal $A$ such that the string does not slack.




*Solution.* Let $x_0$ be the extension of the spring at equilibrium.




Therefore, at equilibrium,

$$T_2 = kx_0$$
$$T_1 = m_1g$$
$$T_1 = T_2$$

Therefore,

$$m_1g = kx_0$$

When $m_1$ is pulled down some $x$,

$$\begin{aligned}
T_2 &= k(x_0 + x) \\
m_1\ddot{x} &= m_1g - T_1 \\
T_1R - T_2R &= \frac{1}{2}m_2R^2\alpha_2 \\
&= \frac{1}{2}m_2R\ddot{x} \\
\therefore T_1 - T_2 &= \frac{1}{2}m_2\ddot{x} \\
\therefore -m_1\ddot{x} + m_1g - k(x_0 + x) &= \frac{1}{2}m_2\ddot{x}
\end{aligned}$$

Solving,

$$\omega = \sqrt{\frac{k}{m_1 + {}^1\!/\!_2 \cdot m_2}}$$

For the string to never slack, $T_1 > 0$ and $T_2 > 0$. Therefore,

$$\begin{aligned} m_1 g - m_2 \ddot{x} &> 0 \\ \therefore m_1 g - m_1 A\omega^2 \cos \omega t &> 0 \end{aligned}$$

$$\begin{aligned} kx_0 + kx &> 0 \\ \therefore m_1 g + kA \cos \omega t &> 0 \end{aligned}$$

Solving,

$$A \leq \frac{m_1 g}{k}$$

and

$$A \leq \frac{g\left(m_1 + {}^1\!/\!_2 \cdot m_2\right)}{k}$$

Therefore,

$$A_{\text{max}} = \frac{m_1 g}{k}$$

**Example 5.** Let a rigid body of mass $m$ be pivoted at a point at a distance $d$ from its centre of mass. Let the moment of inertia of the body about the axis be $I_0$. Find the angular frequency of its oscillations.

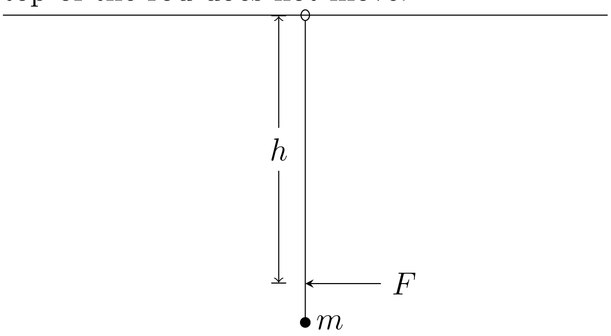
*Solution.* Let the angle between the line joining the centre of mass and the pivot point, and the vertical be $\theta$.

$$\begin{aligned} I_0 \ddot{\theta} &= -d \cdot mg \cdot \sin \theta \\ \therefore \ddot{\theta} + \frac{dmg}{I_0} \sin \theta &= 0 \end{aligned}$$

If $\theta$ is very small, $\sin \theta \approx \theta$

$$\begin{aligned} \therefore \ddot{\theta} + \frac{dmg}{I_0} \theta &= 0 \\ \therefore \omega &= \sqrt{\frac{dmg}{I_0}} \end{aligned}$$

**Example 6.** A rod of mass $m$ and length $L$ has a particle of mass $m$ attached at its bottom. It has a loop at its top through which a string is threaded. A force $F$ is acting on it, for time $\Delta t$, at $h$ from the top. Find $h$ such that the top of the rod does not move.




*Solution.* Let $d$ be the distance between the top and the centre of mass.

$$\begin{aligned} d &= \frac{\frac{L}{2}m + Lm}{2m} \\ &= \frac{3}{4}L \\ I_{\text{top}} &= \frac{4}{3}mL^2 \\ \omega^2 &= \frac{dmg}{L} \\ \therefore \omega^2 &= \frac{9}{8} \cdot \frac{g}{L} \end{aligned}$$

For the top of the rod to be stationary,

$$\begin{aligned} hF\Delta t &= I_{\text{top}}\omega \\ \therefore hF\Delta t &= \frac{4}{3}mL^2\omega \\ \therefore \omega &= \frac{3hF\Delta t}{4mL^2} \end{aligned}$$

and

$$\begin{aligned} v_{\text{COM}} &= \omega d \\ \therefore v_{\text{COM}} &= \frac{F\Delta t}{2m} \end{aligned}$$

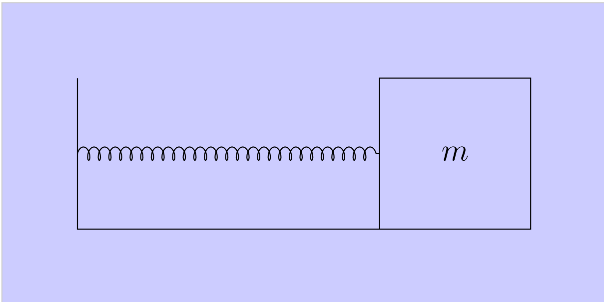
Therefore,

$$h = \frac{8}{9}L$$

# 2 Damped Oscillations

A spring mass oscillator is submerged in water such that the friction between the mass and the water is

$$\overrightarrow{f} = -\beta \overrightarrow{v}$$




Therefore,

$$m\ddot{x} = -kx - \beta\dot{x}$$

$$\therefore \ddot{x} + \frac{\beta}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\therefore \ddot{x} + \frac{\beta}{m}\dot{x} + {\omega_0}^2 x = 0$$

Therefore, the characteristic equation is

$$\lambda^2 + \frac{\beta}{m}\lambda + {\omega_0}^2 = 0$$

$$\therefore \lambda = \frac{-\frac{\beta}{m} \pm \sqrt{\frac{\beta^2}{m^2} - 4{\omega_0}^2}}{2}$$

$$= -\frac{\beta}{2m} \pm \sqrt{\left(\frac{\beta}{2m}\right)^2 - {\omega_0}^2}$$

| | |
|---|---|
| Strong damping | $\frac{\beta}{2m} > \omega_0$ |
| Critical damping | $\frac{\beta}{2m} = \omega_0$ |
| Weak damping | $\frac{\beta}{2m} < \omega_0$ |

Oscillations occur in case of weak damping.

Therefore,

$$\lambda = -\frac{\beta}{2m} \pm i\sqrt{{\omega_0}^2 - \left(\frac{\beta}{2m}\right)^2}$$

Let $\omega_1 = \sqrt{{\omega_0}^2 - \left(\frac{\beta}{2m}\right)^2}$

$$\therefore x = e^{-\beta/2m \cdot t}\left(\widetilde{A}\cos\omega_1 t + \widetilde{B}\sin\omega_1 t\right)$$

Let

$$x(t=0) = 0$$
$$\dot{x}(t=0) = v_0$$

Solving,

$$\widetilde{A} = 0$$
$$\widetilde{B} = \frac{v_0}{\omega_1}$$

Therefore,

$$x(t) = \frac{v_0}{\omega_1} e^{-\beta/2m \cdot t}\sin\omega_1 t$$
$$x(t) = \frac{v_0}{\sqrt{{\omega_0}^2 - \left(\frac{\beta}{2m}\right)^2}} e^{-\beta/2m \cdot t}\sin\left(\sqrt{{\omega_0}^2 - \left(\frac{\beta}{2m}\right)^2}\, t\right)$$

In case of critical damping,

$$x = e^{-\beta/2m \cdot t}(\widetilde{A} + \widetilde{B}t)$$

Let

$$x(t=0) = 0$$
$$\dot{x}(t=0) = v_0$$

Solving,

$$\widetilde{A} = 0$$
$$\widetilde{B} = v_0$$

Therefore,

$$x = v_0 t e^{-\beta/2m \cdot t}$$

In case of strong damping,

$$x = \widetilde{A} e^{\left(-\beta/2m + \sqrt{\left(\beta/2m\right) - \omega_0{}^2}\right)t} + \widetilde{B} e^{\left(-\beta/2m - \sqrt{\left(\beta/2m\right) - \omega_0{}^2}\right)t}$$