Lecture 8

Aakash Jog

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Contents

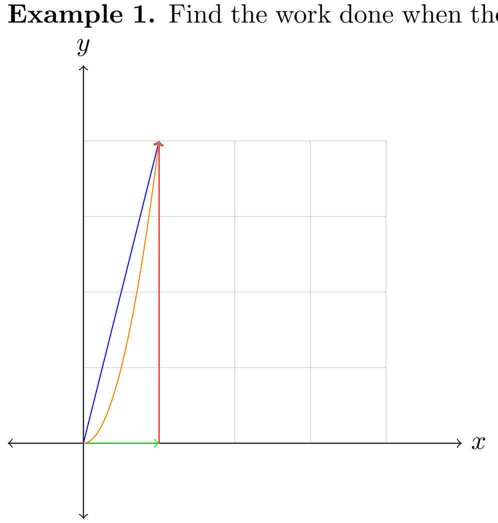
1 Power

Power is defined to be the rate of work done.

$$
\begin{aligned}
P &\doteq \frac{\mathrm{d}W}{\mathrm{d}t} \\
&= \frac{\overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r}}{\mathrm{d}t} \\
&= \overrightarrow{F} \cdot \frac{\mathrm{d}\overrightarrow{r}}{\mathrm{d}t} \\
&= \overrightarrow{F} \cdot \overrightarrow{v}
\end{aligned}
$$

2 Conservative and Non-Conservative Forces

Example 1. Find the work done when the body moves along the paths shown.



$$\overrightarrow{F} = 3x^2y^2\hat{x} + 2x^3y\hat{y}$$

Solution.

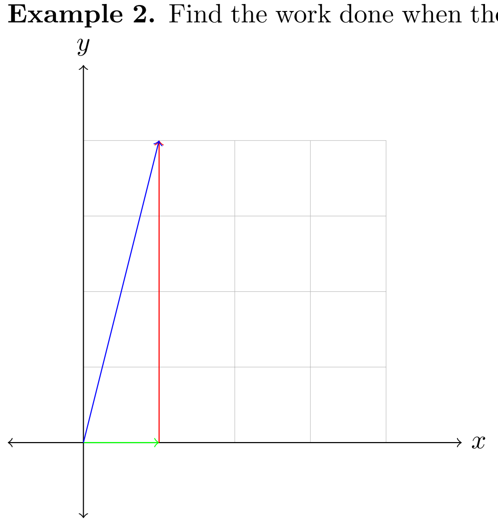
$$
\begin{aligned}
\overrightarrow{F} &= 3x^2y^2\hat{x} + 2x^3y\hat{y} \\
\mathrm{d}\overrightarrow{r} &= (\mathrm{d}x, \mathrm{d}y) \\
\therefore \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} &= 3x^2y^2\,\mathrm{d}x + 2x^3y\,\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
W_1 &= \int\limits_{(0,0)}^{(1,4)} \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} \\
&= \int\limits_0^1 3 \cdot x^2 \cdot 0^2 \cdot \mathrm{d}x + \int\limits_0^4 2 \cdot 1^3 \cdot y \cdot \mathrm{d}y \\
&= 0 + 16 \\
&= 16
\end{aligned}
$$

$$
\begin{aligned}
W_2 &= \int\limits_{(0,0)}^{(1,4)} \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} \\
y &= 4x \\
\therefore W_2 &= \int\limits_0^1 3x^2(4x)^2 \,\mathrm{d}x + 2x^3(4x)(4\,\mathrm{d}x) \\
&= 80 \int\limits_0^1 x^4 \,\mathrm{d}x \\
&= 80 \frac{1^5}{5} \\
&= 16
\end{aligned}
$$

$$
\begin{aligned}
y &= 4x^2 \\
\therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= 8x \\
\therefore \mathrm{d}y &= 8x\,\mathrm{d}x \\
W_2 &= \int\limits_{(0,0)}^{(1,4)} \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} \\
&= \int\limits_0^1 3x^2(4x^2)^2\,\mathrm{d}x + 2x^3(4x^2)(8x\,\mathrm{d}x) \\
&= \int\limits_0^1 112x^6\,\mathrm{d}x \\
&= 112\frac{1^7}{7} \\
&= 16
\end{aligned}
$$

Example 2. Find the work done when the body moves along the paths shown.



$$\overrightarrow{F} = 3x^2\hat{x} + 2x^3y\hat{y}$$

Solution.

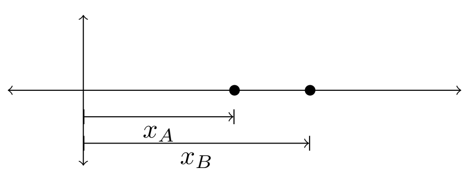
$$
\begin{aligned}
\overrightarrow{F} &= 3x^2\hat{x} + 2x^3y\hat{y} \\
\mathrm{d}\overrightarrow{r} &= (\mathrm{d}x, \mathrm{d}y) \\
\therefore \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} &= 3x^2\,\mathrm{d}x + 2x^3y\,\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
W_1 &= \int_{(0,0)}^{(1,4)} \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} \\
&= \int_0^1 3 \cdot x^2 \cdot \mathrm{d}x + \int_0^4 2 \cdot 1^3 \cdot y \cdot \mathrm{d}y \\
&= 1 + 16 \\
&= 16
\end{aligned}
$$

$$
\begin{aligned}
W_2 &= \int_{(0,0)}^{(1,4)} \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} \\
y &= 4x \\
\therefore W_2 &= \int_0^1 3x^2 \,\mathrm{d}x + 2x^3(4x)(4\,\mathrm{d}x) \\
&= \int_0^1 (3x^2 + 32x^4)\,\mathrm{d}x \\
&= \frac{37}{5}
\end{aligned}
$$

It is evident that in this case, the work done is dependant on the path.

2.1 Potential Energy Corresponding to a 1D Force



$$
\begin{aligned}
W &= \int_{x_A}^{x_B} F \,\mathrm{d}x \\
&= (-U(x_B)) - (-U(x_A)) \\
&= U(x_A) - U(x_B) \\
\therefore \frac{\mathrm{d}(-U(x))}{\mathrm{d}x} &= F \\
\therefore F &= -\frac{\mathrm{d}U}{\mathrm{d}x} \\
\therefore \overrightarrow{F} &= -\frac{\mathrm{d}U}{\mathrm{d}x}\,\hat{x}
\end{aligned}
$$

2.2 Potential Energy Corresponding to a General Force

$$
\begin{aligned}
\overrightarrow{F} &= (F_x(x,y,z), F_y(x,y,z), F_z(x,y,z)) \\
\mathrm{d}\overrightarrow{r} &= (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)
\end{aligned}
$$

$$
\int_{(x_A,y_A,z_A)}^{(x_B,y_B,z_B)} \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} = U_A - U_B
$$

$$
\begin{aligned}
\therefore \int_{(x_A,y_A,z_A)}^{(x_B,y_A,z_A)} F_x \,\mathrm{d}x &= (-U(x_A,y_A,z_A)) - (-U(x_B,y_A,z_A)) \\
\therefore F_x &= \frac{\partial(-U)}{\partial x} \\
&= -\frac{\partial U}{\partial x}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \int_{(x_A,y_A,z_A)}^{(x_B,y_B,z_A)} F_y \,\mathrm{d}y &= (-U(x_A,y_A,z_A)) - (-U(x_B,y_B,z_A)) \\
\therefore F_y &= \frac{\partial(-U)}{\partial y} \\
&= -\frac{\partial U}{\partial y}
\end{aligned}
$$

$$\therefore \int\limits_{(x_A,y_A,z_A)}^{(x_B,y_B,z_B)} F_z \,\mathrm{d}z = (-U(x_A,y_A,z_A)) - (-U(x_B,y_B,z_B))$$

$$\begin{aligned}\therefore F_z &= \frac{\partial(-U)}{\partial z} \\ &= -\frac{\partial U}{\partial z}\end{aligned}$$

Definition 1. $\overrightarrow{F}$ is a conservative force iff

$$\exists U(x,y,z), \text{ s.t. } F_x = -\frac{\partial U}{\partial x}, F_y = -\frac{\partial U}{\partial y}, F_z = -\frac{\partial U}{\partial z}$$

$$\begin{aligned}\overrightarrow{F} &= \left(-\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z}\right) \\ &= -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) U \\ &= -\overrightarrow{\nabla} U\end{aligned}$$

Example 3. Show that $\overrightarrow{F}$ is conservative.

$$\overrightarrow{F}(x,y) = (3x^2y^2, 2x^3y)$$

Solution.

$$\begin{aligned}\frac{\partial U}{\partial x} &= -3x^2y^2 \\ \therefore U &= \int -3x^2y^2 \,\mathrm{d}x \\ &= -3\frac{x^3}{3}y^2 + c(y) \\ &= x^3y^2 + c(y) \\ \frac{\partial U}{\partial y} &= -2x^3y \\ \therefore -x^3(2y) + c'(y) &= -2x^3y \\ \therefore c'(y) &= 0 \\ \therefore c(y) &= \text{constant} \\ \therefore U(x,y,z) &= -x^3y^2 + c\end{aligned}$$

Example 4. Show that $\overrightarrow{F}$ is conservative.

$$\overrightarrow{F}(x,y) = (3x^2, 2x^3y)$$

Solution.

$$
\begin{aligned}
\frac{\partial U}{\partial x} &= -3x^2 \\
\therefore U &= \int -3x^2 \,\mathrm{d}x \\
&= -3\frac{x^3}{3} + c(y) \\
&= x^3 + c(y) \\
\frac{\partial U}{\partial y} &= -2x^3 y \\
\therefore c'(y) &= -2x^3 y
\end{aligned}
$$

Therefore, $\overrightarrow{F}$ is non-conservative.

2.3 Line Integral Over a Closed Path

If a force $\overrightarrow{F}$ is conservative,

$$
\begin{aligned}
\int\limits_{\text{path 1}} \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} &= \int\limits_{\text{path 2}} \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} \\
\therefore \int\limits_{\text{path 1}} \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} - \int\limits_{\text{path 2}} \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} &= 0 \\
\therefore \int\limits_{\text{path 1}} \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} + \int\limits_{\text{path 2}} \overrightarrow{F} \cdot (-\mathrm{d}\overrightarrow{r}) &= 0 \\
\therefore \oint \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} &= 0
\end{aligned}
$$