Lecture 24

Aakash Jog

Tuesday 20th January, 2015

Contents

1

Example 1. A man is pushing a box of mass m on a horizontal surface, with $F = \dfrac{mgt}{\tau}$. The coefficients of friction between the box and the floor are μ_s and μ_k. Find the velocity as a function of time.

Solution. Case I : $t < t_0$

$$\begin{aligned}
\mu_s mg &= \frac{mgt_0}{\tau} \\
\therefore \mu_s &= \frac{t_0}{\tau} \\
\therefore t_0 &= \mu_s \tau
\end{aligned}$$

Case II : $t > t_0$

$$\begin{aligned}
F_{\text{net}} &= \frac{mgt}{\tau} - \mu_k mg \\
&= mg\left(\frac{t}{\tau} - \mu_k\right) \\
\therefore a &= g\left(\frac{t}{\tau} - \mu_k\right) \\
\therefore v &= \frac{gt^2}{2\tau} - \mu_k gt + c \\
v(\mu_s\tau) &= 0 \\
\therefore 0 &= \frac{g{\mu_s}^2\tau^2}{2\tau} - \mu_k g\mu_s\tau + c \\
\therefore c &= \mu_s\mu_k g\tau - \frac{{\mu_s}^2 g\tau}{2} \\
\therefore v &= \frac{gt^2}{2\tau} - \mu_k gt + \mu_s\mu_k g\tau - \frac{{\mu_s}^2 g\tau}{2}
\end{aligned}$$

Therefore,

$$v = \begin{cases} 0 & ; \quad t < \mu_s\tau \\ \dfrac{gt^2}{2\tau} - \mu_k gt + \mu_s\mu_k g\tau - \dfrac{{\mu_s}^2 g\tau}{2} & ; \quad t > \mu_s\tau \end{cases}$$

Example 2. A hollow sphere, a solid sphere, a hollow cylinder, and a solid cylinder of equal masses and radii are rolling without slipping down an incline of angle θ. What is the minimal coefficient of static friction that will assure rolling without slipping for all bodies?

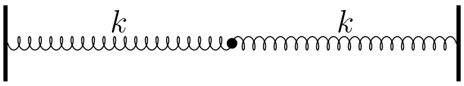
Solution. About the centre of mass,

$$
\begin{aligned}
mg\sin\theta - f &= ma \\
kmR^2\alpha &= fR \\
\therefore mg\sin\theta - kma &= ma \\
\therefore a &= \frac{g\sin\theta}{k+1} \\
\therefore f &= kma \\
&= \frac{k}{k+1} mg\sin\theta
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f_s &\leq \mu mg\cos\theta \\
\therefore \mu_{\min} &= \frac{1}{2}\tan\theta
\end{aligned}
$$

Example 3. A system of a mass m and two springs of spring constant k is arranged as shown. It is submerged in a fluid such that $\overrightarrow{f} = -\gamma\overrightarrow{v}$. Assuming there is no under-damping, what should be k for the mass to return to a point near the origin in minimal time?



Solution.

$$
\begin{aligned}
F &= -2kx - \gamma\dot{x} \\
\therefore m\ddot{x} &= -\gamma\dot{x} - 2kx \\
\therefore \ddot{x} + \frac{\gamma}{m}\dot{x} + \frac{2k}{m}x &= 0
\end{aligned}
$$

Therefore, for minimal k, there must be critical damping

$$
\begin{aligned}
\left(\frac{\gamma}{m}\right)^2 - 4\left(\frac{2k}{m}\right) &= 0 \\
\therefore k &= \frac{\gamma^2}{8m}
\end{aligned}
$$

Example 4. Two rockets start from the same point in space with zero velocity.
The first rocket moves in the x direction, with initial mass m_0. It releases gas with rate λ, with velocity v_0 with respect to it.

The second rocket moves at 30° with respect to the y direction. It releases gas with velocity v_0 with respect to it. Its mass changes as

$$m(t) = m_0 e^{-\alpha t}$$

What is the magnitude of acceleration of rocket 2 with respect to rocket 1, at $t = \dfrac{1}{\alpha}$?

Solution. For rocket 1,

$$\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}t} &= m\,\frac{\mathrm{d}v}{\mathrm{d}t} + v_0\,\frac{\mathrm{d}m}{\mathrm{d}t} \\
\therefore 0 &= (m_0 - \lambda t)a_1 - v_0\lambda \\
\therefore a_1 &= \frac{v_0\lambda}{m_0 - \lambda t}
\end{aligned}$$

For rocket 2,

$$\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}t} &= m\,\frac{\mathrm{d}v}{\mathrm{d}t} + v_0\,\frac{\mathrm{d}m}{\mathrm{d}t} \\
\therefore - &= m_0 e^{-\alpha t} a_2 - v_0\left(\alpha m_0 e^{-\alpha t}\right) \\
\therefore a_2 &= \alpha v_0
\end{aligned}$$

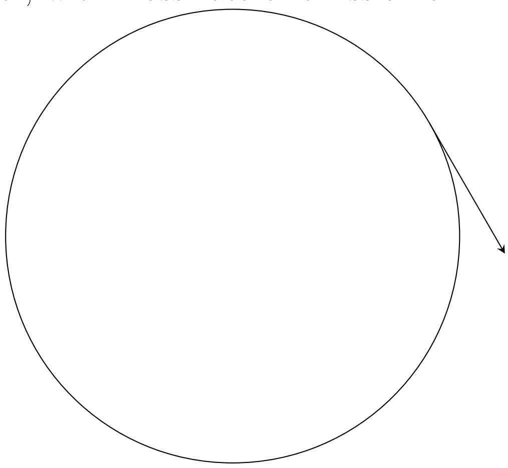
Therefore,

$$\begin{aligned}
\overrightarrow{a_1} &= \left(\frac{v_0\lambda}{m_0 - \lambda t}, 0\right) \\
\overrightarrow{a_2} &= \left(\frac{1}{2}\alpha v_0, \frac{\sqrt{3}}{2}\alpha v_0\right)
\end{aligned}$$

Therefore,

$$a_{2,1} = \sqrt{\left(\frac{1}{2}\alpha v_0 - \frac{v_0\alpha}{m_0 - {}^{\lambda}/_{\alpha}}\right)^2 + \left(\frac{\sqrt{3}}{2}\alpha v_0\right)^2}$$

Example 5. A thin cylinder of radius R is filled with a gas and is fixed at its centre. Its initial mass is m_0. Gas is emitted with velocity u with respect to the cylinder, with mass rate of emission c. Find the angular velocity of

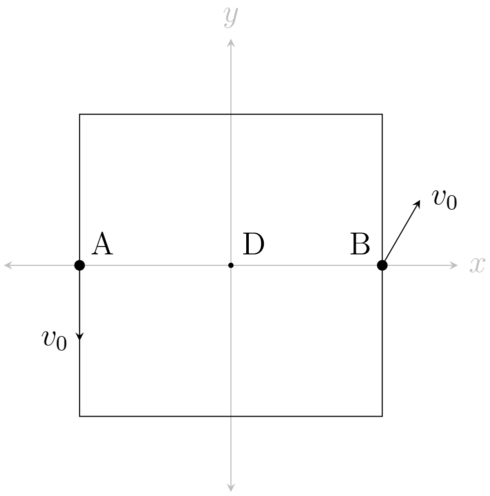


the cylinder.

Solution. About the centre,

$$\begin{aligned}
L(t) &= \frac{1}{2}(m + \mathrm{d}m)R^2\omega + R(\mathrm{d}m)\omega \\
L(t + \mathrm{d}t) &= \frac{1}{2}(m + \mathrm{d}m)R^2(\omega + \mathrm{d}\omega)R^2 + (-\,\mathrm{d}m)(\omega R - u)R \\
\therefore \frac{\mathrm{d}L}{\mathrm{d}t} &= \frac{\frac{1}{2}(m + \mathrm{d}m)R^2\,\mathrm{d}\omega + \mathrm{d}mRu}{\mathrm{d}t} \\
\therefore 0 &= \frac{1}{2}mR^2\frac{\mathrm{d}\omega}{\mathrm{d}t} + \frac{\mathrm{d}m}{\mathrm{d}t}Ru \\
&= \frac{1}{2}(m_0 - ct)R^2\frac{\mathrm{d}\omega}{\mathrm{d}t} - cRu \\
\therefore (m_0 - ct)R^2\frac{\mathrm{d}\omega}{\mathrm{d}t} &= 2cRu \\
\therefore \mathrm{d}\omega &= \frac{2cu\,\mathrm{d}t}{R(m_0 - ct)} \\
\therefore \omega &= \frac{2u}{R}\ln\frac{m_0}{m_0 - ct}
\end{aligned}$$

Example 6. Two birds are sitting on a square body with sides of length $2a$ and mass density σ. Bird A flies away in the negative x direction and Bird B files away at 30°, towards the positive x direction, with respect to the y direction. Find the velocity of the centre of the square and the angular velocity of the square.



Solution. By COLM in the y direction,

$$\begin{aligned}
0 &= -mv_0 + mv_0 \cos\alpha + 4a^2\sigma v_{Dy} \\
\therefore 4a^2\sigma v_{Dy} &= mv_0(1-\cos\alpha) \\
\therefore v_{Dy} &= \frac{mv_0(1-\cos\alpha)}{4a^2\sigma}
\end{aligned}$$

By COLM in the x direction,

$$\begin{aligned}
-4a^2\sigma v_{Dx} &= mv_0 \sin\alpha \\
\therefore v_{Dx} &= -\frac{mv_0 \sin\alpha}{4a^2\sigma}
\end{aligned}$$

Therefore,

$$v_D = -\frac{mv_0 \sin\alpha}{4a^2\sigma}\hat{\imath} + \frac{mv_0(1-\cos\alpha)}{4a^2\sigma}\hat{\jmath}$$

By COAM,

$$\begin{aligned}
mv_0 a + mv_0 \cos\alpha a &= I\omega \\
\therefore \omega &= \frac{mv_0}{I}(1+\cos\alpha) \\
&= \frac{mv_0}{\frac{8}{3}\sigma a^4}(1+\cos\alpha) \\
&= \frac{3mv_0}{8\sigma a^4}(1+\cos\alpha)
\end{aligned}$$

ω is clockwise.

Example 7. A thin box with walls of length a and mass m, each, is placed on a surface. A point cat is at the bottom right corner of the box. The cat jumps up towards the left at an angle α with the horizontal, with velocity v. Assuming there is no friction, find the velocity of the box while the cat is in the air. On reaching the other side, the cat clings to the wall. Assuming no friction, find the height the cat reaches. If friction is infinite, find the height. Consider v_0 and α to be such that the cat reaches the wall at height $\frac{a}{2}$. Assuming the friction to be inifinite, find v_0 such that the box does not topple.

Solution. By COLM in the horizontal direction,

$$\begin{aligned} 0 &= -mv_0 \cos\alpha + 4mv \\ \therefore v &= \frac{v_0 \cos\alpha}{4} \end{aligned}$$

$$\begin{aligned} (v_{\text{cat, box}})_x &= \frac{5}{4} v_0 \cos\alpha \\ \therefore a &= \frac{5}{4} v_0 \cos\alpha t \\ \therefore t &= \frac{4a}{5v_0 \cos\alpha} \end{aligned}$$

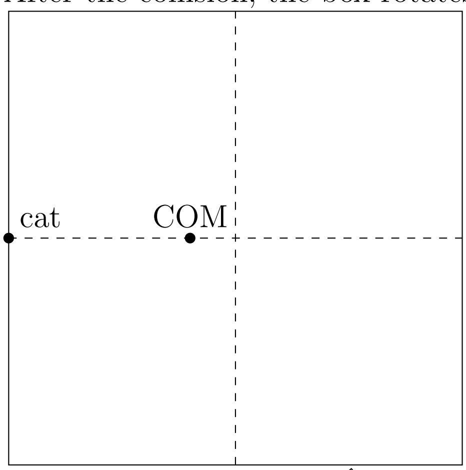
Therefore,

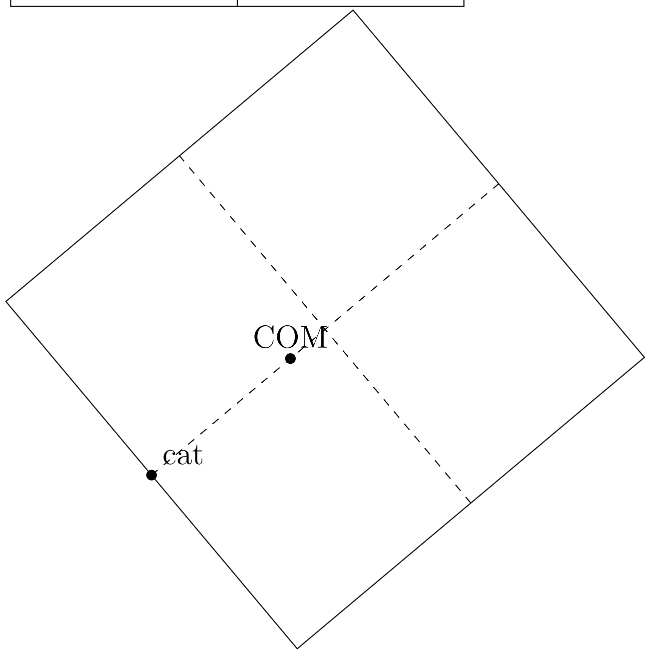
$$\begin{aligned} y &= v_0 \sin\alpha t - \frac{1}{2} g t^2 \\ &= v_0 \sin\alpha \cdot \frac{4a}{5v_0 \cos\alpha} - \frac{1}{2} g \frac{16a^2}{9{v_0}^2 \cos^2\alpha} \\ &= \frac{4a \tan\alpha}{5} - \frac{8a^2 g}{25{v_0}^2 \sec^2\alpha} \end{aligned}$$

If the friction is infinite,

$$\begin{aligned} a &= v_0 \cos\alpha t \\ \therefore t &= \frac{a}{\cos\alpha} \\ \therefore y &= v_0 \sin\alpha t - \frac{1}{2} g t^2 \\ &= v_0 \sin\alpha \frac{a}{\cos\alpha} - \frac{1}{2} g \frac{a^2}{\cos^2\alpha} \\ &= v_0 a \tan\alpha - \frac{a^2 g \sec^2\alpha}{2} \end{aligned}$$

After the collision, the box rotates about point A, i.e. the lower left corner.





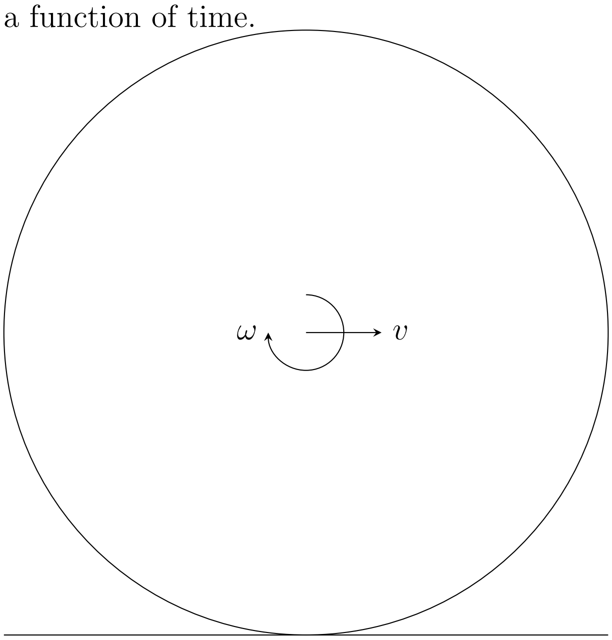
At the instant of collision, angular momentum is conserved. After that, linear and angular momenta are not conserved, but mechanical energy is conserved. By COAM at the time of collision, about the lower left corner,

$$\frac{a}{2} m v_0 \cos\alpha = I_A \omega$$

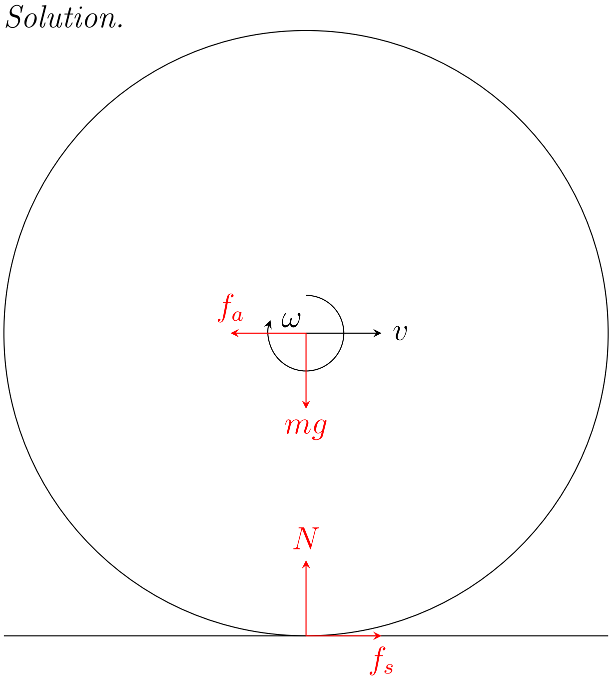
$$\therefore \frac{a}{2} m v_0 \cos\alpha = \left(2 \cdot \frac{1}{2} m a^2 + 2 \left(\frac{1}{12} m a^2 + m \left(a^2 + \frac{a^2}{4} \right) \right) + m \left(\frac{a}{2} \right)^2 \right) \omega$$

Example 8. A body with moment of inertia I_0, radius R and mass m is rolling on a surface with $v_0 = \omega_0 R$. The air friction is given by $\overrightarrow{f_a} = -\gamma \overrightarrow{v}$.

Assume $\mu mg > \gamma v_0$ Find all forces acting on the body. Find the friction due to the ground, as a function of v. Find the velocity of the centre of mass as a function of time.



Solution.



$$
\begin{aligned}
ma &= f_a - f_s \\
\therefore ma &= \gamma v - f_s
\end{aligned}
$$

$$I_0\alpha = f_s R$$

As the body is purely rolling,

$$a = \alpha R$$