Lecture 7

Aakash Jog

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Contents

1 Energy

1.1 Leibenitz's 'Vis Viva'

Based on experimental analyses, Leibenitz defined 'vis viva' as the product of the weight of a body and its height above the ground. This definition contradicted Descartes, who defined the life force as the product of the mass of a body and the magnitude of its velocity.

1.2 Coriolis' Definition of Work

Coriolis defined work as the multiplication of the displacement of a body, and the component of the force responsible for the displacement, in the direction of the displacement, or vice versa.

$$\begin{aligned}
\mathrm{d}W &\doteq \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} \\
\therefore W &\doteq \int_{\overrightarrow{r_A}}^{\overrightarrow{r_B}} \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r}
\end{aligned}$$

For a general body,

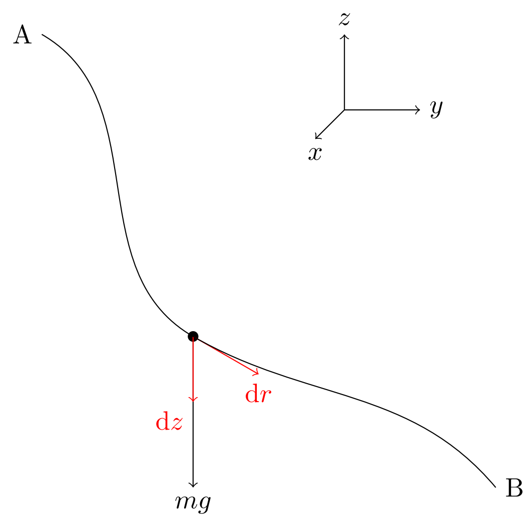
$$\begin{aligned}
W &= \int_{\overrightarrow{r_A}}^{\overrightarrow{r_B}} \overrightarrow{F_{\text{total}}} \cdot \mathrm{d}\overrightarrow{r} \\
&= \int_{\overrightarrow{r_A}}^{\overrightarrow{r_B}} m \frac{\mathrm{d}\overrightarrow{v}}{\mathrm{d}t} \cdot \mathrm{d}\overrightarrow{r}
\end{aligned}$$

In Leibenitz's notation, $\mathrm{d}t$ is a scalar. Therefore, we can move it around.

$$\begin{aligned}
\therefore W &= \int_{\overrightarrow{r_A}}^{\overrightarrow{r_B}} m \,\mathrm{d}\overrightarrow{v} \cdot \frac{\mathrm{d}\overrightarrow{r}}{\mathrm{d}t} \\
&= \int_{\overrightarrow{v_A}}^{\overrightarrow{v_B}} m \,\mathrm{d}\overrightarrow{v} \cdot \overrightarrow{v} \\
\mathrm{d}(v^2) &= 2\overrightarrow{v}\,\mathrm{d}\overrightarrow{v} \\
\therefore W &= \int_{\overrightarrow{v_A}}^{\overrightarrow{v_B}} \frac{1}{2} m \,\mathrm{d}(v^2) \\
&= \frac{1}{2} m (v_B^2 - v_A^2) \\
&= \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2 \\
&\doteq E_K(B) - E_K(A)
\end{aligned}$$

1.3 Gravitational Potential Energy

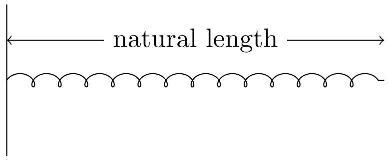
$$\overrightarrow{F_{\text{total}}} = \overrightarrow{F_{\text{gravitation}}} + \overrightarrow{F_{\text{other}}}$$
$$\overrightarrow{F_{\text{total}}} = m\overrightarrow{g} + \overrightarrow{F_{\text{other}}}$$

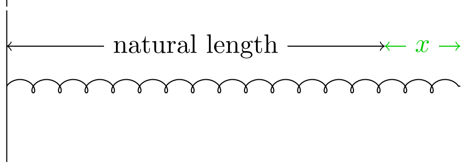


$$\begin{aligned} W(m\overrightarrow{g}) &= \int_{\overrightarrow{r_A}}^{\overrightarrow{r_B}} m\overrightarrow{g} \cdot \mathrm{d}\overrightarrow{r} \\ &= \int_{\overrightarrow{z_B}}^{\overrightarrow{z_A}} mg(\mathrm{d}z) \\ &= mg(z_A - z_B) \\ &= mgz_a - mgz_B \end{aligned}$$

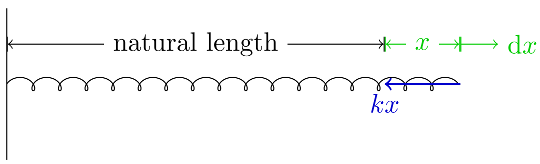
$$\begin{aligned} W = \int_B^A \overrightarrow{F_{\text{total}}} \cdot \mathrm{d}\overrightarrow{r} &= E_K(B) - E_K(A) \\ \therefore \int_B^A \overrightarrow{F_{\text{other}}} \cdot \mathrm{d}\overrightarrow{r} + W(m\overrightarrow{g}) &= E_K(B) - E_K(A) \\ \therefore \int_B^A \overrightarrow{F_{\text{other}}} \cdot \mathrm{d}\overrightarrow{r} &= (E_K(B) + mgz_B) - (E_K(A) + mgz_A) \\ &= E_M(B) - E_M(A) \end{aligned}$$

1.4 Spring Potential Energy





$$F = kx$$



The work done by the spring force is

$$\begin{aligned}
W(k) &= \int_{x_A}^{x_B} -kx \, \mathrm{d}x \\
&= \frac{1}{2}kx_A^2 - \frac{1}{2}kx_B^2
\end{aligned}$$

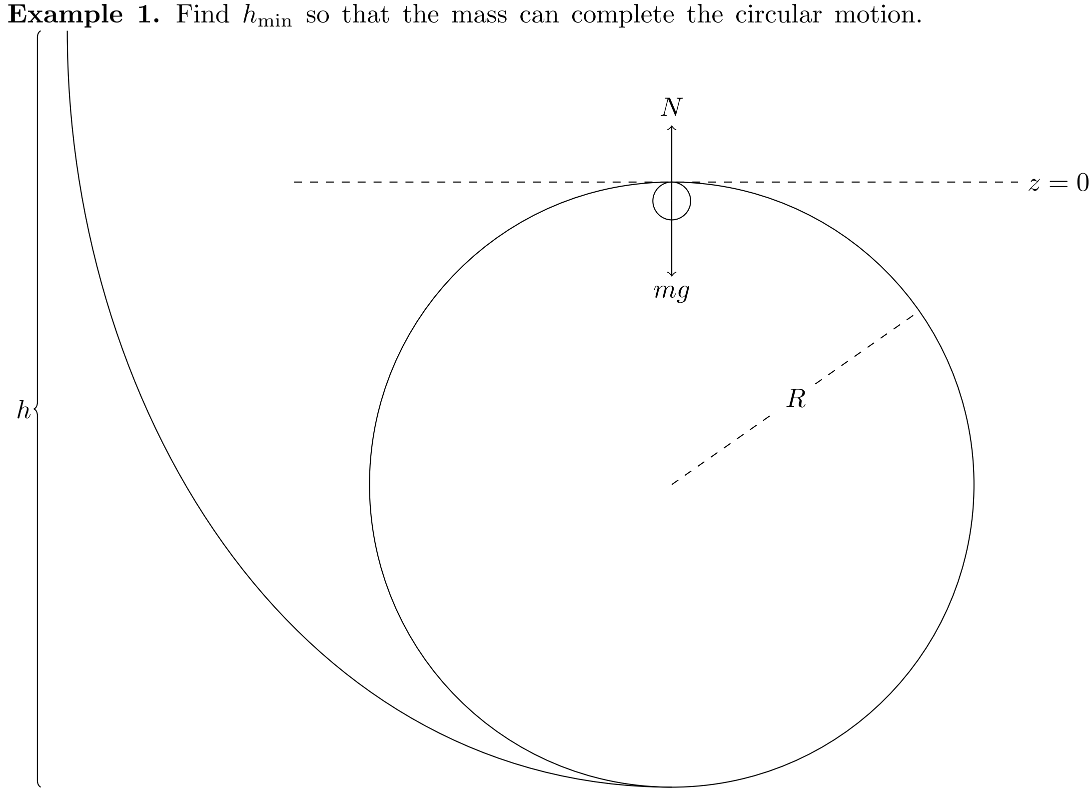
$$W = \int_B^A \overrightarrow{F_{\text{total}}} \cdot \mathrm{d}\overrightarrow{r} = E_K(B) - E_K(A)$$

$$\therefore \int_B^A \overrightarrow{F_{\text{other}}} \cdot \mathrm{d}\overrightarrow{r} + W(m\overrightarrow{g}) + W(k\overrightarrow{x}) = E_K(B) - E_K(A)$$

$$\begin{aligned}
\therefore \int_B^A \overrightarrow{F_{\text{other}}} \cdot \mathrm{d}\overrightarrow{r} &= (E_K(B) + mgz_B + \frac{1}{2}kx_B^2) - (E_K(A) + mgz_A + \frac{1}{2}kx_A^2) \\
&= E_M(B) - E_M(A)
\end{aligned}$$

1.5 Examples

Example 1. Find $h_{\min}$ so that the mass can complete the circular motion.



Solution.

$$mg + N = m\frac{v^2}{R}$$

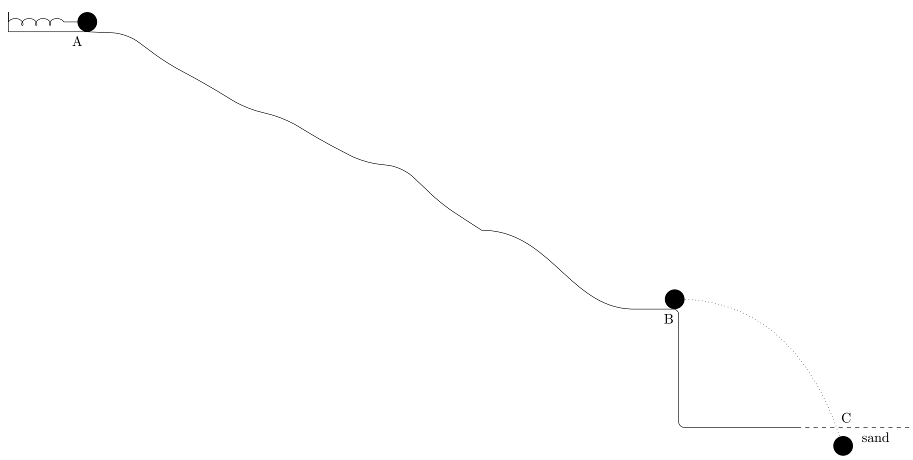
At $h_{\min}, N = 0$

$$\therefore v = \sqrt{gR}$$

$E_M =$ constant
Therefore, comparing mechanical energy at A and B,

$$mg(h - 2R) = \frac{1}{2}mv^2$$
$$\therefore h_{\min} = \frac{5}{2}R$$

Example 2. Find the average force exerted by the sand on the body.



$$
\begin{aligned}
m &= 2\,\mathrm{kg} \\
h_1 &= 3\,\mathrm{m} \\
h_2 &= 3\,\mathrm{m} \\
l &= 10\,\mathrm{m} \\
k &= 1000\,\mathrm{N\,m^{-1}} \\
x &= 40\,\mathrm{cm} \\
\mu_k &= 0.2
\end{aligned}
$$

Consider the datum of gravitational potential energy at A.

$$
\begin{aligned}
W(\overrightarrow{F_{\text{other}}}) &= E_M(B) - E_M(A) \\
\therefore 0 - \mu_k mgl &= (\frac{1}{2}mv_B^2 + mg(-h1) + 0) - (0 + 0 + \frac{1}{2}kx^2) \\
\therefore v_B &= 10 \text{ m/s}
\end{aligned}
$$

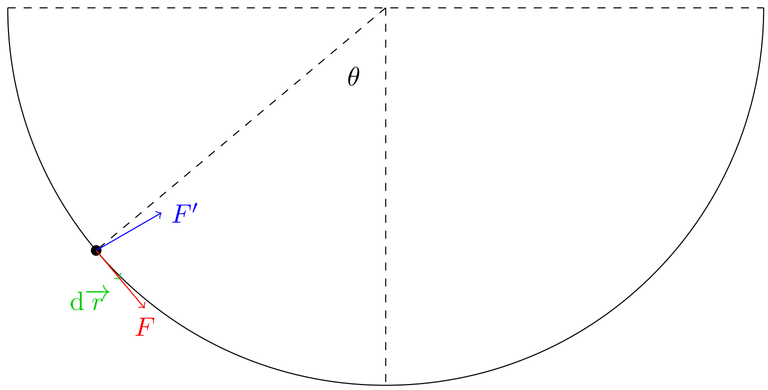
At point C,

$$
\begin{aligned}
E_M &= \frac{1}{2}kx^2 + mg(h_1 + h_2) - \mu_k mgl \\
&= \frac{1}{2}(1000)(0.4)^2 + (2)(10)(6) - (0.2)(2)(10)(10) \\
&= 160 \text{ J}
\end{aligned}
$$

This energy is is dissipated by the action of the force of the sand.

$$
\begin{aligned}
(F_{\text{sand}})(0.1) &= 160 \\
\therefore F_{\text{sand}} &= 1600 \text{ N}
\end{aligned}
$$

Example 3. $\overrightarrow{F'}$ is always at 30 degrees from horizontal.
$\overrightarrow{F}$ is always tangential to the hemisphere.



Find the work done by both forces.

Solution.

$$
\begin{aligned}
W(F) &= \int_A^B \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} \\
&= \int_A^B F \,\mathrm{d}r \\
&= F \int_A^B \mathrm{d}r \\
&= F(\pi r)
\end{aligned}
$$

$$
\begin{aligned}
W(F') &= \int_A^B \overrightarrow{F}\,\mathrm{d}\overrightarrow{r} \\
&= \int_A^B F' \,\mathrm{d}r \cos\left(\theta + \frac{\pi}{6}\right) \\
&= F' \int_A^B \cos\left(\theta + \frac{\pi}{6}\right) \mathrm{d}r \\
&= F' \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \cos\left(\theta + \frac{\pi}{6}\right) (-R\,\mathrm{d}\theta) \\
&= F'R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(\theta + \frac{\pi}{6}\right) \mathrm{d}\theta \\
&= F'R \sin\left(\frac{2\pi}{3}\right) - F'R \sin\left(\frac{-\pi}{3}\right) \\
&= F'R\sqrt{3}
\end{aligned}
$$

The work done by F' is the same, even if the body is taken directly from A to B.

1.6 Work Done by Friction

$$\begin{aligned} W(\overrightarrow{F_{\text{total}}}) &= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \\ \int_{x_A}^{x^B} -f \,\mathrm{d}x &= \int_0^x \mu_k mg \,\mathrm{d}x \\ \therefore -\mu_k mgx &= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \end{aligned}$$

1.7 Friction on an Inclined Plane

$$\begin{aligned} \int \overrightarrow{F_{\text{other}}} \cdot \mathrm{d}\overrightarrow{r} &= E_M(f) - E_M(i) \\ \therefore W(\overrightarrow{N}) + W(\overrightarrow{f}) &= \frac{1}{2}mv_f^2 - mgl\sin\theta \\ \therefore -\mu_k mg\cos\theta l &= \frac{1}{2}mv_f^2 - mgl\sin\theta \\ \therefore -\mu_k mgx &= \frac{1}{2}mv_f^2 - mgl\sin\theta \end{aligned}$$