Lecture 13

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Contents

1 Variable Mass Systems

Example 1. At $t = 0$, rain starts falling vertically into a cart of mass m_0, moving with a velocity v_0, with rate k. Find the velocity of the cart as a function of time.

Solution.

$$\begin{aligned}\frac{\mathrm{d}m}{\mathrm{d}t} &= k\\ \therefore m - m_0 &= kt\\ \therefore m &= m_0 + kt\end{aligned}$$

$$\begin{aligned}p_x(t) &= mv\\ p_x(t + \mathrm{d}t) &= (m + \mathrm{d}m)(v + \mathrm{d}v)\\ &= mv + m\,\mathrm{d}v + \mathrm{d}mv + \mathrm{d}m\,\mathrm{d}v\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}p_x}{\mathrm{d}t} &= 0\\ \therefore 0 &= \frac{p_x(t + \mathrm{d}t) - p_x(t)}{\mathrm{d}t}\\ &= m\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\mathrm{d}m}{\mathrm{d}t}v + \cancelto{0}{\frac{\mathrm{d}m\,\mathrm{d}v}{\mathrm{d}t}}\\ &= (m_0 + kt)\frac{\mathrm{d}v}{\mathrm{d}t} + kb\\ \therefore v &= \frac{m_0v_0}{m_0 + kt}\end{aligned}$$

Example 2. A cart of mass m_0, filled with sand of mass m_1 is moving with v_0. At $t = 0$, sand starts to drop through a hole in the bottom, with a rate k. Find the velocity of the cart as a function of time.

Solution.

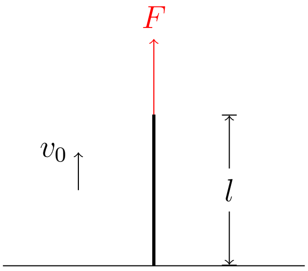
$$\begin{aligned}\frac{\mathrm{d}m}{\mathrm{d}t} &= -k\\ \therefore m - (m_0 + m_1) &= -kt\\ \therefore m &= (m_0 + m_1) - kt\end{aligned}$$

$$\begin{aligned}p_x(t) &= mv\\ p_x(t + \mathrm{d}t) &= (m + \mathrm{d}m)(v + \mathrm{d}v) + (-\,\mathrm{d}mv)\\ &= mv + m\,\mathrm{d}v + \cancel{\mathrm{d}mv} + \mathrm{d}m\,\mathrm{d}v - \cancel{\mathrm{d}mv}\end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{d}p_x}{\mathrm{d}t} &= 0 \\
\therefore 0 &= \frac{p_x(t+\mathrm{d}t) - p_x(t)}{\mathrm{d}t} \\
&= m\frac{\mathrm{d}v}{\mathrm{d}t} + \cancelto{0}{\frac{\mathrm{d}m\,\mathrm{d}v}{\mathrm{d}t}} \\
\therefore \frac{\mathrm{d}v}{\mathrm{d}t} &= 0 \\
\therefore v &= v_0
\end{aligned}
$$

Example 3. A piece of rope with length L and mass M_0 is lying on a table, with one end being pulled up by a force F s.t. it moves upwards with a constant v. Find F as a function of l, i.e. the length of the rope above the table, and the power of this force.

Solution.



$$
\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}t} &= F(l) - \frac{M_0}{L}lg \\
\therefore \frac{\mathrm{d}m}{\mathrm{d}t}v + m\cancelto{0}{\frac{\mathrm{d}v}{\mathrm{d}t}} &= F(l) - \frac{M_0}{L}lg \\
\therefore \frac{\mathrm{d}m}{\mathrm{d}t}v_0 &= F(l) - \frac{M_0}{L}lg
\end{aligned}
$$

$$
\begin{aligned}
m(l) &= \frac{M_0}{L}l \\
\therefore m(t) &= \frac{M_0}{L}v_0t \\
\therefore \frac{\mathrm{d}m}{\mathrm{d}t} &= \frac{M_0v_0}{L}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
F(l) &= \frac{M_0}{L}(v_0^2 + lg) \\
\therefore P_F &= F(l) \cdot v \\
&= \frac{M_0}{L}(v_0^3 + lgv_0)
\end{aligned}
$$

$$
\begin{aligned}
E_K &= \frac{1}{2}m(l)v_0^2 \\
&= \frac{M_0}{2L}lv_0^2 \\
U &= m(l)g\frac{l}{2} \\
&= \frac{M_0}{2L}l^2g \\
E &= E_K + U \\
&= \frac{M_0}{2L}(lv_0^2 + l^2g) \\
&= \frac{M_0}{2L}(v_0^3t + v_0^2gt^2) \\
\therefore \frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{M_0v_0^3}{2L} + \frac{M_0v_0^2gt}{L}
\end{aligned}
$$

Therefore, the power supplied by the force is greater than the rate of change of energy of the rope. This is because power is required to provide energy to the parts of the rope which are lying on the table to set them into motion.

Example 4. A body is thrown upwards with v_0. It experiences gravitational force and friction due to air $F = -bmv$, where $b > 0$ is a constant. After what time will the body reach its maximum height?

Solution.

$$\begin{aligned} m\,\frac{\mathrm{d}v}{\mathrm{d}t} &= -mg - bmv \\ \therefore \int\limits_{v_0}^{0} \frac{\mathrm{d}v}{g+bv} &= -\int\limits_{0}^{t_{\text{top}}} \mathrm{d}t \\ \therefore \frac{1}{b}\int\limits_{v_0}^{0} \frac{b\,\mathrm{d}v}{g+bv} &= -\int\limits_{0}^{t_{\text{top}}} \mathrm{d}t \\ \therefore \frac{1}{b}\ln\left(\frac{g}{g+bv_0}\right) &= -t_{\text{top}} \\ \therefore t_{\text{top}} &= \frac{1}{b}\ln\left(1+\frac{bv_0}{g}\right) \end{aligned}$$

$$v(t) = \left(\frac{g}{b}+v_0\right)e^{-bt} - \frac{g}{b}$$

Also,

$$\begin{aligned} \ddot{y} + b\dot{y} &= 0 \\ \therefore y &= Ae^{0t} + Be^{-bt} \\ &= A + Be^{-bt} \end{aligned}$$

At $t = 0$,

$$\begin{aligned} y &= 0 \\ \dot{y} &= v_0 \\ y_P &= -\frac{g}{b}t \end{aligned}$$

where y_P is a particular solution.
Solving,

$$\begin{aligned} A &= \frac{g}{b^2} + \frac{v_0}{b} \\ B &= -\frac{g}{b^2} - \frac{v_0}{b} \end{aligned}$$