Lecture 17

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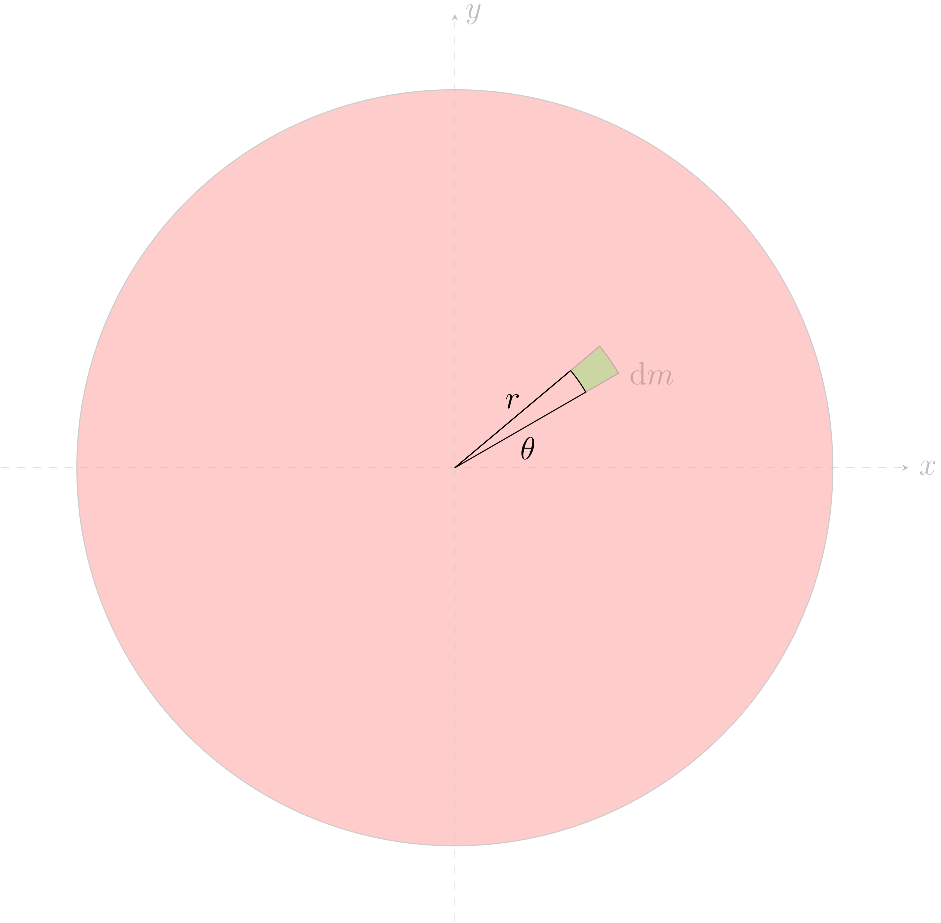
Contents

1 Moment of Inertia

Example 1. Find the moment of inertia of a disk of radius R and mass m.

Solution.

$$\sigma = \frac{m}{\pi R^2}$$

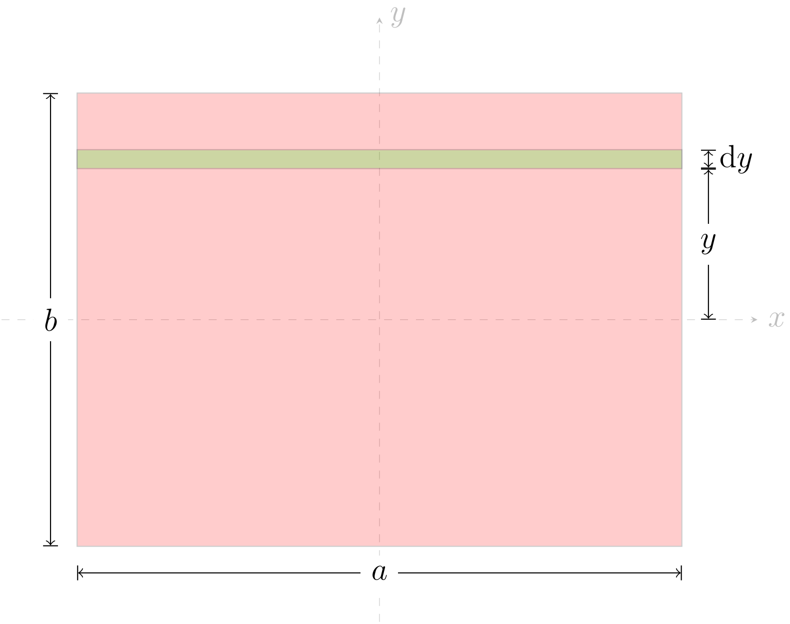


$$
\begin{aligned}
I_z &= \int r^2 \,\mathrm{d}m \\
&= \int_0^R \int_0^{2\pi} r^2 r \,\mathrm{d}\theta \,\mathrm{d}r \sigma \\
&= \sigma \int_0^R r^3 \,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \\
&= \sigma \frac{R^4}{4} \cdot 2\pi \\
&= \frac{1}{2}\pi\sigma R^4 \\
&= \frac{1}{2}(\pi R^2 \sigma) R^2 \\
&= \frac{1}{2} m R^2
\end{aligned}
$$

Example 2. Find the moment of inertia of a rectangular body.

Solution.

$$\sigma = \frac{ab}{m}$$



$$\begin{aligned} I_x &= \iint y^2 \,\mathrm{d}m \\ &= \int_{-b/2}^{b/2} y^2 a \,\mathrm{d}y\sigma \\ &= \frac{1}{12}ab^3\sigma \\ &= \frac{1}{12}mb^2 \end{aligned}$$

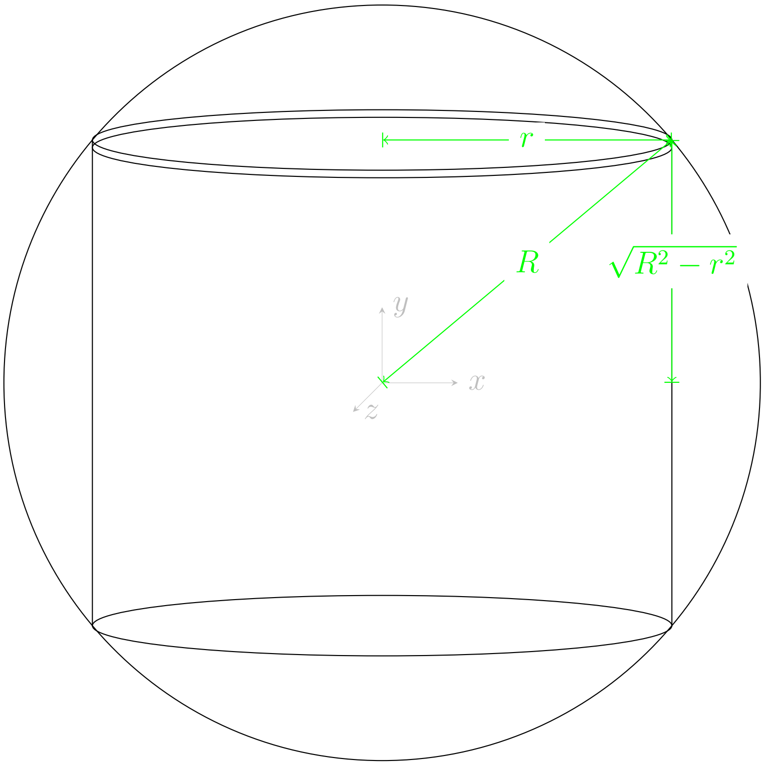
Similarly,

$$I_y = \frac{1}{12}ma^2$$

Example 3. Find the moment of inertia of a sphere

Solution. [Using cylindrical coordinates]

$$\rho = \frac{m}{4/3 \cdot \pi r^3}$$



$$
\begin{aligned}
I &= \int_0^R r^2 \,\mathrm{d}m \\
&= \int_0^R r^2 \cdot 2\pi r h \,\mathrm{d}r \rho \\
&= \frac{2}{3} m R^2
\end{aligned}
$$

Solution. [Using spherical coordinates]

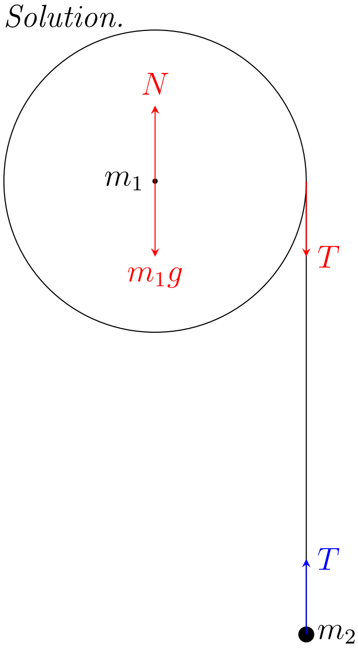
$$\rho = \frac{m}{{}^4\!/_3 \cdot \pi R^3}$$

$$
\begin{aligned}
I &= \int (r')^2 \,\mathrm{d}m \\
&= \int_0^{\pi}\int_0^{2\pi}\int_0^{R} r^2 \sin^2\theta \rho r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\theta \\
&= 2\pi\rho \int_0^{\pi} \sin^2\theta \sin\theta \,\mathrm{d}\theta \int_0^{R} r^4 \,\mathrm{d}r \\
&= \frac{2}{5} m R^2
\end{aligned}
$$

2 Rigid Body Dynamics

Example 4. A rope is wound on a cylindrical pulley of mass m_1, and a mass m_2 is attached to the free end of the rope. The pulley is fixed through its centre. The system is released from rest. Find the acceleration of m_2.

Solution.

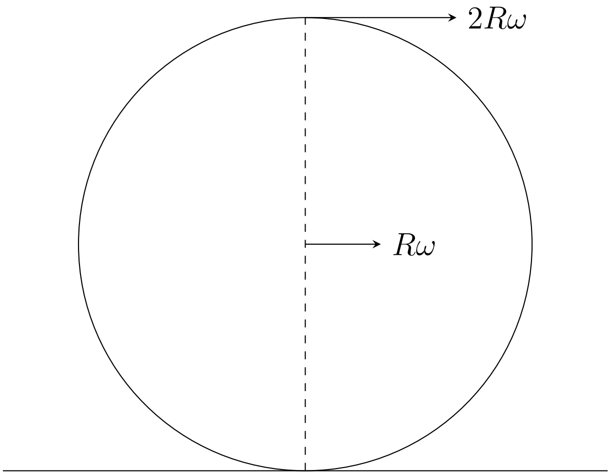


$$
\begin{aligned}
N &= m_1 g + T \\
m_2 g - T &= m_2 a \\
RT &= \frac{1}{2} m_1 R^2 \alpha \\
&= \frac{1}{2} m_1 R^2 \frac{a}{R}
\end{aligned}
$$

Solving,

$$a = \frac{m_2 g}{1/2 \cdot m_1 + m_2}$$

2.1 Pure Rolling



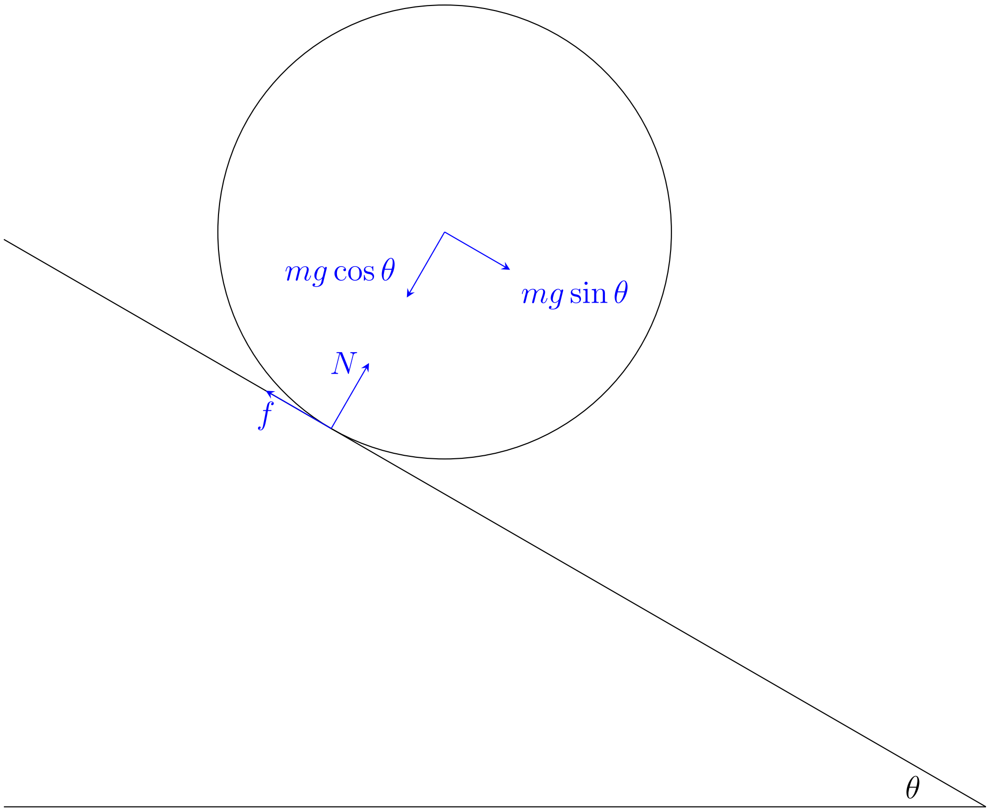
$$\begin{aligned} x &= R\theta \\ \therefore v_{\text{COM}} &= R\omega \end{aligned}$$

The point of contact is the instantaneous axis of pure rotation.

$$\begin{aligned} \overrightarrow{L_0} &= \cancelto{0}{\overrightarrow{r_{\text{COM}}}} \times (m\overrightarrow{v_{\text{COM}}}) + \left(\frac{1}{2}mR^2\omega\right)(-\hat{z}) \\ &= -\frac{1}{2}mR^2\omega\hat{z} \end{aligned}$$

Example 5. A body is rolling down an inclined plane. Find the acceleration of the body.

Solution. [Using COM axis] Friction must exist for the motion to be purely rolling.



As the body is purely rolling,

$$v_{\text{COM}} = \omega R$$
$$a_{\text{COM}} = \alpha R$$

$$\begin{aligned} Rf &= I_{\text{COM}}\alpha \\ &= kmR^2\alpha \\ &= kmRa_{\text{COM}} \end{aligned}$$

$$\begin{aligned} f &= kma_{\text{COM}} \\ mg\sin\theta - f &= ma_{\text{COM}} \end{aligned}$$

Therefore,

$$a_{\text{COM}} = \frac{g\sin\theta}{1+k}$$

Solution. [Using IAOR] About the IAOR,

$$Rmg\sin\theta = (kmR^2 + mR^2)\alpha$$
$$\therefore a_{\text{COM}} = \frac{g\sin\theta}{1+k}$$