LECTURE 14

AAKASH JOG

Contents

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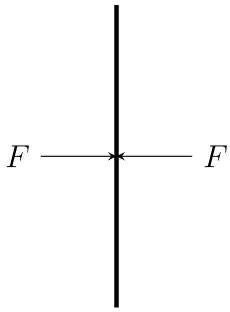
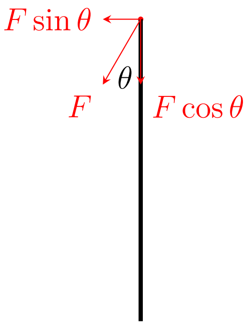
1. Rigid Body Mechanics

1.1. Definitions.



$$
\begin{aligned}
M_{\text{total}}\overrightarrow{a_{\text{COM}}} &= \sum \overrightarrow{F_{\text{ext}}} \\
\therefore v_{\text{COM}} &= \text{ constant}
\end{aligned}
$$



Definition 1 (Torque)**.**

$$
\begin{aligned}
\tau &\doteq rF\sin\theta \\
\overrightarrow{\tau} &\doteq \overrightarrow{r} \times \overrightarrow{F}
\end{aligned}
$$

Definition 2 (Angular momentum)**.**

$$
\overrightarrow{L} \doteq \overrightarrow{r} \times \overrightarrow{p}
$$

$$
\begin{aligned}
\overrightarrow{\tau_A} &= 0 \\
\therefore \frac{\mathrm{d}\overrightarrow{L_A}}{\mathrm{d}t} &= 0 \\
\therefore \overrightarrow{L_A} &= \text{ constant} \\
&= rmv\hat{z} \\
&= mr^2\omega\hat{z} \\
&= (mr^2)\overrightarrow{\omega}
\end{aligned}
$$

Definition 3 (Moment of Inertia)**.**

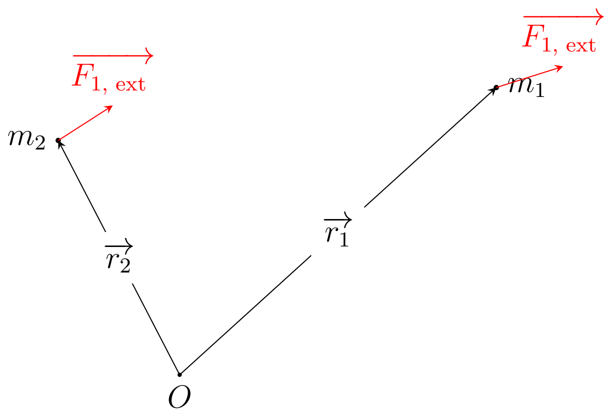
$$I \doteq mr^2$$

$$\begin{aligned} E_k &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}mr^2\omega^2 \\ &= \frac{1}{2}I\omega^2 \end{aligned}$$

Example 1. A particle attached to a string is on a horizontal table, moving in a circle of radius r_0 with v_0. The other end on the string goes through the table, through a hole in the centre of the circle. It is pulled down by a force F. Find the velocity of the particle as a function of the radius.

Solution.

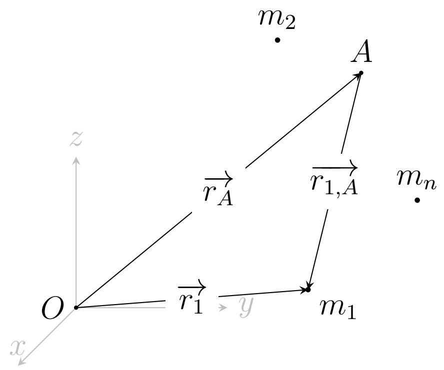
$$\begin{aligned} \overrightarrow{\tau} &= 0 \\ \therefore \overrightarrow{L_A} &= r_0 m v_0 \hat{z} \\ \therefore \overrightarrow{L} &= rmv\hat{z} \end{aligned}$$



Example 2.

Solution.

$$\begin{aligned} \overrightarrow{F_{1,2}} &= -\overrightarrow{F_{2,1}} \\ \overrightarrow{\tau_{\text{total },0}} &= \overrightarrow{\tau_{1,0}} + \overrightarrow{\tau_{2,0}} \\ &= \overrightarrow{r_1} \times \overrightarrow{F_{1,\text{ ext}}} + \overrightarrow{r_1} \times \overrightarrow{F_{1,2}} + \overrightarrow{r_2} \times \overrightarrow{F_{2,\text{ ext}}} + \overrightarrow{r_2} \times \overrightarrow{F_{2,1}} \\ &= \overrightarrow{r_1} \times \overrightarrow{F_{1,\text{ ext}}} + \overrightarrow{r_2} \times \overrightarrow{F_{2,\text{ ext}}} + \cancelto{0}{(\overrightarrow{r_1} - \overrightarrow{r_2}) \times \overrightarrow{F_{1,2}}} \\ &= \overrightarrow{\tau_{\text{total},O,\text{ ext}}} \end{aligned}$$



1.2. **Conditions for Equilibrium.**

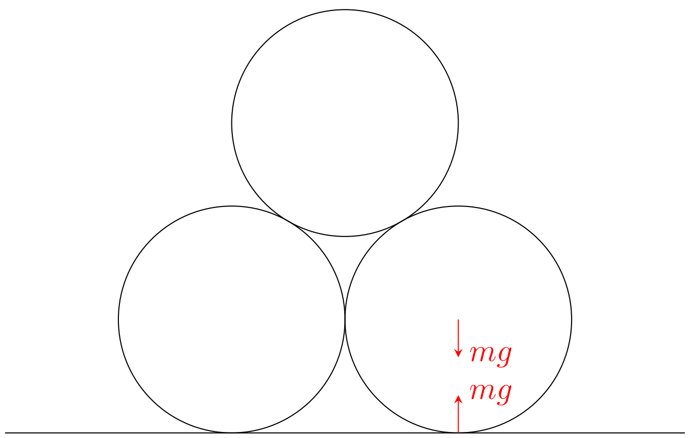
$$\sum \overrightarrow{F_{i,\text{ ext}}} = 0$$
$$\sum \overrightarrow{\tau_{i,O,\text{ ext}}} = 0$$

$$\begin{aligned}
\sum \overrightarrow{\tau_{i,A,\text{ ext}}} &= \sum \overrightarrow{r_{i,A}} \times \overrightarrow{F_{i,\text{ ext}}} \\
&= \sum \overrightarrow{r_{i,O}} \times \overrightarrow{F_{i,\text{ ext}}} - \sum \overrightarrow{r_A} \times \overrightarrow{F_{i,\text{ ext}}} \\
&= 0
\end{aligned}$$

Therefore, equilibrium is independent of the point of reference.
Hence the conditions for equilibrium are

$$\sum \overrightarrow{F_{i,\text{ ext}}} = 0$$
$$\sum \overrightarrow{\tau_{i,\text{ ext}}} = 0$$

Example 3. Find the minimum μ for which the bodies remain at rest.



Solution.

$$\begin{aligned}
\sum \overrightarrow{F_{i,\text{ ext}}} &= 0 \\
\therefore 2N_f &= 3mg \\
\therefore N_f &= \frac{3}{2}mg
\end{aligned}$$

With respect to an axis passing through A,

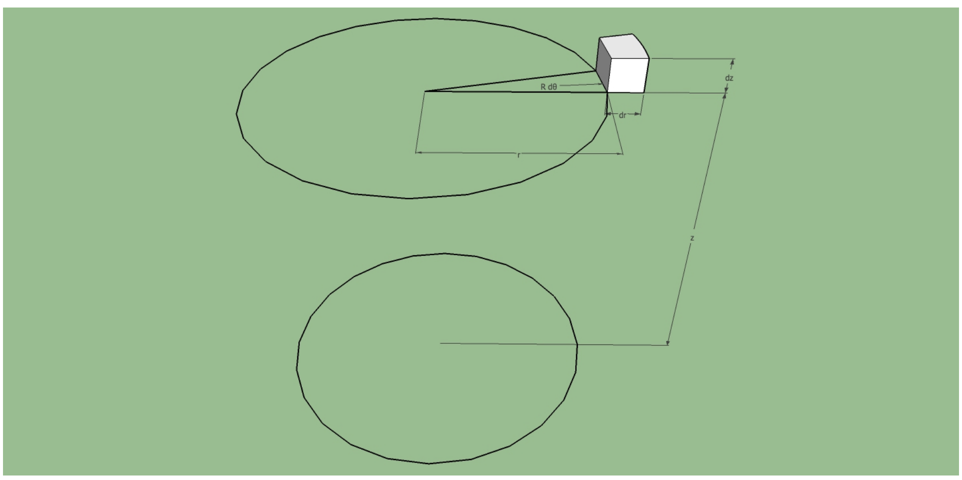
$$\begin{aligned}
\sum \overrightarrow{\tau_{i,\text{ ext}}} &= 0 \\
\therefore (-\hat{z})dmg + \hat{z}d \cdot \frac{3}{2}mg + (-\hat{z})Hf_f &= 0 \\
\therefore f_f &= \frac{1/2 \cdot mgd}{H} \\
&= \frac{1/2 \cdot mgR\cos 60}{R + R\sin 60} \\
&\le \mu N_f \\
\therefore \frac{1/2 \cdot mg \cdot 1/2}{1 + \sqrt{3}/2} &= \mu \cdot 3/2 \cdot mg \\
\therefore \mu &\ge \frac{1/6}{1 + \sqrt{3}/2}
\end{aligned}$$

2. Coordinate Systems

2.1. **Cylindrical Coordinates.**

$$(x, y, z) \to (r, \theta, z)$$

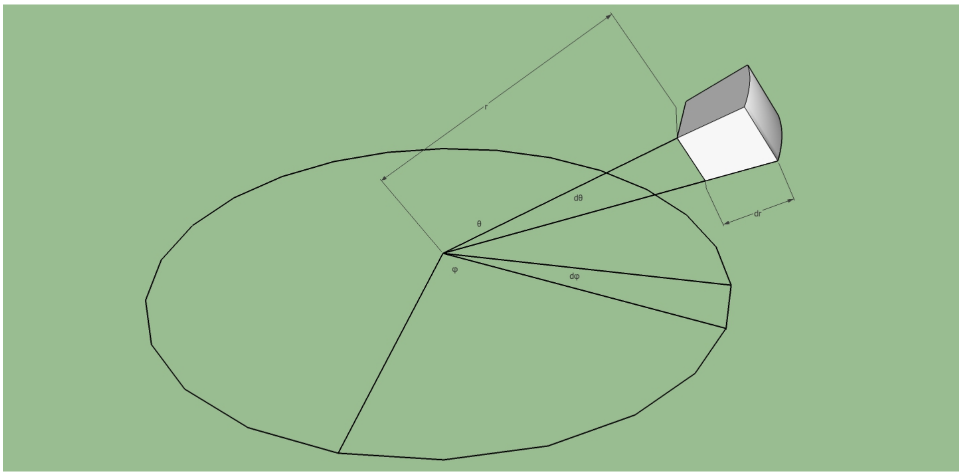
$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z
\end{aligned}$$



$$\mathrm{d}V = r\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}z$$

2.2. Spherical Coordinates.

$$(x, y, z) \to (r, \theta, \varphi)$$



$$\begin{aligned}\mathrm{d}V &= r\,\mathrm{d}\theta \cdot r\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}r \\ &= r^2\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}r\,\mathrm{d}\theta\end{aligned}$$