PHYSICS 2 : ASSIGNMENT 3

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Exercise 1.
Find the potential at distance s from an infinitely long straight wire that carries a uniform line charge λ. Compute the gradient of your potential, and check that it yields the correct field.

Solution 1.
For an infinite line of charge, the charge at infinity is not zero. Therefore, it is wrong to assume that the electric potential at infinity is zero.

$$\begin{aligned}
\varphi(s) - \varphi(r_0) &= -\int\limits_{r_0}^{s} \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{r} \\
&= -\int\limits_{r_0}^{s} E \,\mathrm{d}r \\
&= -\int\limits_{r_0}^{s} \frac{\lambda}{2\pi\varepsilon_0 r} \,\mathrm{d}r \\
&= -\frac{\lambda}{2\pi\varepsilon_0} \ln r \Big|_{r_0}^{s} \\
&= \frac{\lambda}{2\pi\varepsilon_0} (\ln r_0 - \ln s) \\
\therefore \varphi(s) &= \varphi(r_0) + \frac{\lambda}{2\pi\varepsilon_0} (\ln r_0 - \ln s)
\end{aligned}$$

If $\varphi(r_0)$, where $r_0 \neq 0$, $r_0 \neq \infty$, is set to be 0, then,

$$\varphi(s) = \frac{\lambda}{2\pi\varepsilon_0} \left(\ln \frac{r_0}{s} \right)$$

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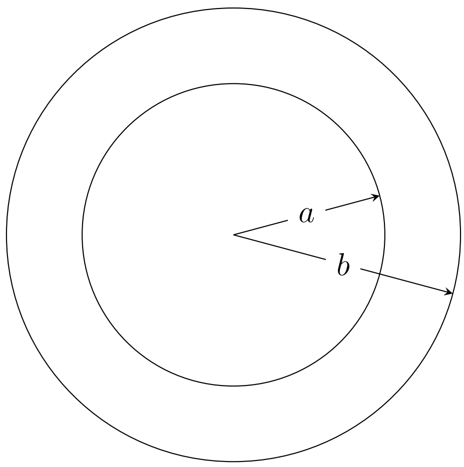
$$\begin{aligned}
\nabla\varphi &= \nabla\left(\frac{\lambda}{2\pi\varepsilon_0}\left(\ln\frac{r_0}{s}\right)\right) \\
&= \frac{\lambda}{2\pi\varepsilon_0}\frac{\mathrm{d}}{\mathrm{d}r}\left(\ln\frac{r_0}{s}\right) \\
&= -\frac{\lambda}{2\pi\varepsilon_0}\frac{\mathrm{d}}{\mathrm{d}r}\left(\ln\frac{s}{r_0}\right) \\
\therefore -E &= -\frac{\lambda}{2\pi\varepsilon_0}\frac{1}{s} \\
\therefore E &= \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{s}
\end{aligned}$$

This matches the known value of the electric field.

Exercise 2.

A hollow spherical shell carries charge density $\rho = \dfrac{k}{r^2}$ in the region $a \leq r \leq b$. Find the potential at the centre, using infinity as your reference point.



Solution 2.

As $\varphi(\infty) = 0$, and as $E = \dfrac{\mathrm{d}\varphi}{\mathrm{d}r}$

$$\begin{aligned}
\varphi(0) &= -\int\limits_{\infty}^{0} E \,\mathrm{d}r \\
&= \int\limits_{0}^{a} 0 \,\mathrm{d}r + \int\limits_{a}^{b} \left(\frac{k}{\varepsilon_0}\frac{r-a}{r^2}\right) \mathrm{d}r + \int\limits_{b}^{\infty} \left(\frac{k}{\varepsilon_0}\frac{b-a}{r^2}\right) \mathrm{d}r \\
&= 0 + \frac{k}{\varepsilon_0}\frac{b-a}{b} - \frac{k}{\varepsilon_0}\left(\ln\left(\frac{a}{b}\right) + 1 - \frac{a}{b}\right) \\
&= \frac{k}{\varepsilon_0} \ln \frac{b}{a}
\end{aligned}$$

Exercise 3.
A long coaxial cable carries a uniform volume charge density ρ on the inner cylinder (radius a), and a uniform surface charge density on the outer cylindrical shell (radius b). This surface charge is negative and of just the right magnitude so that the cable as a whole is electrically neutral. Find the potential difference between a point on the axis and a point on the outer cylinder.

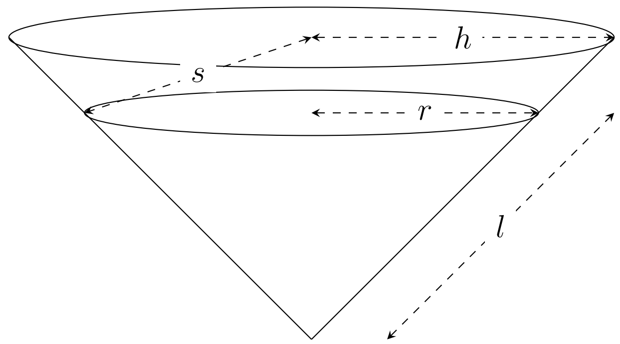
Solution 3.

$$\begin{aligned}
\varphi(b) - \varphi(0) &= -\int\limits_{0}^{b} E \,\mathrm{d}r \\
&= -\int\limits_{0}^{a} E \,\mathrm{d}r - \int\limits_{a}^{b} E \,\mathrm{d}r \\
&= -\int\limits_{0}^{a} \frac{\rho r}{2\varepsilon_0} \,\mathrm{d}r - \int\limits_{a}^{b} \frac{\rho a^2}{2\varepsilon_0 r} \,\mathrm{d}r \\
&= -\frac{\rho}{2\varepsilon_0}\left(\frac{a^2}{2}\right) - \frac{\rho a^2}{2\varepsilon_0}(\ln b - \ln a) \\
&= -\frac{\rho a^2}{4\varepsilon_0}\left(1 + 2\ln\frac{b}{a}\right)
\end{aligned}$$

Exercise 4.
A conical surface (an empty ice-cream cone) carries a uniform surface charge σ. The height of the cone is h, as is the radius of the top. Find

the potential difference between points a (the vertex) and b (the centre of the top).

Solution 4.



Consider an elemental ring of radius r and thickness $\mathrm{d}r$ as shown.
Let s be the distance from the centre of the base to any point on the elemental ring.
Therefore,

$$
\begin{aligned}
\varphi(a) &= \int \frac{k\,\mathrm{d}q}{l} \\
&= \int_0^{\sqrt{2}h} \frac{k\sigma \cdot 2\pi r}{l}\,\mathrm{d}l \\
&= \int_0^{\sqrt{2}h} \frac{2k\sigma\pi r}{\sqrt{2}r}\,\mathrm{d}l \\
&= \frac{2k\sigma\pi}{\sqrt{2}} \cdot \sqrt{2}h \\
&= \frac{\sigma h}{2\varepsilon_0}
\end{aligned}
$$

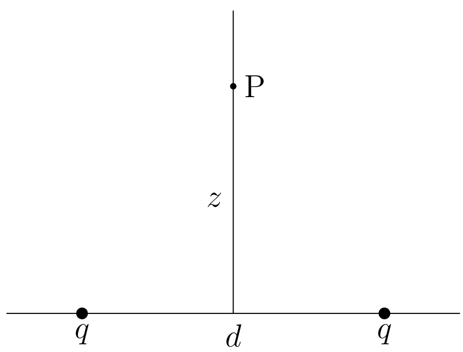
$$
\begin{aligned}
\varphi(b) &= \int \frac{k\,\mathrm{d}q}{s} \\
&= \int_0^{\sqrt{2}h} \frac{k\sigma \cdot 2\pi r}{\sqrt{h^2 + l^2 - \sqrt{2}hl}}\,\mathrm{d}l \\
&= \frac{2k\sigma\pi}{\sqrt{2}} \left(\sqrt{h^2 + l^2 - \sqrt{2}hl} \right) \Bigg|_0^{\sqrt{2}} \\
&\quad + \frac{2k\sigma\pi}{\sqrt{2}} \left(\frac{h}{\sqrt{2}} \ln \left(2\sqrt{h^2 + l^2 - \sqrt{2}hl} + 2l - \sqrt{2}h \right) \right) \Bigg|_0^{\sqrt{2}} \\
&= \frac{\sigma h}{4\varepsilon_0} \ln \left(3 + 2\sqrt{2} \right) \\
&= \frac{\sigma h}{2\varepsilon_0} \ln \left(1 + \sqrt{2} \right)
\end{aligned}
$$

Therefore,

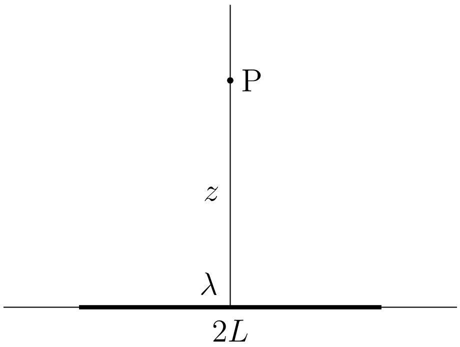
$$
\varphi(a) - \varphi(b) = \frac{\sigma h}{2\varepsilon_0} \left(1 - \ln \left(1 + \sqrt{2} \right) \right)
$$

Exercise 5.

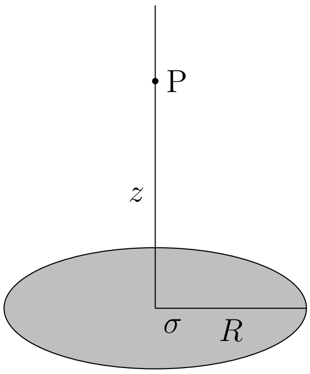
Find the potential at a distance z above the centre of the charge distributions shown. In each case, compute $\overrightarrow{E} = \overrightarrow{\nabla}\varphi$, and compare your answers with the fields computed last week. Suppose that we changed the right-hand charge in the first figure to $-q$; what then is the potential at P? What field does that suggest? Explain.



(1)



(2)



(3)

Solution 5.

(1)

$$\varphi = 2 \cdot \frac{1}{4\pi\varepsilon_0} \frac{q}{\sqrt{d^2/4 + z^2}}$$

By symmetry,

$E_x = E_y = 0$

Therefore,

$$\begin{aligned} E_z &= -\frac{\mathrm{d}}{\mathrm{d}z}\,\varphi \\ &= -\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{4\pi\varepsilon_0} \frac{2q}{\sqrt{d^2/4 + z^2}}\right) \\ &= \frac{1}{4\pi\varepsilon_0} \frac{2q}{\left(d^2/4 + z^2\right)^{3/2}} \end{aligned}$$

This is consistent with the known electric field.

If the right-hand charge is changed to $-q$,

$$\begin{aligned}\varphi &= \frac{1}{4\pi\varepsilon} q\sqrt{d^2/4 + z^2} + \frac{1}{4\pi\varepsilon} -q\sqrt{d^2/4 + z^2} \\ &= 0\end{aligned}$$

Therefore, $E_z = 0$, therefore, the field must be in the horizontal direction only.

(2)

$$\begin{aligned}\varphi &= \int_{-L}^{L} \frac{1}{4\pi\varepsilon_0} \frac{\lambda \, \mathrm{d}x}{\sqrt{x^2 + z^2}} \\ &= \frac{\lambda}{4\pi\varepsilon_0} \ln\left(x + \sqrt{x^2 + z^2}\right)\Big|_{-L} L \\ &= \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{L + \sqrt{L^2 + z^2}}{z}\right)\end{aligned}$$

By symmetry,

$E_x = E_y = 0$

Therefore,

$$\begin{aligned}E_z &= -\frac{\mathrm{d}}{\mathrm{d}z} \varphi \\ &= -\frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{L + \sqrt{L^2 + z^2}}{z}\right)\right) \\ &= \frac{\lambda}{2\pi\varepsilon_0} \frac{L}{z\sqrt{z^2 + L^2}}\end{aligned}$$

This is consistent with the known electric field.

(3) Considering the ring to be made up of elemental rings,

$$\begin{aligned}\varphi &= \int_{0}^{R} \frac{1}{4\pi\varepsilon_0} \frac{\sigma \cdot 2\pi r \, \mathrm{d}r}{\sqrt{r^2 + z^2}} \\ &= \frac{\sigma}{2\varepsilon_0} \int_{0}^{R} \frac{r}{\sqrt{r^2 + z^2}} \, \mathrm{d}r \\ &= \frac{\sigma}{2\varepsilon_0} \sqrt{r^2 + z^2}\Big|_{0}^{R} \\ &= \frac{\sigma}{2\varepsilon_0} \left(\sqrt{R^2 + z^2} - z\right)\end{aligned}$$

By symmetry,

$E_x = E_y = 0$

Therefore,

$$\begin{aligned}
E_z &= -\frac{\mathrm{d}}{\mathrm{d}z}\,\varphi \\
&= -\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+z^2}-z\right)\right) \\
&= \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right)
\end{aligned}$$

This is consistent with the known electric field.