PHYSICS 2 : ASSIGNMENT 5

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Exercise 1.
Find the electric field at height z above the center of a square sheet (side a) carrying a uniform charge density σ. Check your results for the limiting case $a \to \infty$.

Solution 1.
Consider a square loop of side l and thickness $\mathrm{d}l$.

$$\mathrm{d}E = 2k \frac{\sigma l z \,\mathrm{d}l}{\left(\frac{l^2}{4} + z^2\right)\left(\frac{l^2}{4} + \frac{l^2}{4} + z^2\right)^{\frac{1}{2}}}$$

$$\begin{aligned}
\therefore E &= \int\limits_0^a \frac{1}{2\pi\varepsilon_0} \frac{\sigma l z \,\mathrm{d}l}{\left(\frac{l^2}{4} + z^2\right)\left(\frac{l^2}{4} + \frac{l^2}{4} + z^2\right)^{\frac{1}{2}}} \\
&= 4\sigma \tan^{-1}\left(\frac{\sqrt{l^2 + 2z^2}}{\sqrt{2}z}\right)\Bigg|_0^a \\
&= 4\sigma \tan^{-1}\left(\frac{\sqrt{a^2 + 2z^2}}{\sqrt{2}z}\right) - 4\sigma \tan^{-1}\left(\frac{\sqrt{2z^2}}{\sqrt{2}z}\right) \\
&= 4\sigma \tan^{-1}\left(\frac{\sqrt{a^2 + 2z^2}}{\sqrt{2}z}\right) - \sigma\pi
\end{aligned}$$

If $a \to \infty$,

$$\begin{aligned}
E &= \lim_{a \to \infty} 4\sigma \tan^{-1}\left(\frac{\sqrt{a^2 + 2z^2}}{\sqrt{2}z}\right) - 4\sigma \tan^{-1}\left(\frac{\sqrt{2z^2}}{\sqrt{2}z}\right) \\
&= 4\sigma \frac{\pi}{2} - \sigma\pi \\
&= \sigma\pi
\end{aligned}$$

Exercise 2.
If the electric field in some region (in spherical coordinates) is given by the expression

$$\overrightarrow{E} = \frac{A\hat{r} + B\sin(\theta)\cos(\varphi)\hat{\varphi}}{r}$$

where A and B are constants. What is the charge density?

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Solution 2.

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\varepsilon_0} \\
\therefore \rho &= \varepsilon_0 \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{A}{r} \right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \left(\frac{B \sin\theta \cos\varphi}{r} \right) \right) \\
&= \varepsilon_0 \left(\frac{A}{r^2} + \frac{1}{r \sin\theta} \frac{B \sin\theta}{r} (-\sin\theta) \right) \\
&= \frac{\varepsilon_0}{r^2} (A - B \sin\varphi)
\end{aligned}$$

Exercise 3.
An inverted hemispherical bowl of radius R is carrying a uniform surface charge density σ. Find the potential difference between the "north pole" and the center.

Solution 3.
If the hemispherical shell was complete, the potential at the centre would be

$$\begin{aligned}
\varphi_{\text{full sphere}} &= \frac{1}{4\pi\varepsilon_0} \frac{Q}{R} \\
&= \frac{1}{4\pi\varepsilon_0} \frac{\sigma \cdot 4\pi R^2}{R} \\
&= \frac{\sigma R}{\varepsilon_0}
\end{aligned}$$

The potential at the centre due to the hemispherical shell is half of that due to the entire shell.
Therefore,

$$\begin{aligned}
\varphi_{\text{centre}} &= \frac{1}{2} \left(\frac{1}{4\pi\varepsilon_0} \frac{\sigma \cdot 2\pi R^2}{R} \right) \\
&= \frac{\sigma R}{2\varepsilon_0}
\end{aligned}$$

Consider an elemental ring of radius r at height z from the pole.
Therefore,

$$\begin{aligned}
\mathrm{d}\varphi_{\text{pole}} &= \frac{1}{4\pi\varepsilon_0} \frac{\sigma \cdot 2\pi r \, \mathrm{d}r}{\sqrt{r^2 + z^2}} \\
\therefore \varphi_{\text{pole}} &= \frac{\sigma}{2\varepsilon_0} \int_0^R \frac{r \, \mathrm{d}r}{\sqrt{r^2 + z^2}} \\
&= \frac{\sigma R}{\sqrt{2}\varepsilon_0}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\varphi_{\text{pole}} - \varphi_{\text{centre}} &= \frac{\sigma R}{\varepsilon_0} - \frac{\sigma R}{\sqrt{2} - \varepsilon_0} \\
&= \frac{\sigma R}{\varepsilon_0}\left(\sqrt{2} - 1\right)
\end{aligned}
$$