Physics 2

Aakash Jog

2014-15

Contents

1 Lecturer Information

Dr. Erez Pyetan

Office: Sharet 325
Telephone: 7565
E-mail: erezpyet@mail.tau.ac.il

2 Textbooks

1. D. Halliday, R. Resnick, and K. S. Krane: *Physics*, 5th edition, vol. 2 (Wiley)
2. D.J. Griffiths: *Introduction to Electrodynamics*

Part I
Electrostatics

1 Coulomb's Law

Law 1 (Coulomb's Law)**.** *The force between two charged particles is directly proportional to the product of the charges of the particles, and inversely proportional to the square of the distance between them.*

$$\begin{aligned} F &\propto \frac{q_1 q_2}{r^2} \\ F &= k\frac{q_1 q_2}{r^2} \\ &= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r^2} \end{aligned}$$

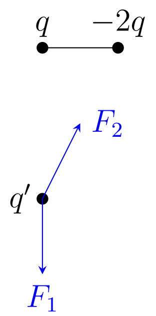
The constant of proportionality is $k = 8.99 \times 10^9 \mathrm{N\,m^2\,C^{-2}}$.
$\varepsilon_0 = 8.8541878162 \times 10^{-12} \mathrm{C^2\,N^{-1}\,m^{-2}}$ *is called the permittivity of free space. In vector notation,*

$$\overrightarrow{F_{21}} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{{r_{12}}^2} \hat{r_{12}}$$

Charge is defined according to this law.

Exercise 1.
A charge q is placed at the origin. A charge $-2q$ is placed at 1 m from it, in the x direction. Find a point on the y-axis where the total force acting on a charge q' will be parallel to the x-axis.

Solution 1.



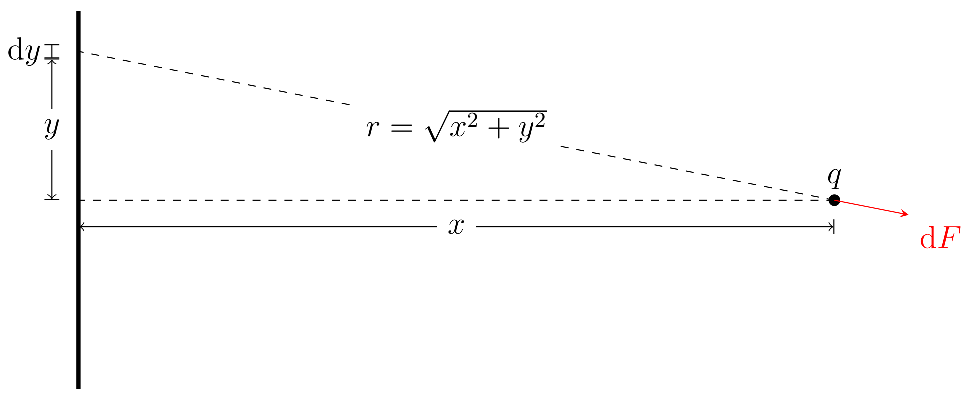
For the net force to be in the x direction, the components of F_1 and F_2 in the y direction must cancel each other out.

$$
\begin{aligned}
F_1 &= F_2 \sin\theta \\
\therefore \cancel{k} \cdot \frac{(\cancel{q})(-2q)}{y^2+1} \cdot \frac{y}{\sqrt{y^2+1}} &= \cancel{k} \cdot \frac{(q)(\cancel{q})}{y^2} \\
\therefore \frac{-2y}{(y^2+1)^{3/2}} &= \frac{1}{y^2} \\
\therefore y &= \pm\sqrt{\frac{1}{2^{2/3}-1}}
\end{aligned}
$$

Exercise 2.

A rod of length L has a uniformly distributed charge Q, with line charge density $\lambda = \dfrac{Q}{L}$. A point charge q is kept at a distance x as shown.

Solution 2.



The y components of the forces of the elemental charges at y and $-y$ on q are cancelled out. Therefore, the net force is in the x direction only.

$$
\begin{aligned}
\mathrm{d}F &= k\frac{(\mathrm{d}Q)(q)}{r^2} \\
\mathrm{d}F_x &= \mathrm{d}F\cos\theta \\
&= k\frac{(\mathrm{d}Q)(q)}{r^2}\cos\theta \\
&= k\frac{(\lambda\,\mathrm{d}y)(q)}{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}} \\
&= k\lambda q x\frac{\mathrm{d}y}{(x^2+y^2)^{3/2}} \\
\therefore\ \overrightarrow{F} &= \hat{x}\int \mathrm{d}F_x \\
&= \hat{k}\lambda q x\int\limits_{-L/2}^{L/2}\frac{\mathrm{d}y}{(x^2+y^2)^{3/2}}
\end{aligned}
$$

Substituting $y = x\tan\theta$ and $\mathrm{d}y = x\sec^2\theta\,\mathrm{d}\theta$

$$
\begin{aligned}
\overrightarrow{F} &= \hat{x}\lambda q k x\int\limits_{-\theta_0}^{\theta_0}\frac{1}{x^2}\cos\theta\,\mathrm{d}\theta \\
&= \hat{x}\frac{\lambda q k}{x}\int\limits_{-\theta_0}^{\theta_0}\cos\theta\,\mathrm{d}\theta
\end{aligned}
$$

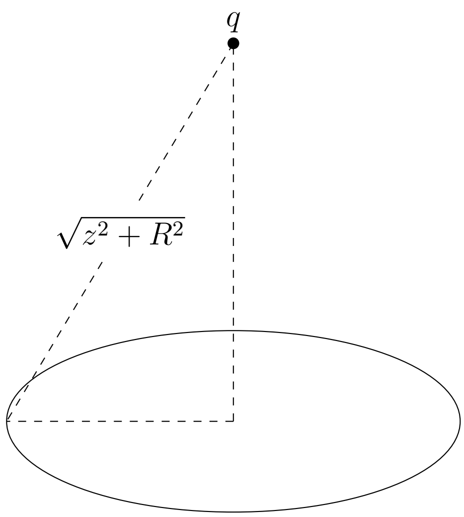
Therefore,

$$
\begin{aligned}
\overrightarrow{F} &= \hat{x}\frac{2\lambda qk}{x}\sin\theta_0 \\
&= \hat{x}\frac{2\lambda qk}{x}\frac{\dfrac{L}{2}}{\left(\left(\dfrac{L}{2}\right)^2 + x^2\right)^{1/2}} \\
&= \hat{x}\frac{2\left(\dfrac{Q}{L}\right)qk}{x}\cdot\frac{\dfrac{L}{2}}{\left(\left(\dfrac{L}{2}\right)^2 + x^2\right)^{1/2}} \\
&= k\frac{Qq}{x\left(\left(\dfrac{L}{2}\right)^2 + x^2\right)^{1/2}}\hat{x}
\end{aligned}
$$

Exercise 3.

A point charge q is kept at a distance z above a ring of radius R charged with $Q = 2\pi R\lambda$, where λ is the linear charge density. Find the force acting on q.

Solution 3.



Due to the symmetry of the ring, the net force acting on q is in the z direction only.

$$\begin{aligned}
\mathrm{d}F_z &= \mathrm{d}F\cos\theta \\
&= k\frac{(\mathrm{d}Q)(q)}{z^2+R^2}\cos\theta \\
&= k\frac{(\mathrm{d}Q)(q)}{z^2+R^2}\frac{z}{\sqrt{z^2+R^2}} \\
&= kqz\frac{\mathrm{d}Q}{\left(z^2+R^2\right)^{3/2}} \\
\therefore \overrightarrow{F} &= \hat{z}\int \mathrm{d}F_z \\
&= \hat{z}kqz\frac{1}{\left(z^2+R^2\right)^{3/2}}\int_0^Q \mathrm{d}Q \\
&= k\frac{Qqz}{\left(z^2+R^2\right)^{3/2}}\overrightarrow{z}
\end{aligned}$$

Exercise 4.

A point charge q is kept at a distance z above a disk of radius R charged with $Q = \pi R^2\sigma$, where σ is the surface charge density. Find the force acting on q.

Solution 4.

The disk can be considered to be made up of elemental rings, with radii varying from 0 to R.
Therefore,

$$\begin{aligned}
\mathrm{d}\overrightarrow{F} &= k\frac{qQ_{\text{ring}}}{\left(z^2+R^2\right)^{3/2}}\hat{z} \\
&= k\frac{q(\sigma\cdot 2\pi r\cdot \mathrm{d}r)}{\left(z^2+R^2\right)^{3/2}}z\hat{z}
\end{aligned}$$

Hence,

$$\begin{aligned}
\overrightarrow{F} &= \int \mathrm{d}\overrightarrow{F} \\
&= \hat{z} \int_0^R k \frac{q\sigma \cdot 2\pi r z \cdot \mathrm{d}r}{(z^2 + R^2)^{3/2}} \\
&= 2kzq\sigma\pi \left(\frac{1}{|z|} - \frac{1}{\sqrt{z^2 + R^2}} \right) \hat{z}
\end{aligned}$$

If $z << R$, i.e. for an infinite sheet,

$$F = 2q\sigma\pi k$$

2 Electric Field

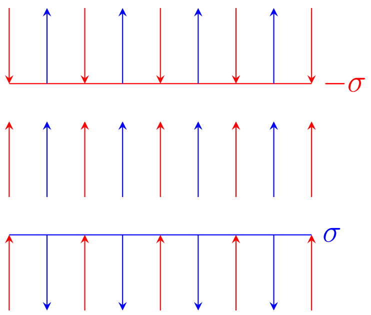
Definition 1 (Electric field)**.** The electric field at a point in space is the electric force felt by a charge of 1 C had it been kept there.

2.1 Standard Electric Fields

Line of charge	$\frac{1}{4\pi\varepsilon_0} \frac{\lambda L}{r\sqrt{r^2 + \frac{L^2}{4}}}$
Infinite line of charge	$\frac{\lambda}{2\pi\varepsilon_0 r}$
Ring of charge	$\frac{\lambda R z}{2\varepsilon_0 (z^2 + R^2)^{3/2}}$
Infinite plane of charge	$\frac{\sigma}{2\varepsilon_0}$

2.2 Capacitors

A parallel plate capacitor is constructed by arranging two infinite plates with surface charge density σ and $-\sigma$ respectively.

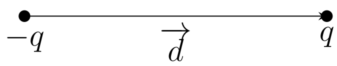


The electric field due to the plates are as shown. Therefore, the fields between the plates add up and the fields outside the plates cancel out. Therefore, the net field inside the capacitor is

$$\frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

3 Electric Dipoles

Definition 2 (Electric dipole)**.** Two charges, q and $-q$, separated by a distance d is called an electric dipole.



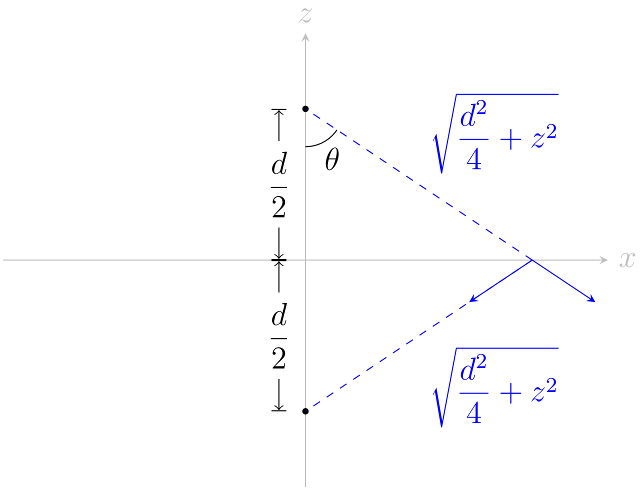
Definition 3 (Dipole moment)**.** If two charges q and $-q$ are separated by a distance d, the dipole moment is defined as

$$\vec{P} \doteq q \cdot \vec{d}$$

where $\vec{d}$ is the vector of length d pointing from $-q$ to q.

3.1 Electric Field Due to Electric Dipoles

3.1.1 Electric Field



$$
\begin{aligned}
\overrightarrow{F} &= 2E_+ \cos\theta(-\hat{z}) \\
&= 2 \cdot \frac{1}{4\pi\varepsilon_0} \frac{q}{\left(\frac{d}{2}\right)^2 + x^2} \cdot \frac{\frac{d}{2}}{\left(\left(\frac{d}{2}\right)^2 + x^2\right)^{1/2}}(-\hat{z}) \\
&= -\frac{1}{4\pi\varepsilon_0} \frac{\overrightarrow{P}}{\left(\left(\frac{d}{2}\right)^2 + x^2\right)^{3/2}}
\end{aligned}
$$

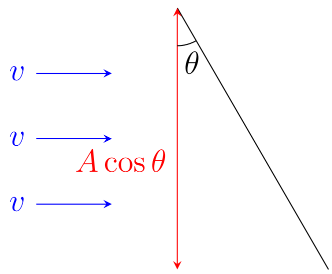
4 Gauss' Law

Definition 4 (Electric flux)**.** Electric flux is defined as the dot product of the electric field passing through a surface, and the area vector of the surface.

$$\Phi = \overrightarrow{E} \cdot \overrightarrow{A}$$

where the magnitude of the area vector is proportional to the area of the surface and the direction is perpendicular to the surface.

This can be modelled as water passing through a surface.



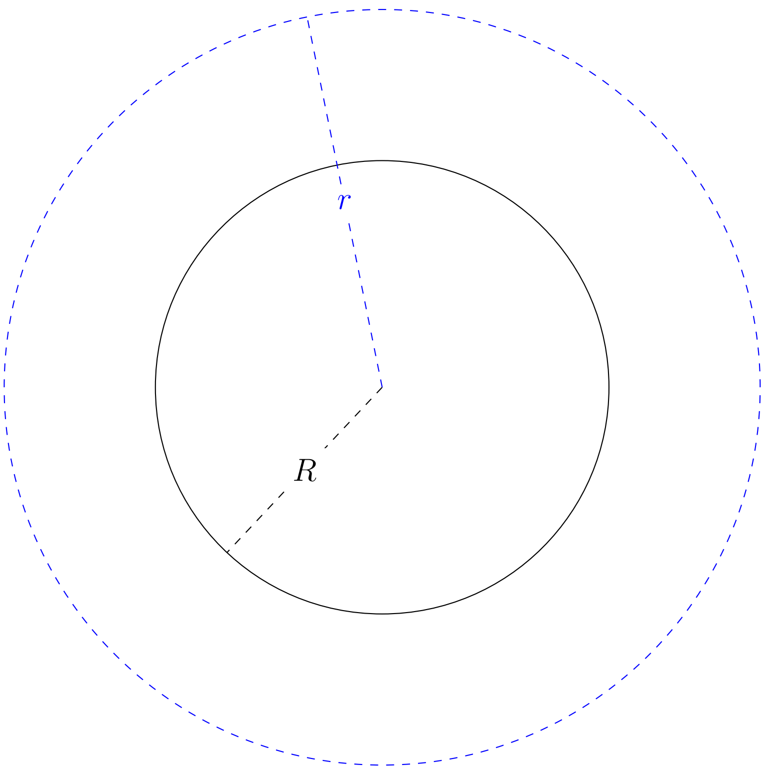
The flux of the water passing through the area A is $Av\cos\theta$.

Theorem 1 (Gauss' Law)**.**

$$\oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A} = \frac{Q_{\text{inside}}}{\varepsilon_0}$$

Exercise 5.
A hollow sphere of radius R has surface charge density σ. Find the field at a point at distance r from the centre of the sphere.

Solution 5.
Consider the imaginary Gaussian surface as a sphere with radius r.



Using Gauss' Law over the Gaussian surface,

$$\oiint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{\text{total}}}{\varepsilon_0}$$
$$\therefore \oiint E \, \mathrm{d}A = \frac{Q_{\text{total}}}{\varepsilon_0}$$
$$\therefore E \oiint \mathrm{d}A = \frac{Q_{\text{total}}}{\varepsilon_0}$$
$$\therefore E \cdot 4\pi r^2 = \frac{Q_{\text{total}}}{\varepsilon_0}$$
$$\therefore \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q_{\text{total}}}{r^2} \hat{r}$$

Similarly for $r < R$, $E = 0$.

5 Electric Potential

Definition 5 (Electrical Potential)**.** The electric potential due to a point charge q is

$$\varphi\left(\vec{r}\right) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r} + c$$

If a charge q in moved from point A to B,

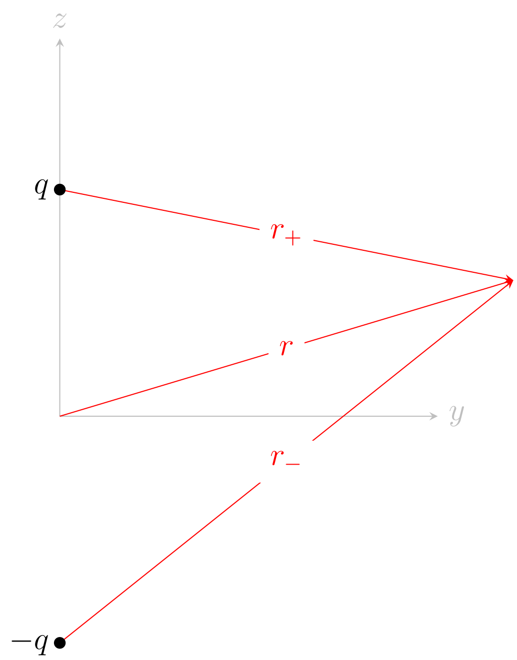
$$\begin{aligned} W_{\text{A}\to\text{B}} &= \int\limits_{\vec{r_{\text{A}}}}^{\vec{r_{\text{B}}}} \vec{E} \cdot \mathrm{d}\vec{r} \\ &= \int\limits_{r_{\text{A}}}^{r_{\text{B}}} \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \, \mathrm{d}r \\ &= -\frac{1}{4\pi\varepsilon_0} \frac{q}{r} \bigg|_{r_{\text{A}}}^{r_{\text{B}}} \\ &= \frac{1}{4\pi\varepsilon_0} \frac{q}{r_{\text{A}}} - \frac{1}{4\pi\varepsilon_0} \frac{q}{r_{\text{B}}} \end{aligned}$$

Therefore,

$$W_{\text{A}\to\text{B}} = \varphi\left(\vec{r_A}\right) - \varphi\left(\vec{r_B}\right)$$
$$\therefore \varphi\left(\vec{r_B} - \vec{r_A}\right) = -\int\limits_{\vec{r_A}}^{\vec{r_B}} \vec{E} \cdot \mathrm{d}\vec{r}$$

Exercise 6.
An electric dipole with charges q and $-q$ is placed on the z-axis with distance d between the charges. Find the field at a general point in space. Find points at which the electric potential is zero.

Solution 6.



Let the electric potential at infinity be zero.

$$\begin{aligned}\varphi\left(\overrightarrow{r}\right) &= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+}+\frac{-q}{r_-}\right)\\ &= \frac{1}{4\pi\varepsilon_0}\frac{q(r_- - r_+)}{r_-\cdot r_+}\end{aligned}$$

Therefore, the potential is zero only if $r_+ = r_-$. Therefore, for all points on the xy-plane, the potential is zero.

Exercise 7.
Find the electric potential at a point at distance r on the equator of a line of charge of length L and uniform line charge density λ.

Solution 7.

Consider an elemental charge $\mathrm{d}q$ with length $\mathrm{d}z$ at a distance z from the centre of the line.

$$\mathrm{d}q = \lambda \, \mathrm{d}z$$

Therefore,

$$\begin{aligned}
\varphi(r) &= \int \frac{1}{4\pi\varepsilon_0} \frac{q}{\sqrt{z^2 + r^2}} \\
&= \int_{-L/2}^{L/2} \frac{1}{4\pi\varepsilon_0} \frac{\lambda \, \mathrm{d}z}{\sqrt{z^2 + r^2}} \\
&= \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{\sqrt{L^2 + 4r^2} + L}{\sqrt{L^2 + 4r^2} - L} \right)
\end{aligned}$$

Exercise 8.

Find the electric potential due to an infinite line of charge.

Solution 8.

For an infinite line of charge, the charge at infinity is not zero. Therefore, it is wrong to assume that the electric potential at infinity is zero. Therefore, the result for a finite line of charge cannot be used to find the potential due to an infinite line of charge.

Therefore, the potential needs to be calculated using the electric field.

$$\begin{aligned}
\varphi(r) - \varphi(r_0) &= -\int\limits_{r_0}^{r} \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{r} \\
&= -\int\limits_{r_0}^{r} E \,\mathrm{d}r \\
&= -\int\limits_{r_0}^{r} \frac{\lambda}{2\pi\varepsilon_0 r} \,\mathrm{d}r \\
&= -\frac{\lambda}{2\pi\varepsilon_0} \ln r \bigg|_{r_0}^{r} \\
&= \frac{\lambda}{2\pi\varepsilon_0} (\ln r_0 - \ln r) \\
\therefore \varphi(r) &= \varphi(r_0) + \frac{\lambda}{2\pi\varepsilon_0} (\ln r_0 - \ln r)
\end{aligned}$$

Exercise 9.

Find the electric potential due to a hollow sphere with surface charge density σ and radius R.

Solution 9.

Let the electric potential at infinity be zero.

$$\overrightarrow{E} = \begin{cases} 0 & ; \quad r < R \\ \dfrac{1}{4\pi\varepsilon_0} \dfrac{q}{r^2} & ; \quad r > R \end{cases}$$

Therefore, if $r > R$,

$$\begin{aligned}
\varphi(r) - \cancelto{0}{\varphi(\infty)} &= -\int\limits_{\infty}^{r} \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{r} \\
&= \int\limits_{r}^{\infty} E \, \mathrm{d}r \\
&= \int\limits_{r}^{\infty} \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \, \mathrm{d}r \\
&= -\frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \bigg|_{r}^{\infty} \\
\therefore \varphi(r) &= \frac{1}{4\pi\varepsilon_0} \frac{q}{r}
\end{aligned}$$

If $r < R$,

$$\begin{aligned}
\varphi(r) - \varphi(R) &= \int\limits_{R}^{r} \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{r} \\
&= \int\limits_{R}^{r} E \, \mathrm{d}r \\
&= \int\limits_{R}^{r} 0 \, \mathrm{d}r \\
\therefore \varphi(r) &= \varphi(R) \\
&= \frac{1}{4\pi\varepsilon_0} \frac{q}{R}
\end{aligned}$$

Therefore, the potential is constant.
Therefore,

$$\varphi = \begin{cases} \dfrac{1}{4\pi\varepsilon_0} \dfrac{q}{R} & ; \quad r \leq R \\ \dfrac{1}{4\pi\varepsilon_0} \dfrac{q}{r} & ; \quad r \geq R \end{cases}$$

Exercise 10.
Find the electric potential due a ring of charge with radius R and charge q, at a distance z from its centre, on its axis of symmetry.

Solution 10.
Let the electric potential at infinity be zero.

$$
\begin{aligned}
\varphi_{\text{ring}} &= \int \frac{1}{4\pi\varepsilon_0} \frac{\mathrm{d}q}{r} \\
&= \int_0^q \frac{1}{4\pi\varepsilon_0} \frac{\mathrm{d}q}{\sqrt{R^2+z^2}} \\
&= \frac{1}{4\pi\varepsilon_0} \frac{1}{\sqrt{R^2+z^2}}
\end{aligned}
$$

Exercise 11.
Find the electric potential due a disk of charge with radius R and charge q, at a distance z from its centre, on its axis of symmetry.

Solution 11.
Let the electric potential at infinity be zero.

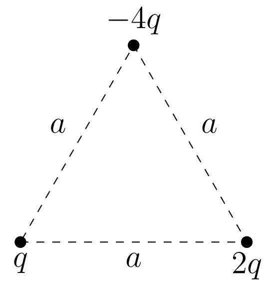
Consider an elemental ring of thickness $\mathrm{d}r$ and radius r. Therefore,

$$
\begin{aligned}
\varphi_{\text{disk}} &= \int \mathrm{d}\varphi_{\text{ring}} \\
&= \int \frac{1}{4\pi\varepsilon_0} \frac{\mathrm{d}q_{\text{ring}}}{\sqrt{r^2+z^2}} \\
&= \int_0^R \frac{1}{4\pi\varepsilon_0} \frac{2\pi r\,\mathrm{d}r\sigma}{\sqrt{r^2+z^2}} \\
&= \frac{\sigma}{2\varepsilon}\left(\sqrt{R^2+z^2} - |z|\right)
\end{aligned}
$$

6 Electrical Potential Energy

Exercise 12.
Three charges, q, $-4q$, $2q$ are placed on the vertices of an equilateral triangle of side a. Find the energy in the system.

Solution 12.



The energy in the system is the amount of energy required to build the system by bringing each of the charges from infinity to its position, one by one.
Let the positions of q, $2q$ and $-4q$ be A, B and C respectively.
The energy required to bring the first charge, q, from infinity to A is zero, as there are no forces acting on it.
The energy required to bring the second charge, $2q$, from infinity to B is

$$\begin{aligned}
U_{2q} &= -\int_{\infty}^{\vec{r}_{\mathrm{B}}} \vec{F} \cdot \mathrm{d}\vec{r} \\
&= -\int_{\infty}^{\vec{r}_{\mathrm{B}}} (2q) \cdot \vec{E} \cdot \mathrm{d}\vec{r} \\
&= -(2q) \int_{\infty}^{\vec{r}_{\mathrm{B}}} \vec{E} \cdot \mathrm{d}\vec{r} \\
&= (2q)\left(\varphi(\mathrm{B}) - \varphi(\infty)\right) \\
&= (2q) \cdot \varphi(\mathrm{B})
\end{aligned}$$

where $\varphi(\mathrm{B})$ is potential at point B due to the existing charges, i.e. q. Similarly, the energy required to bring the third charge, $-4q$, from infinity to C is $(-4q) \cdot \varphi(\mathrm{C})$, where $\varphi(\mathrm{C})$ is the potential at point C due to the existing charges, i.e. q and $2q$.
Therefore, the total energy required is

$$U = (0) + (2q)\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{a^2}\right) + (-4q)\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{a^2} + \frac{1}{4\pi\varepsilon_0}\frac{2q}{a^2}\right)$$

Exercise 13.
Find the potential energy in a solid sphere of charge, with charge density ρ and radius R.

Solution 13.
Consider a solid sphere of charge with ρ and r. Consider an elemental shell of thickness $\mathrm{d}r$ on this sphere.
Therefore,

$$\begin{aligned}\mathrm{d}V &= 4\pi r^2 \,\mathrm{d}r \\ \therefore \mathrm{d}q &= \rho \cdot \mathrm{d}V \\ &= 4\pi\rho r^2 \,\mathrm{d}r\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{d}U &= \frac{1}{4\pi\varepsilon_0}\frac{q_{\text{inside}}}{r}\,\mathrm{d}q \\ &= \frac{1}{4\pi\varepsilon_0}\frac{\frac{4}{3}\pi r^3\rho}{r} \cdot 4\pi r^2 \,\mathrm{d}r\rho \\ \therefore U &= \int\limits_0^R \frac{4\pi\rho^2}{3\varepsilon_0} r^4 \,\mathrm{d}r \\ &= \frac{4\pi\rho^2 R^5}{15\varepsilon_0}\end{aligned}$$

7 Integral Form of Gauss' Law

The volume of the elemental body used for integration is denoted by $\mathrm{d}^3 r$.
For Cartesian coordinate systems, $\mathrm{d}^3 r = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.
For cylindrical coordinate systems, $\mathrm{d}^3 r = r\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}z$.
For spherical coordinate systems, $\mathrm{d}^3 r = r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi$.

$$\begin{aligned}\iint\limits_{\partial V} \vec{E} \cdot \mathrm{d}\vec{A} &= \frac{1}{\varepsilon_0} Q_{\text{inside}} \\ &= \frac{1}{\varepsilon_0}\iiint\limits_V \rho \,\mathrm{d}^3 r^1\end{aligned}$$

If a body with volume V and surface area S is cut into two parts, with volumes V_1 and V_2 and surface area S_1 and S_2 respectively,

$$V = V_1 + V_2$$

However, the surface area increases,

$$S = S_1 + S_2 + 2A \neq S_1 + S_2$$

where A is the area of the new surface created due to the cut.
Therefore,

$$\iint\limits_{S_1} \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{A} + \iint\limits_{S_2} \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{A} = \iint\limits_{\partial V} \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{A}$$

Consider a small cuboid of sides $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$. Let the vertex of the cube, nearest to the origin be (x, y, z).

Therefore,

$$
\begin{aligned}
\Phi = \quad & E_z\left(x+\frac{\mathrm{d}x}{2}, y+\frac{\mathrm{d}y}{2}, z+\mathrm{d}z\right)\mathrm{d}x\,\mathrm{d}y \\
& - E_z\left(x+\frac{\mathrm{d}x}{2}, y+\frac{\mathrm{d}y}{2}, z\right)\mathrm{d}x\,\mathrm{d}y \\
& + E_y\left(x+\frac{\mathrm{d}x}{2}, y+\mathrm{d}y, z+\frac{\mathrm{d}z}{2}\right)\mathrm{d}x\,\mathrm{d}z \\
& - E_y\left(x+\frac{\mathrm{d}x}{2}, y, z+\frac{\mathrm{d}z}{2}\right)\mathrm{d}x\,\mathrm{d}z \\
& + E_x\left(x+\mathrm{d}x, y+\frac{\mathrm{d}y}{2}, z+\frac{\mathrm{d}z}{2}\right)\mathrm{d}y\,\mathrm{d}z \\
& - E_x\left(x, y+\frac{\mathrm{d}y}{2}, z+\frac{\mathrm{d}z}{2}\right)\mathrm{d}y\,\mathrm{d}z \\
= \quad & \frac{E_z\left(x+\frac{\mathrm{d}x}{2}, y+\frac{\mathrm{d}y}{2}, z+\mathrm{d}z\right) - E_z\left(x+\frac{\mathrm{d}x}{2}, y+\frac{\mathrm{d}y}{2}, z\right)}{\mathrm{d}z}\mathrm{d}x\,\mathrm{d}y \\
& + \frac{E_y\left(x+\frac{\mathrm{d}x}{2}, y+\mathrm{d}y, z+\frac{\mathrm{d}z}{2}\right) - E_y\left(x+\frac{\mathrm{d}x}{2}, y, z+\frac{\mathrm{d}z}{2}\right)}{\mathrm{d}y}\mathrm{d}x\,\mathrm{d}z \\
& + \frac{E_x\left(x+\mathrm{d}x, y+\frac{\mathrm{d}y}{2}, z+\frac{\mathrm{d}z}{2}\right) - E_x\left(x, y+\frac{\mathrm{d}y}{2}, z+\frac{\mathrm{d}z}{2}\right)}{\mathrm{d}x}\mathrm{d}y\,\mathrm{d}z \\
= \quad & \left(\left.\frac{\partial E_x}{\partial x}\right|_{\left(x+\frac{\mathrm{d}x}{2}, y+\frac{\mathrm{d}y}{2}, z\right)} \right. \\
& \quad + \left.\frac{\partial E_y}{\partial y}\right|_{\left(x+\frac{\mathrm{d}x}{2}, y, z+\frac{\mathrm{d}z}{2}\right)} \\
& \quad \left. + \left.\frac{\partial E_z}{\partial z}\right|_{\left(x+\frac{\mathrm{d}x}{2}, y+\frac{\mathrm{d}y}{2}, z\right)} \right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z
\end{aligned}
$$

Therefore

$$\begin{aligned}
\operatorname{div} \overrightarrow{E} &= \lim_{\mathrm{d}x\to 0, \mathrm{d}y \to 0, \mathrm{d}z \to 0} \frac{\Phi}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z} \\
&= \left.\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\right|_{(x,y,z)} \\
&= \overrightarrow{\nabla} \cdot \overrightarrow{E}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{E} &= -\overrightarrow{\nabla}\varphi \\
\therefore \operatorname{div} \overrightarrow{E} &= \overrightarrow{\nabla} \cdot \left(-\overrightarrow{\nabla}\varphi\right) \\
&= \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial x^2} \\
&= \nabla^2 \varphi = \Delta \varphi
\end{aligned}$$

Definition 6 (Laplacian)**.** ∇^2 is called the Laplacian.

8 Vector Analyses in Cylindrical and Spherical Coordinate Systems

8.1 Cylindrical Coordinates

$$\begin{aligned}
\overrightarrow{\nabla} f &= \frac{\partial f}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\theta} + \frac{\partial f}{\partial z}\,\hat{z} \\
\overrightarrow{\nabla} \cdot \overrightarrow{F} &= \frac{1}{r}\frac{\partial}{\partial r}\left(r F_r\right) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z} \\
\overrightarrow{\nabla} \times \overrightarrow{F} &= \left(\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right)\hat{r} + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r F_\theta\right) - \frac{\partial F_r}{\partial \theta}\right)\hat{z} \\
\nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

8.2 Spherical Coordinates

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\,\hat{\varphi}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(F_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial f}$$

$$\begin{aligned}\vec{\nabla}\times\vec{F} = \quad & \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(F_\varphi\sin\theta\right) - \frac{\partial F_\theta}{\partial\varphi}\right)\hat{r} \\ & + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(rF_\varphi\right)\right)\hat{\theta} \\ & + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rF_\theta\right) - \frac{\partial F_r}{\partial\theta}\right)\hat{\varphi}\end{aligned}$$

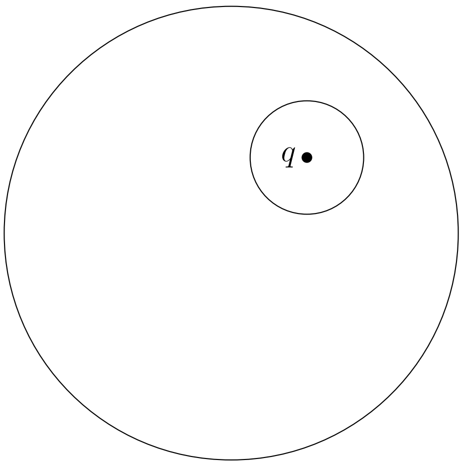
$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

9 Conductors

In an electrostatic condition, the field inside a conductor is zero. If it is not, as the conductor allows movement of charged particles, there will be a current and the condition will not be electrostatic.

Exercise 14.

A point charge q is kept inside a cavity in a conducting sphere. Find the charge on the surfaces of the sphere.



Solution 14.
As the sphere is neutral, $\varphi = 0$.
Therefore, $-\dfrac{\rho}{\varepsilon_0} = 0$.
Therefore, by the Poisson equation,

$$\begin{aligned}
\nabla^2 \varphi &= 0 \\
\therefore \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r} \right) &= 0 \\
\therefore \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r} \right) &= 0 \\
\therefore r^2 \frac{\partial \varphi}{\partial r} &= c \\
\therefore \frac{\partial \varphi}{\partial r} &= \frac{c}{r^2} \\
\therefore \varphi &= \int \frac{c}{r^2} \, \mathrm{d}r \\
&= -\frac{c}{r} + d
\end{aligned}$$

As $\varphi(\infty) = 0$, $d = 0$. Therefore,

$$\varphi = -\frac{c}{r}$$

Therefore,

$$\begin{aligned}
\varphi(R) &= -\frac{c}{R} \\
\therefore c &= -R\varphi(R) \\
\therefore \overrightarrow{E} &= -\overrightarrow{\nabla} \varphi(r) \\
&= -\frac{\partial \varphi}{\partial r} \hat{r} \\
&= -\left(-\frac{R\varphi(R)}{r^2} \right) \hat{r} \\
&= \frac{R\varphi(R)}{r^2} \hat{r}
\end{aligned}$$

Therefore, the field is constant all over the outer surface.
Therefore,

$$\therefore \sigma = \frac{q}{4\pi R^2}$$

10 Capacitors

A capacitor is constructed by arranging two conductors, charged with opposite charges.
Suppose the charges on the conductors are $+Q$ and $-Q$. Therefore,

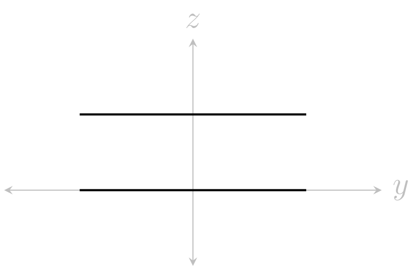
$$\begin{aligned}
V &= \varphi(+) - \varphi(-) \\
&= -\int_{(-)}^{(+)} \overrightarrow{E} \cdot \mathrm{d}\,\overrightarrow{l} \\
&= \int_{(+)}^{(-)} \overrightarrow{E} \cdot \mathrm{d}\,\overrightarrow{l} \\
\therefore V &\propto Q
\end{aligned}$$

Definition 7 (Capacitance)**.** Let the charges on the opposite terminals of a capacitor be $+Q$ and $-Q$ respectively. The ratio between Q and the potential difference between the terminals is called the capacitance of the capacitor.

$$C = \frac{Q}{V}$$

10.1 Parallel Plate Capacitors

Consider a capacitor made of two conducting plates of surface area A each. Let the distance between them be d. Let the charge distribution on the plates be σ and $-\sigma$ respectively.



If d is very small compared to A, the plates can be considered to be infinite.
Therefore,

$$\overrightarrow{E} = \begin{cases} \dfrac{\sigma}{\varepsilon_0}\hat{z} & ; \quad 0 < z < d \\ 0 & ; \quad z < 0 \text{ or } z > d \end{cases}$$

Therefore,

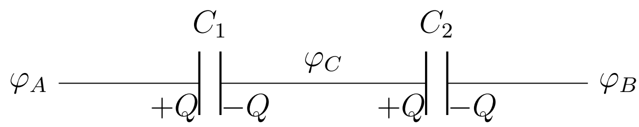
$$
\begin{aligned}
C &= \frac{Q}{V} \\
&= \frac{\sigma A}{Ed} \\
&= \frac{\sigma A}{\frac{\sigma}{\varepsilon_0} \cdot d} \\
&= \frac{A\varepsilon_0}{d}
\end{aligned}
$$

10.2 Concentric Spherical Capacitor

Consider a conducting sphere, with radius R_1 and charge $+Q$, surrounded by a concentric conducting shell of radius R_2 and charge $-Q$. Therefore, the potential difference between the any point at R_1 from the centre and any point at R_2 from the centre is

$$\begin{aligned}
V &= \varphi(R_1) - \varphi(R_2) \\
&= -\int_{R_2}^{R_1} \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{r} \\
&= \int_{R_1}^{R_2} \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{r} \\
&= \int_{R_1}^{R_2} E \, \mathrm{d}r \\
&= \int_{R_1}^{R_2} \frac{Q}{4\pi\varepsilon_0 r^2} \, \mathrm{d}r \\
&= -\frac{Q}{4\pi\varepsilon_0 r}\bigg|_{R_1}^{R_2} \\
&= \frac{Q}{4\pi\varepsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \\
\therefore C &= \frac{Q}{V} \\
&= \frac{4\pi\varepsilon_0}{\dfrac{1}{R_1} - \dfrac{1}{R_2}} \\
&= 4\pi\varepsilon_0 \frac{R_1 R_2}{R_2 - R_1}
\end{aligned}$$

10.3 Capacitors in Series



$$\begin{aligned}
V &= (\varphi_A - \varphi_C) + (\varphi_C - \varphi_B) \\
&= \frac{Q}{C_1} + \frac{Q}{C_2} \\
\therefore \frac{Q}{C_{\text{eq}}} &= \frac{Q}{C_1} + \frac{Q}{C_2} \\
\therefore \frac{1}{C_{\text{eq}}} &= \frac{1}{C_1} + \frac{1}{C_2}
\end{aligned}$$

10.4 Capacitors in Parallel

$$C_{\text{eq}} = C_1 + C_2$$

10.5 Energy Stored in a Capacitor

Consider a capacitor with potential difference V and charge Q.
The energy required to move an elemental charge $\mathrm{d}q$ from the negatively charged plate to the positively charged plate is

$$\begin{aligned}
\mathrm{d}U &= V\,\mathrm{d}q \\
&= \frac{q}{C}\,\mathrm{d}q \\
\therefore U &= \int\limits_0^Q \frac{q}{C}\,\mathrm{d}q \\
&= \frac{Q^2}{2C} \\
&= \frac{1}{2}QV \\
&= \frac{1}{2}CV^2
\end{aligned}$$

10.6 Energy Density

Consider a parallel plate capacitor, with plates of area A, with distance d between them and charge $+Q$ and $-Q$ respectively. The electric field between them is

$$\begin{aligned}
E &= \frac{\sigma}{\varepsilon_0} \\
&= \frac{Q}{\varepsilon_0 A} \\
\therefore Ed &= \frac{Qd}{\varepsilon_0 A}
\end{aligned}$$

As $V = Ed$,

$$\begin{aligned}
V &= \frac{Qd}{\varepsilon_0 A} \\
\therefore \frac{A\varepsilon_0}{d} &= \frac{Q}{V} \\
\therefore \frac{A\varepsilon_0}{d} &= C
\end{aligned}$$

Therefore,

$$\begin{aligned}
U &= \frac{1}{2}CV^2 \\
&= \frac{1}{2}\varepsilon_0 \frac{A}{d} E^2 d^2 \\
&= \frac{1}{2}(Ad)\varepsilon_0 E^2 \\
\therefore \frac{U}{Ad} &= \frac{1}{2}\varepsilon_0 E^2
\end{aligned}$$

Therefore, as Ad is the volume of the space between the plates of the capacitor, $\frac{U}{Ad}$ is called the energy density. Therefore,

$$u = \frac{1}{2}\varepsilon_0 E^2$$

In general, for a charged body with volume charge density ρ, assuming $\varphi(\infty) = 0$,

$$U = \iiint\limits_V \frac{1}{2}\rho\varphi \, \mathrm{d}^3 r$$

As $\rho = -\varepsilon_0 \nabla^2 \varphi$,

$$U = \iiint\limits_V \frac{1}{2}(-\varepsilon_0 \nabla^2 \varphi)\varphi \, \mathrm{d}^3 r$$

$$
\begin{aligned}
\overrightarrow{E} &= -\overrightarrow{\nabla}\varphi \\
\therefore \nabla^2\varphi &= \overrightarrow{\nabla} \cdot \left(\overrightarrow{\nabla}\varphi\right) \\
\therefore \left(\nabla^2\varphi\right)\varphi &= \overrightarrow{\nabla} \cdot \left(\overrightarrow{\nabla}\varphi\right)\varphi
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{\nabla} \cdot \left(\varphi\left(\overrightarrow{\nabla}\varphi\right)\right) &= \left(\overrightarrow{\nabla}\varphi\right)\left(\overrightarrow{\nabla}\varphi\right) + \varphi\overrightarrow{\nabla} \cdot \left(\overrightarrow{\nabla}\varphi\right) \\
&= \left(-\overrightarrow{E}\right)\left(-\overrightarrow{E}\right) + \varphi\nabla^2\varphi \\
\therefore \nabla^2\varphi &= \overrightarrow{\nabla} \cdot \left(-\varphi\overrightarrow{E}\right) - E^2
\end{aligned}
$$

Therefore, substituting in U,

$$
\begin{aligned}
U &= \iiint\limits_V \left(-\frac{1}{2}\varepsilon_0\right)\left(\overrightarrow{\nabla} \cdot \left(-\varphi\overrightarrow{E}\right) - E^2\right) \mathrm{d}^3 r \\
&= \frac{1}{2}\varepsilon_0 \iiint\limits_V \overrightarrow{\nabla} \cdot \left(\varphi\overrightarrow{E}\right) \mathrm{d}^3 r + \iiint\limits_V \frac{1}{2}\varepsilon_0 E^2 \,\mathrm{d}^3 r \\
&= \frac{1}{2}\varepsilon_0 \iint\limits_{\partial V} \varphi\overrightarrow{E} \cdot \mathrm{d}\overrightarrow{A} + \iiint\limits_V \frac{1}{2}\varepsilon_0 E^2 \,\mathrm{d}^3 r
\end{aligned}
$$

As there are no charges at infinity, the electric field at infinity is zero. Therefore, the flux through the large surface is zero. Therefore, $\varepsilon_0 \iint\limits_{\partial V} \varphi\overrightarrow{E} \cdot \mathrm{d}\overrightarrow{A} = 0$. Also, according to the initial assumption, $\varphi(\infty) = 0$.

$$
\begin{aligned}
\therefore U &= \iiint\limits_V \frac{1}{2}\varepsilon_0 E^2 \,\mathrm{d}^3 r \\
\therefore u &= \frac{1}{2}\varepsilon_0 E^2
\end{aligned}
$$

Exercise 15.

Find the potential energy contained in a sphere of charge with radius R.

Solution 15.

$$E = \begin{cases} \dfrac{\rho r}{3\varepsilon_0} & ; \quad r < R \\ \dfrac{\frac{4}{3}\pi R^3 \rho}{4\pi\varepsilon_0 r^2} & ; \quad r > R \end{cases}$$

Therefore

$$\begin{aligned}
U &= \iiint_V \frac{1}{2}\varepsilon_0 E^2 \, \mathrm{d}^3 r \\
&= \int_0^R \frac{1}{2}\varepsilon_0 \left(\frac{\rho r}{3\varepsilon_0}\right)^2 \cdot 4\pi r^2 \, \mathrm{d}r + \int_R^\infty \frac{1}{2}\varepsilon_0 \left(\frac{R^3\rho}{3\varepsilon_0 r^2}\right) 4\pi r^2 \, \mathrm{d}r \\
&= \frac{2\pi\rho^2}{9\varepsilon_0} \int_0^R r^4 \, \mathrm{d}r + \frac{2\pi\rho^2 R^6}{9\varepsilon_0} \int_R^\infty \frac{1}{r^2} \, \mathrm{d}r \\
&= \frac{2\pi\rho^2}{9\varepsilon_0} \left(\frac{R^5}{5}\right) + \frac{2\pi\rho^2 R^6}{9\varepsilon_0} \left(-\frac{1}{r}\right)\Bigg|_R^\infty \\
&= \frac{2\pi\rho^2 R^5}{9\varepsilon_0} \left(\frac{1}{5} + 1\right) \\
&= \frac{2\pi\rho^2 R^5}{9\varepsilon_0} \cdot \frac{6}{5} \\
&= \frac{4\pi\rho^2 R^5}{15\varepsilon_0}
\end{aligned}$$

11 Dielectric Materials

Definition 8 (Dielectric constant or relative permittivity)**.** If the electric field in a dielectric material across some voltage and some distance is E, and the electric field in an identical setup, with a vacuum is E_0, the ratio between E_0 and E is called the dielectric constant of the material or the relative permittivity of the material.

$$\kappa_E = \frac{E_0}{E}$$

Consider a parallel plate capacitor with a dielectric slab of κ_E inserted between its plates.

Let the surface charge densities on each of the plates be $+\sigma_{\text{free}}$ and $-\sigma_{\text{free}}$, and charges $+Q$ and $-Q$.
Let the field inside the capacitor, if it the dielectric slab is absent, be E_0, and the field inside the capacitor, if the dielectric slab is present be E.
Consider a cuboid Gaussian surface at the interface of the dielectric slab and the plate with charge density $+\sigma_{\text{free}}$.
As the electric field entering the Gaussian surface is more than the electric field exiting it, there must be some negative charge on the interface.
Let the surface charge density of this bound charge be $-\sigma_{\text{bound}}$.
Therefore, the net field inside the capacitor is

$$E = \frac{\sigma_{\text{free}} - \sigma_{\text{bound}}}{\varepsilon_0}$$

Definition 9 (Permittivity of a dielectric)**.**

$$\begin{aligned}
\frac{C}{C_0} &= \frac{\frac{Q}{V}}{\frac{Q}{V_0}} \\
&= \frac{V_0}{V} \\
&= \frac{E_0 d}{E d} \\
&= \frac{E_0}{E} \\
&= \kappa_E \\
\therefore C &= C_0 \kappa_E \\
&= \varepsilon_0 \kappa_E \frac{A}{d}
\end{aligned}$$

$\varepsilon = \varepsilon_0 \kappa_E$ is called the permittivity of the dielectric.

11.1 Gauss' Law in Dielectric Materials

$$\begin{aligned}
\iint \overrightarrow{E_0} \cdot \mathrm{d}\overrightarrow{A} &= \frac{Q_{\text{free}}}{\varepsilon_0} \\
\therefore \iint \kappa_E \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{A} &= \frac{Q_{\text{free}}}{\varepsilon_0} \\
\therefore \iint \kappa_E \varepsilon_0 \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{A} &= Q_{\text{free}}
\end{aligned}$$

Definition 10 (Electric displacement field)**.**

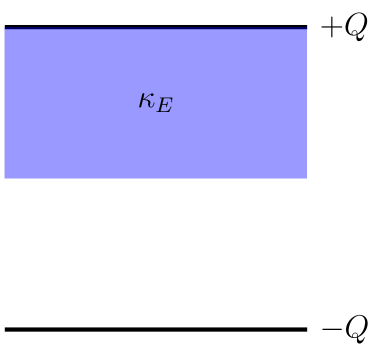
$$\overrightarrow{D} = \varepsilon_0 \kappa_E \overrightarrow{E} = \varepsilon \overrightarrow{E}$$

is called the electric displacement field.

Therefore,

$$\oiint_{\partial V} \overrightarrow{D} \cdot \mathrm{d}\overrightarrow{A} = Q_{\text{free}}$$
$$\therefore \iiint_V \overrightarrow{\nabla} \cdot \overrightarrow{D} \, \mathrm{d}^3 r = \rho_{\text{free}}$$
$$\therefore \overrightarrow{\nabla} \cdot \overrightarrow{D} = \rho_{\text{free}}$$

Exercise 16.
A slab of dielectric material, with thickness x is inserted in a parallel plate capacitor, with plates of surface area A and distance d between them, as shown.



Find the charge density on the interface of the dielectric slab and the effective capacitance.

Solution 16.
As the charge on the plates is $+Q$ and $-Q$ respectively, and as the area of the plates is A,

$$\sigma_{\text{free}} = \frac{Q}{A}$$

Consider a cuboid Gaussian surface at the interface the dielectric slab and the free space, inside the capacitor. As the field entering the Gaussian surface is less than the field exiting it, there must a bound positive charge at the interface. Let the surface charge density of this bound charge be $+\sigma_b$.

$$\begin{aligned} E_2 S - E_1 S &= \frac{\sigma_b A}{\varepsilon_0} \\ \therefore \frac{Q}{\varepsilon_0 A} - \frac{Q}{\varepsilon_0 \kappa_E A} &= \frac{\sigma_b}{\varepsilon_0} \\ \therefore \sigma_b &= \frac{Q}{A}\left(1 - \frac{1}{\kappa_E}\right) \\ &= \frac{Q}{A} \cdot \frac{\kappa_E - 1}{\kappa_E} \\ \therefore \sigma_b &\leq \sigma_{\text{free}} \end{aligned}$$

$$\begin{aligned} C &= \frac{Q}{V} \\ &= \frac{Q}{E_1 x + E_2(d-x)} \\ &= \frac{\cancel{Q}}{\frac{\cancel{Q}}{\varepsilon_0 \kappa A} x + \frac{\cancel{Q}}{\varepsilon_0 A}(d-x)} \\ &= \frac{\varepsilon_0 A}{\frac{x}{k} + d - x} \\ &= \varepsilon_0 \frac{A}{d - x\left(1 - \frac{1}{k}\right)} \end{aligned}$$

Part II
Electrodynamics

1 Currents

Definition 11 (Current)**.** Current is defined as the rate of charge passing through a cross-sectional area.

$$I = \frac{\mathrm{d}q}{\mathrm{d}t}$$

Definition 12 (Current density)**.**

$$\vec{J} = \frac{I}{A}$$

$\vec{J}$ is called the current density.
Therefore, the current is the flux of $\vec{J}$.

1.1 Continuity Law

$$\begin{aligned} Q &= \iiint\limits_V \rho(x, y, z, t) \, \mathrm{d}^3 r \\ \therefore \frac{\mathrm{d}Q}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_V \rho(x, y, z, t) \, \mathrm{d}^3 r \\ &= \iiint\limits_V \frac{\mathrm{d}\rho}{\mathrm{d}t} \, \mathrm{d}^3 r \end{aligned}$$

$$I = \iint\limits_{\partial V} \vec{J} \, \mathrm{d}\vec{a}$$

$$\begin{aligned} I &= -\frac{\mathrm{d}Q}{\mathrm{d}t} \\ \therefore \iint\limits_{\partial V} \vec{J} \cdot \mathrm{d}\vec{A} &= \iiint\limits_V -\frac{\partial \rho}{\partial t} \, \mathrm{d}^3 r \end{aligned}$$

Therefore, by the divergence theorem,

$$\iiint\limits_V \vec{\nabla} \cdot \vec{J} \, \mathrm{d}^3 r = \iiint\limits_V -\frac{\partial \rho}{\partial t} \, \mathrm{d}^3 r$$
$$\therefore \vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

1.2 Charge Carriers

Current density is the product of the total charge and the net drift velocity of the charge carriers.

$$\vec{J} = nq\overrightarrow{v_{\text{drift}}}$$

where n is the number of charge carriers and q is the charge on each of them.

$$\overrightarrow{v_{\text{drift}}} = \frac{1}{n} \sum \vec{v_i}$$

where $\vec{v_i}$ is the velocity of the i^{th} charge carrier.

2 Ohm's Law

Law 2 (Ohm's Law)**.** *The voltage across two points on a conductor is directly proportional to the current through the conductor. The constant of proportionality is called the resistance of the conductor.*

$$\frac{V}{I} = R$$

3 Resistors

3.1 Resistivity

Consider a conductor of cross-sectional area A and length L connected to a potential difference of V. Let the current passing through it be I.
Therefore,

$$I = JA$$
$$V = EL$$

Therefore,

$$
\begin{aligned}
R &= \frac{V}{I} \\
&= \frac{EL}{JA}
\end{aligned}
$$

As $E \propto J$, E can be written as ρJ. Therefore,

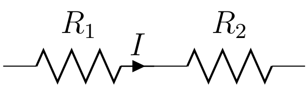
$$
\begin{aligned}
R &= \frac{\rho JL}{JA} \\
&= \rho\frac{L}{A}
\end{aligned}
$$

Definition 13 (Resistivity)**.** If

$$
R = \rho\frac{L}{A}
$$

ρ is called the resistivity of the conductor.
$\sigma = \frac{1}{\rho}$ is called the conductivity of the conductor.
They are constant for a particular material.

3.2 Resistors in Series



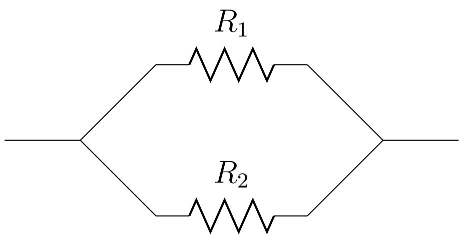
Let V_1 and V_2 be the voltages across R_1 and R_2 respectively.
As the resistors are in series,

$$
\begin{aligned}
V &= V_1 + V_2 \\
&= IR_1 + IR_2 \\
&= I(R_1 + R_2) \\
&= IR_{\text{equivalent}}
\end{aligned}
$$

Therefore,

$$
R_{\text{equivalent}} = R_1 + R_2
$$

3.3 Resistors in Parallel



Let I_1 and I_2 be the currents through R_1 and R_2 respectively.
As the resistors are in parallel,

$$\begin{aligned} I &= I_1 + I_2 \\ &= \frac{V}{R_1} + \frac{V}{R_2} \\ &= V\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \\ &= \frac{V}{R_{\text{equivalent}}} \end{aligned}$$

Therefore,

$$\frac{1}{R_{\text{equivalent}}} = \frac{1}{R_1} + \frac{1}{R_2}$$

4 Power and Energy

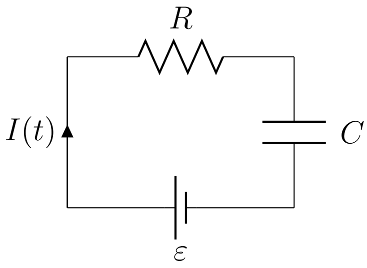
Consider a resistor of resistance R connected to a source of voltage V.
The energy needed to transfer a charge $\mathrm{d}q$ across V is

$$\mathrm{d}U = V\,\mathrm{d}q$$

Therefore the power dissipated across the resistor is

$$\begin{aligned} P &= \frac{\mathrm{d}U}{\mathrm{d}t} \\ &= V\,\frac{\mathrm{d}q}{\mathrm{d}t} \\ &= VI \\ &= I^2R \end{aligned}$$

5 RC Series Circuit



5.1 Charge on the Capacitor

$$
\begin{aligned}
\oint \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{l} &= 0 \\
\therefore -\varepsilon + IR + \frac{Q}{C} &= 0 \\
\therefore IR + \frac{Q}{C} &= \varepsilon \\
\therefore I + \frac{Q}{RC} &= \frac{\varepsilon}{R} \\
\therefore \frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{RC}Q &= \frac{\varepsilon}{R} \\
\therefore Q &= \frac{1}{e^{\int \frac{1}{RC}\,\mathrm{d}t}} \int e^{\int \frac{1}{RC}\,\mathrm{d}t} \frac{\varepsilon}{R} \\
&= \frac{1}{e^{\frac{t}{RC}}} \varepsilon C e^{\frac{t}{RC}} + D
\end{aligned}
$$

As $Q(t = 0) = 0$, $D = -\varepsilon C$.
Therefore,

$$
Q = \varepsilon C \left(1 - e^{-\frac{t}{RC}}\right)
$$

Definition 14 (Time constant of RC circuit)**.** $\tau = RC$ is called the time constant of a circuit with a resistor of resistance R and a capacitor of capacitance C connected in series.

5.2 Current in the Circuit

$$
\begin{aligned}
I &= \frac{\mathrm{d}Q}{\mathrm{d}t} \\
&= \frac{\varepsilon}{R} e^{-\frac{t}{RC}}
\end{aligned}
$$

5.3 Power and Energy

The power supplied by the battery is

$$\begin{aligned} P_\varepsilon &= \varepsilon I \\ &= \frac{\varepsilon^2}{R} e^{-\frac{t}{RC}} \end{aligned}$$

The power dissipated at the resistor is

$$\begin{aligned} P_R &= I^2 R \\ &= \frac{\varepsilon^2}{R} e^{-\frac{2t}{RC}} \end{aligned}$$

The energy stored in the capacitor is

$$\begin{aligned} U_C &= \frac{Q^2}{2C} \\ &= \frac{\varepsilon^2 C}{2} \left(1 - e^{-\frac{t}{RC}}\right)^2 \end{aligned}$$

Therefore the power in the capacitor is

$$\begin{aligned} P_C &= \frac{\mathrm{d}U_C}{\mathrm{d}t} \\ &= \varepsilon^2 C \left(1 - e^{-\frac{t}{RC}}\right) \frac{e^{-\frac{t}{RC}}}{RC} \\ &= \frac{\varepsilon^2}{R} e^{-\frac{2t}{RC}} \end{aligned}$$

Therefore,

$$P_\varepsilon = P_R + P_C$$

Therefore, the power is conserved.

Exercise 17.

A thick conducting, cylindrical shell with inner radius a, outer radius b, and height h is charged with conductivity $\sigma(r) = \alpha r^2$. Assuming $h << a$,

1. Find the net resistance if the flat faces are connected to terminals of a battery.
2. Find the net resistance if the inner and outer surfaces are connected to terminals of a battery.

Solution 17.

1. Consider an elemental resistor, i.e. a thin cylindrical shell of radius $a < r < b$ and thickness $\mathrm{d}r$.
   Therefore,

$$\begin{aligned}\mathrm{d}R &= \frac{1}{\sigma}\frac{L}{A} \\ &= \frac{1}{\alpha r^2}\frac{h}{2\pi r\,\mathrm{d}r} \\ &= \frac{h}{2\pi\alpha r^3\,\mathrm{d}r}\end{aligned}$$

   As the elemental resistors are connected in parallel,

$$\begin{aligned}\frac{1}{R} &= \int \frac{1}{\mathrm{d}R} \\ &= \int_a^b \frac{2\pi\alpha r^3\,\mathrm{d}r}{h} \\ &= \frac{\pi\alpha}{2h}(b^4 - a^4) \\ \therefore R &= \frac{2h}{\pi\alpha(b^4 - a^4)}\end{aligned}$$

   Alternatively,

$$\begin{aligned}J &= \sigma E \\ &= \sigma\frac{V}{h} \\ &= \alpha r^2\frac{V}{h}\end{aligned}$$

$$\begin{aligned}I &= \iint \overrightarrow{J} \cdot \overrightarrow{\mathrm{d}A} \\ &= \int_a^b \alpha r^2\frac{V}{h} \cdot 2\pi r\,\mathrm{d}r \\ &= \frac{\pi\alpha}{2h}(b^4 - a^4)V\end{aligned}$$

Therefore,

$$\begin{aligned} R &= \frac{V}{I} \\ &= \frac{2h}{\pi\alpha(b^4 - a^4)} \end{aligned}$$

The charge density in the body is

$$\begin{aligned} \varrho &= \varepsilon_0 \overrightarrow{\nabla} \cdot \overrightarrow{E} \\ &= 0 \end{aligned}$$

2. Consider an elemental resistor, i.e. a thin cylindrical shell of radius $a < r < b$ and thickness $\mathrm{d}r$.
Therefore,

$$\begin{aligned} \mathrm{d}R &= \frac{1}{\sigma}\frac{L}{A} \\ &= \frac{1}{\alpha r^2}\frac{\mathrm{d}r}{2\pi r h} \\ &= \frac{\mathrm{d}r}{2\pi\alpha r^3 h} \end{aligned}$$

As the elemental resistors are connected in series,

$$\begin{aligned} R &= \int \mathrm{d}R \\ &= \int\limits_a^b \frac{\mathrm{d}r}{2\pi\alpha r^3 h} \\ &= \frac{1}{2\pi\alpha h}\left(-\frac{1}{2b^2} + \frac{1}{2a^2}\right) \\ &= \frac{1}{4\pi\alpha h}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) \end{aligned}$$

Alternatively,

$$\begin{aligned} J &= \sigma E \\ \therefore \frac{I}{2\pi r h} &= \alpha r^2 E \\ \therefore E &= \frac{I}{2\pi\alpha r^3 h} \end{aligned}$$

Therefore,

$$\begin{aligned} V &= \int_a^b E \, \mathrm{d}r \\ &= \int_a^b \frac{I \, \mathrm{d}r}{2\pi\alpha r^3 h} \left(\frac{1}{a^2} - \frac{1}{b^2} \right) \end{aligned}$$

Therefore,

$$\begin{aligned} R &= \frac{V}{I} \\ &= \frac{1}{4\pi\alpha h} \left(\frac{1}{a^2} - \frac{1}{b^2} \right) \end{aligned}$$

The charge density in the body is

$$\begin{aligned} \varrho &= \varepsilon_0 \overrightarrow{\nabla} \cdot \overrightarrow{E} \\ &= \varepsilon_0 \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} (r E_r) \\ &= \varepsilon_0 \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{I}{2\pi\alpha r^3 h} \right) \\ &= -\frac{\varepsilon_0 I}{\pi\alpha r^4 h} \end{aligned}$$

Therefore, the charge in the body is not zero.
However, it is constant in time. Therefore, in the steady state, it does not violate Kirchoff's Current Law.
Consider a cuboid Gaussian surface which contains part of the elemental ring in it.
As σ is varying, the electric field decreases as r increases.
Therefore by Gauss' Law, as the electric field entering the Gaussian surface is more than the electric field exiting the Gaussian surface. Hence, there must be a negative charge inside the Gaussian surface.