Physics 2 : Recitations

Aakash Jog

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Contents

1 Instructor Information

Dr. Richard Spitzberg

Office: Ma Aabadot 119
E-mail: rms9999@gmail.com

Part I
Electrostatics

1 Gravitation and Electromagnetism

Gravitation	Electromagnetism
$F_G = G\dfrac{m_1 m_2}{r^2}$	$F_E = k\dfrac{q_1 q_2}{r^2}$
$G = 6.7 \times 10^{11}\mathrm{N\,m^2\,kg^{-2}}$	$8.99 \times 10^{9}\mathrm{N\,m^2\,C^{-2}}$

2 Coulomb's Law

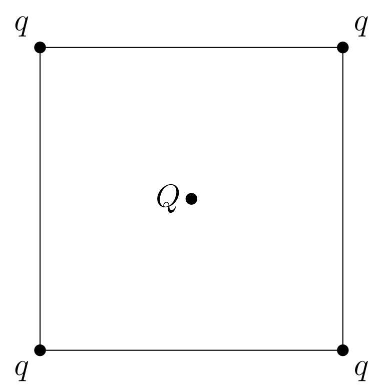
Recitation 1 – Exercise 1.

Four identical charges q are placed in the corners of a square of length a. A fifth charge Q is free to move along the straight line perpendicular to the square plane and passing through its centre. When the charge Q is in the same plane as the other charges, all the forces in the system cancel out.

1. Calculate Q for a given q and a.

2. Find the force $\overrightarrow{F(z)}$ acting on the charge Q when it is at height z above the square.

Recitation 1 – Solution 1.



Consider q on the top right corner of the square. The total force acting on it is 0. Therefore

$$\begin{aligned} 0 &= \frac{kq^2}{a^2}\cos\frac{\pi}{4} + \frac{kq^2}{a^2}\cos\frac{\pi}{4} + \frac{kq^2}{2a^2} + \frac{2kQq}{a^2} \\ \therefore Q &= -\frac{1+2\sqrt{2}}{4}q \end{aligned}$$

If Q is at a height z from the plane, the distance between each q and Q is

$$r = \sqrt{z^2 + \left(\frac{a}{\sqrt{2}}\right)^2}$$

Therefore the force of each q on Q is $\dfrac{kQq}{r^2}$.
Due to symmetry, the components of the forces in the z direction will add up, and all other components will cancel out.
Let the angle between the z direction and the line joining q and Q be φ.
Therefore, the net force is

$$\begin{aligned} F &= 4\frac{kQq}{r^2}\cos\varphi \\ &= 4\frac{kQq}{r^2}\frac{z}{r} \\ &= 4\frac{kQq}{z^2\left(1+\frac{a^2}{2z^2}\right)^{3/2}} \end{aligned}$$

Recitation 1 – Exercise 2.

1. A wire of length 3 metre is charged with $2\,\mathrm{C\,m^{-1}}$. What is the wire's total charge?

Recitation 1 – Solution 2.

$$\begin{aligned} \lambda &= \frac{Q}{L} \\ \therefore Q &= L\lambda \\ &= 6\mathrm{C} \end{aligned}$$

Recitation 1 – Exercise 3.

A wire of length L has the following charge distribution: $\lambda = \lambda_0 \cos \frac{\pi x}{L}$, where x is the distance from the wire's edge. What is the wire's total charge?

Recitation 1 – Solution 3.

$$\begin{aligned} \lambda &= \frac{\mathrm{d}q}{\mathrm{d}x} \\ \therefore \frac{\mathrm{d}q}{\mathrm{d}x} &= \lambda_0 \cos \frac{\pi x}{L} \\ \therefore q &= \int_0^L \lambda_0 \cos \frac{\pi x}{L} \,\mathrm{d}x \\ &= 0 \end{aligned}$$

Recitation 1 – Exercise 4.

A hollow sphere of radius R is uniformly charged with a charge Q. Calculate the charge distribution on the surface of the sphere.

Recitation 1 – Solution 4.

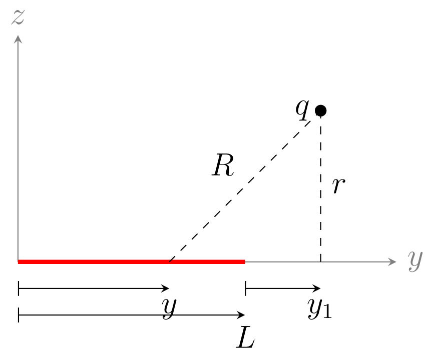
$$\begin{aligned} \sigma &= \frac{Q}{A} \\ &= \frac{Q}{4\pi R^2} \end{aligned}$$

Recitation 1 – Exercise 5.

A straight thin wire is uniformly charged with distribution λ. A charge q is positioned at distance y_1 beneath the wire and r away form it.

1. Find the force acting on the charge q.
2. Show that when the charge is positioned in front of the centre of the wire the $\hat{y}$ component of the force is cancelled.
3. Calculate the force an infinite straight wire will exert on the charge q.

Recitation 1 – Solution 5.



Consider an elemental charge dQ of length dy, at distance y as shown. Let the angle between the line joining dQ and q and the y direction be θ.

$$F_y = F\cos\theta$$
$$F_z = F\sin\theta$$

Let

$$a = L + y_1$$
$$\therefore R = \sqrt{r^2 + (a-y)^2}$$

Therefore,

$$\cos\theta = \frac{a-y}{R}$$
$$\sin\theta = \frac{r}{R}$$

Therefore,

$$\begin{aligned} F_y &= kq\int_0^L \frac{\lambda\,\mathrm{d}y}{R^2}\frac{(a-y)}{R} \\ &= kq\lambda\int_0^L \frac{\mathrm{d}y(a-y)}{\left((a-y)^2+r^2\right)^{3/2}} \\ &= kq\lambda\left(\frac{1}{\sqrt{{y_1}^2+r^2}} - \frac{1}{\sqrt{a^2+r^2}}\right) \end{aligned}$$

$$
\begin{aligned}
F_z &= kq \int\limits_0^L \frac{\lambda \,\mathrm{d}y}{R^2} \frac{r}{R} \\
&= kq\lambda \int\limits_0^L \frac{r \,\mathrm{d}y}{\left((a-y)^2 + r^2\right)^{3/2}} \\
&= \frac{kq\lambda}{r} \left(\frac{a}{\sqrt{r^2 + a^2}} - \frac{y_1}{\sqrt{r^2 + {y_1}^2}} \right)
\end{aligned}
$$

When the charge is positioned above the centre of the wire,

$$
\begin{aligned}
y_1 &= -\frac{L}{2} \\
\therefore a &= \frac{L}{2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
F_y &= kq\lambda \left(\frac{1}{\sqrt{{y_1}^2 + r^2}} - \frac{1}{\sqrt{a^2 + r^2}} \right) \\
&= kq\lambda \left(\frac{1}{\sqrt{-\frac{L}{2}^2 + r^2}} - \frac{1}{\sqrt{\frac{L}{2}^2 + r^2}} \right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
F_z &= \frac{kq\lambda}{r}\left(\frac{a}{\sqrt{r^2+a^2}} - \frac{y_1}{\sqrt{r^2+{y_1}^2}}\right) \\
&= \frac{kq\lambda}{r}\left(\frac{\frac{L}{2}}{\sqrt{r^2+\frac{L}{2}^2}} - \frac{-\frac{L}{2}}{\sqrt{r^2+-\frac{L}{2}^2}}\right) \\
&= \frac{kq\lambda}{r}\left(\frac{L}{\sqrt{r^2+\frac{L}{2}^2}}\right) \\
&= \frac{kq\lambda}{r}\left(\frac{1}{\sqrt{\frac{1}{4}+\left(\frac{r}{L}\right)^2}}\right)
\end{aligned}
$$

If the line is infinite, $L \to \infty$. Therefore

$$
\begin{aligned}
F_z &= \frac{kq\lambda}{r}\left(\frac{1}{\sqrt{\frac{1}{4}+\left(\frac{r}{L}\right)^2}}\right) \\
&= \frac{2kq\lambda}{r}
\end{aligned}
$$

3 Gauss' Law

Recitation 2 – Exercise 2.

A ball of radius a is charged with distribution $\rho = \rho_0 \frac{r}{a}$. Find the electric field everywhere.

Recitation 2 – Solution 2.
Consider a spherical Gaussian surface of radius r.
If $r \leq a$, the charge in the interior of the Gaussian surface is

$$\begin{aligned} q(r) &= \int\limits_0^r \frac{\rho_0 r}{a} \cdot 4\pi r^2 \, \mathrm{d}r \\ &= \frac{\rho_0}{a} \pi r^4 \end{aligned}$$

Therefore, by Gauss' Law,

$$\begin{aligned} E \cdot 4\pi r^2 &= \frac{q(r)}{\varepsilon_0} \\ \therefore E &= \frac{\rho_0 \pi r^4}{4\pi a r^2} \\ &= \frac{\rho_0 r^2}{4a\varepsilon_0} \end{aligned}$$

If $r \geq a$, the entire ball of charge is in the interior of the Gaussian surface. Therefore,

$$\begin{aligned} Q &= q(a) \\ &= \frac{\rho_0}{a} \cdot \pi a^4 \\ &= \rho_0 \pi a^3 \end{aligned}$$

Therefore, by Gauss' Law,

$$\begin{aligned} E \cdot 4\pi r^2 &= \frac{Q}{\varepsilon_0} \\ \therefore E &= \frac{Q}{4\pi r^2 \varepsilon_0} \\ &= \frac{\rho_0 a^3}{4r^2 \varepsilon_0} \end{aligned}$$

Therefore,

$$E = \begin{cases} \dfrac{\rho_0 r^2}{4a\varepsilon_0} & ; \quad r \leq a \\ \dfrac{\rho_0 a^3}{4r^2 \varepsilon} & ; \quad r \geq a \end{cases}$$

Recitation 2 – Exercise 3.

An infinitely long cylinder of radius a is charged with distribution $\rho = \rho_0 \dfrac{r}{a}$. Find the electric field everywhere.

Recitation 2 – Solution 3.

Consider a infinite cylindrical Gaussian surface with radius r.
If $r \leq a$, the charge in the interior of the Gaussian surface is

$$\begin{aligned} q(r) &= \int_0^r \frac{\rho_0 r}{a} \pi r^2 \, \mathrm{d}r \\ &= \frac{2\pi\rho_0 L r^3}{3a} \end{aligned}$$

Therefore, by Gauss' Law,

$$\begin{aligned} E \cdot 2\pi r L &= \frac{2\pi\rho_0 L r^3}{3a\varepsilon_0} \\ \therefore E &= \frac{\rho_0 r^2}{3a\varepsilon_0} \end{aligned}$$

If $r \geq a$, the entire cylinder of charge is in the interior of the Gaussian surface. Therefore,

$$\begin{aligned} Q &= q(a) \\ &= \frac{2\pi\rho_0 L a^3}{3a} \\ &= \frac{2\pi\rho_0 L a^2}{3} \end{aligned}$$

Therefore, by Gauss' Law,

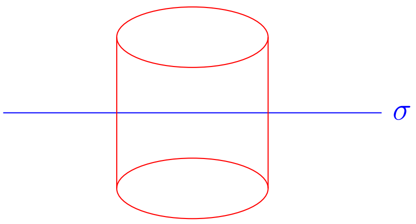
$$\begin{aligned} E \cdot 2\pi r L &= \frac{2\pi\rho_0 L a^2}{3\varepsilon_0} \\ \therefore E &= \frac{\rho_0 a^2}{3\varepsilon_0 r} \end{aligned}$$

Therefore,

$$E = \begin{cases} \dfrac{\rho_0 r^2}{3a\varepsilon_0} & ; \quad r \leq a \\ \dfrac{\rho_0 a^2}{3\varepsilon_0 r} & ; \quad r \geq a \end{cases}$$

Recitation 2 – Exercise 4.
Find the electric field due to a thin infinite plane of uniform charge distribution σ.

Recitation 2 – Solution 4.
Consider a cylindrical Gaussian surface, with ends of area A, as shown.



The charge in the interior of the surface is

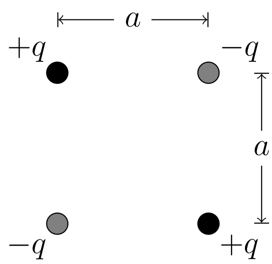
$$\mathrm{d}q = A\sigma$$

Therefore, by Gauss' Law,

$$\begin{aligned} E_1 \cdot A_1 + E_2 \cdot A_2 &= \frac{A\sigma}{\varepsilon_0} \\ \therefore 2EA &= \frac{A\sigma}{\varepsilon_0} \\ \therefore E &= \frac{\sigma}{2\varepsilon_0} \end{aligned}$$

4 Electric Potential

Recitation 3 – Exercise 2.
A system of four charges is constructed as shown.



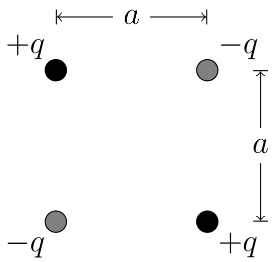
a) Calculate the work needed to build this system.

b) What is the potential in the centre of the system?

c) Calculate the potential in each of the corners (calculate as if there is no charge in the corner you are calculating for).

Recitation 3 – Solution 2.

a) Let the positions of the charges be A, B, C, D.



The work done to bring the first charge from infinity to A is

$$W_{\mathrm{A}} = 0$$

The work done to bring the first charge from infinity to B is

$$W_{\mathrm{B}} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a\sqrt{2}}$$

Similarly for the other two charges.
Therefore,

$$\begin{aligned} W &= 0 + \frac{q^2}{4\sqrt{2}\pi\varepsilon_0} + \frac{-2q^2}{4\pi\varepsilon_0 a} + \left(\frac{-2q^2}{4\pi\varepsilon_0 a} + \frac{q^2}{4\sqrt{2}\pi\varepsilon_0}\right) \\ &= \frac{q^2}{2\sqrt{2}\pi\varepsilon_0} - \frac{q^2}{\pi\varepsilon_0 a} \end{aligned}$$

b)

$$\begin{aligned} V_{\text{centre}} &= V_q + V_q + V_{-q} + V_{-q} \\ &= \frac{q}{4\pi\varepsilon_0 \left(\frac{a}{\sqrt{2}}\right)} + \frac{q}{4\pi\varepsilon_0 \left(\frac{a}{\sqrt{2}}\right)} - \frac{q}{4\pi\varepsilon_0 \left(\frac{a}{\sqrt{2}}\right)} - \frac{q}{4\pi\varepsilon_0 \left(\frac{a}{\sqrt{2}}\right)} \\ &= 0 \end{aligned}$$

c)

$$\begin{aligned} V_{\text{A}} &= \frac{1}{4\pi\varepsilon_0}\frac{-q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{2}a} \\ V_{\text{B}} &= \frac{1}{4\pi\varepsilon_0}\frac{-q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{2}a} \\ V_{\text{C}} &= \frac{1}{4\pi\varepsilon_0}\frac{q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{\sqrt{2}a} \\ V_{\text{D}} &= \frac{1}{4\pi\varepsilon_0}\frac{q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{\sqrt{2}a} \end{aligned}$$

Recitation 3 – Exercise 3.

A ring of radius R is charged with total charge Q.

a) Calculate the electric field in the centre of the ring.

b) Calculate the potential in the centre of the ring by integrating the contributions of the infinitesimal charge elements of the ring.

Recitation 3 – Solution 3.

a) Due to the symmetry of the ring, the field due to every elemental charge $\mathrm{d}q$ will be cancelled out by the field due to a elemental charge diametrically opposite to $\mathrm{d}q$.
Therefore,

$$\overrightarrow{E} = 0$$

b)

$$\begin{aligned} \mathrm{d}V &= \frac{\mathrm{d}q}{4\pi\varepsilon_0 R} \\ \therefore V &= \frac{Q}{4\pi\varepsilon_0 R} \end{aligned}$$

Recitation 3 – Exercise 4.

Calculate the potential resulting from a ball charged with constant volume distribution ρ. Use the expression

$$\varphi(r_2) - \varphi(r_1) = -\int_{r_1}^{r_2} E(r)\,\mathrm{d}r$$

Repeat the calculation twice:

a) Set $\varphi(r = R) = 0$

b) Set $\varphi(r = \infty) = 0$

Recitation 3 – Solution 4.

a) Let $\varphi(r=\infty)=0$.

If $r>R$,

$$
\begin{aligned}
E &= -\frac{\mathrm{d}\varphi}{\mathrm{d}r} \\
\therefore -\frac{\mathrm{d}\varphi}{\mathrm{d}r} &= \frac{Q}{4\pi\varepsilon_0 r^2} \\
\therefore \int \mathrm{d}\varphi &= \int\limits_{\infty}^{r} \frac{Q}{4\pi\varepsilon_0 r^2}\,\mathrm{d}r \\
\therefore \varphi(r)-\varphi(\infty) &= \frac{Q}{4\pi\varepsilon_0 r} - 0 \\
\therefore \varphi(r) &= \frac{Q}{4\pi\varepsilon_0 r}
\end{aligned}
$$

If $r<R$,

$$
\begin{aligned}
E &= \frac{q(r)}{4\pi\varepsilon_0 r^2} \\
&= \frac{\rho\cdot\frac{4}{3}\pi r^3}{4\pi\varepsilon_0 r^2} \\
&= \frac{\rho r}{3\varepsilon_0}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
E &= -\frac{\mathrm{d}\varphi}{\mathrm{d}r} \\
\therefore \int \mathrm{d}\varphi &= \int\limits_{r}^{R} \frac{\rho}{3\varepsilon_0} r\,\mathrm{d}r \\
\therefore \varphi(R)-\varphi(r) &= \frac{\rho}{6\varepsilon_0}\left(r^2-R^2\right) \\
\therefore \varphi(r) &= \frac{Q}{4\pi\varepsilon_0 R} + \frac{\rho\left(R^2-r^2\right)}{6\varepsilon_0} \\
&= \frac{Q}{8\pi\varepsilon_0 R}\left(3-\frac{r^2}{R^2}\right)
\end{aligned}
$$

b) Let $\varphi(r = R) = 0$.

$$\begin{aligned}
\therefore \varphi(R) - \varphi(r) &= \frac{\rho}{6\varepsilon_0}\left(r^2 - R^2\right) \\
\therefore \varphi(r) &= \frac{\rho}{6\varepsilon_0}\left(R^2 - r^2\right) \\
&= \frac{Q}{8\pi R\varepsilon_0}\left(1 - \frac{r^2}{R^2}\right)
\end{aligned}$$

Recitation 4 – Exercise 1.

A point charge Q is surrounded by a spherical grounded shell of radius R_1.

1. What is the charge accumulated on the shell? Where did it come from?
2. The entire system is covered with another spherical shell of radius R_2 and charged with q. What will be the charge accumulated on the grounded shell?

Recitation 4 – Solution 1.

1. The charge accumulated on the shell comes from the ground.
   Let the charge on the shell be Q_1.
   As the shell is grounded, the net potential on it must be zero.

$$\begin{aligned}
\varphi &= \varphi \text{ due to } Q + \varphi \text{ due to } Q' \\
\therefore 0 &= \frac{Q}{4\pi\varepsilon_0 R_1} + \frac{Q_1}{4\pi\varepsilon_0 R_1} \\
\therefore Q_1 &= -Q
\end{aligned}$$

2. Let the charge on the grounded shell be Q_2.

$$\begin{aligned}
\varphi &= \varphi \text{ due to } Q + \varphi \text{ due to } Q_2 + \varphi \text{ due to } q \\
\therefore 0 &= \frac{Q}{4\pi\varepsilon_0 R_1} + \frac{Q_2}{4\pi\varepsilon_0 R_1} + \frac{q}{4\pi\varepsilon_0 R_2} \\
\therefore Q_2 &= Q - \frac{qR_1}{R_2}
\end{aligned}$$

Recitation 4 – Exercise 3.

A point charge Q is set in the center (same distance from all corners) of a perfect tetrahedron. The bottom face of the tetrahedron is uniformly charged with charge density σ.

Recitation 4 – Solution 3.

1. Consider a spherical Gaussian surface passing through the vertices of the tetrahedron.
   Therefore by Gauss' Law, the total flux passing through the sphere, due to Q is $\oiint \overrightarrow{E} \cdot \mathrm{d}A$.
   Hence, by symmetry, the flux through every surface of the tetrahedron, due to Q is

$$\begin{aligned}\Phi &= \frac{1}{4} \oiint \overrightarrow{E} \cdot \mathrm{d}A \\ &= \frac{1}{4} \frac{Q}{\varepsilon_0}\end{aligned}$$

   The flux on the bottom face due to the bottom face itself is zero. Therefore the total flux through the bottom face is the flux due to Q only.

2. As the charge on Q and the bottom face of the tetrahedron are similar in charge, the force between them is repulsive in nature. Hence, the force on the bottom face is directed downwards.

3. Let the area of the bottom face

$$\begin{aligned}F &= (\sigma A)E \\ &= \sigma(EA)\end{aligned}$$

   As Q is exactly above the centre of the bottom face, due to symmetry, $\oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A} = EA$

$$\begin{aligned}\therefore F &= \sigma \left(\oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A} \right) \\ &= F \left(\frac{Q}{4\varepsilon_0} \right)\end{aligned}$$

Recitation 4 – Exercise 4.
The following electric field is given:

$$\overrightarrow{E} = \alpha y^2 \hat{\imath} + \alpha(2xy + z^2)\hat{\jmath} + 2\alpha yz\hat{k}$$

Calculate the potential at âČŮ$r = (x, y, z)$, set the potential at the origin to be zero.

Recitation 4 – Solution 4.
Let $\varphi(0, 0, 0) = 0$.
Therefore,

$$\begin{aligned}
E_x &= -\frac{\mathrm{d}\varphi}{\mathrm{d}x} \\
&= \alpha y^2 \\
\therefore \varphi &= -\alpha xy^2 + f_1(y, z) \\
E_y &= -\frac{\mathrm{d}\varphi}{\mathrm{d}y} \\
&= \alpha(2xy + z^2) \\
\therefore \varphi &= -\alpha xy^2 - \alpha z^2 y + f_1(x, z) \\
E_z &= -\frac{\mathrm{d}\varphi}{\mathrm{d}z} \\
&= 2\alpha yz \\
\therefore \varphi &= -\alpha yz^2 + f_2(x, y)
\end{aligned}$$

Comparing the three expressions of φ,

$$\varphi = -\alpha xy^2 - \alpha yz^2 + c$$

As $\varphi(0, 0, 0) = 0$, $c = 0$.
Therefore,

$$\varphi = -\alpha xy^2 - \alpha yz^2$$

Recitation 4 – Exercise 5.
A thin rod of length L is charged with a uniform charge density λ is laid along the x-axis.

1. Calculate the electric potential along the x-axis (where $x > L$)
2. Calculate the electric field along the x-axis (where $x > L$)

3. A second identical thin rod is placed along the x-axis at distance L from the edge of the first rod. Calculate the force due to left rod, acting on the right one.

Recitation 4 – Solution 5.

1. Consider an elemental charge $\mathrm{d}q$ with length $\mathrm{d}x$ at a distance x from the origin.
   Therefore, the potential at a distance d from the origin is

$$\begin{aligned}
\mathrm{d}\varphi &= \frac{\mathrm{d}q}{4\pi\varepsilon_0(L+d-x} \\
\therefore \int \mathrm{d}\varphi &= \int_0^L \frac{\lambda\,\mathrm{d}x}{4\pi\varepsilon_0(L+d-x)} \\
\therefore \varphi &= \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{L+d}{d}\right)
\end{aligned}$$

2.

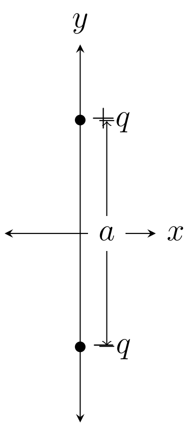
$$\begin{aligned}
\mathrm{d}E &= \frac{\mathrm{d}\varphi}{\mathrm{d}r} \\
\therefore \mathrm{d}E &= \frac{\mathrm{d}q}{4\pi\varepsilon_0(L+d-x)^2} \\
\therefore E &= \frac{\lambda}{4\pi\varepsilon_0}\int_{L+d}^{d}\frac{\mathrm{d}u}{u^2} \\
&= \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{L}{(L+d)(d)}\right)
\end{aligned}$$

3. Consider an elemental charge $\mathrm{d}q$with length $\mathrm{d}x$, on the second rod, at a distance x from the end of the first rod.

$$
\begin{aligned}
\mathrm{d}F &= \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{L}{(L+x)(x)}\right)\mathrm{d}q \\
&= \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{L}{(L+x)(x)}\right)\lambda\,\mathrm{d}x \\
\therefore F &= \int\limits_{L}^{2L}\frac{\lambda^2}{4\pi\varepsilon_0}\left(\frac{L}{(L+x)(x)}\right)\mathrm{d}x \\
&= \frac{\lambda^2}{4\pi\varepsilon_0}\ln\frac{x}{L+x}\bigg|_{L}^{2L} \\
&= \frac{\lambda^2}{4\pi\varepsilon_0}\left(\ln\frac{2}{3}-\ln\frac{1}{2}\right) \\
&= \frac{\lambda^2}{4\pi\varepsilon_0}\left(\ln\frac{4}{3}\right)
\end{aligned}
$$

Recitation 5 – Exercise 1.

An electric dipole is comprised of two opposite charges q and $-q$ positioned at distance a from each other as shown.



1. Calculate the electric field along the y-axis.
2. What is the electric field when $y >> a$?
3. Repeat the previous sub-questions for points along the x-axis.

Recitation 5 – Solution 1.

1.

$$\overrightarrow{E_y} = \frac{q}{4\pi\varepsilon_0\left(y-\frac{a}{2}\right)^2}\hat{j} - \frac{q}{4\pi\varepsilon_0\left(y+\frac{a}{2}\right)^2}\hat{j}$$
$$= \frac{q}{4\pi\varepsilon_0 y^2}\left(\left(1-\frac{d}{2y}\right)^{-2} - \left(1-\frac{d}{2y}\right)^{-2}\right)\hat{j}$$

2. By the Binomial theorem, if $x << 1$, $(1 \pm x)^n \approx 1 \pm nx)$.
   Therefore, as $y << d$,

$$\left(1 \pm \frac{d}{2y}\right)^2 = \left(1 \pm \frac{2d}{2y}\right)$$

   Therefore,

$$\overrightarrow{E_y} = \frac{q}{4\pi\varepsilon_0 y^2}\left(1+\frac{d}{y}-1+\frac{d}{y}\right)\hat{j}$$
$$= \frac{2\left(q\hat{j}d\right)}{4\pi\varepsilon_0 y^2 \cdot y}$$
$$= \frac{2\overrightarrow{p}}{4\pi\varepsilon_0 y^3}$$

3. Let the distance between a point $(x, 0)$ and each of the charges be r. Let the angle between the x-axis and the line joining a point $(x, 0)$ and $+q$ or $-q$ be θ.
   Therefore,

$$\sin(\theta) = \frac{d}{2r}$$

$$\overrightarrow{E_x} = -2\frac{q}{4\pi\varepsilon_0 r^2}\sin(\theta)\hat{j}$$
$$= -\frac{qd}{4\pi\varepsilon_0 r^3}\hat{j}$$
$$= -\frac{\overrightarrow{p}}{4\pi\varepsilon_0 r^3}$$

   If $x >> d$, $r = d$. Therefore,

$$\overrightarrow{E_x} = -\frac{\overrightarrow{p}}{4\pi\varepsilon_0 r^3}$$

Recitation 5 – Exercise 2.
Calculate the following expressions using both spherical and Cartesian coordinates

1. $\overrightarrow{\nabla} r = \hat{r}$
2. $\overrightarrow{\nabla} \frac{1}{r} = -\frac{\hat{r}}{r^2}$

Recitation 5 – Solution 2.
1. Using Cartesian coordinates,

$$\begin{aligned}\overrightarrow{r} &= x\hat{\imath} + y\hat{\jmath} + z\hat{k} \\ \therefore r &= \sqrt{x^2+y^2+z^2}\end{aligned}$$

Therefore,

$$\begin{aligned}\overrightarrow{\nabla}(r) &= \hat{\imath}\,\frac{\partial r}{\partial x} + \hat{\jmath}\,\frac{\partial r}{\partial y} + \hat{k}\,\frac{\partial r}{\partial z} \\ &= \hat{\imath}\,\frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} + \hat{\jmath}\,\frac{\partial\sqrt{x^2+y^2+z^2}}{\partial y} + \hat{k}\,\frac{\partial\sqrt{x^2+y^2+z^2}}{\partial z} \\ &= \hat{\imath}\frac{x}{\sqrt{x^2+y^2+z^2}} + \hat{\jmath}\frac{y}{\sqrt{x^2+y^2+z^2}} + \hat{k}\frac{z}{\sqrt{x^2+y^2+z^2}} \\ &= \frac{\overrightarrow{r}}{r} \\ &= \hat{r}\end{aligned}$$

Using spherical coordinates,

$$\begin{aligned}\overrightarrow{\nabla}(r) &= \hat{r}\cancelto{1}{\frac{\partial r}{\partial r}} + \hat{\theta}\frac{1}{r}\cancelto{0}{\frac{\partial r}{\partial \theta}} + \hat{\varphi}\frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial r}{\partial \varphi}} \\ &= \hat{r}\end{aligned}$$

2. Using Cartesian coordinates,

$$\begin{aligned}\overrightarrow{\nabla}\left(\frac{1}{r}\right) &= \hat{\imath}\,\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + \hat{\jmath}\,\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + \hat{k}\,\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) \\ &= -\left(\hat{\imath}\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \hat{\jmath}\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \hat{k}\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) \\ &= -\frac{\overrightarrow{r}}{r^3} \\ &= -\frac{\hat{r}}{r^2}\end{aligned}$$

Using spherical coordinates,

$$\vec{\nabla}\left(\frac{1}{r}\right) = \hat{r}\,\frac{\partial}{\partial r}\left(\frac{1}{r}\right) + \hat{\theta}\frac{1}{r}\cancelto{0}{\frac{\partial}{\partial\theta}\left(\frac{1}{r}\right)} + \hat{\varphi}\frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial}{\partial\varphi}\left(\frac{1}{r}\right)}$$

$$= \hat{r}\frac{-1}{r^2}$$

$$= -\frac{\hat{r}}{r^2}$$

5 Differential Form of Gauss' Law

Recitation 5 – Exercise 3.

The following potential is given in cylindrical coordinates

$$\varphi(r) = \begin{cases} -\frac{\rho_0 r^2}{8\varepsilon_0} - \frac{\rho_0 r R}{4\varepsilon_0} & ; \quad r < R \\ -\frac{\rho_0 R^2}{2\varepsilon_0}\ln\left(\frac{r}{R}\right) + c & ; \quad r > R \end{cases}$$

1. Find the constant c.
2. Calculate the electric field $\vec{E}$ everywhere.
3. Calculate the total charge, Q, inside a section with height H.
4. Calculate the charge density ρ using the differential Gauss' Law.
5. Show that an integral on the density you found , i.e. ρ, gives the total charge you calculated above.

Recitation 5 – Solution 3.

1. In cylindrical coordinates,

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$z = z$$

where φ is the angle between $\vec{r}$ and the x-axis.
Let

$$-\frac{\rho_0 r^2}{8\varepsilon_0} - \frac{\rho_0 r R}{4\varepsilon_0} = \varphi_1(r)$$

$$-\frac{\rho_0 R^2}{2\varepsilon_0}\ln\left(\frac{r}{R}\right) + c = \varphi_2(r)$$

Therefore,

$$\varphi(r) = \begin{cases} \varphi_1(r) & ; \quad r < R \\ \varphi_2(r) & ; \quad r > R \end{cases}$$

At the surface the values of φ_1 and φ_2 must be equal. Therefore,

$$\begin{aligned} \varphi_1(R) &= \varphi_2(R) \\ \therefore -\frac{\rho_0 R^2}{8\varepsilon_0} - \frac{\rho_0 R \cdot R}{4\varepsilon_0} &= -\frac{\rho_0 R^2}{2\varepsilon_0} \ln\left(\frac{R}{R}\right) + c \\ \therefore c &= -\frac{3\rho_0 R^2}{8\varepsilon_0} \end{aligned}$$

2.

$$\begin{aligned} E_1 &= -\frac{\mathrm{d}\varphi_1}{\mathrm{d}r} \\ &= \frac{\rho_0 (r+R)}{4\varepsilon_0} \\ E_2 &= -\frac{\mathrm{d}\varphi_2}{\mathrm{d}r} \\ &= \frac{\rho_0 R^2}{2\varepsilon_0 r} \end{aligned}$$

3. Consider a cylindrical Gaussian surface with radius just larger than R and height H.
Therefore by Gauss' Law,

$$\begin{aligned} \oiint \vec{E}\,\mathrm{d}\vec{S} &= \frac{Q}{\varepsilon_0} \\ \therefore \frac{\rho_0 R^2}{2\varepsilon_0 R} \cdot 2\pi R^2 H &= \frac{Q}{\varepsilon_0} \\ \therefore Q &= \frac{\rho_0 \pi R^2 H}{\varepsilon_0} \end{aligned}$$

4. By the differential form of Gauss' Law,

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\varepsilon_0} \\
\vec{\nabla} \cdot \vec{E} &= \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial}{\partial \varphi}(E_\varphi) + \frac{\partial}{\partial z}(E_z) \\
\therefore \vec{\nabla} \cdot \vec{E_1} &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\rho_0}{4\varepsilon_0}(r+R)\right) + \frac{1}{r}\frac{\partial}{\partial \varphi}\cancelto{0}{(E_\varphi)} + \frac{\partial}{\partial z}\cancelto{0}{(E_z)} \\
\therefore \frac{\rho}{\varepsilon_0} &= \frac{\rho_0}{4\varepsilon_0 r}(2r+R) \\
\therefore \rho &= \frac{\rho_0}{4}\left(2 + \frac{R}{h}\right)
\end{aligned}$$

5.

$$\begin{aligned}
Q &= \int_0^R \rho(r) \cdot 2\pi r H \,\mathrm{d}r \\
&= \frac{2\rho_0 \pi H}{4}\left(\frac{(2r)^2}{2} + Rr\right)\Bigg|_0^R \\
&= \pi R^2 H \rho_0
\end{aligned}$$

Recitation 5 – Exercise 4.

1. Calculate the potential resulting from a solid ball of radius R charged uniformly with constant distribution ρ.

2. Calculate the potential along the z-axis resulting from a solid cylinder of radius a and height L charged uniformly with constant distribution $-\rho$.

Recitation 5 – Solution 4.

1.

$$q(r) = \begin{cases} \rho_0 \cdot \frac{4}{3}\pi r^3 & ; \quad r < R \\ \rho_0 \cdot \frac{4}{3}\pi R^3 & ; \quad r > R \end{cases} \therefore \varphi(r) \quad = \begin{cases} \frac{Q}{4\pi\varepsilon_0 r} & ; \quad r > R \\ \frac{Q}{8\pi\varepsilon_0 R^3}(3R^2 - r^2 & ; \quad r < R \end{cases}$$

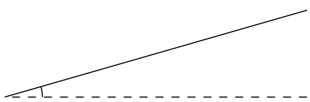
2. Consider an elemental disk of charge $\mathrm{d}^2 q$ at height z from the centre of the cylinder.

Therefore, the potential along the z-axis is

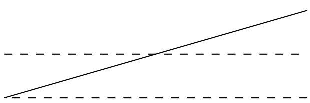
$$V(z') = \int\limits_{r=0}^{r=a} \int\limits_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \frac{\mathrm{d}^2 q}{4\pi\varepsilon_0 (z' - z)}$$

6 Capacitors

Recitation 6 – Exercise 1.
A plate capacitor which is made of square plates of sides a fell and a little angle θ formed between its plates as shown. The smallest distance between the plates is d. Calculate the new capacitance.



Recitation 6 – Solution 1.
The tilted capacitor plate can be considered to be approximately equivalent to a parallel plate at height $d + \frac{a \tan\theta}{2}$, i.e. the tilted plate can be considered to be made parallel to the other plate by pivoting it at the axis through its midpoint.



The capacitance of the original capacitor is

$$\begin{aligned} C &= \frac{Q}{V} \\ &= \frac{\varepsilon_0 A}{d} \end{aligned}$$

Therefore, the capacitance of the tilted capacitor is

$$C' = \frac{\varepsilon_0 a^2}{\left(d + \frac{a \tan\theta}{2}\right)}$$

As $\theta << 1$, $\tan\theta \approx \theta$.

$$\therefore C' = \frac{\varepsilon_0 A}{d}\left(1 + \frac{a\theta}{2d}\right)^{-1}$$

Alternatively, the tilted capacitor can be considered to be a capacitor with capacitance varying with x.
Considering the origin to be at the left end of the lower plate, the equation of the tilted plate is

$$\begin{aligned} y &= d + mx \\ &= d + \tan\theta x \end{aligned}$$

As $\theta << 1$,

$$y = d + \theta x$$

Thererefore,

$$\begin{aligned} C &= \frac{\int\limits_0^L \frac{\varepsilon_0 A}{y}}{\int\limits_0^L \mathrm{d}x} \\ &= \frac{\int\limits_0^L \frac{\varepsilon_0 a\,\mathrm{d}x}{d+\theta x}}{L} \\ &= \frac{\frac{\varepsilon_0 A}{\theta}\int\limits_0^L \frac{\mathrm{d}u}{u}}{L} \\ &= \frac{\varepsilon_0 A}{a\theta}\ln\left(\frac{d + a\theta}{d}\right) \\ &= \frac{\varepsilon_0 A}{d}\left(\left(\frac{a\theta}{d}\right) - \frac{1}{2}\left(\frac{a\theta}{d}\right)^2 + \dots\right) \\ &= \frac{\varepsilon_0 A}{d} - \frac{\varepsilon_A}{2}\frac{a\theta}{d^2} + \dots \end{aligned}$$

Recitation 6 – Exercise 2.

A cylindrical capacitor is comprised of two concentric cylinders of length L and radii a and b (where $L >> a, b$ and $a < b$). The inner cylinder (radius a) carries total charge Q and the outer cylinder (radius b) is grounded. Assume vacuum inside the system and all bodies to be conducting.

1. Calculate the electric field everywhere.
2. Calculate the capacitance per unit length.
3. Calculate the energy density everywhere.

Recitation 6 – Solution 2.

1. As the bodies are condicting, the field inside the inner cylinder is 0. Consider a cylindrical Gaussian surface with radius $a < r < b$. Therefore, by Gauss' Law,

$$\oiint \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{S} = \frac{Q}{\varepsilon_0}$$
$$\therefore E = \frac{Q}{2\pi\varepsilon_0 L r}$$

As the outer cylinder is grounded, and due to the charge on the inner cylinder, the charge on its inner surface is $-Q$.
Therefore, by Gauss's Law, the field outside must be 0.

2.

$$E = \frac{Q}{2\pi\varepsilon_0 L r}$$
$$\therefore V = \int_a^b \frac{Q}{2\pi\varepsilon_0 L r}\,\mathrm{d}r$$
$$= \frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right)$$
$$\therefore C = \frac{2\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)}$$

3.

$$U = \frac{CV^2}{2}$$
$$= \frac{1}{2}\frac{2\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)}\left(\frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right)\right)^2$$

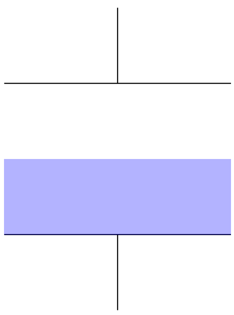
Therefore,

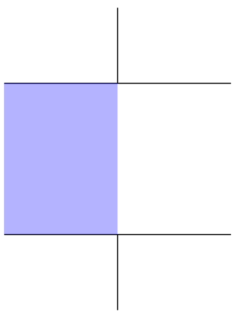
$$u = \frac{U}{\pi L(b^2 - a^2)}$$

7 Dielectric Materials

Recitation 6 – Exercise 3.

The capacitance of an empty plate capacitor (vacuum between the plates) is C_0. Half of the capacitor volume is filled with a dielectric material of constant κ in two different ways as shown.

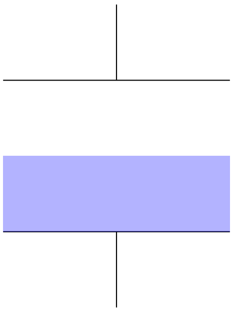




Calculate the new capacitance in the two cases.

Recitation 6 – Solution 3.

The arrangements are equivalent to connection of capacitors in series and parallel respectively.



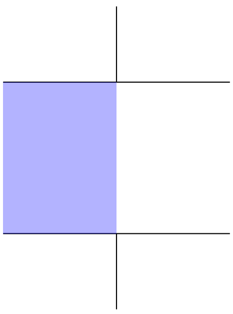
$\equiv$

In this case,

$$C_1 = \frac{\varepsilon_0 A}{\frac{d}{2}}$$
$$C_2 = \frac{\kappa \varepsilon_0 A}{\frac{d}{2}}$$

Therefore,

$$C_{\text{equivalent}} = \frac{C_1 C_2}{C_1 + C_2}$$



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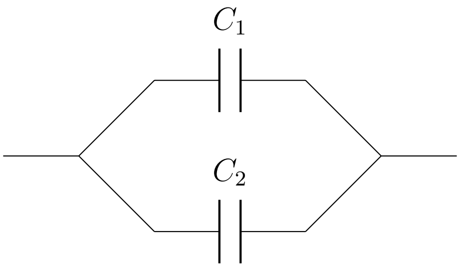


In this case,

$$C_1 = \frac{\varepsilon_0 \frac{A}{2}}{d}$$

$$C_2 = \frac{\kappa \varepsilon_0 \frac{A}{2}}{d}$$

Therefore,

$$C_{\text{equivalent}} = C_1 + C_2$$

Recitation 7 – Exercise 1.
A plate capacitor of area A and distance d is connected to a potential difference V. A dielectric material of constant $\varepsilon = \kappa_e \varepsilon_0$ is inserted into the capacitor (while connected to the voltage source).

1. Calculate the new capacitance.
2. How did the free charge on the capacitor plates change in the process?
3. How did the energy stored in the capacitor change?

Now, all the dielectric material is removed. Afterward, the capacitor is disconnected from the voltage source. A dielectric material of constant $\varepsilon = \kappa_e \varepsilon_0$ is inserted into the capacitor (while disconnected from the voltage source).

4. Calculate the new capacitance.
5. How did the potential difference between the plates change?
6. How did the energy stored in the capacitor change?

For both cases, was the dielectric material attracted or repelled by the capacitor?

Recitation 7 – Solution 1.

If there is no dielectric in the capacitor, the capacitance is

$$\begin{aligned} C_0 &= \frac{Q_0}{V_0} \\ &= \frac{Q_0}{E_0 d} \\ &= \frac{\sigma_0 A}{\frac{\sigma_0}{\varepsilon_0} d} \\ &= \frac{\varepsilon_0 A}{d} \end{aligned}$$

1. After the dielectric material is inserted,

$$C_1 = \frac{Q_1}{V_1}$$

As the battery is connected, the voltage will remain constant. Therefore, $V_0 = V_1$.

$$\begin{aligned} \therefore C_1 &= \frac{Q_1}{V_0} \\ &= \frac{\sigma_1 A}{\frac{\sigma_1}{\kappa_e \varepsilon_0} d} \\ &= \frac{\kappa_e \varepsilon_0 A}{d} \\ &= \kappa_e C_0 \end{aligned}$$

2.

$$\begin{aligned} C_1 &= \frac{Q_1}{V_1} \\ \therefore \kappa_e C_0 &= \frac{Q_1}{V_0} \\ \therefore Q_1 &= \kappa_e C_0 V_0 \\ &= \kappa_e Q_0 \end{aligned}$$

3.

$$\begin{aligned} U_1 &= \frac{1}{2} C_1 {V_1}^2 \\ &= \frac{1}{2} \kappa_e C_0 {V_0}^2 \\ &= \kappa_e U_0 \end{aligned}$$

4.

$$C_2 = \frac{Q_2}{V_2}$$

As the battery is disconnected, the charge cannot flow out and must remain constant. Therefore, $Q_0 = Q_1$.

$$\begin{aligned}
\therefore C_2 &= \frac{Q_0}{V_2} \\
&= \frac{\sigma_2 A}{\frac{\sigma_2}{\kappa_e \varepsilon_0} d} \\
&= \frac{\kappa_e \varepsilon_0 A}{d} \\
&= \kappa_e C_0
\end{aligned}$$

5.

$$\begin{aligned}
C_2 &= \frac{Q_2}{V_2} \\
\therefore \kappa_e C_0 &= \frac{Q_0}{V_2} \\
\therefore V_2 &= \frac{Q_0}{\kappa_e C_0} \\
&= \frac{V_0}{\kappa_e}
\end{aligned}$$

6.

$$\begin{aligned}
U_2 &= \frac{1}{2} C_2 {V_2}^2 \\
&= \frac{1}{2} \kappa_e C_0 \frac{{V_0}^2}{{\kappa_e}^2} \\
&= \frac{U_0}{\kappa_e}
\end{aligned}$$

As $U_1 > U_0$, the process of inserting the dielectric increases the potential. Therefore, the dielectric is repelled by the plates.
As $U_2 < U_0$, the process of inserting the dielectric decreases the potential. Therefore, the dielectric is attracted by the plates.

Part II
Electrodynamics

Recitation 7 – Exercise 2.
The current and current density are related by the relations

$$\begin{aligned}
\overrightarrow{j} &= \rho\overrightarrow{v} \\
\overrightarrow{k} &= \sigma\overrightarrow{v} \\
I &= \lambda v
\end{aligned}$$

1. Calculate the current in a cylinder of radius R carrying uniform charge density ρ moving in velocity v along the cylinder axis.
2. Calculate the current on a segment of length L in an infinite thin plane carrying uniform charge density σ moving in velocity v along the plane surface.
3. Calculate the current in a thin ring of radius R carrying uniform charge distribution λ rotating around its axis with period T.

Recitation 7 – Solution 2.
1.

$$\begin{aligned}
I &= \iint \overrightarrow{j} \cdot \overrightarrow{\mathrm{d}A} \\
&= \rho v A
\end{aligned}$$

2.

$$\begin{aligned}
I &= \int \overrightarrow{k} \cdot \overrightarrow{\mathrm{d}l} \\
&= \sigma v l
\end{aligned}$$

3.

$$\begin{aligned}
I &= \lambda v \\
&= \lambda\omega R \\
&= \lambda \cdot \frac{2\pi}{T} \cdot R \\
&= \frac{2\pi\lambda R}{T}
\end{aligned}$$

Recitation 7 – Exercise 3.

A cylinder of length L and cross-section A is made of metallic material with conductivity $\sigma = \sigma_0 \frac{L}{x}$. The bases of the cylinder are connected to potential difference V.

1. Calculate the resistivity.
2. Calculate the current density in the cylinder.
3. Calculate the electric field inside the metal.

Recitation 7 – Solution 3.

1.

$$\begin{aligned} \sigma &= \sigma_0 \frac{L}{x} \\ \therefore \rho &= \frac{1}{\sigma} \\ &= \frac{x}{\sigma_0 L} \end{aligned}$$

2. The cylinder can be considered to be a resistor made up of elemental disk resistors in series.
   Consider an elemental disk of thickness $\mathrm{d}x$ at a distance x from the origin.
   Therefore, the resistance of the disk is

$$\begin{aligned} \mathrm{d}R &= \frac{\rho \, \mathrm{d}x}{A} \\ &= \frac{x \, \mathrm{d}x}{\sigma_0 L A} \\ \therefore R &= \int_0^L \frac{x \, \mathrm{d}x}{\sigma_0 L A} \\ &= \frac{L^2}{2\sigma_0 L A} \\ &= \frac{L}{2\sigma_0 A} \end{aligned}$$

Therefore,

$$\begin{aligned}
V &= IR \\
&= I\frac{L}{2\sigma_0 A} \\
\therefore I &= \frac{2\sigma_0 AV}{L} \\
\therefore \frac{I}{A} &= \frac{2\sigma_0 V}{L} \\
\therefore j &= \frac{2\sigma_0 V}{L}
\end{aligned}$$

3.

$$\begin{aligned}
j &= \sigma E \\
\therefore \frac{2\sigma_0 V}{L} &= \sigma E \\
\therefore E &= \frac{2Vx}{L^2}
\end{aligned}$$

Recitation 7 – Exercise 5.

A cylinder of length L and radius a ($L >> a$) is made of metallic material with resistivity $\rho = \rho_0 \frac{r}{a}$. The bases of the cylinder are made of ideal conductor and are connected to potential difference V.

1. Calculate the resistance.

2. Calculate the current in the cylinder.

3. Calculate the electric field, $\overrightarrow{E}$, inside the metal.

4. Calculate the current density, $\overrightarrow{j}$, and make sure that $\int \overrightarrow{j} \cdot \overrightarrow{\mathrm{d}s} = I$.

Recitation 7 – Solution 5.

1. The cylinder can be considered to be a resistor made up of elemental cylindrical shell disk resistors in parallel.
   Consider an elemental cylindrical shell of radius r thickness $\mathrm{d}r$.

Therefore, the resistance of the shell is

$$
\begin{aligned}
\mathrm{d}R &= \frac{\rho L}{2\pi r \,\mathrm{d}r} \\
&= \frac{\rho_0 r L}{2\pi a r \,\mathrm{d}r} \\
&= \frac{\rho_0 L}{2\pi a \,\mathrm{d}r} \\
\therefore \frac{1}{R} &= \int \frac{1}{\mathrm{d}R} \\
&= \int\limits_0^a \frac{2\pi a \,\mathrm{d}r}{\rho_0 L} \\
&= \frac{2\pi a^2}{\rho_0 L} \\
\therefore R &= \frac{\rho_0 L}{2\pi a^2}
\end{aligned}
$$

2.

$$
\begin{aligned}
V &= IR \\
\therefore V &= I \frac{\rho_0 L}{2\pi a^2} \\
\therefore I &= \frac{2\pi a^2 V}{\rho_0 L}
\end{aligned}
$$

3.

$$
\begin{aligned}
V &= EL \\
\therefore E &= \frac{V}{L}
\end{aligned}
$$

4.

$$
\begin{aligned}
j &= \frac{I}{A} \\
&= \frac{\frac{2\pi a^2 V}{\rho_0 L}}{\pi a^2} \\
&= \frac{2V}{\rho_0 L}
\end{aligned}
$$

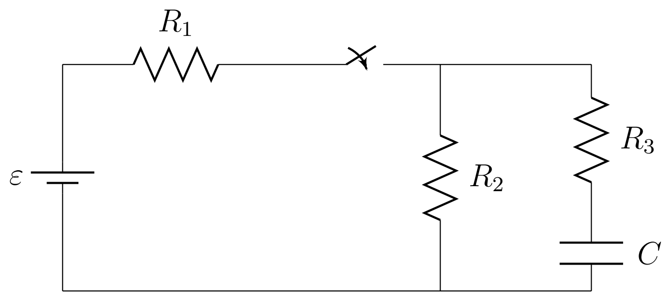
Therefore,

$$\begin{aligned}\int \vec{j} \cdot \vec{\mathrm{d}s} &= \int_0^a \frac{2V}{\rho_0 L} \cdot 2\pi r \, \mathrm{d}r \\ &= \frac{2\pi a^2 V}{\rho_0 L} \\ &= I\end{aligned}$$

Therefore, $\int \vec{j} \cdot \vec{\mathrm{d}s} = I$.

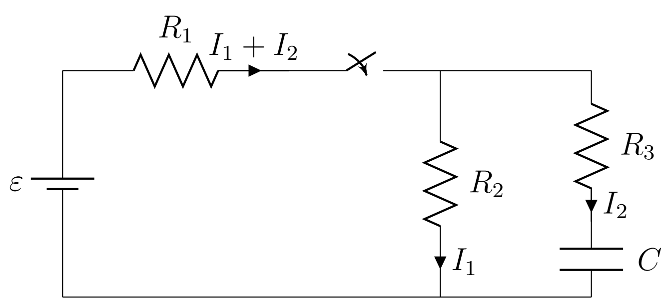
Recitation 8 – Exercise 2.

A circuit is comprised of a voltage source (ε), three identical resistors ($R_1 = R_2 = R_3 = R$), a switch (S), and an empty capacitor (C). At $t = 0$ the switch is closed and current starts to flow.



1. Calculate the current flowing in each of the resistors after a very short time ($t \to 0^+$).
2. Calculate the current flowing in each of the resistors after a very long time ($t \to \infty$).
3. What is short/long time in this case?

Recitation 8 – Solution 2.



By KVL on the left loop,

$$\varepsilon - (I_1 + I_2)R - I_1 R = 0$$

By KVL on the larger loop,

$$\varepsilon - (I_1 + I_2)R - I_2 R - \frac{Q}{C} = 0$$

Therefore, solving,

$$\begin{aligned}
3I_2R + \frac{2Q}{C} &= \varepsilon \\
\therefore 3R\,\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{2Q}{C} &= \varepsilon \\
\therefore \frac{\mathrm{d}Q}{\mathrm{d}t} &= \left(\frac{1}{3R}\right)\left(\varepsilon - \frac{2Q}{C}\right) \\
\therefore \frac{\mathrm{d}Q}{Q - \frac{C\varepsilon}{2}} &= -\frac{1}{\frac{3RC}{2}}\,\mathrm{d}t
\end{aligned}$$

Let $\tau = \frac{3RC}{2}$. Therefore,

$$\begin{aligned}
\therefore \frac{\mathrm{d}Q}{Q - \frac{C\varepsilon}{2}} &= -\frac{1}{\tau}\,\mathrm{d}t \\
\therefore \int\limits_{-\frac{C\varepsilon}{2}}^{Q-\frac{C\varepsilon}{2}} \frac{\mathrm{d}Q}{Q - \frac{C\varepsilon}{2}} &= \int\limits_{0}^{t} -\frac{1}{\tau}\,\mathrm{d}t \\
\therefore Q &= \frac{C\varepsilon}{2}\left(1 - e^{-\frac{t}{\tau}}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_2(t) &= \frac{\mathrm{d}Q}{\mathrm{d}t} \\
&= \frac{C\varepsilon}{2}\left(\frac{1}{\tau}\right)e^{-\frac{t}{\tau}} \\
&= \frac{\varepsilon}{3R}e^{-\frac{t}{\tau}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_1 &= \frac{\varepsilon - I_2R}{2R} \\
&= \frac{\varepsilon}{2R} - \frac{I_2}{2}
\end{aligned}$$

1. If $t \to 0^+$,

$$\begin{aligned}
I_2 &= \lim_{t \to 0^+} \frac{\varepsilon}{3R} e^{-\frac{t}{\tau}} \\
&= \frac{\varepsilon}{3R} e^0 \\
&= \frac{\varepsilon}{3R}
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_1 &= \frac{\varepsilon}{2R} - \frac{I_2}{2} \\
&= \frac{\varepsilon}{2R} - \frac{\frac{\varepsilon}{3R}}{2} \\
&= \frac{\varepsilon}{2R} - \frac{\varepsilon}{6R} \\
&= \frac{\varepsilon}{3R}
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_0 &= I_1 + I_2 \\
&= \frac{\varepsilon}{3R} + \frac{\varepsilon}{3R} \\
&= \frac{2\varepsilon}{3R}
\end{aligned}$$

2. If $t \to \infty$,

$$\begin{aligned}
I_2 &= \lim_{t \to \infty} \frac{\varepsilon}{3R} e^{-\frac{t}{\tau}} \\
&= \frac{\varepsilon}{3R} \cdot 0 \\
&= 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_1 &= \frac{\varepsilon}{2R} - \frac{I_2}{2} \\
&= \frac{\varepsilon}{2R} - 0 \\
&= \frac{\varepsilon}{2R}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
I_0 &= I_1 + I_2 \\
&= \frac{\varepsilon}{2R} + 0 \\
&= \frac{\varepsilon}{2R}
\end{aligned}
$$

3. These short and long times are in comparison to the time constant of the circuit, i.e. $\tau = \frac{3RC}{2}$.

Part III
Magnetism

1 Lorentz Force

Recitation 8 – Exercise 4.
A particle of mass m, charge q and velocity $\overrightarrow{v}$ is entering a region of a constant uniform magnetic field $\overrightarrow{B}$. The angle between the particle's velocity and the magnetic field is θ.

1. Describe the trajectory of the particle.
2. Calculate the cyclotron radius R.
3. Calculate the cyclotron frequency ω and period T.
4. Calculate the longitudinal distance the particle traverses in one time period.

Recitation 8 – Solution 4.

1. The magnetic force acting on the particle is

$$\overrightarrow{F} = q\overrightarrow{v} \times \overrightarrow{B}$$

Therefore, the force is always perpendicular to the velocity.
Therefore, it changes the direction of the component of the velocity perpendicular to the magnetic field, but not its magnitude.
However the component of the velocity in the direction of $\overrightarrow{B}$ is unaffected by the force.
Therefore, the particle goes in a circle, due to the component of the velocity perpendicular to $\overrightarrow{B}$ and in a straight line due to the component of the velocity parallel to $\overrightarrow{B}$. Hence, it goes in a helical path.

2. Let the direction of $\overrightarrow{B}$ be $\hat{k}$.

$$\begin{aligned}
\overrightarrow{F} &= q\overrightarrow{v} \times \overrightarrow{B} \\
\therefore \overrightarrow{F} &= qv\left(\hat{j}\sin\theta + \hat{k}\cos\theta\right) \times \hat{k}B \\
&= qvB\sin\theta\hat{i}
\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{mv^2\sin^2\theta}{R} &= qvB\sin\theta \\ \therefore R &= \frac{mv\sin\theta}{qB}\end{aligned}$$

3.

$$\begin{aligned}\omega &= \frac{v\sin\theta}{R} \\ &= \frac{v\sin\theta}{\frac{mv\sin\theta}{qB}} \\ &= \frac{qB}{m}\end{aligned}$$

Therefore,

$$\begin{aligned}T = \frac{2\pi}{\omega} \\ &= \frac{2\pi}{\frac{qB}{m}} \\ &= \frac{2\pi m}{qB}\end{aligned}$$

4. The component of the velocity in the direction of $\overrightarrow{B}$ is responsible for the longitudinal movement of the particle.
   Therefore, the distance traversed by the particle in one time period is,

$$\begin{aligned}d &= v\cos\theta\cdot T \\ &= v\cos\theta\frac{2\pi m}{qB} \\ &= \frac{2\pi mv\cos\theta}{qB}\end{aligned}$$

2 Biot-Savart Law

Recitation 8 – Exercise 6.
Given an arc of radius R and angle α, the arc carries current I. Calculate the magnetic field along the arc's axis.

Recitation 8 – Solution 6.

$$\mathrm{d}l = R\,\mathrm{d}\alpha$$

Let $\overrightarrow{r}$ be the vector joining $\mathrm{d}l$ and a point at a distance z on the axis. Therefore,

$$\overrightarrow{\mathrm{d}B} = \frac{\mu_0}{4\pi}\frac{I\overrightarrow{\mathrm{d}l} \times \hat{r}}{r^2}$$

Therefore, $\overrightarrow{B}$ must be perpendicular to both $\hat{r}$ and $\overrightarrow{\mathrm{d}l}$.

$$\begin{aligned}\mathrm{d}B_z &= \frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}l}{r^2}\cos\theta \\ \therefore B_z &= \frac{\mu_0 I}{4\pi}\int_0^{\alpha}\frac{R\,\mathrm{d}\alpha}{r^2}\cos\theta \\ &= \frac{\mu_0}{4\pi}\frac{IR^2\cdot\alpha}{(z^2+R^2)^{\frac{3}{2}}}\end{aligned}$$

Recitation 9 – Exercise 1.
The electric potential is given in cylindrical coordinates by

$$V(r) = \begin{cases} A(a^2-r^2) & ; \quad r<a \\ -Aa^2\ln\frac{r}{a}+C & ; \quad a<r<b \\ -(Aa^2+B)\ln\frac{r}{b}+D & ; \quad b<r \end{cases}$$

where r is the distance from the z axis. The constants $A \neq 0$ and $B \neq 0$ are also given.

1. Calculate the electric field everywhere.
2. Find the charge distributions that create this field.
3. Find the surface charge density at $r = a$ and at $r = 2b$.
4. Write an expression for C and D as functions of the given parameters a, b, A and B.

Recitation 9 – Solution 1.

1. If $r < a$,

$$\begin{aligned} E &= -\frac{\partial V}{\partial r}\,\hat{r} \\ &= -\frac{\partial}{\partial r}\left(A(a^2 - r^2)\right)\hat{r} \\ &= 2Ar\hat{r} \end{aligned}$$

If $a < r < b$,

$$\begin{aligned} E &= -\frac{\partial V}{\partial r}\,\hat{r} \\ &= -\frac{\partial}{\partial r}\left(-Aa^2 \ln\frac{r}{a} + C\right)\hat{r} \\ &= \frac{Aa^2}{r}\hat{r} \end{aligned}$$

If $b < r$,

$$\begin{aligned} E &= -\frac{\partial V}{\partial r}\,\hat{r} \\ &= -\frac{\partial}{\partial r}\left(- - (Aa^2 + B)\ln\frac{r}{b} + D\right)\hat{r} \\ &= \frac{Aa^2 + B}{r}\hat{r} \end{aligned}$$

2.

$$\begin{aligned} \overrightarrow{\nabla} \cdot \overrightarrow{E} &= \frac{\rho}{\varepsilon_0} \\ \therefore \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_z}{\partial z} &= \frac{\rho}{\varepsilon_0} \end{aligned}$$

Therefore, if $r < a$,

$$\begin{aligned} \overrightarrow{\nabla} \cdot \overrightarrow{E} &= \frac{1}{r}\frac{\partial}{\partial r}(rE) \\ &= \frac{1}{r}\frac{\partial}{\partial r}(r \cdot 2Ar) \\ \therefore \frac{\rho}{\varepsilon_0} &= 4A \\ \therefore \rho &= 4A\varepsilon_0 \end{aligned}$$

Therefore, if $a < r < b$,

$$\begin{aligned}
\overrightarrow{\nabla} \cdot \overrightarrow{E} &= \frac{1}{r}\frac{\partial}{\partial r}(rE) \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{Aa^2}{r}\right) \\
\therefore \frac{\rho}{\varepsilon_0} &= 0 \\
\therefore \rho &= 0
\end{aligned}$$

Therefore, if $b < r$,

$$\begin{aligned}
\overrightarrow{\nabla} \cdot \overrightarrow{E} &= \frac{1}{r}\frac{\partial}{\partial r}(rE) \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{Aa^2 + B}{r}\right) \\
\therefore \frac{\rho}{\varepsilon_0} &= 0 \\
\therefore \rho &= 0
\end{aligned}$$

3. Consider a cylindrical Gaussian surface with radius r and height H.
   Therefore, by Gauss' Law,

$$\begin{aligned}
\int \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{S} &= \frac{Q}{\varepsilon_0} \\
&= \int (2Ar)\,\mathrm{d}S \\
&= (2Ar)(2\pi rH)
\end{aligned}$$

   Therefore,

$$\begin{aligned}
\sigma(r = a) &= \frac{Q}{2\pi aH} \\
\sigma(r = b) &= \frac{Q}{2\pi bH}
\end{aligned}$$

4. The potential must be continuous at $r = a$ and $r = b$.
   Therefore, comparing the expressions of the potential,

$$\begin{aligned}
C &= 0 \\
D &= Aa^2 \ln\frac{a}{b}
\end{aligned}$$

3 Magnetic Dipole Moment

Recitation 9 – Exercise 3.
A thin disk of radius R carries surface charge density σ. The disk is rotating around its axis at angular frequency ω in the x-y plane.

1. Calculate the disk's magnetic dipole moment.
2. Calculate the magnetic field in the center of the disk.

Recitation 9 – Solution 3.

1. Consider an elemental ring of radius r and thickness dr. Therefore, the current due to the rotating elemental ring is

$$\begin{aligned} \mathrm{d}I &= \frac{\mathrm{d}q}{2\pi} \cdot \omega \\ &= \frac{\omega}{2\pi}\,\mathrm{d}q \\ \therefore I &= \int \frac{\omega}{2\pi}\,\mathrm{d}q \\ &= \frac{Q\omega}{2\pi} \end{aligned}$$

The magnetic dipole moment of the disk is

$$\begin{aligned} \vec{\mu} &= I\vec{A} \\ \therefore \mu &= \frac{Q\omega}{2\pi} A \\ &= \frac{Q\omega}{2\pi} \cdot \pi R^2 \\ &= \frac{Q\omega R^2}{2} \end{aligned}$$

The magnetic dipole moment is directed perpendicular to the disk.

2. The magnetic field due to the moving elemental ring is

$$\begin{aligned} \mathrm{d}B &= \frac{\mu_0\,\mathrm{d}I}{2r} \\ \therefore B &= \frac{\mu_0 \omega \sigma}{2} \int_0^R \mathrm{d}r \\ &= \frac{\mu_0 \omega \sigma R}{2} \end{aligned}$$

4 Ampere's Law

Recitation 9 – Exercise 4.
A hollow cylinder of inner radius a and outer radius b carries uniform current density âČŮ $\overrightarrow{j}$ parallel to the cylinder's axis. Calculate the magnetic field $\overrightarrow{B}$ everywhere.

Recitation 9 – Solution 4.
Consider a closed, virtual Ampere loop of radius r.
Therefore, by Ampere's law,

$$\int \overrightarrow{B} \cdot \mathrm{d}\overrightarrow{l} = \mu_0 I_{\text{enclosed}}$$

If $b < r$,

$$\begin{aligned}
\int \overrightarrow{B} \cdot \mathrm{d}\overrightarrow{l} &= \mu_0 I_{\text{enclosed}} \\
\therefore B \cdot 2\pi r &= \mu_0 j \pi (b^2 - a^2) \\
\therefore B &= \frac{\mu_0 j (b^2 - a^2)\pi}{2\pi r} \\
&= \frac{\mu_0 j (b^2 - a^2)}{2r}
\end{aligned}$$

If $a < r < b$,

$$\begin{aligned}
\int \overrightarrow{B} \cdot \mathrm{d}\overrightarrow{l} &= \mu_0 I_{\text{enclosed}} \\
\therefore B \cdot 2\pi r &= \mu_0 j \pi (r^2 - a^2) \\
\therefore B &= \frac{\mu_0 j (r^2 - a^2)\pi}{2\pi r} \\
&= \frac{\mu_0 j (r^2 - a^2)}{2r}
\end{aligned}$$

If $r < a$,

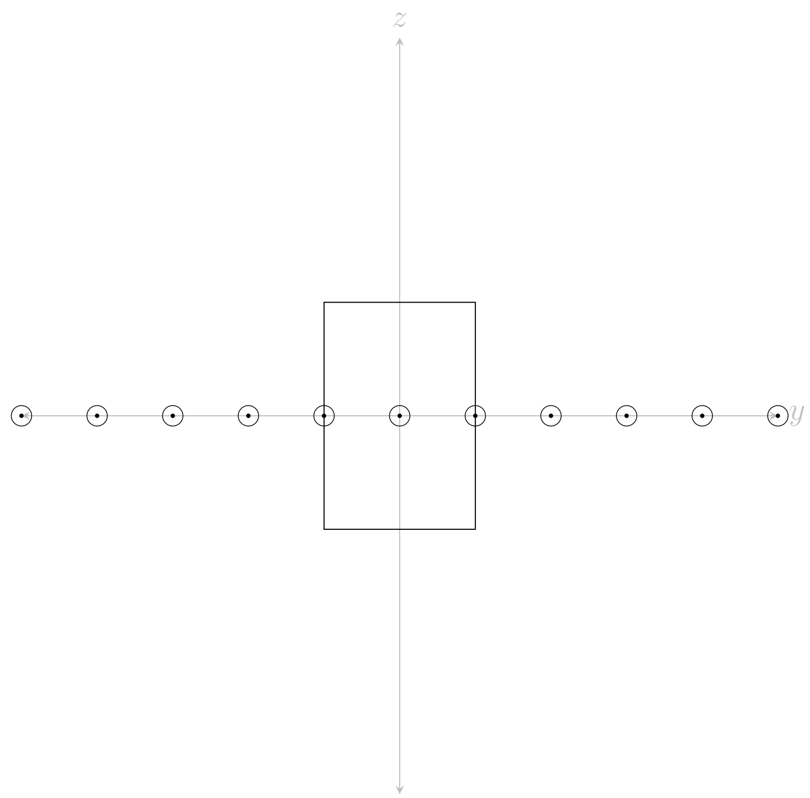
$$\begin{aligned}
\int \overrightarrow{B} \cdot \mathrm{d}\overrightarrow{l} &= \mu_0 I_{\text{enclosed}} \\
\therefore B \cdot 2\pi r &= 0 \\
\therefore B &= 0
\end{aligned}$$

Recitation 9 – Exercise 6.

A uniform surface charge density σ is spread on the entire x-y plane. This charge density is moving at constant velocity $\overrightarrow{v} = v\hat{x}$. Calculate the magnetic field everywhere.

Recitation 9 – Solution 6.

Consider a square virtual Ampere loop, directed anti-clockwise, as shown.



Let

$$\overrightarrow{k} = \sigma \overrightarrow{v}$$
$$\therefore I = kl$$

Therefore by Ampere's Law,

$$\oint \overrightarrow{B} \cdot \mathrm{d}\overrightarrow{l} = \mu_0 I_{\text{enclosed}}$$
$$\therefore 2|B|l = \mu_0 kl$$
$$\therefore |B| = \frac{\mu_0 k}{2}$$

Therefore,

$$\overrightarrow{B} = \begin{cases} -\frac{\mu_0 k}{2}\hat{y} & ; \quad z > 0 \\ \frac{\mu_0 k}{2}\hat{y} & ; \quad z < 0 \end{cases}$$
$$= \begin{cases} -\frac{\mu_0 \sigma v}{2}\hat{y} & ; \quad z > 0 \\ \frac{\mu_0 \sigma v}{2}\hat{y} & ; \quad z < 0 \end{cases}$$

Recitation 10 – Exercise 3.
A parallel plate capacitor is made of two circular disks of radius a with distance d between them. The capacitor carries charge $\pm Q_0$ on its plates. At $t = 0$, the capacitor is connected to a resistor R.

1. Find the time dependent current $I(t)$, in the circuit.
2. Find the surface current density $\overrightarrow{k}(r,t)$, on the plates of the capacitor.

The capacitor is charged again, with charge $\pm Q_0$ on its plates. At $t = 0$, the capacitor is filled with a material of resistivity ρ_0.

3. Find the time dependent current $I(t)$, in the circuit.
4. Find the surface current density $\overrightarrow{k}(r,t)$, on the plates of the capacitor.

Recitation 10 – Solution 3.
1. As the capacitor is being discharged,

$$I = \frac{Q_0}{RC}e^{-\frac{t}{RC}}$$

2. The area of the plates of the capacitor is

$$A = \pi a^2$$

The charge on an annular area with inner radius r and outer radius a is

$$q = \left(\frac{Q}{\pi a^2}\right)\left(\pi a^2 - \pi r^2\right)$$
$$= Q\left(1 - \frac{r^2}{a^2}\right)$$

Therefore, the current through that annular area is

$$\begin{aligned}
I &= \frac{\mathrm{d}q}{\mathrm{d}t} \\
\therefore k \cdot 2\pi r &= \frac{\mathrm{d}q}{\mathrm{d}t} \\
\therefore 2\pi r k &= \frac{\mathrm{d}Q}{\mathrm{d}t}\left(1 - \frac{r^2}{a^2}\right) \\
\therefore k &= \frac{I}{2\pi r}\left(1 - \frac{r^2}{a^2}\right)
\end{aligned}$$

3. The capacitor filled with the resisting material is equivalent to the capacitor connected to an equivalent resistor.
   The resistance due to the resisting material is

$$\begin{aligned}
R &= \rho_0 \frac{d}{A} \\
&= \frac{\rho_0 d}{\pi a^2}
\end{aligned}$$

   Therefore,

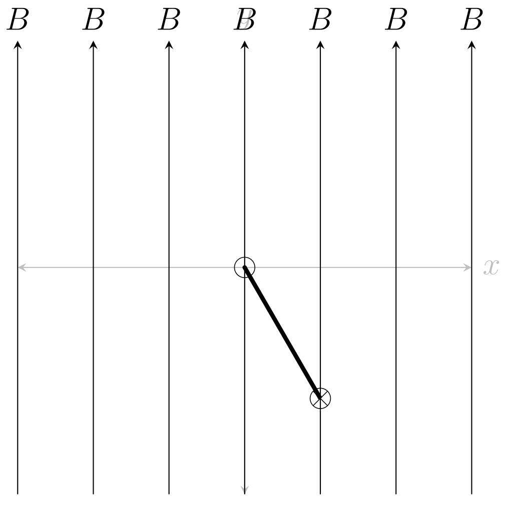
$$\begin{aligned}
I &= \frac{Q_0}{RC} e^{-\frac{t}{RC}} \\
&= \frac{Q_0}{\frac{\rho_0 d}{\pi a^2} C} e^{-\frac{t}{\frac{\rho_0 d}{\pi a^2} C}} \\
&= \frac{Q_0 \pi a^2}{\rho_0 d C} e^{\frac{\pi a^2 t}{\rho_0 d C}}
\end{aligned}$$

4.

$$k = \frac{I}{2\pi r}\left(1 - \frac{r^2}{a^2}\right)$$

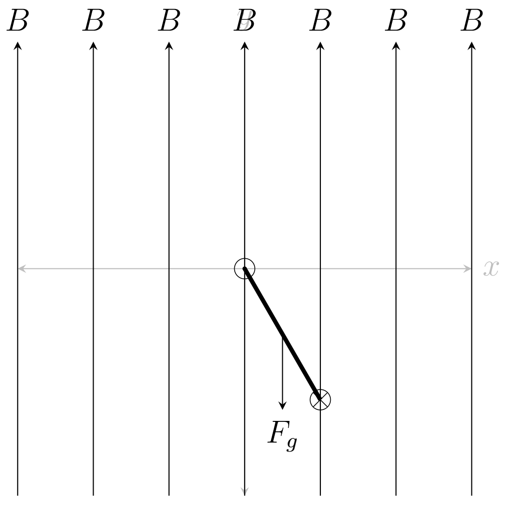
Recitation 10 – Exercise 4.

A square loop of sides a and mass m, carrying current I, is hung on the z axis with frictionless hinges as shown.



The gravitational force is in the $-\hat{y}$ direction. Calculate the magnetic field $\overrightarrow{B} = B_y\hat{y}$ needed in order to maintain the square loop at angle θ with the y-z plane.

Recitation 10 – Solution 4.



$$\begin{aligned}
\overrightarrow{\tau_g} &= \overrightarrow{r} \times \overrightarrow{F_g} \\
&= \left(\frac{a}{2}\right)\left(\hat{\imath}\sin\theta - \hat{\jmath}\cos\theta\right) \times (mg)\left(-\hat{\jmath}\right) \\
&= \frac{a}{2} \cdot mg\sin\theta\left(-\hat{k}\right)
\end{aligned}$$

$$
\begin{aligned}
\overrightarrow{\tau_B} &= \overrightarrow{r} \times \overrightarrow{F_B} \\
&= a\left(\hat{\imath}\sin\theta - \hat{\jmath}\cos\theta\right)\left(IaB_0\hat{\imath}\right) \\
&= Ia^2B\cos\theta\hat{k}
\end{aligned}
$$

For the rod to be at equilibrium,

$$
\begin{aligned}
\tau_g &= \tau_B \\
\therefore B &= \frac{mg}{2Ia}\tan\theta
\end{aligned}
$$

Recitation 11 – Exercise 1.
A hollow cylinder of inner radius a and outer radius b is given. The following magnetic field is measured

$$
B = \begin{cases} 0\hat{\theta} & ; \quad r < a \\ C\left(r - \frac{a^2}{r}\right)\hat{\theta} & ; \quad a < r < b \\ \frac{A}{r}\hat{\theta} & ; \quad b < r \end{cases}
$$

where r is the distance from the z axis. The constants A and C are also given.
Calculate the current density $\overrightarrow{j}$ everywhere.

Recitation 11 – Solution 1.
By the differential form of Ampere's Law,

$$
\overrightarrow{\nabla} \times \overrightarrow{B} = \mu_0\overrightarrow{j}
$$

In cylindrical coordinates,

$$
\begin{aligned}
\overrightarrow{\nabla} \times \overrightarrow{B} = \quad & \left(\frac{1}{r}\frac{\partial B_z}{\partial\theta} - \frac{\partial B_\theta}{\partial z}\right)\hat{r} \\
& + \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right)\hat{\theta} \\
& + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rB_\theta\right) - \frac{\partial B_r}{\partial\theta}\right)\hat{z}
\end{aligned}
$$

Therefore,

If $r < a$,

$$\begin{aligned}
\overrightarrow{\nabla} \times \overrightarrow{B} &= 0 \\
\therefore \mu_0 \overrightarrow{j} &= 0 \\
\therefore \overrightarrow{j} &= 0
\end{aligned}$$

If $a < r < b$,

$$\begin{aligned}
\overrightarrow{\nabla} \times \overrightarrow{B} &= \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)\hat{z} \\
\therefore \mu_0 \hat{j} &= \frac{c}{r}\frac{\partial}{\partial r}\left(r^2 - a^2\right)\hat{z} \\
&= \frac{2Cr}{r}\hat{z} \\
&= 2C\hat{z} \\
\therefore \overrightarrow{j} &= \frac{2C}{\mu_0}
\end{aligned}$$

If $b < r$,

$$\begin{aligned}
\overrightarrow{\nabla} \times \overrightarrow{B} &= \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)\hat{z} \\
\therefore \mu_0 \hat{j} &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{A}{r}\right)\hat{z} \\
&= 0 \\
\therefore \hat{j} &= 0
\end{aligned}$$

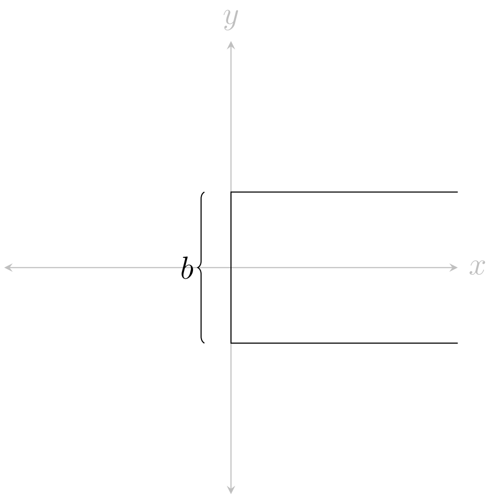
5 Faraday's Law

Recitation 11 – Exercise 3.

A long rectangular wire loop of side b is made of thin ideal conductor. The far end of the rectangular loop is not connected. A rod of mass m and resistance R is placed on the loop, and can slide on it without friction. A uniform, constant magnetic field $\overrightarrow{B} = B_0\hat{z}$, perpendicular to the plane of the loop, is given. The velocity of the rod when it is placed at $x = 0$, the left end of the loop, is $\overrightarrow{v} = v_0\hat{x}$.

Calculate the emf ε, the current I induced in the rectangular wire, the velocity $\overrightarrow{v}$, and the force $\overrightarrow{F}$ acting on the rod as a function of x.

Recitation 11 – Solution 3.



$$
\begin{aligned}
\Phi_B &= \int \overrightarrow{V} \cdot \mathrm{d}\overrightarrow{A} \\
&= BA \\
&= Bbl
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} &= Bb\,\frac{\mathrm{d}l}{\mathrm{d}t} \\
&= B_0 bv
\end{aligned}
$$

Therefore, by Faraday's Law,

$$
\begin{aligned}
\varepsilon &= -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} \\
&= -B_0 bv
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\varepsilon &= IR \\
\therefore I &= \frac{B_0 bv}{R}
\end{aligned}
$$

As the magnetic flux is increasing in the $\hat{z}$ direction, i.e. inwards, by Lenz's Law, the current is directed clockwise.

$$\begin{aligned}
\overrightarrow{F} &= -Ib\hat{j} \times B_0\hat{k} \\
&= -IbB_0\hat{i} \\
\therefore m\frac{\mathrm{d}v}{\mathrm{d}t} &= -IbB_0 \\
\therefore \frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{IbB_0}{m} \\
&= -\frac{{B_0}^2b^2v}{mR} \\
\therefore \frac{\mathrm{d}v}{v} &= -\frac{{B_0}^2b^2}{mR} \\
\therefore \int\limits_{v_0}^{v} \frac{\mathrm{d}v}{v} &= \int\limits_{0}^{t} -\frac{{B_0}^2b^2}{mR} \\
\therefore \ln\left(\frac{v}{v_0}\right) &= -\frac{{B_0}^2b^2}{mR}t \\
\therefore v &= v_0e^{-\frac{{B_0}^2b^2}{mR}t}
\end{aligned}$$

Recitation 11 – Exercise 4.
A metallic rod of length R is rotating in the x-y plane, around one of its ends, at angular velocity ω in an area with constant magnetic field $\overrightarrow{B} = B_0\hat{x}$. Calculate the integral $\int \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}l}$ between the edges of the reod in two ways.

1. Using integration over the Lorentz force acting on the charges in the rod.

2. Using Faraday's Law.

Recitation 11 – Solution 4.
1.

$$\begin{aligned}
q\overrightarrow{v} \times \overrightarrow{B} &= q\overrightarrow{E} \\
\therefore qvB &= qE \\
\therefore B_0q\omega r &= qE \\
\therefore E &= B_0\omega r
\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^a \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}r} &= \int_0^a B_0 \omega r \,\mathrm{d}r \\ &= \frac{B_0 \omega a^2}{2}\end{aligned}$$

2. The area swept by the rod is,

$$\begin{aligned}\mathrm{d}A &= \int_0^a r\theta \,\mathrm{d}r \\ &= \frac{a^2\theta}{2}\end{aligned}$$

By Faraday's Law,

$$\begin{aligned}\varepsilon &= -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\left(\overrightarrow{B} \cdot \overrightarrow{A}\right) \\ &= -B_0 \frac{\mathrm{d}A}{\mathrm{d}t} \\ &= -B_0 \omega \frac{a^2}{2}\end{aligned}$$

Therefore,

$$\int \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}l} = \frac{B_0 \omega a^2}{2}$$

Recitation 12 – Exercise 2.

The magetic field inside an infinite long solenoid of radius a is $\overrightarrow{B} = B_z(t)\hat{z}$.

1. Calculate the electric field everywhere.
2. Show that the differential and the integral forms of Faraday's Law give the same result.

Recitation 12 – Solution 2.

$$\begin{aligned} B &= B_z(t) \\ &= \mu_0 n I(t) \end{aligned}$$

Consider a virtual Amperian loop of radius r, coaxial to the solenoid. Therefore, the magnetic flux through it is,

$$\Phi_B = \begin{cases} \mu_0 \pi r^2 n I(t) & ; \quad r < a \\ \mu_0 \pi a^2 n I(t) & ; \quad r > a \end{cases}$$

Therefore,

$$\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = \begin{cases} \mu_0 \pi r^2 n \dot{I}(t) & ; \quad r < a \\ \mu_0 \pi a^2 n \dot{I}(t) & ; \quad r > a \end{cases}$$

Therefore, by the integral form of Faraday's Law,

$$\begin{aligned} \oint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}l} &= -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} \\ \therefore E \cdot 2\pi r &= \begin{cases} -\mu_0 \pi r^2 n \dot{I}(t) & ; \quad r < a \\ -\mu_0 \pi a^2 n \dot{I}(t) & ; \quad r > a \end{cases} \\ \therefore E &= \begin{cases} -\frac{\mu_0 r n}{2} \dot{I}(t) & ; \quad r < a \\ -\frac{\mu_0 a^2 n}{2r} \dot{I}(t) & ; \quad r > a \end{cases} \end{aligned}$$

By the differential form of Faraday's Law,

$$\begin{aligned} \overrightarrow{\nabla} \times \overrightarrow{E} &= -\frac{\partial \overrightarrow{B}}{\partial t} \\ \therefore \left(\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} (r E_\theta) \hat{k} - \cancelto{0}{\frac{\partial E_r}{\partial \theta}} \right) \hat{k} &= -\mu_0 n \frac{\mathrm{d}I}{\mathrm{d}t} \hat{k} \\ \therefore \frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) &= -\mu_0 n \dot{I}(t) \\ \therefore \partial (r E_\theta) &= -\mu_0 n \dot{I}(t) r \partial r \end{aligned}$$

Therefore,

$$\begin{aligned} r E_\theta &= \begin{cases} -\mu_0 n \dot{I} \frac{r^2}{2} & ; \quad r < a \\ -\mu_0 n \dot{I} \frac{a^2}{2} & ; \quad r > a \end{cases} \\ \therefore E &= \begin{cases} -\frac{\mu_0 r n}{2} \dot{I}(t) & ; \quad r < a \\ -\frac{\mu_0 a^2 n}{2r} \dot{I}(t) & ; \quad r > a \end{cases} \end{aligned}$$

Recitation 12 – Exercise 5.
A short solenoid (length l and radius a, with n_1 turns per unit length) lies on the axis of a very long solenoid (radius $b > a$, n_2 turns per unit length). A current I_1 flows in the short solenoid. What is the flux through the long solenoid?

Recitation 12 – Solution 5.
If the current in the larger solenoid is I_2 and the current in the smaller solenoid is 0,

$$B = \mu_0 n_2 I_2$$

Therefore, the magnetic flux through the short solenoid is,

$$\begin{aligned}\Phi_{B1} &= n_1 l B A \\ &= n_1 l \cdot \mu_0 n_2 I_2 \cdot \pi a^2\end{aligned}$$

As $M = \frac{\Phi_{B1}}{I_2} = \frac{\Phi_{B2}}{I_1}$,

$$\begin{aligned}\frac{\Phi_{B2}}{I_1} &= \frac{\Phi_{B1}}{I_2} \\ \therefore \frac{\Phi_{B2}}{I_1} &= \frac{n_1 l \cdot \mu_0 n_2 I_2 \cdot \pi a^2}{I_2} \\ &= \mu_0 \pi n_1 n_2 l a^2 \\ \therefore \Phi_{B2} &= \mu_0 \pi n_1 n_2 l a^2 I_1\end{aligned}$$

Therefore, the flux through the long solenoid is $\mu_0 \pi n_1 n_2 l a^2 I_1$.