Physics 2

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Contents

1 Lecturer Information

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2 Textbooks

1. D. Halliday, R. Resnick, and K. S. Krane: *Physics*, 5th edition, vol. 2 (Wiley)
2. D.J. Griffiths: *Introduction to Electrodynamics*

Part I
Electrostatics

1 Coulomb's Law

Law 1 (Coulomb's Law)**.** *The force between two charged particles is directly proportional to the product of the charges of the particles, and inversely proportional to the square of the distance between them.*

$$F \propto \frac{q_1 q_2}{r^2}$$
$$F = k\frac{q_1 q_2}{r^2}$$
$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r^2}$$

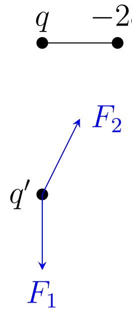
The constant of proportionality is $k = 8.99 \times 10^9 \mathrm{N\,m^2\,C^{-2}}$.
$\varepsilon_0 = 8.8541878162 \times 10^{-12} \mathrm{C^2\,N^{-1}\,m^{-2}}$ *is called the permittivity of free space.*
In vector notation,

$$\overrightarrow{F_{21}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{{r_{12}}^2}\hat{r_{12}}$$

Charge is defined according to this law.

Exercise 1.
A charge q is placed at the origin. A charge $-2q$ is placed at 1 m from it, in the x direction. Find a point on the y-axis where the total force acting on a charge q' will be parallel to the x-axis.

Solution 1.

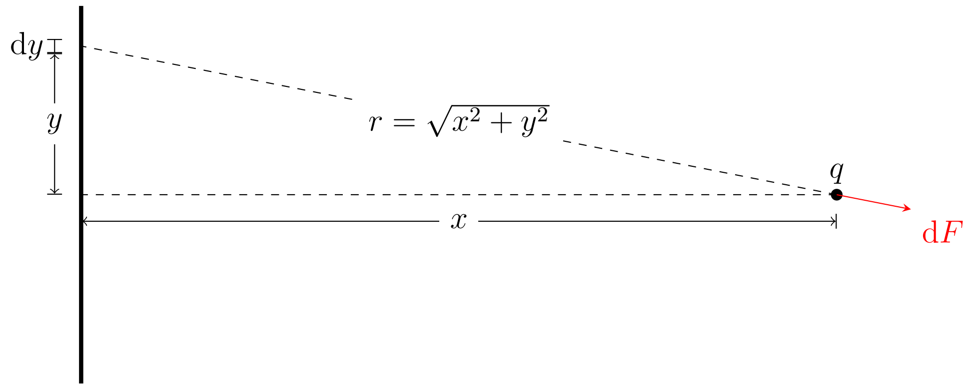


For the net force to be in the x direction, the components of F_1 and F_2 in the y direction must cancel each other out.

$$\begin{aligned}
F_1 &= F_2 \sin\theta \\
\therefore \cancel{k} \cdot \frac{(\cancel{q})(-2\cancel{q})}{y^2+1} \cdot \frac{y}{\sqrt{y^2+1}} &= \cancel{k} \cdot \frac{(\cancel{q})(\cancel{q})}{y^2} \\
\therefore \frac{-2y}{(y^2+1)^{3/2}} &= \frac{1}{y^2} \\
\therefore y &= \pm\sqrt{\frac{1}{2^{2/3}-1}}
\end{aligned}$$

Exercise 2.
A rod of length L has a uniformly distributed charge Q, with line charge density $\lambda = \dfrac{Q}{L}$. A point charge q is kept at a distance x as shown.

Solution 2.



The y components of the forces of the elemental charges at y and $-y$ on q are cancelled out. Therefore, the net force is in the x direction only.

$$
\begin{aligned}
\mathrm{d}F &= k\frac{(\mathrm{d}Q)(q)}{r^2} \\
\mathrm{d}F_x &= \mathrm{d}F\cos\theta \\
&= k\frac{(\mathrm{d}Q)(q)}{r^2}\cos\theta \\
&= k\frac{(\lambda\,\mathrm{d}y)(q)}{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}} \\
&= k\lambda qx\frac{\mathrm{d}y}{(x^2+y^2)^{3/2}} \\
\therefore\ \overrightarrow{F} &= \hat{x}\int \mathrm{d}F_x \\
&= \hat{k}\lambda qx\int_{-L/2}^{L/2}\frac{\mathrm{d}y}{(x^2+y^2)^{3/2}}
\end{aligned}
$$

Substituting $y = x\tan\theta$ and $\mathrm{d}y = x\sec^2\theta\,\mathrm{d}\theta$

$$
\begin{aligned}
\overrightarrow{F} &= \hat{x}\lambda qkx\int_{-\theta_0}^{\theta_0}\frac{1}{x^2}\cos\theta\,\mathrm{d}\theta \\
&= \hat{x}\frac{\lambda qk}{x}\int_{-\theta_0}^{\theta_0}\cos\theta\,\mathrm{d}\theta
\end{aligned}
$$

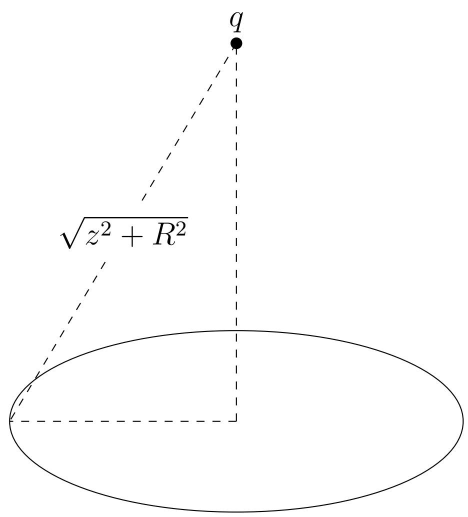
Therefore,

$$\begin{aligned}
\overrightarrow{F} &= \hat{x}\frac{2\lambda qk}{x}\sin\theta_0 \\
&= \hat{x}\frac{2\lambda qk}{x}\frac{\frac{L}{2}}{\left(\left(\frac{L}{2}\right)^2+x^2\right)^{1/2}} \\
&= \hat{x}\frac{2\left(\frac{Q}{L}\right)qk}{x}\cdot\frac{\frac{L}{2}}{\left(\left(\frac{L}{2}\right)^2+x^2\right)^{1/2}} \\
&= k\frac{Qq}{x\left(\left(\frac{L}{2}\right)^2+x^2\right)^{1/2}}\hat{x}
\end{aligned}$$

Exercise 3.

A point charge q is kept at a distance z above a ring of radius R charged with $Q = 2\pi R\lambda$, where λ is the linear charge density. Find the force acting on q.

Solution 3.



Due to the symmetry of the ring, the net force acting on q is in the z direction only.

$$\begin{aligned}
\mathrm{d}F_z &= \mathrm{d}F\cos\theta \\
&= k\frac{(\mathrm{d}Q)(q)}{z^2+R^2}\cos\theta \\
&= k\frac{(\mathrm{d}Q)(q)}{z^2+R^2}\frac{z}{\sqrt{z^2+R^2}} \\
&= kqz\frac{\mathrm{d}Q}{\left(z^2+R^2\right)^{3/2}} \\
\therefore \overrightarrow{F} &= \hat{z}\int \mathrm{d}F_z \\
&= \hat{z}kqz\frac{1}{\left(z^2+R^2\right)^{3/2}}\int_0^Q \mathrm{d}Q \\
&= k\frac{Qqz}{\left(z^2+R^2\right)^{3/2}}\overrightarrow{z}
\end{aligned}$$

Exercise 4.

A point charge q is kept at a distance z above a disk of radius R charged with $Q = \pi R^2\sigma$, where σ is the surface charge density. Find the force acting on q.

Solution 4.

The disk can be considered to be made up of elemental rings, with radii varying from 0 to R.
Therefore,

$$\begin{aligned}
\mathrm{d}\overrightarrow{F} &= k\frac{qQ_{\text{ring}}}{\left(z^2+R^2\right)^{3/2}}\hat{z} \\
&= k\frac{q(\sigma\cdot 2\pi r\cdot \mathrm{d}r)}{\left(z^2+R^2\right)^{3/2}}z\hat{z}
\end{aligned}$$

Hence,

$$
\begin{aligned}
\vec{F} &= \int \mathrm{d}\vec{F} \\
&= \hat{z} \int_0^R k \frac{q\sigma \cdot 2\pi r z \cdot \mathrm{d}r}{\left(z^2 + R^2\right)^{3/2}} \\
&= 2kzq\sigma\pi \left(\frac{1}{|z|} - \frac{1}{\sqrt{z^2 + R^2}} \right) \hat{z}
\end{aligned}
$$

If $z << R$, i.e. for an infinite sheet,

$$F = 2q\sigma\pi k$$

2 Electric Field

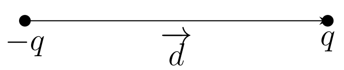
Definition 1 (Electric field)**.** The electric field at a point in space is the electric force felt by a charge of 1 C had it been kept there.

2.1 Standard Electric Fields

Line of charge	$\dfrac{1}{4\pi\varepsilon_0} \dfrac{\lambda L}{r\sqrt{r^2 + \dfrac{L^2}{4}}}$
Infinite line of charge	$\dfrac{\lambda}{2\pi\varepsilon_0 r}$
Ring of charge	$\dfrac{\lambda R z}{2\varepsilon_0 \left(z^2 + R^2\right)^{3/2}}$
Infinite plane of charge	$\dfrac{\sigma}{2\varepsilon_0}$

3 Electric Dipoles

Definition 2 (Electric dipole)**.** Two charges, q and $-q$, separated by a distance d is called an electric dipole.



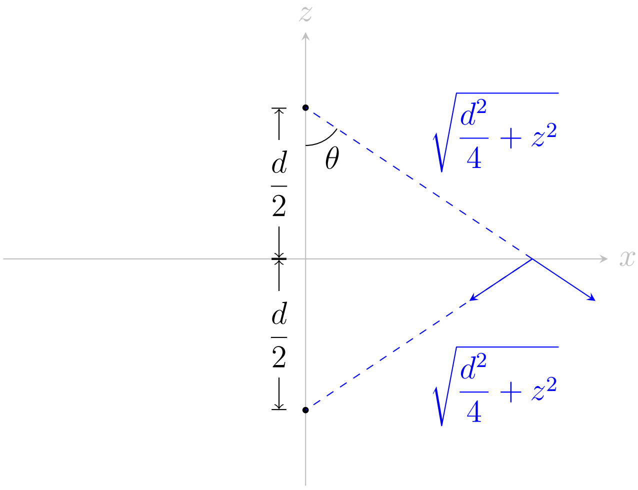
Definition 3 (Dipole moment)**.** If two charges q and $-q$ are separated by a distance d, the dipole moment is defined as

$$\overrightarrow{P} \doteq q \cdot \overrightarrow{d}$$

where $\overrightarrow{d}$ is the vector of length d pointing from $-q$ to q.

3.1 Electric Field Due to Electric Dipoles

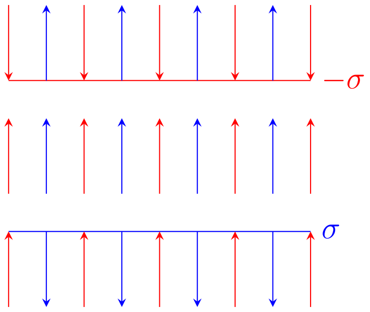
3.1.1 Electric Field



$$\begin{aligned}\overrightarrow{F} &= 2E_+ \cos\theta(-\hat{z}) \\ &= 2 \cdot \frac{1}{4\pi\varepsilon_0} \frac{q}{\left(\frac{d}{2}\right)^2 + x^2} \cdot \frac{\frac{d}{2}}{\left(\left(\frac{d}{2}\right)^2 + x^2\right)^{1/2}}(-\hat{z}) \\ &= -\frac{1}{4\pi\varepsilon_0} \frac{\overrightarrow{P}}{\left(\left(\frac{d}{2}\right)^2 + x^2\right)^{3/2}}\end{aligned}$$

4 Capacitors

A parallel plate capacitor is constructed by arranging two infinite plates with surface charge density σ and $-\sigma$ respectively.



The electric field due to the plates are as shown. Therefore, the fields between the plates add up and the fields outside the plates cancel out. Therefore, the net field inside the capacitor is

$$\frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

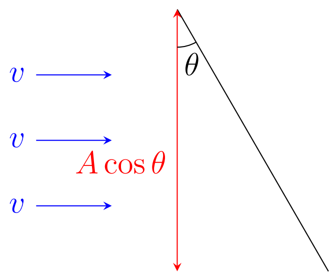
5 Gauss' Law

Definition 4 (Electric flux)**.** Electric flux is defined as the dot product of the electric field passing through a surface, and the area vector of the surface.

$$\Phi = \overrightarrow{E} \cdot \overrightarrow{A}$$

where the magnitude of the area vector is proportional to the area of the surface and the direction is perpendicular to the surface.

This can be modelled as water passing through a surface.



The flux of the water passing through the area A is $Av\cos\theta$.

Theorem 1 (Gauss' Law)**.**

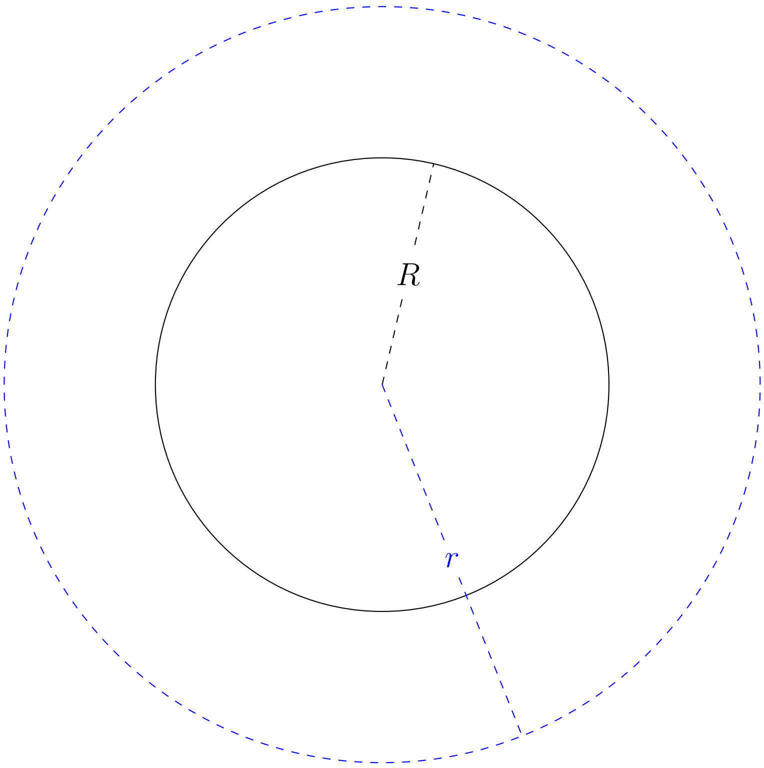
$$\oiint \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{A} = \frac{Q_{\text{inside}}}{\varepsilon_0}$$

Exercise 5.

A hollow sphere of radius R has surface charge density σ. Find the field at a point at distance r from the centre of the sphere.

Solution 5.

Consider the imaginary Gaussian surface as a sphere with radius r.



Using Gauss' Law over the Gaussian surface,

$$\begin{aligned}
\oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A} &= \frac{Q_{\text{total}}}{\varepsilon_0} \\
\therefore \oiint E \, \mathrm{d}A &= \frac{Q_{\text{total}}}{\varepsilon_0} \\
\therefore E \oiint \mathrm{d}A &= \frac{Q_{\text{total}}}{\varepsilon_0} \\
\therefore E \cdot 4\pi r^2 &= \frac{Q_{\text{total}}}{\varepsilon_0} \\
\therefore \overrightarrow{E} &= \frac{1}{4\pi\varepsilon_0} \frac{Q_{\text{total}}}{r^2} \hat{r}
\end{aligned}$$

Similarly for $r < R$, $E = 0$.

6 Electric Potential

Definition 5 (Electrical Potential)**.** The electric potential due to a point charge q is

$$\varphi\left(\overrightarrow{r}\right) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} + c$$

If a charge q in moved from point A to B,

$$\begin{aligned}
W_{\mathrm{A}\to\mathrm{B}} &= \int\limits_{\overrightarrow{r_\mathrm{A}}}^{\overrightarrow{r_\mathrm{B}}} \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{r} \\
&= \int\limits_{r_\mathrm{A}}^{r_\mathrm{B}} \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,\mathrm{d}r \\
&= -\frac{1}{4\pi\varepsilon_0}\frac{q}{r}\bigg|_{r_\mathrm{A}}^{r_\mathrm{B}} \\
&= \frac{1}{4\pi\varepsilon_0}\frac{q}{r_\mathrm{A}} - \frac{1}{4\pi\varepsilon_0}\frac{q}{r_\mathrm{B}}
\end{aligned}$$

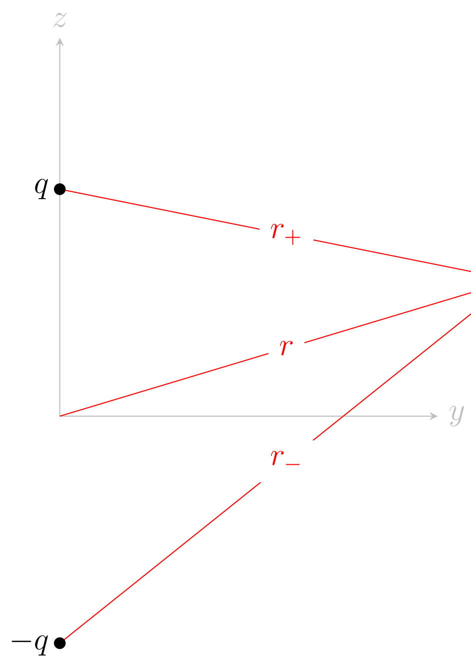
Therefore,

$$\begin{aligned}
W_{\mathrm{A}\to\mathrm{B}} &= \varphi\left(\overrightarrow{r_A}\right) - \varphi\left(\overrightarrow{r_B}\right) \\
\therefore \varphi\left(\overrightarrow{r_B} - \overrightarrow{r_A}\right) &= -\int\limits_{\overrightarrow{r_A}}^{\overrightarrow{r_B}} \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{r}
\end{aligned}$$

Exercise 6.

An electric dipole with charges q and $-q$ is placed on the z-axis with distance d between the charges. Find the field at a general point in space. Find points at which the electric potential is zero.

Solution 6.



Let the electric potential at infinity be zero.

$$\begin{aligned}\varphi\left(\vec{r}\right) &= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+}+\frac{-q}{r_-}\right)\\ &= \frac{1}{4\pi\varepsilon_0}\frac{q(r_- - r_+)}{r_-\cdot r_+}\end{aligned}$$

Therefore, the potential is zero only if $r_+ = r_-$. Therefore, for all points on the xy-plane, the potential is zero.

Exercise 7.

Find the electric potential at a point at distance r on the equator of a line of charge of length L and uniform line charge density λ.

Solution 7.

Consider an elemental charge $\mathrm{d}q$ with length $\mathrm{d}z$ at a distance z from the centre of the line.

$$\mathrm{d}q = \lambda\,\mathrm{d}z$$

Therefore,

$$\begin{aligned}
\varphi(r) &= \int \frac{1}{4\pi\varepsilon_0} \frac{q}{\sqrt{z^2+r^2}} \\
&= \int\limits_{-L/2}^{L/2} \frac{1}{4\pi\varepsilon_0} \frac{\lambda \,\mathrm{d}z}{\sqrt{z^2+r^2}} \\
&= \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{\sqrt{L^2+4r^2}+L}{\sqrt{L^2+4r^2}-L}\right)
\end{aligned}$$

Exercise 8.
Find the electric potential due to an infinite line of charge.

Solution 8.
For an infinite line of charge, the charge at infinity is not zero. Therefore, it is wrong to assume that the electric potential at infinity is zero. Therefore, the result for a finite line of charge cannot be used to find the potential due to an infinite line of charge.
Therefore, the potential needs to be calculated using the electric field.

$$\begin{aligned}
\varphi(r) - \varphi(r_0) &= -\int\limits_{r_0}^{r} \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{r} \\
&= -\int\limits_{r_0}^{r} E \,\mathrm{d}r \\
&= -\int\limits_{r_0}^{r} \frac{\lambda}{2\pi\varepsilon_0 r} \,\mathrm{d}r \\
&= -\frac{\lambda}{2\pi\varepsilon_0} \ln r \bigg|_{r_0}^{r} \\
&= \frac{\lambda}{2\pi\varepsilon_0} (\ln r_0 - \ln r) \\
\therefore \varphi(r) &= \varphi(r_0) + \frac{\lambda}{2\pi\varepsilon_0} (\ln r_0 - \ln r)
\end{aligned}$$

Exercise 9.

Find the electric potential due to a hollow sphere with surface charge density σ and radius R.

Solution 9.

Let the electric potential at infinity be zero.

$$\overrightarrow{E} = \begin{cases} 0 & ; \quad r < R \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r^2} & ; \quad r > R \end{cases}$$

Therefore, if $r > R$,

$$\begin{aligned} \varphi(r) - \cancelto{0}{\varphi(\infty)} &= -\int_\infty^r \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{r} \\ &= \int_r^\infty E \,\mathrm{d}r \\ &= \int_r^\infty \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,\mathrm{d}r \\ &= -\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\bigg|_r^\infty \\ \therefore \varphi(r) &= \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \end{aligned}$$

If $r < R$,

$$\begin{aligned}
\varphi(r) - \varphi(R) &= \int\limits_R^r \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{r} \\
&= \int\limits_R^r E \,\mathrm{d}r \\
&= \int\limits_R^r 0 \,\mathrm{d}r \\
\therefore \varphi(r) &= \varphi(R) \\
&= \frac{1}{4\pi\varepsilon_0} \frac{q}{R}
\end{aligned}$$

Therefore, the potential is constant.
Therefore,

$$\varphi = \begin{cases} \dfrac{1}{4\pi\varepsilon_0} \dfrac{q}{R} & ; \quad r \leq R \\ \dfrac{1}{4\pi\varepsilon_0} \dfrac{q}{r} & ; \quad r \geq R \end{cases}$$

Exercise 10.
Find the electric potential due a ring of charge with radius R and charge q, at a distance z from its centre, on its axis of symmetry.

Solution 10.
Let the electric potential at infinity be zero.

$$\begin{aligned}
\varphi_{\text{ring}} &= \int \frac{1}{4\pi\varepsilon_0} \frac{\mathrm{d}q}{r} \\
&= \int\limits_0^q \frac{1}{4\pi\varepsilon_0} \frac{\mathrm{d}q}{\sqrt{R^2 + z^2}} \\
&= \frac{1}{4\pi\varepsilon_0} \frac{1}{\sqrt{R^2 + z^2}}
\end{aligned}$$

Exercise 11.
Find the electric potential due a disk of charge with radius R and charge q, at a distance z from its centre, on its axis of symmetry.

Solution 11.
Let the electric potential at infinity be zero.

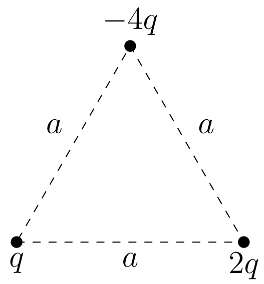
Consider an elemental ring of thickness $\mathrm{d}r$ and radius r. Therefore,

$$\begin{aligned}
\varphi_{\text{disk}} &= \int \mathrm{d}\varphi_{\text{ring}} \\
&= \int \frac{1}{4\pi\varepsilon_0} \frac{\mathrm{d}q_{\text{ring}}}{\sqrt{r^2+z^2}} \\
&= \int_0^R \frac{1}{4\pi\varepsilon_0} \frac{2\pi r\,\mathrm{d}r\sigma}{\sqrt{r^2+z^2}} \\
&= \frac{\sigma}{2\varepsilon}\left(\sqrt{R^2+z^2}-|z|\right)
\end{aligned}$$

7 Electrical Potential Energy

Exercise 12.
Three charges, q, $-4q$, $2q$ are placed on the vertices of an equilateral triangle of side a. Find the energy in the system.

Solution 12.



The energy in the system is the amount of energy required to build the system by bringing each of the charges from infinity to its position, one by one.
Let the positions of q, $2q$ and $-4q$ be A, B and C respectively.
The energy required to bring the first charge, q, from infinity to A is zero, as

there are no forces acting on it.
The energy required to bring the second charge, $2q$, from infinity to B is

$$\begin{aligned}
U_{2q} &= -\int_{\infty}^{\vec{r}_{\mathrm{B}}} \vec{F} \cdot \mathrm{d}\vec{r} \\
&= -\int_{\infty}^{\vec{r}_{\mathrm{B}}} (2q) \cdot \vec{E} \cdot \mathrm{d}\vec{r} \\
&= -(2q) \int_{\infty}^{\vec{r}_{\mathrm{B}}} \vec{E} \cdot \mathrm{d}\vec{r} \\
&= (2q)\left(\varphi(\mathrm{B}) - \varphi(\infty)\right) \\
&= (2q) \cdot \varphi(\mathrm{B})
\end{aligned}$$

where $\varphi(\mathrm{B})$ is potential at point B due to the existing charges, i.e. q. Similarly, the energy required to bring the third charge, $-4q$, from infinity to C is $(-4q) \cdot \varphi(\mathrm{C})$, where $\varphi(\mathrm{C})$ is the potential at point C due to the existing charges, i.e. q and $2q$.
Therefore, the total energy required is

$$U = (0) + (2q)\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{a^2}\right) + (-4q)\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{a^2} + \frac{1}{4\pi\varepsilon_0}\frac{2q}{a^2}\right)$$

Exercise 13.

Find the potential energy in a solid sphere of charge, with charge density ρ and radius R.

Solution 13.

Consider a solid sphere of charge with ρ and r. Consider an elemental shell of thickness $\mathrm{d}r$ on this sphere.
Therefore,

$$\begin{aligned}
\mathrm{d}V &= 4\pi r^2 \,\mathrm{d}r \\
\therefore \mathrm{d}q &= \rho \cdot \mathrm{d}V \\
&= 4\pi\rho r^2 \,\mathrm{d}r
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\mathrm{d}U &= \frac{1}{4\pi\varepsilon_0}\frac{q_{\text{inside}}}{r}\,\mathrm{d}q \\
&= \frac{1}{4\pi\varepsilon_0}\frac{\frac{4}{3}\pi r^3\rho}{r}\cdot 4\pi r^2\,\mathrm{d}r\rho \\
\therefore U &= \int_0^R \frac{4\pi\rho^2}{3\varepsilon_0} r^4\,\mathrm{d}r \\
&= \frac{4\pi\rho^2 R^5}{15\varepsilon_0}
\end{aligned}
$$

8 Integral Form of Gauss' Law

The volume of the elemental body used for integration is denoted by $\mathrm{d}^3 r$.
For Cartesian coordinate systems, $\mathrm{d}^3 r = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.
For cylindrical coordinate systems, $\mathrm{d}^3 r = r\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}z$.
For spherical coordinate systems, $\mathrm{d}^3 r = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi$.

$$
\begin{aligned}
\iint\limits_{\partial V} \vec{E}\cdot\mathrm{d}\vec{A} &= \frac{1}{\varepsilon_0}Q_{\text{inside}} \\
&= \frac{1}{\varepsilon_0}\iiint\limits_{V}\rho\,\mathrm{d}^3 r^1
\end{aligned}
$$

If a body with volume V and surface area S is cut into two parts, with volumes V_1 and V_2 and surface area S_1 and S_2 respectively,

$$V = V_1 + V_2$$

However, the surface area increases,

$$S = S_1 + S_2 + 2A \neq S_1 + S_2$$

where A is the area of the new surface created due to the cut.
Therefore,

$$\iint\limits_{S_1}\vec{E}\cdot\mathrm{d}\vec{A} + \iint\limits_{S_2}\vec{E}\cdot\mathrm{d}\vec{A} = \iint\limits_{\partial V}\vec{E}\cdot\mathrm{d}\vec{A}$$

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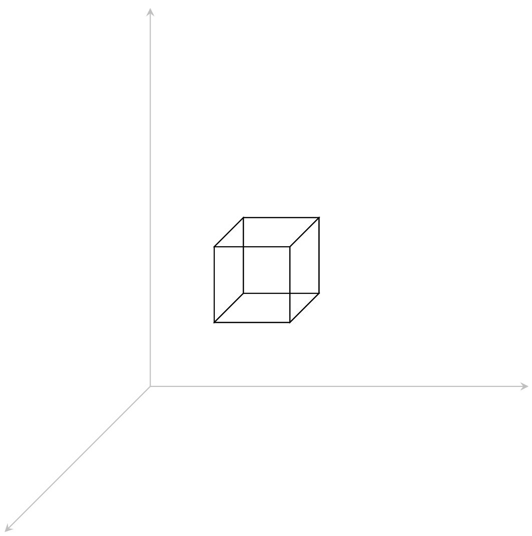

$$
\begin{aligned}
\Phi = \quad & E_z\left(x+\frac{\mathrm{d}x}{2}, y+\frac{\mathrm{d}y}{2}, z+\mathrm{d}z\right)\mathrm{d}x\,\mathrm{d}y \\
& - E_z\left(x+\frac{\mathrm{d}x}{2}, y+\frac{\mathrm{d}y}{2}, z\right)\mathrm{d}x\,\mathrm{d}y \\
& + E_y\left(x+\frac{\mathrm{d}x}{2}, y+\mathrm{d}y, z+\frac{\mathrm{d}z}{2}\right)\mathrm{d}x\,\mathrm{d}z \\
& - E_y\left(x+\frac{\mathrm{d}x}{2}, y, z+\frac{\mathrm{d}z}{2}\right)\mathrm{d}x\,\mathrm{d}z \\
& + E_x\left(x+\mathrm{d}x, y+\frac{\mathrm{d}y}{2}, z+\frac{\mathrm{d}z}{2}\right)\mathrm{d}y\,\mathrm{d}z \\
& - E_x\left(x, y+\frac{\mathrm{d}y}{2}, z+\frac{\mathrm{d}z}{2}\right)\mathrm{d}y\,\mathrm{d}z \\
= \quad & \frac{E_z\left(x+\frac{\mathrm{d}x}{2}, y+\frac{\mathrm{d}y}{2}, z+\mathrm{d}z\right) - E_z\left(x+\frac{\mathrm{d}x}{2}, y+\frac{\mathrm{d}y}{2}, z\right)}{\mathrm{d}z}\mathrm{d}x\,\mathrm{d}y \\
& + \frac{E_y\left(x+\frac{\mathrm{d}x}{2}, y+\mathrm{d}y, z+\frac{\mathrm{d}z}{2}\right) - E_y\left(x+\frac{\mathrm{d}x}{2}, y, z+\frac{\mathrm{d}z}{2}\right)}{\mathrm{d}y}\mathrm{d}x\,\mathrm{d}z \\
& + \frac{E_x\left(x+\mathrm{d}x, y+\frac{\mathrm{d}y}{2}, z+\frac{\mathrm{d}z}{2}\right) - E_x\left(x, y+\frac{\mathrm{d}y}{2}, z+\frac{\mathrm{d}z}{2}\right)}{\mathrm{d}x}\mathrm{d}y\,\mathrm{d}z \\
= \quad & \left(\left.\frac{\partial E_x}{\partial x}\right|_{\left(x+\frac{\mathrm{d}x}{2}, y+\frac{\mathrm{d}y}{2}, z\right)} \right. \\
& \quad + \left.\frac{\partial E_y}{\partial y}\right|_{\left(x+\frac{\mathrm{d}x}{2}, y, z+\frac{\mathrm{d}z}{2}\right)} \\
& \quad \left. + \left.\frac{\partial E_z}{\partial z}\right|_{\left(x+\frac{\mathrm{d}x}{2}, y+\frac{\mathrm{d}y}{2}, z\right)} \right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z
\end{aligned}
$$

Therefore

$$\begin{aligned}
\operatorname{div} \overrightarrow{E} &= \lim_{\mathrm{d}x\to 0, \mathrm{d}y\to 0, \mathrm{d}z\to 0} \frac{\Phi}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z} \\
&= \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\Bigg|_{(x,y,z)} \\
&= \overrightarrow{\nabla} \cdot \overrightarrow{E}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{E} &= -\overrightarrow{\nabla}\varphi \\
\therefore \operatorname{div} \overrightarrow{E} &= \overrightarrow{\nabla} \cdot \left(-\overrightarrow{\nabla}\varphi\right) \\
&= \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial x^2} \\
&= \nabla^2 \varphi = \Delta\varphi
\end{aligned}$$

Definition 6 (Laplacian)**.** ∇^2 is called the Laplacian.

9 Conductors

In an electrostatic condition, the field inside a conductor is zero. If it is not, as the conductor allows movement of charged particles, there will be a current and the condition will not be electrostatic.