Physics 2 : Recitations

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1 Instructor Information

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Part I
Electrostatics

1 Gravitation and Electromagnetism

Gravitation	Electromagnetism
$F_G = G\dfrac{m_1 m_2}{r^2}$	$F_E = k\dfrac{q_1 q_2}{r^2}$
$G = 6.7 \times 10^{11}\,\mathrm{N\,m^2\,kg^{-2}}$	$8.99 \times 10^{9}\,\mathrm{N\,m^2\,C^{-2}}$

2 Coulomb's Law

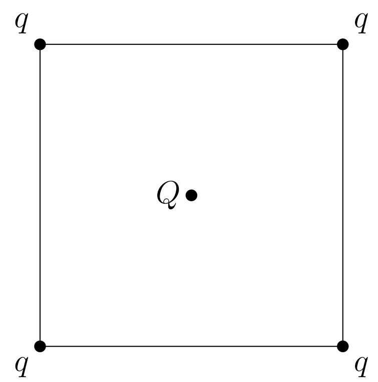
Recitation 1 – Exercise 1.

Four identical charges q are placed in the corners of a square of length a. A fifth charge Q is free to move along the straight line perpendicular to the square plane and passing through its centre. When the charge Q is in the same plane as the other charges, all the forces in the system cancel out.

1. Calculate Q for a given q and a.

2. Find the force $\overrightarrow{F(z)}$ acting on the charge Q when it is at height z above the square.

Recitation 1 – Solution 1.



Consider q on the top right corner of the square. The total force acting on it is 0. Therefore

$$\begin{aligned}
0 &= \frac{kq^2}{a^2}\cos\frac{\pi}{4} + \frac{kq^2}{a^2}\cos\frac{\pi}{4} + \frac{kq^2}{2a^2} + \frac{2kQq}{a^2} \\
\therefore Q &= -\frac{1+2\sqrt{2}}{4}q
\end{aligned}$$

If Q is at a height z from the plane, the distance between each q and Q is

$$r = \sqrt{z^2 + \left(\frac{a}{\sqrt{2}}\right)^2}$$

Therefore the force of each q on Q is $\dfrac{kQq}{r^2}$.
Due to symmetry, the components of the forces in the z direction will add up, and all other components will cancel out.
Let the angle between the z direction and the line joining q and Q be φ.
Therefore, the net force is

$$\begin{aligned}
F &= 4\frac{kQq}{r^2}\cos\varphi \\
&= 4\frac{kQq}{r^2}\frac{z}{r} \\
&= 4\frac{kQq}{z^2\left(1+\dfrac{a^2}{2z^2}\right)^{3/2}}
\end{aligned}$$

Recitation 1 – Exercise 2.

1. A wire of length 3 metre is charged with 2 $\mathrm{C\,m^{-1}}$. What is the wire's total charge?

Recitation 1 – Solution 2.

$$\begin{aligned}
\lambda &= \frac{Q}{L} \\
\therefore Q &= L\lambda \\
&= 6\mathrm{C}
\end{aligned}$$

Recitation 1 – Exercise 3.

A wire of length L has the following charge distribution: $\lambda = \lambda_0 \cos \frac{\pi x}{L}$, where x is the distance from the wire's edge. What is the wire's total charge?

Recitation 1 – Solution 3.

$$\begin{aligned}
\lambda &= \frac{\mathrm{d}q}{\mathrm{d}x} \\
\therefore \frac{\mathrm{d}q}{\mathrm{d}x} &= \lambda_0 \cos \frac{\pi x}{L} \\
\therefore q &= \int\limits_0^L \lambda_0 \cos \frac{\pi x}{L}\, \mathrm{d}x \\
&= 0
\end{aligned}$$

Recitation 1 – Exercise 4.

A hollow sphere of radius R is uniformly charged with a charge Q. Calculate the charge distribution on the surface of the sphere.

Recitation 1 – Solution 4.

$$\begin{aligned}
\sigma &= \frac{Q}{A} \\
&= \frac{Q}{4\pi R^2}
\end{aligned}$$

Recitation 1 – Exercise 5.

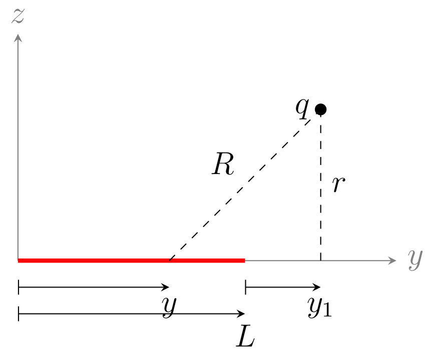
A straight thin wire is uniformly charged with distribution λ. A charge q is positioned at distance y_1 beneath the wire and r away form it.

1. Find the force acting on the charge q.

2. Show that when the charge is positioned in front of the centre of the wire the $\hat{y}$ component of the force is cancelled.

3. Calculate the force an infinite straight wire will exert on the charge q.

Recitation 1 – Solution 5.



Consider an elemental charge dQ of length dy, at distance y as shown. Let the angle between the line joining dQ and q and the y direction be θ.

$$F_y = F\cos\theta$$
$$F_z = F\sin\theta$$

Let

$$a = L + y_1$$
$$\therefore R = \sqrt{r^2 + (a-y)^2}$$

Therefore,

$$\cos\theta = \frac{a-y}{R}$$
$$\sin\theta = \frac{r}{R}$$

Therefore,

$$\begin{aligned} F_y &= kq\int_0^L \frac{\lambda\,\mathrm{d}y}{R^2}\frac{(a-y)}{R} \\ &= kq\lambda\int_0^L \frac{\mathrm{d}y(a-y)}{\left((a-y)^2+r^2\right)^{3/2}} \\ &= kq\lambda\left(\frac{1}{\sqrt{{y_1}^2+r^2}} - \frac{1}{\sqrt{a^2+r^2}}\right) \end{aligned}$$

$$
\begin{aligned}
F_z &= kq \int_0^L \frac{\lambda \,\mathrm{d}y}{R^2} \frac{r}{R} \\
&= kq\lambda \int_0^L \frac{r \,\mathrm{d}y}{\left((a-y)^2 + r^2\right)^{3/2}} \\
&= \frac{kq\lambda}{r} \left(\frac{a}{\sqrt{r^2 + a^2}} - \frac{y_1}{\sqrt{r^2 + {y_1}^2}} \right)
\end{aligned}
$$

When the charge is positioned above the centre of the wire,

$$
\begin{aligned}
y_1 &= -\frac{L}{2} \\
\therefore a &= \frac{L}{2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
F_y &= kq\lambda \left(\frac{1}{\sqrt{{y_1}^2 + r^2}} - \frac{1}{\sqrt{a^2 + r^2}} \right) \\
&= kq\lambda \left(\frac{1}{\sqrt{-\frac{L}{2}^2 + r^2}} - \frac{1}{\sqrt{\frac{L}{2}^2 + r^2}} \right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
F_z &= \frac{kq\lambda}{r}\left(\frac{a}{\sqrt{r^2+a^2}} - \frac{y_1}{\sqrt{r^2+{y_1}^2}}\right) \\
&= \frac{kq\lambda}{r}\left(\frac{\frac{L}{2}}{\sqrt{r^2+\frac{L}{2}^2}} - \frac{-\frac{L}{2}}{\sqrt{r^2+-\frac{L}{2}^2}}\right) \\
&= \frac{kq\lambda}{r}\left(\frac{L}{\sqrt{r^2+\frac{L}{2}^2}}\right) \\
&= \frac{kq\lambda}{r}\left(\frac{1}{\sqrt{\frac{1}{4}+\left(\frac{r}{L}\right)^2}}\right)
\end{aligned}
$$

If the line is infinite, $L \to \infty$. Therefore

$$
\begin{aligned}
F_z &= \frac{kq\lambda}{r}\left(\frac{1}{\sqrt{\frac{1}{4}+\left(\frac{r}{L}\right)^2}}\right) \\
&= \frac{2kq\lambda}{r}
\end{aligned}
$$

3 Gauss' Law

Recitation 2 – Exercise 2.

A ball of radius a is charged with distribution $\rho = \rho_0 \frac{r}{a}$. Find the electric field everywhere.

Recitation 2 – Solution 2.
Consider a spherical Gaussian surface of radius r.
If $r \leq a$, the charge in the interior of the Gaussian surface is

$$\begin{aligned} q(r) &= \int_0^r \frac{\rho_0 r}{a} \cdot 4\pi r^2 \, \mathrm{d}r \\ &= \frac{\rho_0}{a} \pi r^4 \end{aligned}$$

Therefore, by Gauss' Law,

$$\begin{aligned} E \cdot 4\pi r^2 &= \frac{q(r)}{\varepsilon_0} \\ \therefore E &= \frac{\rho_0 \pi r^4}{4\pi a r^2} \\ &= \frac{\rho_0 r^2}{4a\varepsilon_0} \end{aligned}$$

If $r \geq a$, the entire ball of charge is in the interior of the Gaussian surface. Therefore,

$$\begin{aligned} Q &= q(a) \\ &= \frac{\rho_0}{a} \cdot \pi a^4 \\ &= \rho_0 \pi a^3 \end{aligned}$$

Therefore, by Gauss' Law,

$$\begin{aligned} E \cdot 4\pi r^2 &= \frac{Q}{\varepsilon_0} \\ \therefore E &= \frac{Q}{4\pi r^2 \varepsilon_0} \\ &= \frac{\rho_0 a^3}{4r^2 \varepsilon_0} \end{aligned}$$

Therefore,

$$E = \begin{cases} \dfrac{\rho_0 r^2}{4a\varepsilon_0} & ; \quad r \leq a \\ \dfrac{\rho_0 a^3}{4r^2\varepsilon} & ; \quad r \geq a \end{cases}$$

Recitation 2 – Exercise 3.

An infinitely long cylinder of radius a is charged with distribution $\rho = \rho_0 \dfrac{r}{a}$. Find the electric field everywhere.

Recitation 2 – Solution 3.

Consider a infinite cylindrical Gaussian surface with radius r.
If $r \leq a$, the charge in the interior of the Gaussian surface is

$$\begin{aligned} q(r) &= \int_0^r \frac{\rho_0 r}{a} \pi r^2 \, \mathrm{d}r \\ &= \frac{2\pi\rho_0 L r^3}{3a} \end{aligned}$$

Therefore, by Gauss' Law,

$$\begin{aligned} E \cdot 2\pi r L &= \frac{2\pi\rho_0 L r^3}{3a\varepsilon_0} \\ \therefore E &= \frac{\rho_0 r^2}{3a\varepsilon_0} \end{aligned}$$

If $r \geq a$, the entire cylinder of charge is in the interior of the Gaussian surface. Therefore,

$$\begin{aligned} Q &= q(a) \\ &= \frac{2\pi\rho_0 L a^3}{3a} \\ &= \frac{2\pi\rho_0 L a^2}{3} \end{aligned}$$

Therefore, by Gauss' Law,

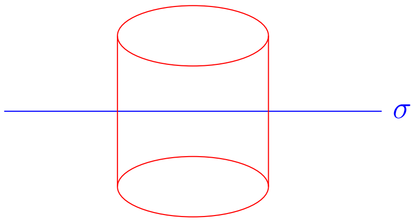
$$\begin{aligned} E \cdot 2\pi r L &= \frac{2\pi\rho_0 L a^2}{3\varepsilon_0} \\ \therefore E &= \frac{\rho_0 a^2}{3\varepsilon_0 r} \end{aligned}$$

Therefore,

$$E = \begin{cases} \dfrac{\rho_0 r^2}{3a\varepsilon_0} & ; \quad r \leq a \\ \dfrac{\rho_0 a^2}{3\varepsilon_0 r} & ; \quad r \geq a \end{cases}$$

Recitation 2 – Exercise 4.
Find the electric field due to a thin infinite plane of uniform charge distribution σ.

Recitation 2 – Solution 4.
Consider a cylindrical Gaussian surface, with ends of area A, as shown.



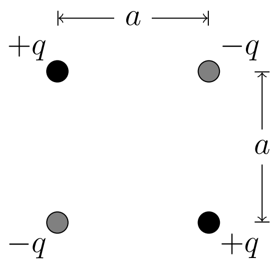
The charge in the interior of the surface is

$$\mathrm{d}q = A\sigma$$

Therefore, by Gauss' Law,

$$\begin{aligned} E_1 \cdot A_1 + E_2 \cdot A_2 &= \frac{A\sigma}{\varepsilon_0} \\ \therefore 2EA &= \frac{A\sigma}{\varepsilon_0} \\ \therefore E &= \frac{\sigma}{2\varepsilon_0} \end{aligned}$$

Recitation 3 – Exercise 2.
A system of four charges is constructed as shown.



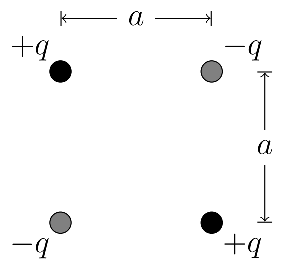
a) Calculate the work needed to build this system.

b) What is the potential in the centre of the system?

c) Calculate the potential in each of the corners (calculate as if there is no charge in the corner you are calculating for).

Recitation 3 – Solution 2.

a) Let the positions of the charges be A, B, C, D.



The work done to bring the first charge from infinity to A is

$$W_{\mathrm{A}} = 0$$

The work done to bring the first charge from infinity to B is

$$W_{\mathrm{B}} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a\sqrt{2}}$$

Similarly for the other two charges.
Therefore,

$$\begin{aligned} W &= 0 + \frac{q^2}{4\sqrt{2}\pi\varepsilon_0} + \frac{-2q^2}{4\pi\varepsilon_0 a} + \left(\frac{-2q^2}{4\pi\varepsilon_0 a} + \frac{q^2}{4\sqrt{2}\pi\varepsilon_0}\right) \\ &= \frac{q^2}{2\sqrt{2}\pi\varepsilon_0} - \frac{q^2}{\pi\varepsilon_0 a} \end{aligned}$$

b)

$$\begin{aligned} V_{\text{centre}} &= V_q + V_q + V_{-q} + V_{-q} \\ &= \frac{q}{4\pi\varepsilon_0\left(\frac{a}{\sqrt{2}}\right)} + \frac{q}{4\pi\varepsilon_0\left(\frac{a}{\sqrt{2}}\right)} - \frac{q}{4\pi\varepsilon_0\left(\frac{a}{\sqrt{2}}\right)} - \frac{q}{4\pi\varepsilon_0\left(\frac{a}{\sqrt{2}}\right)} \\ &= 0 \end{aligned}$$

c)

$$V_{\mathrm{A}} = \frac{1}{4\pi\varepsilon_0}\frac{-q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{2}a}$$
$$V_{\mathrm{B}} = \frac{1}{4\pi\varepsilon_0}\frac{-q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{2}a}$$
$$V_{\mathrm{C}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{\sqrt{2}a}$$
$$V_{\mathrm{D}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{\sqrt{2}a}$$

Recitation 3 – Exercise 3.
A ring of radius R is charged with total charge Q.

a) Calculate the electric field in the centre of the ring.

b) Calculate the potential in the centre of the ring by integrating the contributions of the infinitesimal charge elements of the ring.

Recitation 3 – Solution 3.

a) Due to the symmetry of the ring, the field due to every elemental charge dq will be cancelled out by the field due to a elemental charge diametrically opposite to dq.
Therefore,

$$\overrightarrow{E} = 0$$

b)

$$\mathrm{d}V = \frac{\mathrm{d}q}{4\pi\varepsilon_0 R}$$
$$\therefore V = \frac{Q}{4\pi\varepsilon_0 R}$$

Recitation 3 – Exercise 4.

Calculate the potential resulting from a ball charged with constant volume distribution ρ. Use the expression

$$\varphi(r_2) - \varphi(r_1) = -\int_{r_1}^{r_2} E(r)\,\mathrm{d}r$$

Repeat the calculation twice:

a) Set $\varphi(r = R) = 0$

b) Set $\varphi(r = \infty) = 0$

Recitation 3 – Solution 4.

a) Let $\varphi(r = \infty) = 0$.

If $r > R$,

$$\begin{aligned}
E &= -\frac{\mathrm{d}\varphi}{\mathrm{d}r} \\
\therefore -\frac{\mathrm{d}\varphi}{\mathrm{d}r} &= \frac{Q}{4\pi\varepsilon_0 r^2} \\
\therefore \int \mathrm{d}\varphi &= \int\limits_{\infty}^{r} \frac{Q}{4\pi\varepsilon_0 r^2}\,\mathrm{d}r \\
\therefore \varphi(r) - \varphi(\infty) &= \frac{Q}{4\pi\varepsilon_0 r} - 0 \\
\therefore \varphi(r) &= \frac{Q}{4\pi\varepsilon_0 r}
\end{aligned}$$

If $r < R$,

$$\begin{aligned}
E &= \frac{q(r)}{4\pi\varepsilon_0 r^2} \\
&= \frac{\rho \cdot \frac{4}{3}\pi r^3}{4\pi\varepsilon_0 r^2} \\
&= \frac{\rho r}{3\varepsilon_0}
\end{aligned}$$

Therefore

$$\begin{aligned}
E &= -\frac{\mathrm{d}\varphi}{\mathrm{d}r} \\
\therefore \int \mathrm{d}\varphi &= \int\limits_{r}^{R} \frac{\rho}{3\varepsilon_0} r\,\mathrm{d}r \\
\therefore \varphi(R) - \varphi(r) &= \frac{\rho}{6\varepsilon_0}\left(r^2 - R^2\right) \\
\therefore \varphi(r) &= \frac{Q}{4\pi\varepsilon_0 R} + \frac{\rho\left(R^2 - r^2\right)}{6\varepsilon_0} \\
&= \frac{Q}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)
\end{aligned}$$

b) Let $\varphi(r = R) = 0$.

$$\begin{aligned}
\therefore \varphi(R) - \varphi(r) &= \frac{\rho}{6\varepsilon_0}\left(r^2 - R^2\right) \\
\therefore \varphi(r) &= \frac{\rho}{6\varepsilon_0}\left(R^2 - r^2\right) \\
&= \frac{Q}{8\pi R\varepsilon_0}\left(1 - \frac{r^2}{R^2}\right)
\end{aligned}$$