Physics 2 : Recitations

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Contents

1 Instructor Information

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Part I
Electrostatics

1 Gravitation and Electromagnetism

Gravitation	Electromagnetism
$F_G = G\dfrac{m_1 m_2}{r^2}$	$F_E = k\dfrac{q_1 q_2}{r^2}$
$G = 6.7 \times 10^{11}\mathrm{N\,m^2\,kg^{-2}}$	$8.99 \times 10^{9}\mathrm{N\,m^2\,C^{-2}}$

2 Coulomb's Law

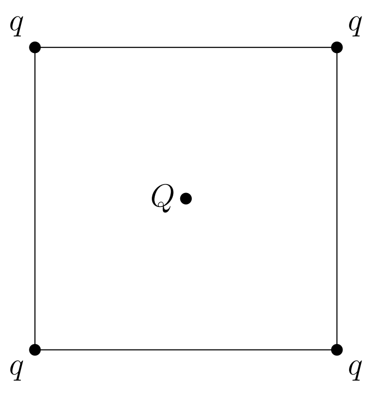
Recitation 1 – Exercise 1.

Four identical charges q are placed in the corners of a square of length a. A fifth charge Q is free to move along the straight line perpendicular to the square plane and passing through its centre. When the charge Q is in the same plane as the other charges, all the forces in the system cancel out.

1. Calculate Q for a given q and a.

2. Find the force $\overrightarrow{F(z)}$ acting on the charge Q when it is at height z above the square.

Recitation 1 – Solution 1.



Consider q on the top right corner of the square. The total force acting on it is 0. Therefore

$$\begin{aligned}
0 &= \frac{kq^2}{a^2}\cos\frac{\pi}{4} + \frac{kq^2}{a^2}\cos\frac{\pi}{4} + \frac{kq^2}{2a^2} + \frac{2kQq}{a^2} \\
\therefore Q &= -\frac{1+2\sqrt{2}}{4}q
\end{aligned}$$

If Q is at a height z from the plane, the distance between each q and Q is

$$r = \sqrt{z^2 + \left(\frac{a}{\sqrt{2}}\right)^2}$$

Therefore the force of each q on Q is $\dfrac{kQq}{r^2}$.
Due to symmetry, the components of the forces in the z direction will add up, and all other components will cancel out.
Let the angle between the z direction and the line joining q and Q be φ.
Therefore, the net force is

$$\begin{aligned}
F &= 4\frac{kQq}{r^2}\cos\varphi \\
&= 4\frac{kQq}{r^2}\frac{z}{r} \\
&= 4\frac{kQq}{z^2\left(1+\dfrac{a^2}{2z^2}\right)^{3/2}}
\end{aligned}$$

Recitation 1 – Exercise 2.

1. A wire of length 3 metre is charged with $2\,\mathrm{C\,m^{-1}}$. What is the wire's total charge?

Recitation 1 – Solution 2.

$$\begin{aligned}
\lambda &= \frac{Q}{L} \\
\therefore Q &= L\lambda \\
&= 6\mathrm{C}
\end{aligned}$$

Recitation 1 – Exercise 3.

A wire of length L has the following charge distribution: $\lambda = \lambda_0 \cos \frac{\pi x}{L}$, where x is the distance from the wire's edge. What is the wire's total charge?

Recitation 1 – Solution 3.

$$\begin{aligned}
\lambda &= \frac{\mathrm{d}q}{\mathrm{d}x} \\
\therefore \frac{\mathrm{d}q}{\mathrm{d}x} &= \lambda_0 \cos \frac{\pi x}{L} \\
\therefore q &= \int_0^L \lambda_0 \cos \frac{\pi x}{L} \, \mathrm{d}x \\
&= 0
\end{aligned}$$

Recitation 1 – Exercise 4.

A hollow sphere of radius R is uniformly charged with a charge Q. Calculate the charge distribution on the surface of the sphere.

Recitation 1 – Solution 4.

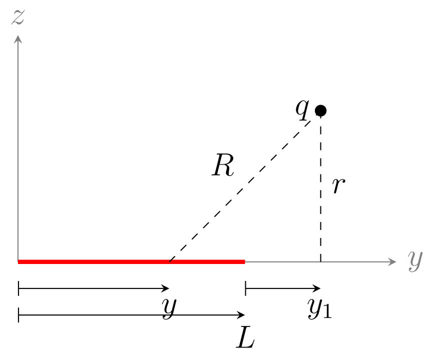
$$\begin{aligned}
\sigma &= \frac{Q}{A} \\
&= \frac{Q}{4\pi R^2}
\end{aligned}$$

Recitation 1 – Exercise 5.

A straight thin wire is uniformly charged with distribution λ. A charge q is positioned at distance y_1 beneath the wire and r away form it.

1. Find the force acting on the charge q.
2. Show that when the charge is positioned in front of the centre of the wire the $\hat{y}$ component of the force is cancelled.
3. Calculate the force an infinite straight wire will exert on the charge q.

Recitation 1 – Solution 5.



Consider an elemental charge $\mathrm{d}Q$ of length $\mathrm{d}y$, at distance y as shown. Let the angle between the line joining $\mathrm{d}Q$ and q and the y direction be θ.

$$F_y = F\cos\theta$$
$$F_z = F\sin\theta$$

Let

$$a = L + y_1$$
$$\therefore R = \sqrt{r^2 + (a-y)^2}$$

Therefore,

$$\cos\theta = \frac{a-y}{R}$$
$$\sin\theta = \frac{r}{R}$$

Therefore,

$$\begin{aligned} F_y &= kq\int_0^L \frac{\lambda\,\mathrm{d}y}{R^2}\frac{(a-y)}{R} \\ &= kq\lambda\int_0^L \frac{\mathrm{d}y(a-y)}{\left((a-y)^2+r^2\right)^{3/2}} \\ &= kq\lambda\left(\frac{1}{\sqrt{{y_1}^2+r^2}} - \frac{1}{\sqrt{a^2+r^2}}\right) \end{aligned}$$

$$
\begin{aligned}
F_z &= kq \int_0^L \frac{\lambda \,\mathrm{d}y}{R^2} \frac{r}{R} \\
&= kq\lambda \int_0^L \frac{r \,\mathrm{d}y}{\left((a-y)^2 + r^2\right)^{3/2}} \\
&= \frac{kq\lambda}{r} \left(\frac{a}{\sqrt{r^2 + a^2}} - \frac{y_1}{\sqrt{r^2 + {y_1}^2}} \right)
\end{aligned}
$$

When the charge is positioned above the centre of the wire,

$$
\begin{aligned}
y_1 &= -\frac{L}{2} \\
\therefore a &= \frac{L}{2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
F_y &= kq\lambda \left(\frac{1}{\sqrt{{y_1}^2 + r^2}} - \frac{1}{\sqrt{a^2 + r^2}} \right) \\
&= kq\lambda \left(\frac{1}{\sqrt{-\frac{L}{2}^2 + r^2}} - \frac{1}{\sqrt{\frac{L}{2}^2 + r^2}} \right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
F_z &= \frac{kq\lambda}{r}\left(\frac{a}{\sqrt{r^2+a^2}} - \frac{y_1}{\sqrt{r^2+{y_1}^2}}\right) \\
&= \frac{kq\lambda}{r}\left(\frac{\frac{L}{2}}{\sqrt{r^2+\frac{L}{2}^2}} - \frac{-\frac{L}{2}}{\sqrt{r^2+-\frac{L}{2}^2}}\right) \\
&= \frac{kq\lambda}{r}\left(\frac{L}{\sqrt{r^2+\frac{L}{2}^2}}\right) \\
&= \frac{kq\lambda}{r}\left(\frac{1}{\sqrt{\frac{1}{4}+\left(\frac{r}{L}\right)^2}}\right)
\end{aligned}
$$

If the line is infinite, $L \to \infty$. Therefore

$$
\begin{aligned}
F_z &= \frac{kq\lambda}{r}\left(\frac{1}{\sqrt{\frac{1}{4}+\left(\frac{r}{L}\right)^2}}\right) \\
&= \frac{2kq\lambda}{r}
\end{aligned}
$$

3 Gauss' Law

Recitation 2 – Exercise 2.

A ball of radius a is charged with distribution $\rho = \rho_0 \frac{r}{a}$. Find the electric field everywhere.

Recitation 2 – Solution 2.
Consider a spherical Gaussian surface of radius r.
If $r \leq a$, the charge in the interior of the Gaussian surface is

$$\begin{aligned} q(r) &= \int_0^r \frac{\rho_0 r}{a} \cdot 4\pi r^2 \, \mathrm{d}r \\ &= \frac{\rho_0}{a} \pi r^4 \end{aligned}$$

Therefore, by Gauss' Law,

$$\begin{aligned} E \cdot 4\pi r^2 &= \frac{q(r)}{\varepsilon_0} \\ \therefore E &= \frac{\rho_0 \pi r^4}{4\pi a r^2} \\ &= \frac{\rho_0 r^2}{4a\varepsilon_0} \end{aligned}$$

If $r \geq a$, the entire ball of charge is in the interior of the Gaussian surface. Therefore,

$$\begin{aligned} Q &= q(a) \\ &= \frac{\rho_0}{a} \cdot \pi a^4 \\ &= \rho_0 \pi a^3 \end{aligned}$$

Therefore, by Gauss' Law,

$$\begin{aligned} E \cdot 4\pi r^2 &= \frac{Q}{\varepsilon_0} \\ \therefore E &= \frac{Q}{4\pi r^2 \varepsilon_0} \\ &= \frac{\rho_0 a^3}{4r^2 \varepsilon_0} \end{aligned}$$

Therefore,

$$E = \begin{cases} \dfrac{\rho_0 r^2}{4a\varepsilon_0} & ; \quad r \leq a \\ \dfrac{\rho_0 a^3}{4r^2\varepsilon} & ; \quad r \geq a \end{cases}$$

Recitation 2 – Exercise 3.

An infinitely long cylinder of radius a is charged with distribution $\rho = \rho_0 \dfrac{r}{a}$. Find the electric field everywhere.

Recitation 2 – Solution 3.

Consider a infinite cylindrical Gaussian surface with radius r.
If $r \leq a$, the charge in the interior of the Gaussian surface is

$$\begin{aligned} q(r) &= \int_0^r \frac{\rho_0 r}{a} \pi r^2 \, \mathrm{d}r \\ &= \frac{2\pi\rho_0 L r^3}{3a} \end{aligned}$$

Therefore, by Gauss' Law,

$$\begin{aligned} E \cdot 2\pi r L &= \frac{2\pi\rho_0 L r^3}{3a\varepsilon_0} \\ \therefore E &= \frac{\rho_0 r^2}{3a\varepsilon_0} \end{aligned}$$

If $r \geq a$, the entire cylinder of charge is in the interior of the Gaussian surface. Therefore,

$$\begin{aligned} Q &= q(a) \\ &= \frac{2\pi\rho_0 L a^3}{3a} \\ &= \frac{2\pi\rho_0 L a^2}{3} \end{aligned}$$

Therefore, by Gauss' Law,

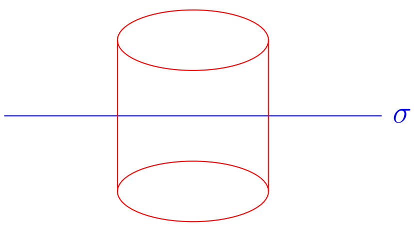
$$\begin{aligned} E \cdot 2\pi r L &= \frac{2\pi\rho_0 L a^2}{3\varepsilon_0} \\ \therefore E &= \frac{\rho_0 a^2}{3\varepsilon_0 r} \end{aligned}$$

Therefore,

$$E = \begin{cases} \dfrac{\rho_0 r^2}{3a\varepsilon_0} & ; \quad r \leq a \\ \dfrac{\rho_0 a^2}{3\varepsilon_0 r} & ; \quad r \geq a \end{cases}$$

Recitation 2 – Exercise 4.
Find the electric field due to a thin infinite plane of uniform charge distribution σ.

Recitation 2 – Solution 4.
Consider a cylindrical Gaussian surface, with ends of area A, as shown.



The charge in the interior of the surface is

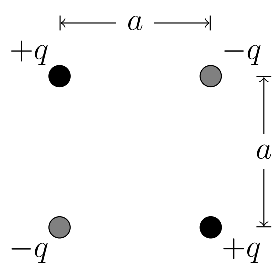
$$\mathrm{d}q = A\sigma$$

Therefore, by Gauss' Law,

$$\begin{aligned} E_1 \cdot A_1 + E_2 \cdot A_2 &= \frac{A\sigma}{\varepsilon_0} \\ \therefore 2EA &= \frac{A\sigma}{\varepsilon_0} \\ \therefore E &= \frac{\sigma}{2\varepsilon_0} \end{aligned}$$

4 Electric Potential

Recitation 3 – Exercise 2.
A system of four charges is constructed as shown.



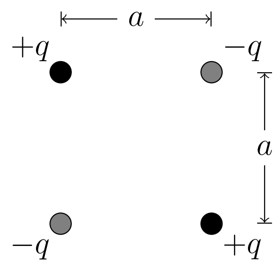
a) Calculate the work needed to build this system.

b) What is the potential in the centre of the system?

c) Calculate the potential in each of the corners (calculate as if there is no charge in the corner you are calculating for).

Recitation 3 – Solution 2.

a) Let the positions of the charges be A, B, C, D.



The work done to bring the first charge from infinity to A is

$$W_{\mathrm{A}} = 0$$

The work done to bring the first charge from infinity to B is

$$W_{\mathrm{B}} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a\sqrt{2}}$$

Similarly for the other two charges.
Therefore,

$$\begin{aligned} W &= 0 + \frac{q^2}{4\sqrt{2}\pi\varepsilon_0} + \frac{-2q^2}{4\pi\varepsilon_0 a} + \left(\frac{-2q^2}{4\pi\varepsilon_0 a} + \frac{q^2}{4\sqrt{2}\pi\varepsilon_0}\right) \\ &= \frac{q^2}{2\sqrt{2}\pi\varepsilon_0} - \frac{q^2}{\pi\varepsilon_0 a} \end{aligned}$$

b)

$$
\begin{aligned}
V_{\text{centre}} &= V_q + V_q + V_{-q} + V_{-q} \\
&= \frac{q}{4\pi\varepsilon_0\left(\dfrac{a}{\sqrt{2}}\right)} + \frac{q}{4\pi\varepsilon_0\left(\dfrac{a}{\sqrt{2}}\right)} - \frac{q}{4\pi\varepsilon_0\left(\dfrac{a}{\sqrt{2}}\right)} - \frac{q}{4\pi\varepsilon_0\left(\dfrac{a}{\sqrt{2}}\right)} \\
&= 0
\end{aligned}
$$

c)

$$
\begin{aligned}
V_{\text{A}} &= \frac{1}{4\pi\varepsilon_0}\frac{-q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{2}a} \\
V_{\text{B}} &= \frac{1}{4\pi\varepsilon_0}\frac{-q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{2}a} \\
V_{\text{C}} &= \frac{1}{4\pi\varepsilon_0}\frac{q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{\sqrt{2}a} \\
V_{\text{D}} &= \frac{1}{4\pi\varepsilon_0}\frac{q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{\sqrt{2}a}
\end{aligned}
$$

Recitation 3 – Exercise 3.

A ring of radius R is charged with total charge Q.

a) Calculate the electric field in the centre of the ring.

b) Calculate the potential in the centre of the ring by integrating the contributions of the infinitesimal charge elements of the ring.

Recitation 3 – Solution 3.

a) Due to the symmetry of the ring, the field due to every elemental charge dq will be cancelled out by the field due to a elemental charge diametrically opposite to dq.
Therefore,

$$\vec{E} = 0$$

b)

$$
\begin{aligned}
\mathrm{d}V &= \frac{\mathrm{d}q}{4\pi\varepsilon_0 R} \\
\therefore V &= \frac{Q}{4\pi\varepsilon_0 R}
\end{aligned}
$$

Recitation 3 – Exercise 4.

Calculate the potential resulting from a ball charged with constant volume distribution ρ. Use the expression

$$\varphi(r_2) - \varphi(r_1) = -\int_{r_1}^{r_2} E(r)\,\mathrm{d}r$$

Repeat the calculation twice:

a) Set $\varphi(r = R) = 0$

b) Set $\varphi(r = \infty) = 0$

Recitation 3 – Solution 4.

a) Let $\varphi(r=\infty)=0$.

If $r>R$,

$$
\begin{aligned}
E &= -\frac{\mathrm{d}\varphi}{\mathrm{d}r} \\
\therefore -\frac{\mathrm{d}\varphi}{\mathrm{d}r} &= \frac{Q}{4\pi\varepsilon_0 r^2} \\
\therefore \int \mathrm{d}\varphi &= \int\limits_{\infty}^{r} \frac{Q}{4\pi\varepsilon_0 r^2}\,\mathrm{d}r \\
\therefore \varphi(r)-\varphi(\infty) &= \frac{Q}{4\pi\varepsilon_0 r} - 0 \\
\therefore \varphi(r) &= \frac{Q}{4\pi\varepsilon_0 r}
\end{aligned}
$$

If $r<R$,

$$
\begin{aligned}
E &= \frac{q(r)}{4\pi\varepsilon_0 r^2} \\
&= \frac{\rho\cdot\frac{4}{3}\pi r^3}{4\pi\varepsilon_0 r^2} \\
&= \frac{\rho r}{3\varepsilon_0}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
E &= -\frac{\mathrm{d}\varphi}{\mathrm{d}r} \\
\therefore \int \mathrm{d}\varphi &= \int\limits_{r}^{R} \frac{\rho}{3\varepsilon_0} r\,\mathrm{d}r \\
\therefore \varphi(R)-\varphi(r) &= \frac{\rho}{6\varepsilon_0}\left(r^2-R^2\right) \\
\therefore \varphi(r) &= \frac{Q}{4\pi\varepsilon_0 R} + \frac{\rho\left(R^2-r^2\right)}{6\varepsilon_0} \\
&= \frac{Q}{8\pi\varepsilon_0 R}\left(3-\frac{r^2}{R^2}\right)
\end{aligned}
$$

b) Let $\varphi(r = R) = 0$.

$$\begin{aligned}
\therefore \varphi(R) - \varphi(r) &= \frac{\rho}{6\varepsilon_0}\left(r^2 - R^2\right) \\
\therefore \varphi(r) &= \frac{\rho}{6\varepsilon_0}\left(R^2 - r^2\right) \\
&= \frac{Q}{8\pi R\varepsilon_0}\left(1 - \frac{r^2}{R^2}\right)
\end{aligned}$$

Recitation 4 – Exercise 1.
A point charge Q is surrounded by a spherical grounded shell of radius R_1.

1. What is the charge accumulated on the shell? Where did it come from?
2. The entire system is covered with another spherical shell of radius R_2 and charged with q. What will be the charge accumulated on the grounded shell?

Recitation 4 – Solution 1.
1. The charge accumulated on the shell comes from the ground.
   Let the charge on the shell be Q_1.
   As the shell is grounded, the net potential on it must be zero.

$$\begin{aligned}
\varphi &= \varphi \text{ due to } Q + \varphi \text{ due to } Q' \\
\therefore 0 &= \frac{Q}{4\pi\varepsilon_0 R_1} + \frac{Q_1}{4\pi\varepsilon_0 R_1} \\
\therefore Q_1 &= -Q
\end{aligned}$$

2. Let the charge on the grounded shell be Q_2.

$$\begin{aligned}
\varphi &= \varphi \text{ due to } Q + \varphi \text{ due to } Q_2 + \varphi \text{ due to } q \\
\therefore 0 &= \frac{Q}{4\pi\varepsilon_0 R_1} + \frac{Q_2}{4\pi\varepsilon_0 R_1} + \frac{q}{4\pi\varepsilon_0 R_2} \\
\therefore Q_2 &= Q - \frac{qR_1}{R_2}
\end{aligned}$$

Recitation 4 – Exercise 3.

A point charge Q is set in the center (same distance from all corners) of a perfect tetrahedron. The bottom face of the tetrahedron is uniformly charged with charge density σ.

Recitation 4 – Solution 3.

1. Consider a spherical Gaussian surface passing through the vertices of the tetrahedron.
   Therefore by Gauss' Law, the total flux passing through the sphere, due to Q is $\oiint \overrightarrow{E} \cdot \mathrm{d}A$.
   Hence, by symmetry, the flux through every surface of the tetrahedron, due to Q is

$$\begin{aligned}\Phi &= \frac{1}{4} \oiint \overrightarrow{E} \cdot \mathrm{d}A \\ &= \frac{1}{4} \frac{Q}{\varepsilon_0}\end{aligned}$$

   The flux on the bottom face due to the bottom face itself is zero. Therefore the total flux through the bottom face is the flux due to Q only.

2. As the charge on Q and the bottom face of the tetrahedron are similar in charge, the force between them is repulsive in nature. Hence, the force on the bottom face is directed downwards.

3. Let the area of the bottom face

$$\begin{aligned}F &= (\sigma A)E \\ &= \sigma(EA)\end{aligned}$$

   As Q is exactly above the centre of the bottom face, due to symmetry, $\oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A} = EA$

$$\begin{aligned}\therefore F &= \sigma \left(\oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A} \right) \\ &= F \left(\frac{Q}{4\varepsilon_0} \right)\end{aligned}$$

Recitation 4 – Exercise 4.
The following electric field is given:

$$\overrightarrow{E} = \alpha y^2\hat{\imath} + \alpha(2xy + z^2)\hat{\jmath} + 2\alpha yz\hat{k}$$

Calculate the potential at âČŮ$r = (x, y, z)$, set the potential at the origin to be zero.

Recitation 4 – Solution 4.
Let $\varphi(0, 0, 0) = 0$.
Therefore,

$$\begin{aligned}
E_x &= -\frac{\mathrm{d}\varphi}{\mathrm{d}x} \\
&= \alpha y^2 \\
\therefore \varphi &= -\alpha xy^2 + f_1(y, z) \\
E_y &= -\frac{\mathrm{d}\varphi}{\mathrm{d}y} \\
&= \alpha(2xy + z^2) \\
\therefore \varphi &= -\alpha xy^2 - \alpha z^2 y + f_1(x, z) \\
E_z &= -\frac{\mathrm{d}\varphi}{\mathrm{d}z} \\
&= 2\alpha yz \\
\therefore \varphi &= -\alpha yz^2 + f_2(x, y)
\end{aligned}$$

Comparing the three expressions of φ,

$$\varphi = -\alpha xy^2 - \alpha yz^2 + c$$

As $\varphi(0, 0, 0) = 0$, $c = 0$.
Therefore,

$$\varphi = -\alpha xy^2 - \alpha yz^2$$

Recitation 4 – Exercise 5.
A thin rod of length L is charged with a uniform charge density λ is laid along the x-axis.

1. Calculate the electric potential along the x-axis (where $x > L$)

2. Calculate the electric field along the x-axis (where $x > L$)

3. A second identical thin rod is placed along the x-axis at distance L from the edge of the first rod. Calculate the force due to left rod, acting on the right one.

Recitation 4 – Solution 5.

1. Consider an elemental charge $\mathrm{d}q$ with length $\mathrm{d}x$ at a distance x from the origin.
   Therefore, the potential at a distance d from the origin is

$$\begin{aligned} \mathrm{d}\varphi &= \frac{\mathrm{d}q}{4\pi\varepsilon_0(L+d-x} \\ \therefore \int \mathrm{d}\varphi &= \int_0^L \frac{\lambda\,\mathrm{d}x}{4\pi\varepsilon_0(L+d-x)} \\ \therefore \varphi &= \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{L+d}{d}\right) \end{aligned}$$

2.

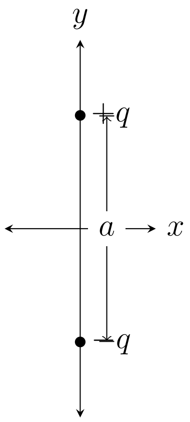
$$\begin{aligned} \mathrm{d}E &= \frac{\mathrm{d}\varphi}{\mathrm{d}r} \\ \therefore \mathrm{d}E &= \frac{\mathrm{d}q}{4\pi\varepsilon_0(L+d-x)^2} \\ \therefore E &= \frac{\lambda}{4\pi\varepsilon_0}\int_{L+d}^{d}\frac{\mathrm{d}u}{u^2} \\ &= \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{L}{(L+d)(d)}\right) \end{aligned}$$

3. Consider an elemental charge $\mathrm{d}q$with length $\mathrm{d}x$, on the second rod, at a distance x from the end of the first rod.

$$
\begin{aligned}
\mathrm{d}F &= \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{L}{(L+x)(x)}\right)\mathrm{d}q \\
&= \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{L}{(L+x)(x)}\right)\lambda\,\mathrm{d}x \\
\therefore F &= \int\limits_{L}^{2L} \frac{\lambda^2}{4\pi\varepsilon_0}\left(\frac{L}{(L+x)(x)}\right)\mathrm{d}x \\
&= \frac{\lambda^2}{4\pi\varepsilon_0}\ln\frac{x}{L+x}\Bigg|_{L}^{2L} \\
&= \frac{\lambda^2}{4\pi\varepsilon_0}\left(\ln\frac{2}{3}-\ln\frac{1}{2}\right) \\
&= \frac{\lambda^2}{4\pi\varepsilon_0}\left(\ln\frac{4}{3}\right)
\end{aligned}
$$

Recitation 5 – Exercise 1.

An electric dipole is comprised of two opposite charges q and $-q$ positioned at distance a from each other as shown.



1. Calculate the electric field along the y-axis.
2. What is the electric field when $y >> a$?
3. Repeat the previous sub-questions for points along the x-axis.

Recitation 5 – Solution 1.

1.

$$\overrightarrow{E_y} = \frac{q}{4\pi\varepsilon_0\left(y-\frac{a}{2}\right)^2}\hat{\jmath} - \frac{q}{4\pi\varepsilon_0\left(y+\frac{a}{2}\right)^2}\hat{\jmath}$$
$$= \frac{q}{4\pi\varepsilon_0 y^2}\left(\left(1-\frac{d}{2y}\right)^{-2} - \left(1-\frac{d}{2y}\right)^{-2}\right)\hat{\jmath}$$

2. By the Binomial theorem, if $x << 1$, $(1 \pm x)^n \approx 1 \pm nx)$.
   Therefore, as $y << d$,

$$\left(1 \pm \frac{d}{2y}\right)^2 = \left(1 \pm \frac{2d}{2y}\right)$$

   Therefore,

$$\overrightarrow{E_y} = \frac{q}{4\pi\varepsilon_0 y^2}\left(1+\frac{d}{y}-1+\frac{d}{y}\right)\hat{\jmath}$$
$$= \frac{2\left(q\hat{\jmath}d\right)}{4\pi\varepsilon_0 y^2\cdot y}$$
$$= \frac{2\overrightarrow{p}}{4\pi\varepsilon_0 y^3}$$

3. Let the distance between a point $(x,0)$ and each of the charges be r. Let the angle between the x-axis and the line joining a point $(x,0)$ and $+q$ or $-q$ be θ.
   Therefore,

$$\sin(\theta) = \frac{d}{2r}$$

$$\overrightarrow{E_x} = -2\frac{q}{4\pi\varepsilon_0 r^2}\sin(\theta)\hat{\jmath}$$
$$= -\frac{qd}{4\pi\varepsilon_0 r^3}\hat{\jmath}$$
$$= -\frac{\overrightarrow{p}}{4\pi\varepsilon_0 r^3}$$

   If $x >> d$, $r = d$. Therefore,

$$\overrightarrow{E_x} = -\frac{\overrightarrow{p}}{4\pi\varepsilon_0 r^3}$$

Recitation 5 – Exercise 2.
Calculate the following expressions using both spherical and Cartesian coordinates

1. $\overrightarrow{\nabla} r = \hat{r}$
2. $\overrightarrow{\nabla} \frac{1}{r} = -\frac{\hat{r}}{r^2}$

Recitation 5 – Solution 2.
1. Using Cartesian coordinates,

$$\begin{aligned}\overrightarrow{r} &= x\hat{\imath} + y\hat{\jmath} + z\hat{k} \\ \therefore r &= \sqrt{x^2+y^2+z^2}\end{aligned}$$

Therefore,

$$\begin{aligned}\overrightarrow{\nabla}(r) &= \hat{\imath}\,\frac{\partial r}{\partial x} + \hat{\jmath}\,\frac{\partial r}{\partial y} + \hat{k}\,\frac{\partial r}{\partial z} \\ &= \hat{\imath}\,\frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} + \hat{\jmath}\,\frac{\partial\sqrt{x^2+y^2+z^2}}{\partial y} + \hat{k}\,\frac{\partial\sqrt{x^2+y^2+z^2}}{\partial z} \\ &= \hat{\imath}\frac{x}{\sqrt{x^2+y^2+z^2}} + \hat{\jmath}\frac{y}{\sqrt{x^2+y^2+z^2}} + \hat{k}\frac{z}{\sqrt{x^2+y^2+z^2}} \\ &= \frac{\overrightarrow{r}}{r} \\ &= \hat{r}\end{aligned}$$

Using spherical coordinates,

$$\begin{aligned}\overrightarrow{\nabla}(r) &= \hat{r}\overbrace{\cancel{\frac{\partial r}{\partial r}}}^{1} + \hat{\theta}\frac{1}{r}\overbrace{\cancel{\frac{\partial r}{\partial \theta}}}^{0} + \hat{\varphi}\frac{1}{r\sin\theta}\overbrace{\cancel{\frac{\partial r}{\partial \varphi}}}^{0} \\ &= \hat{r}\end{aligned}$$

2. Using Cartesian coordinates,

$$\begin{aligned}\overrightarrow{\nabla}\left(\frac{1}{r}\right) &= \hat{\imath}\,\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + \hat{\jmath}\,\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + \hat{k}\,\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) \\ &= -\left(\hat{\imath}\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \hat{\jmath}\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \hat{k}\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) \\ &= -\frac{\overrightarrow{r}}{r^3} \\ &= -\frac{\hat{r}}{r^2}\end{aligned}$$

Using spherical coordinates,

$$\vec{\nabla}\left(\frac{1}{r}\right) = \hat{r}\,\frac{\partial}{\partial r}\left(\frac{1}{r}\right) + \hat{\theta}\frac{1}{r}\cancelto{0}{\frac{\partial}{\partial\theta}\left(\frac{1}{r}\right)} + \hat{\varphi}\frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial}{\partial\varphi}\left(\frac{1}{r}\right)}$$
$$= \hat{r}\frac{-1}{r^2}$$
$$= -\frac{\hat{r}}{r^2}$$

5 Differential Form of Gauss' Law

Recitation 5 – Exercise 3.
The following potential is given in cylindrical coordinates

$$\varphi(r) = \begin{cases} -\frac{\rho_0 r^2}{8\varepsilon_0} - \frac{\rho_0 r R}{4\varepsilon_0} & ; \quad r < R \\ -\frac{\rho_0 R^2}{2\varepsilon_0}\ln\left(\frac{r}{R}\right) + c & ; \quad r > R \end{cases}$$

1. Find the constant c.
2. Calculate the electric field $\vec{E}$ everywhere.
3. Calculate the total charge, Q, inside a section with height H.
4. Calculate the charge density ρ using the differential Gauss' Law.
5. Show that an integral on the density you found , i.e. ρ, gives the total charge you calculated above.

Recitation 5 – Solution 3.
1. In cylindrical coordinates,

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$z = z$$

where φ is the angle between $\vec{r}$ and the x-axis.
Let

$$-\frac{\rho_0 r^2}{8\varepsilon_0} - \frac{\rho_0 r R}{4\varepsilon_0} = \varphi_1(r)$$
$$-\frac{\rho_0 R^2}{2\varepsilon_0}\ln\left(\frac{r}{R}\right) + c = \varphi_2(r)$$

Therefore,

$$\varphi(r) = \begin{cases} \varphi_1(r) & ; \quad r < R \\ \varphi_2(r) & ; \quad r > R \end{cases}$$

At the surface the values of φ_1 and φ_2 must be equal. Therefore,

$$\begin{aligned} \varphi_1(R) &= \varphi_2(R) \\ \therefore -\frac{\rho_0 R^2}{8\varepsilon_0} - \frac{\rho_0 R \cdot R}{4\varepsilon_0} &= -\frac{\rho_0 R^2}{2\varepsilon_0} \ln\left(\frac{R}{R}\right) + c \\ \therefore c &= -\frac{3\rho_0 R^2}{8\varepsilon_0} \end{aligned}$$

2.

$$\begin{aligned} E_1 &= -\frac{\mathrm{d}\varphi_1}{\mathrm{d}r} \\ &= \frac{\rho_0(r+R)}{4\varepsilon_0} \\ E_2 &= -\frac{\mathrm{d}\varphi_2}{\mathrm{d}r} \\ &= \frac{\rho_0 R^2}{2\varepsilon_0 r} \end{aligned}$$

3. Consider a cylindrical Gaussian surface with radius just larger than R and height H.
Therefore by Gauss' Law,

$$\begin{aligned} \oiint \overrightarrow{E}\,\mathrm{d}\overrightarrow{S} &= \frac{Q}{\varepsilon_0} \\ \therefore \frac{\rho_0 R^2}{2\varepsilon_0 R} \cdot 2\pi R^2 H &= \frac{Q}{\varepsilon_0} \\ \therefore Q &= \frac{\rho_0 \pi R^2 H}{\varepsilon_0} \end{aligned}$$

4. By the differential form of Gauss' Law,

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\varepsilon_0} \\
\vec{\nabla} \cdot \vec{E} &= \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial}{\partial \varphi}(E_\varphi) + \frac{\partial}{\partial z}(E_z) \\
\therefore \vec{\nabla} \cdot \vec{E_1} &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\rho_0}{4\varepsilon_0}(r+R)\right) + \frac{1}{r}\frac{\partial}{\partial \varphi}\cancelto{0}{(E_\varphi)} + \frac{\partial}{\partial z}\cancelto{0}{(E_z)} \\
\therefore \frac{\rho}{\varepsilon_0} &= \frac{\rho_0}{4\varepsilon_0 r}(2r+R) \\
\therefore \rho &= \frac{\rho_0}{4}\left(2 + \frac{R}{h}\right)
\end{aligned}
$$

5.

$$
\begin{aligned}
Q &= \int_0^R \rho(r) \cdot 2\pi r H \, \mathrm{d}r \\
&= \frac{2\rho_0 \pi H}{4}\left(\frac{(2r)^2}{2} + Rr\right)\Bigg|_0^R \\
&= \pi R^2 H \rho_0
\end{aligned}
$$

Recitation 5 – Exercise 4.

1. Calculate the potential resulting from a solid ball of radius R charged uniformly with constant distribution ρ.

2. Calculate the potential along the z-axis resulting from a solid cylinder of radius a and height L charged uniformly with constant distribution $-\rho$.

Recitation 5 – Solution 4.

1.

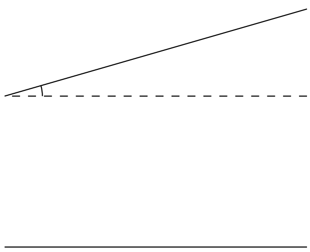
$$
q(r) = \begin{cases} \rho_0 \cdot \frac{4}{3}\pi r^3 & ; \quad r < R \\ \rho_0 \cdot \frac{4}{3}\pi R^3 & ; \quad r > R \end{cases} \therefore \varphi(r) \quad = \begin{cases} \frac{Q}{4\pi\varepsilon_0 r} & ; \quad r > R \\ \frac{Q}{8\pi\varepsilon_0 R^3}(3R^2 - r^2 & ; \quad r < R \end{cases}
$$

2. Consider an elemental disk of charge $\mathrm{d}^2 q$ at height z from the centre of the cylinder.

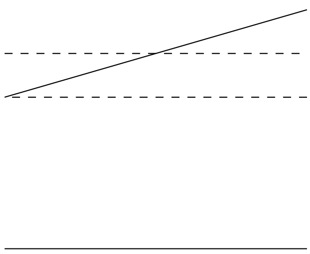
Therefore, the potential along the z-axis is

$$V(z') = \int_{r=0}^{r=a} \int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \frac{\mathrm{d}^2 q}{4\pi\varepsilon_0(z'-z)}$$

Recitation 6 – Exercise 1.
A plate capacitor which is made of square plates of sides a fell and a little angle θ formed between its plates as shown. The smallest distance between the plates is d. Calculate the new capacitance.



Recitation 6 – Solution 1.
The tilted capacitor plate can be considered to be approximately equivalent to a parallel plate at height $d + \frac{a \tan\theta}{2}$, i.e. the tilted plate can be considered to be made parallel to the other plate by pivoting it at the axis through its midpoint.



The capacitance of the original capacitor is

$$\begin{aligned} C &= \frac{Q}{V} \\ &= \frac{\varepsilon_0 A}{d} \end{aligned}$$

Therefore, the capacitance of the tilted capacitor is

$$C' = \frac{\varepsilon_0 a^2}{\left(d + \frac{a\tan\theta}{2}\right)}$$

As $\theta << 1$, $\tan\theta \approx \theta$.

$$\therefore C' = \frac{\varepsilon_0 A}{d}\left(1 + \frac{a\theta}{2d}\right)^{-1}$$

Alternatively, the tilted capacitor can be considered to be a capacitor with capacitance varying with x.
Considering the origin to be at the left end of the lower plate, the equation of the tilted plate is

$$\begin{aligned} y &= d + mx \\ &= d + \tan\theta x \end{aligned}$$

As $\theta << 1$,

$$y = d + \theta x$$

Thererefore,

$$\begin{aligned} C &= \frac{\int\limits_0^L \frac{\varepsilon_0 A}{y}}{\int\limits_0^L \mathrm{d}x} \\ &= \frac{\int\limits_0^L \frac{\varepsilon_0 a\,\mathrm{d}x}{d+\theta x}}{L} \\ &= \frac{\frac{\varepsilon_0 A}{\theta}\int\limits_0^L \frac{\mathrm{d}u}{u}}{L} \\ &= \frac{\varepsilon_0 A}{a\theta}\ln\left(\frac{d+a\theta}{d}\right) \\ &= \frac{\varepsilon_0 A}{d}\left(\left(\frac{a\theta}{d}\right) - \frac{1}{2}\left(\frac{a\theta}{d}\right)^2 + \dots\right) \\ &= \frac{\varepsilon_0 A}{d} - \frac{\varepsilon_A}{2}\frac{a\theta}{d^2} + \dots \end{aligned}$$

Recitation 6 – Exercise 2.

A cylindrical capacitor is comprised of two concentric cylinders of length L and radii a and b (where $L >> a, b$ and $a < b$). The inner cylinder (radius a) carries total charge Q and the outer cylinder (radius b) is grounded. Assume vacuum inside the system and all bodies to be conducting.

1. Calculate the electric field everywhere.
2. Calculate the capacitance per unit length.
3. Calculate the energy density everywhere.

Recitation 6 – Solution 2.

1. As the bodies are condicting, the field inside the inner cylinder is 0. Consider a cylindrical Gaussian surface with radius $a < r < b$. Therefore, by Gauss' Law,

$$\oiint \vec{E} \cdot \mathrm{d}\vec{S} = \frac{Q}{\varepsilon_0}$$
$$\therefore E = \frac{Q}{2\pi\varepsilon_0 L r}$$

   As the outer cylinder is grounded, and due to the charge on the inner cylinder, the charge on its inner surface is $-Q$.
   Therefore, by Gauss's Law, the field outside must be 0.

2.

$$E = \frac{Q}{2\pi\varepsilon_0 L r}$$
$$\therefore V = \int_a^b \frac{Q}{2\pi\varepsilon_0 L r}\,\mathrm{d}r$$
$$= \frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right)$$
$$\therefore C = \frac{2\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)}$$

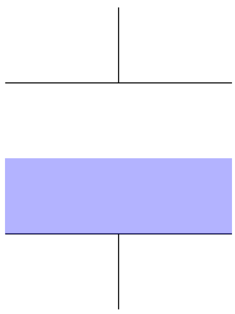
3.

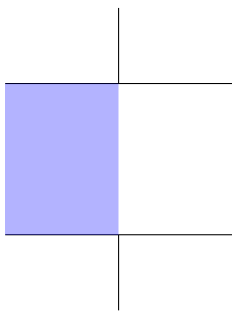
$$U = \frac{CV^2}{2}$$
$$= \frac{1}{2}\frac{2\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)}\left(\frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right)\right)^2$$

Therefore,

$$u = \frac{U}{\pi L(b^2 - a^2)}$$

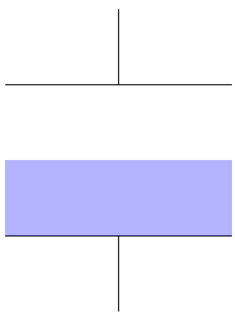
Recitation 6 – Exercise 3.
The capacitance of an empty plate capacitor (vacuum between the plates) is C_0. Half of the capacitor volume is filled with a dielectric material of constant κ in two different ways as shown.





Calculate the new capacitance in the two cases.

Recitation 6 – Solution 3.
The arrangements are equivalent to connection of capacitors in series and parallel respectively.



$\equiv$

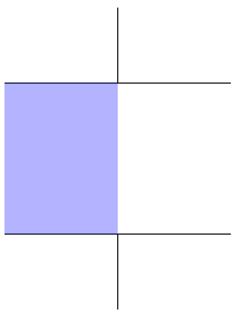
In this case,

$$C_1 = \frac{\varepsilon_0 A}{\frac{d}{2}}$$

$$C_2 = \frac{\kappa \varepsilon_0 A}{\frac{d}{2}}$$

Therefore,

$$C_{\text{equivalent}} = \frac{C_1 C_2}{C_1 + C_2}$$



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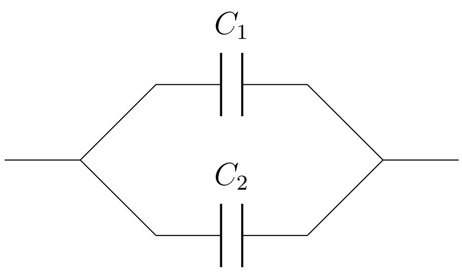

In this case,

$$C_1 = \frac{\varepsilon_0 \frac{A}{2}}{d}$$

$$C_2 = \frac{\kappa \varepsilon_0 \frac{A}{2}}{d}$$

Therefore,

$$C_{\text{equivalent}} = C_1 + C_2$$