QUANTUM AND SOLID STATE PHYSICS : ASSIGNMENT 10

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Exercise 1.

Consider two silicon samples at room temperature, doped P-type with $N_A = 10^{16} \frac{1}{\text{cm}^3}$. Light shines on the left side of the sample, such that a steady state excess carrier concentration of $10^{14} \frac{1}{\text{cm}^3}$ is maintained at $x = 0$, and $G_{\text{optical}} = 0$, throughout the rest of the sample.
Sample 1 is short and has a metal contact at $x = L_1 << L_n$.
Sample 2 is short and has a metal contact at $x = L_2 >> L_n$.

(1) For both samples, which of the following shows the correct directions of electron diffusion and electron diffusion current density $J_{\text{diffusion}_n}$ in the samples.
   (a) electron diffusion $\rightarrow$ $J_{\text{diffusion}_n}$ $\rightarrow$
   (b) electron diffusion $\rightarrow$ $J_{\text{diffusion}_n}$ $\leftarrow$
   (c) electron diffusion $\leftarrow$ $J_{\text{diffusion}_n}$ $\rightarrow$

(2) If 10^{19} EHPs are generated per second, i.e. $G_{\text{optical}} = 10^{19} \frac{1}{\text{cm}^3 \text{s}}$, calculate the minority carrier lifetime in µs.

(3) On the same axes, plot $n(x)$, the minority carrier concentration, in both samples, as a function of position.

(4) If the length of sample 1 is one hundredth of the elecetron diffusion length, i.e.

$$L_n = 100 L_1$$

whet is the correct expression for electron current density, $J_{\text{diffusion}_n}(x)$?
   (a) $J_{\text{diffusion}_n}(x) = -100 q G_{\text{optical}} L_n e^{-\frac{x}{L_n}}$
   (b) $J_{\text{diffusion}_n}(x) = -100 q G_{\text{optical}} {L_n}^2$
   (c) $J_{\text{diffusion}_n}(x) = -\frac{1}{100} q G_{\text{optical}} L_n$
   (d) $J_{\text{diffusion}_n}(x) = -100 q G_{\text{optical}} L_n$

(5) For sample 2, what is the electron diffusion current density at $x = 0$?
   (a) $J_{\text{diffusion}_n}(x = 0) = -q G_{\text{optical}} L_n$
   (b) $J_{\text{diffusion}_n}(x = 0) = -100 q G_{\text{optical}} L_n$
   (c) $J_{\text{diffusion}_n}(x = 0) = -\frac{1}{100} q G_{\text{optical}} L_n$

(6) On the same axes, plot the magnitude of electron diffusion current density $\left| J_{\text{diffusion}_n}(x) \right|$ in both samples as a function of position.

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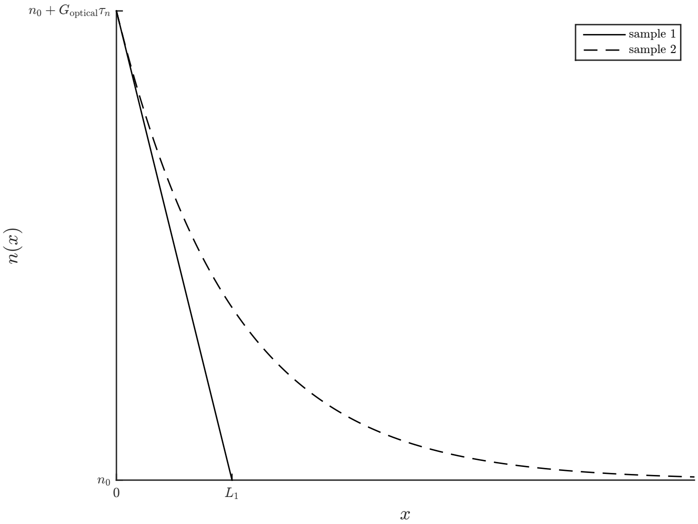
Solution 1.

(1) As the samples are illuminated on the left only, there is an excess of electrons on the left, as compared to the right. Therefore, the direction of electron diffusion will be $\rightarrow$, and hence, the direction of $J_{\text{diffusion}_n}$ will be $\leftarrow$.

(2)

$$
\begin{aligned}
\hat{n} &= G_{\text{optical}} \tau_p \\
\therefore 10^{14} &= 10^{19} \tau_p \\
\therefore \tau_p &= 10^{-5}\text{s} \\
\therefore \tau_p &= 10\text{µs}
\end{aligned}
$$



(3)

(4)

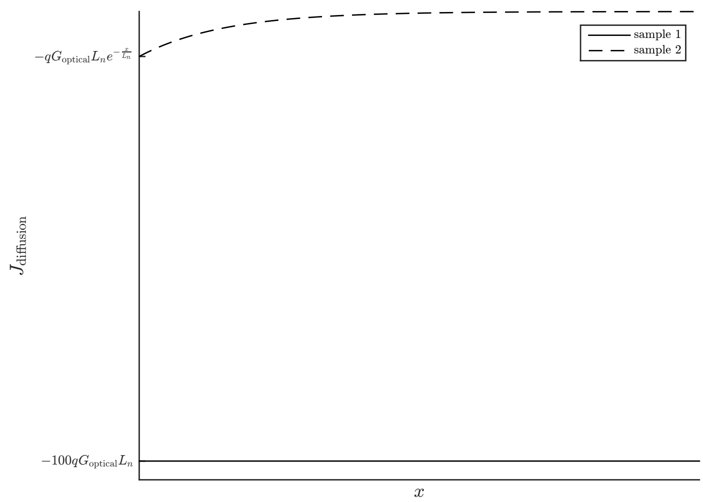
$$
L_n = 100 L_1
$$

Therefore, $\hat{n}(x)$ is linear.
Therefore,

$$
\begin{aligned}
J_{\text{diffusion}_n} &= q D_n \frac{\mathrm{d}\hat{n}}{\mathrm{d}x} \\
&= -q D_n \frac{G_{\text{optical}} \tau_n}{L_1} \\
&= -100 q D_n G_{\text{optical}} \frac{\tau_n}{L_n} \\
&= -100 q G_{\text{optical}} \frac{{L_n}^2}{L_n} \\
&= -100 q G_{\text{optical}} L_n
\end{aligned}
$$

(5) The sample is long. Therefore,

$$
\begin{aligned}
J_{\text{diffusion}_n} &= qD_n \frac{\partial \hat{n}}{\partial x} \\
&= qD_n \frac{\partial}{\partial x} \left(G_{\text{optical}} \tau_n e^{-\frac{x}{L_n}} \right) \\
&= -\frac{q}{L_n} D_n G_{\text{optical}} \tau_n e^{-\frac{x}{L_n}} \\
&= -\frac{q}{L_n} G_{\text{optical}} {L_n}^2 e^{-\frac{x}{L_n}} \\
\therefore J_{\text{diffusion}_n}(0) &= -\frac{q}{L_n} D_n G_{\text{optical}} \tau_n \\
&= -\frac{q}{L_n} G_{\text{optical}} {L_n}^2 \\
&= -qG_{\text{optical}} L_n
\end{aligned}
$$



(6)

Exercise 2.

Two ends of a uniformly doped N-types silicon bar, of length L, are simultaneously illuminated so as to create γN_D excess holes at both $x = 0$ and $x = L$, i.e.,

$$
\begin{aligned}
\hat{p}_n(0) &= \gamma N_D \\
\hat{p}_n(L) &= \gamma N_D
\end{aligned}
$$

where $\gamma = 10^{-3}$.
Light is absorbed only at $x = 0$ and $x = L$, and no light penetrates into the interior of the bar, i.e. for $0 < x < L$, $G_{\text{optical}} = 0$.
Assume that the illumination is steady state, the temperature is 300 K, $N_D >> n_i$ and that the bar is long, i.e. $L >> L_n, L_p$.

(1) Is the carrier generation inside this material low level injection? Explain your answer.
(2) Sketch the general form of the excess concentration profile $\hat{p}(x)$ across the bar.
(3) Write the differential equation from which we can find the solution of $\hat{p}(x)$, and the boundary conditions for this problem.

Solution 2.

(1)

$$n_0 = N_D$$

Therefore,

$$\begin{aligned} p_0 &= \frac{{n_i}^2}{N_D} \\ &= \frac{10^{20}}{N_D} \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{p}(0) &= \gamma N_D \\ &= 10^{-3} N_D \end{aligned}$$

Therefore,

$$\hat{p}(0) < N_D$$
$$\hat{p}(L) < N_D$$

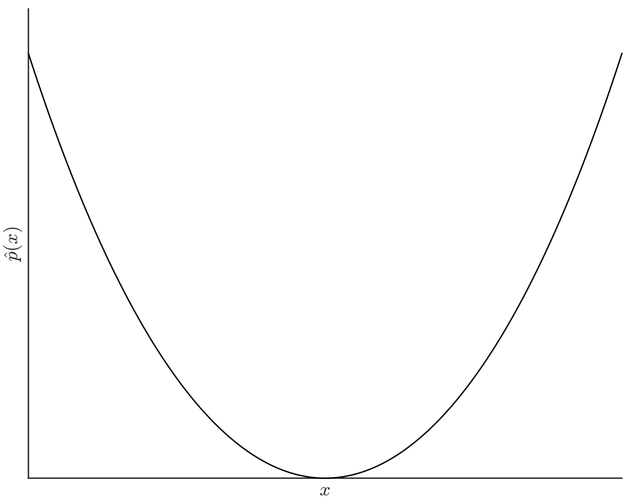
For $0 < x < L$,

$$\hat{p}(x) < \hat{p}(0)$$

Therefore,

$$\hat{p} < N_D$$

Therefore, the carrier generation is low level injection.



(2)
(3)

$$\begin{aligned}
\frac{\partial \hat{p}(x,t)}{\partial t} &= -\frac{1}{q}\frac{\partial J_p(x,t)}{\partial x} + G_{\text{optical}} - \frac{\hat{p}(x,t)}{\tau_p} \\
\therefore 0 &= -\frac{1}{q}\frac{\partial J_p(x,t)}{\partial x} - \frac{\hat{p}(x,t)}{\tau_p} \\
&= -\frac{1}{q}\frac{\partial}{\partial x}\left(-qD_p\frac{\partial \hat{p}}{\partial x}\right) - \frac{\hat{p}}{\tau_p} \\
&= \frac{\partial^2 \hat{p}}{\partial x^2} - \frac{\hat{p}}{\tau_p}
\end{aligned}$$

Exercise 3.

(1) Consider the nth eigenfunction of the one-dimensional harmonic oscillator. Using the lowering and raising operators, calculate the following
  (a) $\langle x \rangle$
  (b) $\langle p \rangle$
  (c) $\langle x^2 \rangle$
  (d) $\langle p^2 \rangle$
  (e) σ_x
  (f) σ_p

  Verify that the uncertainty principle is satisfied.

(2) Consider a particle of mass m in the following potential.

$$V(x) = \frac{1}{2}m\omega^2x^2 + \frac{1}{2}\hbar\omega$$

  Hint: Look at the answer to Question 2 in Homework 5.
  (a) What is the spacing in the ladder of eigenvalues of the energy operator for this potential?
  (b) What is the lowest possible energy value? Write an expression for the energy of the nth eigenfunction in this potential.
  (c) What is the eigenfunction corresponding to the lowest possible energy?

Solution 3.

(1) (a)

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} {\psi_n}^* x \psi_n \,\mathrm{d}x \\
&= \sqrt{\frac{\hbar}{2m\omega}} \int_{-\infty}^{\infty} {\psi_n}^* \left(\hat{a}_+ + \hat{a}_-\right) \psi_n \,\mathrm{d}x
\end{aligned}$$

Therefore, solving,

$$\langle x \rangle = 0$$

(b)

$$\begin{aligned}
\langle p \rangle &= \int_{-\infty}^{\infty} \psi_n{}^* \hat{p} \psi_n \, \mathrm{d}x \\
&= \sqrt{\frac{\hbar}{2m\omega}} \int_{-\infty}^{\infty} \psi_n{}^* \left(\hat{a}_+ - \hat{a}_- \right) \psi_n \, \mathrm{d}x
\end{aligned}$$

Therefore, solving,

$$\langle p \rangle = 0$$

(c)

$$\begin{aligned}
\left\langle x^2 \right\rangle &= \int_{-\infty}^{\infty} \psi_n{}^* x^2 \psi_n \, \mathrm{d}x \\
&= \sqrt{\frac{\hbar}{2m\omega}} \int_{-\infty}^{\infty} \psi_n{}^* \left(\hat{a}_+ + \hat{a}_- \right) \left(\hat{a}_+ + \hat{a}_- \right) \psi_n \, \mathrm{d}x
\end{aligned}$$

Therefore, solving,

$$\left\langle x^2 \right\rangle = \frac{\hbar}{2m\omega}(2n+1)$$

$$\begin{aligned}
\left\langle x^2 \right\rangle &= \int_{-\infty}^{\infty} \psi_n{}^* \hat{p}^2 \psi_n \, \mathrm{d}x \\
&= \sqrt{\frac{\hbar}{2m\omega}} \int_{-\infty}^{\infty} \psi_n{}^* \left(\hat{a}_+ - \hat{a}_- \right) \left(\hat{a}_+ - \hat{a}_- \right) \psi_n \, \mathrm{d}x
\end{aligned}$$

Therefore, solving,

$$\left\langle p^2 \right\rangle = \frac{\hbar m\omega}{2}(2n+1)$$

(d)

$$\begin{aligned}
\sigma_x &= \sqrt{\left\langle x^2 - \right\rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{\hbar}{2m\omega}(2n+1)}
\end{aligned}$$

(e)

$$\begin{aligned}
\sigma_p &= \sqrt{\left\langle p^2 - \right\rangle - \langle p \rangle^2} \\
&= \sqrt{\frac{\hbar}{2m\omega}(2n+1)}
\end{aligned}$$

Therefore,

$$\sigma_x\sigma_p = \frac{\hbar}{2m\omega}(2n+1)$$
$$\therefore \sigma_x\sigma_p \geq \frac{\hbar}{2}$$

Therefore, the uncertainty principle is satisfied.

(2) (a) Let $\psi(x)$ be a solution to

$$V_1(x) = \frac{1}{2}m\omega^2x^2$$

Therefore, as $\frac{1}{2}\hbar\omega$ is constant, $\psi(x)$ is also a solution to

$$V(x) = \frac{1}{2}m\omega^2x^2 + \frac{1}{2}\hbar\omega$$

Therefore, the spacing in the ladder of the eigenvalues of the energy operator for this potential is the same as a standard quantum harmonic oscillator, i.e. $\hbar\omega$.

(b) As the eigenfunctions of $V(x)$ and $V_1(x)$ are common,

$$\psi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$$

Therefore, the eigenvalues are the same as for a quantum harmonic oscillator, with $\frac{1}{2}\hbar\omega$ added. Therefore,

$$E_0 = \frac{1}{2}\hbar\omega + \frac{1}{2}\hbar\omega$$
$$= \hbar\omega$$

Therefore,

$$E_n = (n+1)\hbar w$$

Exercise 4.

(1) Show the following commutation relations.
   (a) $\left[\hat{L}_z, \hat{L}_+\right] = \hbar\hat{L}_+$
   (b) $\left[\hat{L}_z, \hat{L}_-\right] = -\hbar\hat{L}_-$
   (c) $\left[\hat{L}^2, \hat{L}_+\right] = 0$
   (d) $\left[\hat{L}^2, \hat{L}_-\right] = 0$

   where

$$\hat{L}_- = \hat{L}_x - i\hat{L}_y$$
$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y$$

(2) Explain what $\hat{L}_-$ does to an eigenfunction of the angular momentum in the z direction, i.e. $\hat{L}_z$. Hint: Apply $\hat{L}_z$ on $\hat{L}_-Y_{lm}$, where Y_{lm} is an eigenfunction of $\hat{L}_z$, i.e. $\hat{L}_zY_{lm} = \hbar mY_{lm}$.

(3) Explain what $\hat{L}_-$ does to an eigenfunction of the total angular momentum squared operator, i.e. $\hat{L}^2$. Hint: Apply $\hat{L}^2$ on $\hat{L}_-Y_{lm}$, where Y_{lm} is an eigenfunction of $\hat{L}^2$, i.e. $\hat{L}^2Y_{lm} = \hbar^2l(l+1)Y_{lm}$.

Solution 4.

(1) (a)

$$\begin{aligned}\left[\hat{L}_z, \hat{L}_+\right] &= \hat{L}_z\hat{L}_+ - \hat{L}_+\hat{L}_z \\ &= \left[\hat{L}_z, \hat{L}_x\right] + i\left[\hat{L}_z, \hat{L}_y\right] \\ &= i\hbar\hat{L}_y + i\left(-i\hbar\hat{L}_x\right) \\ &= \hbar\left(\hat{L}_x + i\hat{L}_y\right) \\ &= \hbar\hat{L}_+\end{aligned}$$

(b)

$$\begin{aligned}\left[\hat{L}_z, \hat{L}_-\right] &= \hat{L}_z\hat{L}_- - \hat{L}_-\hat{L}_z \\ &= \left[\hat{L}_z, \hat{L}_x\right] - i\left[\hat{L}_z, \hat{L}_y\right] \\ &= i\hbar\hat{L}_y - i\left(-i\hbar\hat{L}_x\right) \\ &= -\hbar\left(\hat{L}_x - i\hat{L}_y\right) \\ &= -\hbar\hat{L}_+\end{aligned}$$

(c)

$$\begin{aligned}\left[\hat{L}^2, \hat{L}_+\right] &= \left[\hat{L}^2, \hat{L}_x + i\hat{L}_y\right] \\ &= \left[\hat{L}^2, \hat{L}_x\right] + i\left[\hat{L}^2, \hat{L}_y\right] \\ &= 0 + 0 \\ &= 0\end{aligned}$$

(d)

$$\begin{aligned}\left[\hat{L}^2, \hat{L}_-\right] &= \left[\hat{L}^2, \hat{L}_x - i\hat{L}_y\right] \\ &= \left[\hat{L}^2, \hat{L}_x\right] - i\left[\hat{L}^2, \hat{L}_y\right] \\ &= 0 - 0 \\ &= 0\end{aligned}$$

(2) Let Y_{lm} be an eigenfunction of $\hat{L}_z$, with eigenvalue m.

$$\begin{aligned}\hat{L}_z\hat{L}_-Y_{lm} &= \left(\left[\hat{L}_z, \hat{L}_-\right] + \hat{L}_-\hat{L}_z\right)Y_{lm} \\ &= \left(-\hbar\hat{L}_- + \hat{L}_-m\hbar\right)Y_{lm} \\ &= \hbar(m-1)\hat{L}_-Y_{lm}\end{aligned}$$

Therefore, $\hat{L}_-Y_{lm}$ is an eigenfunction of $\hat{L}_z$ with eigenvalue $\hbar(m-1)$.

(3) Similarly, let Y_{lm} be an eigenfunction of $\hat{L}_z$, with eigenvalue $\hbar l(l+1)$.

$$\hat{L}^2\hat{L}_-Y_{lm} = \hbar^2 l(l+1)\hat{L}_-Y_{lm}$$

Therefore, $\hat{L}_{-}Y_{lm}$ is also an eigenfunction of $\hat{L}^2$, with the same eigenvalue.