QUANTUM AND SOLID STATE PHYSICS : ASSIGNMENT 11

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Exercise 1.
An N-type silicon sample, from $x = -3L$ to $x = 3L$, is illuminated at steady state, from $x = -L$ to $x - L$. The carrier generation rate for $-L < x < L$ is G_{optical}, and is zero outside this window. There are ohmic contacts at the ends of the sample, at $x = -3L$ and $x = 3L$. Assume $L >> L_p$, low level injection, and no electric field. Set up all the equations and known conditions we would need for solving for $\hat{p}(x)$, across the sample. This should include the differential equations and general solutions in each region, and also continuity and symmetry considerations in order to solve the problem. Hint: Your general solutions, after plugging in boundary conditions, should contain 4 unknowns in total. Draw an approximate plot of $\hat{p}(x)$.

Solution 1.
By the steady state diffusion equation, on the illuminated part,

$$\begin{aligned} 0 &= \frac{\partial \hat{p}}{\partial x} \\ &= -\frac{1}{q}\frac{\partial J_p}{\partial x} + \left(G_{\text{optical}} - \frac{\hat{p}}{\tau_p}\right) \end{aligned}$$

As $\overrightarrow{E} = 0$, $J = J_{\text{diffusion}}$. Therefore, by the transport equations,

$$J_{\text{diffusion}_p} = -qD_p \frac{\partial \hat{p}}{\partial x}$$

Therefore,

$$0 = -\frac{1}{q}\frac{\partial J_{\text{diffusion}_p}}{\partial x} + \left(G_{\text{optical}} - \frac{\hat{p}}{\tau_p}\right)$$

$$\begin{aligned} \therefore D_p \frac{\mathrm{d}^2\hat{p}}{\mathrm{d}x^2} - \frac{\hat{p}}{\tau_p} &= -G_{\text{optical}} \\ \therefore \frac{\mathrm{d}^2\hat{p}}{\mathrm{d}x^2} - \frac{\hat{p}}{D_p\tau_p} &= -\frac{G_{\text{optical}}}{D_p} \\ \therefore \frac{\mathrm{d}^2\hat{p}}{\mathrm{d}x^2} - \frac{\hat{p}}{{L_p}^2} &= -\frac{G_{\text{optical}}}{D_p} \end{aligned}$$

Therefore, for the illuminated part,

$$\hat{p}(x) = Ce^{-\frac{x}{L_p}} + De^{\frac{x}{L_p}} + G_{\text{optical}}\tau_p$$

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Therefore, for the left non-illuminated part,

$$\hat{p}(x) = Ae^{-\frac{x}{L_p}} + Be^{\frac{x}{L_p}}$$

Therefore, for the right non-illuminated part,

$$\hat{p}(x) = Pe^{-\frac{x}{L_p}} + Qe^{\frac{x}{L_p}}$$

As the sample has ohmic contacts at the ends, the concentration at the ends must be zero. Therefore,

$$\hat{p}(-3L) = 0$$
$$\therefore Ae^{\frac{3L}{L_p}} + Be^{-\frac{3L}{L_p}} = 0$$
$$\hat{p}(3L) = 0$$
$$\therefore Pe^{-\frac{3L}{L_p}} + Qe^{\frac{3L}{L_p}} = 0$$

As $L >> L_p$,

$$Be^{-\frac{3L}{L_p}} = 0$$
$$Pe^{-\frac{3L}{L_p}} = 0$$

Therefore,

$$Ae^{\frac{3L}{L_p}} = 0$$
$$Qe^{\frac{3L}{L_p}} = 0$$

Therefore,

$$A = 0$$
$$Q = 0$$

Therefore, for the left non-illuminated part,

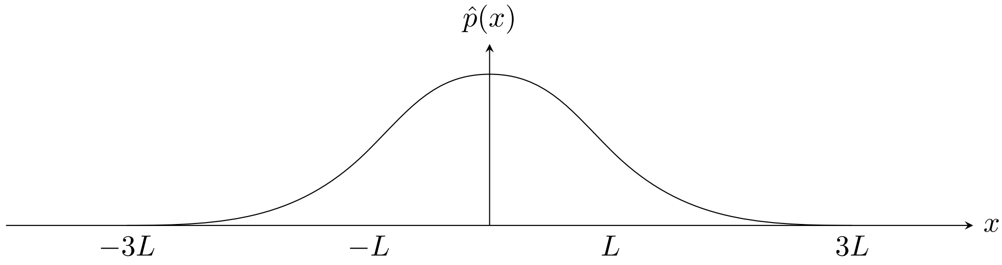
$$\hat{p}(x) = Be^{\frac{x}{L_p}}$$

Therefore, for the right non-illuminated part,

$$\hat{p}(x) = Pe^{-\frac{x}{L_p}}$$

As the concentration profile must be continuous at $x = -L$ and $x = L$,

$$Be^{-\frac{L}{L_p}} = Ce^{\frac{L}{L_p}} + De^{-\frac{L}{L_p}} + G_{\text{optical}}\tau_p$$
$$Pe^{-\frac{L}{L_p}} = Ce^{-\frac{L}{L_p}} + De^{\frac{L}{L_p}} + G_{\text{optical}}\tau_p$$



Exercise 2.
Consider a sample of N-type silicon at room temperature, uniformly doped with

$$N_D = 10^{17} \frac{1}{\mathrm{cm}^3}$$

Light is shining uniformly on the sample, at steady state, with an EHP optical generation rate of

$$G_{\text{optical}} = 10^{20} \frac{1}{\mathrm{cm}^3\,\mathrm{s}}$$

The minority carrier lifetime is

$$\tau_p = 10^{-5}\mathrm{s}$$

The hole mobility is

$$\mu_p = 350 \frac{\mathrm{cm}^2}{\mathrm{V\,s}}$$

(1) What is the steady state hole concentration p across the sample?
(2) Now suppose the following situation.
The semiconductor extends very far in the positive and negative x directions, and there are no metallic contacts at the ends. At $x = 0$, there is an infinitesimally thin layer with very low lifetime that forces the excess minority carrier concentration to go to 0 at $x = 0$, i.e.,

$$\hat{p}(x = 0) = 0$$

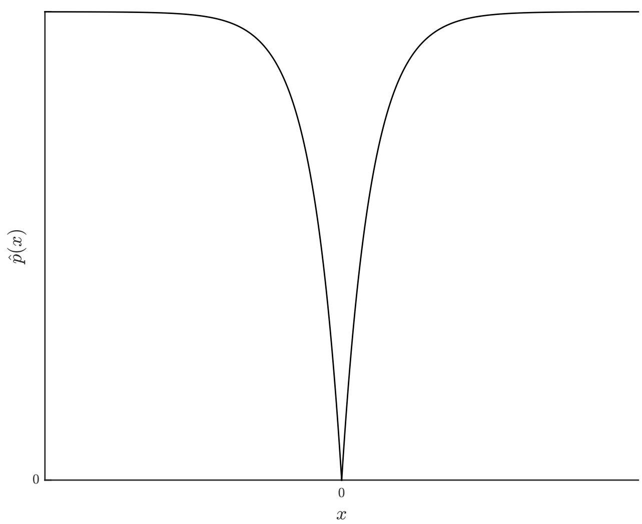
Sketch the form of the expected solution for $\hat{p}(x)$ across the sample. What is the value of $\hat{p}$ as $x \to \pm\infty$? Label it on your drawing.
(3) Approximately how far away from $x = 0$ do we need to go so that $\hat{p}$ is approximately its value at $x = \pm\infty$? Explain with words and numerically. Hint: You should know the behaviour of $\hat{p}(x)$. So think about what approximate value represents the distance at which $\hat{p}$ goes to at $\pm\infty$.

Solution 2.
(1) As the sample is in steady state,

$$\begin{aligned}
\hat{p}(x) &= G_{\text{optical}}\tau_p \\
&= \left(10^{20} \frac{1}{\mathrm{cm}^3\,\mathrm{s}}\right)\left(10^{-5}\mathrm{s}\right) \\
&= 10^{15} \frac{1}{\mathrm{cm}^3}
\end{aligned}$$

Therefore,

$$\begin{aligned} p(x) &= p_0 + \hat{p}(x) \\ &= \frac{{n_i}^2}{N_d} + \hat{p}(x) \\ &= \frac{10^{20}}{10^{17}} + 10^{15} \\ &= 10^3 + 10^{15} \\ &\approx 10^{15} \frac{1}{\text{cm}^3} \end{aligned}$$



(2) For $x \to \pm\infty$, the carrier concentration is not affected by the thin slice at $x = 0$. Therefore, the concentration is as if this slice does not exist. Therefore,

$$\begin{aligned} \lim_{x \to \pm\infty} \hat{p}(x) &= G_{\text{optical}} \tau_p \\ &= \left(10^{20} \frac{1}{\text{cm}^3\,\text{s}}\right) \left(10^{-5}\text{s}\right) \\ &= 10^{15} \frac{1}{\text{cm}^3} \end{aligned}$$

(3) The carrier concentration profile $\hat{p}(x)$ for $x > 0$ is of the form

$$\hat{p}(x) = A\left(1 - e^{-\frac{x}{L_p}}\right)$$

and for $x < 0$ is of the form

$$\hat{p}(x) = A\left(1 - e^{\frac{x}{L_p}}\right)$$

Therefore, the concentration at $x \to \pm\infty$ can be approximated to the concentration at $x = \pm L_p$, where

$$\begin{aligned} L_p &= \sqrt{D_p \tau_p} \\ &= \sqrt{\frac{kT}{q}\mu_p \tau_p} \\ &= \sqrt{\frac{kT}{q}\left(350\frac{\text{cm}^2}{\text{V s}}\right)\left(10^{-5}\text{s}\right)} \\ &= \sqrt{(0.026\text{V})\left(350\frac{\text{cm}^2}{\text{V s}}\right)\left(10^{-5}\text{s}\right)} \\ &= \sqrt{91 \times 10^{-6}\text{cm}^2} \\ &= 9.54 \times 10^{-3}\text{cm} \end{aligned}$$

Exercise 3.

(1) Show that for a ladder of eigenvalues of L_z, for a given l,

$$\hat{L}^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm}$$

the lowest step in the ladder corresponds to

$$m_{\text{minimum}} = -l$$

Remember that

$$\hat{L}_z Y_{lm} = \hbar m Y_{lm}$$

(2) We saw in recitation the following expression.

$$\hat{L}_- Y_{lm} = B_{lm} Y_{l,m-1}$$

Find the coefficient B_{lm} as a function of l and m.

(3) Explain the following commutation relations, for the case of a central potential. You do not need to calculate anything.
   (a) $\left[\hat{L}^2, \hat{H}\right] = 0$
   (b) $\left[\hat{L}_z, \hat{H}\right] = 0$

Solution 3.

(1) For the lowest step in the ladder,

$$\hat{L}_- \hat{L}_z Y_{lm} = 0$$

Therefore,

$$\begin{aligned} \hat{L}_+ 0 &= \hat{L}_+ \hat{L}_- \hat{L}_z Y_{lm} \\ \therefore 0 &= \hat{L}_+ \hat{L}_- Y_{lm} \\ &= \left(\hat{L}^2 - \hat{L}_z^2 + \hbar \hat{L}_z\right) Y_{lm} \\ &= \left(\hbar^2 l(l+1) - \hbar^2 m + \hbar^2 m\right) Y_{lm} \end{aligned}$$

Therefore,

$$\begin{aligned} l(l+1) &= m(m-1) \\ \therefore m &= -l \end{aligned}$$

(2)

$$\hat{L}_{-} Y_{lm} = B_{lm} Y_{l(m-1)}$$

Let

$$\begin{aligned} f &= \hat{L}_i Y_{lm} \\ g &= Y_{lm} \end{aligned}$$

Therefore, the identity

$$\int_{-\infty}^{\infty} f^*(x) \left(\hat{a}_{\pm} g(x)\right) \mathrm{d}x = \int_{-\infty}^{\infty} \left(\hat{a}_{\mp} f(x)\right)^* g(x) \, \mathrm{d}x$$

implies

$$\begin{aligned} &\iint \left(\hat{L}_{+} \hat{L}_{-} Y_{lm}\right)^* Y_{lm} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \\ &= \iint \left(\hbar^2 l(l+1) - \hbar^2 m^2 + \hbar^2 m^2\right) Y_{lm}{}^* Y_{lm} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \\ &= \hbar^2 \left(l(l+1) - m(m-1)\right) \end{aligned}$$

Also,

$$\iint \left(\hat{L}_{-} Y_{lm}\right)^* \left(\hat{L}_{-} Y_{lm}\right) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi = |B_{lm}|^2$$

Therefore,

$$\begin{aligned} |B_{lm}|^2 &= \hbar^2 \left(l(l+1) - m(m-1)\right) \\ \therefore B_{lm} &= h\sqrt{l(l+1) - m(m-1)} \end{aligned}$$

(3) (a)

$$\left[\hat{L}^2, \hat{H}\right] = 0$$

As this commutation relation is zero, the two operators have common eigenfunctions.

(b)

$$\left[\hat{L}_z, \hat{H}\right] = 0$$

As this commutation relation is zero, the two operators have common eigenfunctions.

Exercise 4.
Consider a particle with the following wave function.

$$\Psi(r, \theta, \varphi) = R(r)(2Y_{00} + Y_{11} + 3Y_{10} + 2iY_{1(-1)})$$

where $R(r)$ is a radial function and Y_{lm} are the eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$.

(1) If $\hat{L}^2$ is measured for this state, what values can it have, and with what probabilities?
(2) Find the expectation value of $\hat{L}^2$ for this state.
(3) If $\hat{L}_z$ is measured for this state, what values can it have, and with what probabilities?
(4) Find the expectation value of $\hat{L}_z$ for this state.

Solution 4.
(1)

$$\hat{L}^2 Y_{lm} = l(l+1)\hbar^2 Y_{lm}$$

Therefore, the possible values of L^2 are the values of $l(l+1)\hbar^2$ for $l = 0$ and $l = 1$. Therefore, the possible values of L^2 are 0 and $2\hbar^2$.

Let the constant of normalization for Ψ be A.
The probability corresponding to Y_{00} is

$$\begin{aligned} \frac{|c_1|^2}{A} &= \frac{2^2}{A} \\ &= \frac{4}{A} \end{aligned}$$

The probability corresponding to Y_{11} is

$$\begin{aligned} \frac{|c_2|^2}{A} &= \frac{1^2}{A} \\ &= \frac{1}{A} \end{aligned}$$

The probability corresponding to Y_{10} is

$$\begin{aligned} \frac{|c_3|^2}{A} &= \frac{3^2}{A} \\ &= \frac{9}{A} \end{aligned}$$

The probability corresponding to $Y_{1(-1)}$ is

$$\begin{aligned} \frac{|c_4|^2}{A} &= \frac{|2i|^2}{A} \\ &= \frac{4}{A} \end{aligned}$$

Normalizing,

$$\begin{aligned}
1 &= \sum c_k \\
&= \frac{4+1+9+4}{A} \\
&= \frac{18}{A} \\
\therefore A &= 18
\end{aligned}$$

Therefore,

$$\begin{aligned}
P_1 &= \frac{2}{9} \\
P_2 &= \frac{1}{18} \\
P_3 &= \frac{1}{2} \\
P_4 &= \frac{2}{9}
\end{aligned}$$

(2)

$$\begin{aligned}
\left\langle L^2 \right\rangle &= \sum P_k {L^2}_k \\
&= \frac{2}{9}0 + \frac{1}{18}2\hbar + \frac{1}{2}2\hbar + \frac{2}{9}2\hbar \\
&= \frac{7}{9}2\hbar \\
&= \frac{14\hbar}{9}
\end{aligned}$$

(3)

$$\hat{L}_z Y_{lm} = m\hbar Y_{lm}$$

Therefore, the possible values of L_z are the values of $m\hbar$ for $m = 0$, $m = 1$, and $m = -1$. Therefore, the possible values of L_z are 0, $\hbar$, and $-\hbar$.

Let the constant of normalization for Ψ be A.
The probability corresponding to Y_{00} is

$$\begin{aligned}
\frac{|c_1|^2}{A} &= \frac{2^2}{A} \\
&= \frac{4}{A}
\end{aligned}$$

The probability corresponding to Y_{11} is

$$\begin{aligned}
\frac{|c_2|^2}{A} &= \frac{1^2}{A} \\
&= \frac{1}{A}
\end{aligned}$$

The probability corresponding to Y_{10} is

$$\frac{|c_3|^2}{A} = \frac{3^2}{A} = \frac{9}{A}$$

The probability corresponding to $Y_{1(-1)}$ is

$$\frac{|c_4|^2}{A} = \frac{|2i|^2}{A} = \frac{4}{A}$$

Normalizing,

$$\begin{aligned} 1 &= \sum c_k \\ &= \frac{4+1+9+4}{A} \\ &= \frac{18}{A} \\ \therefore A &= 18 \end{aligned}$$

Therefore,

$$P_1 = \frac{2}{9}$$
$$P_2 = \frac{1}{18}$$
$$P_3 = \frac{1}{2}$$
$$P_4 = \frac{2}{9}$$

(4)

$$\begin{aligned} \langle L_z \rangle &= \sum P_k L_{zk} \\ &= \frac{2}{9}0 + \frac{1}{18}\hbar + \frac{1}{2}0 + \frac{2}{9}(-\hbar) \\ &= \frac{\hbar}{18} - \frac{4\hbar}{18} \\ &= -\frac{3\hbar}{18} \end{aligned}$$