QUANTUM AND SOLID STATE PHYSICS : ASSIGNMENT 9

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Exercise 1.
An N-type silicon sample with the following properties at room temperature is illuminated for a long time under low level injection conditions.

$$\begin{aligned}
N_d &= 10^{16} \frac{1}{\text{cm}^3} \\
\tau_n &= 1\text{µs} \\
\tau_p &= 1\text{µs}
\end{aligned}$$

At steady state, the excess carrier concentration is $\hat{n} = \hat{p} = 10^7 \frac{1}{\text{cm}^3}$. Illumination is stopped at $t = 0$.

(1) What are the electron and hole concentrations n and p for $t < 0$?
(2) Write an expression for the hole concentration, $p(t)$, as a function of time for $t > 0$, and plot your result.
(3) At time $t = 1\text{ms}$, the sample is again illuminated, with the same conditions as for $t < 0$. What is the concentration of holes at $t = 1.001\text{ms}$?

Solution 1.
(1)

$$\begin{aligned}
n &= n_0 + \hat{n} \\
&= N_d + \hat{n} \\
&= 10^{16} + 10^7 \\
&\approx 10^{16} \frac{1}{\text{cm}^3}
\end{aligned}$$

$$\begin{aligned}
p &= p_0 + \hat{p} \\
&= \frac{{n_i}^2}{N_d} + \hat{p} \\
&= \frac{\left(1.5 \times 10^{10}\right)}{10^{16}} + 10^7 \\
&= 2.25 \times 10^4 + 10^7 \\
&\approx 10^7
\end{aligned}$$

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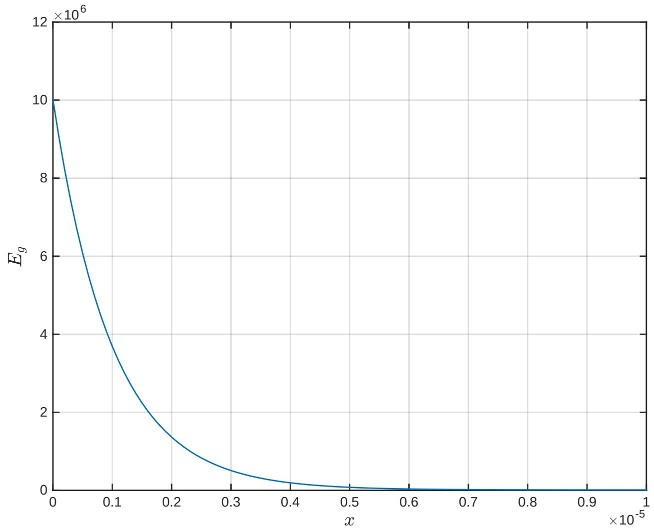
(2)

$$\begin{aligned}\hat{p}(t) &= 10^7 e^{-\frac{t}{\tau_p}}\\ &= 10^7 e^{-\frac{t}{10^{-6}}}\\ &= 10^7 e^{-10^6 t}\frac{1}{\text{cm}^3}\end{aligned}$$

Therefore,

$$\begin{aligned}p &= p_0 + \hat{p}\\ &= 10^4 + 10^7 e^{-10^6 t}\end{aligned}$$

Therefore,



(3)

$$\begin{aligned}\hat{p}(t_0 = 10^{-3}\text{s}) &= 10^7 e^{-10^6 \cdot 10^{-3}}\\ &= 10^7 e^{-10^3}\\ &\approx 0\end{aligned}$$

Therefore,

$$\begin{aligned}\hat{p}(t) &= \left(1 - \frac{1}{e}\right)\hat{p}(0)\\ &= \frac{e-1}{e}10^7\\ &\approx 6.3 \times 10^6 \frac{1}{\text{cm}^3}\end{aligned}$$

Therefore,

$$\begin{aligned} p &= p_0 + \hat{p} \\ &= 10^4 + 6.4 \times 10^6 \\ &\approx 6.3 \times 10^6 \frac{1}{\mathrm{cm}^3} \end{aligned}$$

Exercise 2.
A P-type silicon sample, with the following properties, doping N_a, minority carrier lifetime τ_n, at room temperature is illuminated uniformly throughout the volume of the sample, for a long time, with an optical generation rate of $G_{\text{optical}} \frac{1}{\mathrm{cm}^3\,\mathrm{s}}$. Then, at time $t = 0$, the light intensity is reduced, and for $t > 0$, the optical regeneration rate is half of the value as before, i.e.,

$$G_{\text{optical}}(t > 0) = \frac{1}{2} G_{\text{optical}}(t < 0)$$

Assume low level injection over all time. Determine the equation for excess electron carrier concentration, $\hat{n}(t)$, as a function of time, for $t > 0$.

Solution 2.
For $t > 0$,

$$\frac{\mathrm{d}\hat{n}}{\mathrm{d}t} = \frac{G_{\text{optical}}}{2} - \frac{\hat{n}}{\tau_n}$$

For $t = 0$,

$$\hat{n} = G_{\text{optical}} \tau_n$$

Therefore, solving the ODE,

$$\hat{n} = \frac{G_{\text{optical}} \tau_n}{2} \left(1 + e^{-\frac{t}{\tau_n}} \right)$$

Exercise 3.
Consider the following potential.

$$V(x) = \begin{cases} 0 & ; \quad x < 0 \\ V_0 & ; \quad x > 0 \end{cases}$$

A particle with mass m is approaching the potential from the left. The energy of the particle can be either $0 < E < V_0$, or $E > V_0$.

(1) Write the general solution to the time independent Schrödinger equation for a particle with $E > V_0$.
(2) Write the general solution to the time independent Schrödinger equation for a particle with $0 < E < V_0$.
(3) Write down the boundary conditions required to find the constants from the above parts.
(4) Write an expression for the transmission and reflection coefficients for the case $E > V_0$.
(5) Write an expression for the transmission and reflection coefficients for the case $0 < E < V_0$.

Solution 3.

(1) Let

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$
$$k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$$

For $E > V_0$,

$$\psi(x) = \begin{cases} A_1 e^{ik_1 x} + B_1 e^{-ik_1 x} & ; \quad x < 0 \\ C_1 e^{ik_2 x} & ; \quad x > 0 \end{cases}$$

(2) Let

$$k_3 = \sqrt{\frac{2mE}{\hbar^2}}$$
$$k_4 = \sqrt{\frac{-2m(E - V_0)}{\hbar^2}}$$

For $0 < E < V_0$,

$$\psi(x) = \begin{cases} A_2 e^{ik_3 x} + B_2 e^{-ik_3 x} & ; \quad x < 0 \\ C_2 e^{-k_4 x} & ; \quad x > 0 \end{cases}$$

(3) As the jump at $x = 0$ is finite, ψ and ψ' must be continuous at $x = 0$. Therefore, for $E > V_0$,

$$A_1 + B_1 = C_1$$
$$(A_1 - B_1)k_1 = C_1 k_2$$

Therefore, solving,

$$\frac{B_1}{A_1} = \frac{k_1 - k_2}{k_1 + k_2}$$
$$\frac{C_1}{A_1} = \frac{2k_1}{k_1 + k_2}$$

Therefore, for $0 < E < V_0$,

$$A_2 + B_2 = C_2$$
$$(A_2 - B_2)k_3 = iC_2 k_4$$

Therefore, solving,

$$\frac{B_2}{A_2} = \frac{k_3 - ik_4}{k_3 + ik_4}$$
$$\frac{C_2}{A_2} = \frac{2k_3}{k_3 + ik_4}$$

(4) For $E > V_0$,

$$\begin{aligned} T &= \frac{k_2|C_1|^2}{k_1|A_1|^2} \\ &= \frac{k_2}{k_1}\frac{4{k_1}^2}{(k_1+k_2)^2} \\ &= \frac{4k_1k_2}{(k_1+k_2)^2} \\ R &= \frac{k_2|B_1|^2}{k_1|A_1|^2} \\ &= \frac{(k_1-k_2)^2}{(k_1+k_2)^2} \end{aligned}$$

For $0 < E < V_0$,

$$\begin{aligned} T &= \frac{k_4|C_2|^2}{k_3|A_2|^2} \\ &= \frac{k_4}{k_3}\frac{4{k_3}^2}{{k_3}^2+{k_4}^2} \\ &= \frac{4k_3k_4}{{k_3}^2+{k_4}^2} \\ R &= \frac{k_4|B_2|^2}{k_3|A_2|^2} \\ &= \frac{{k_3}^2+{k_4}^2}{{k_3}^2+{k_4}^2} \\ &= 1 \end{aligned}$$

Therefore, as $R + T = 1$,

$$T = 0$$

Exercise 4.

(1) Prove the following commutation relation.

$$\left[\hat{H}, \hat{a}_-\right] = -\hbar\omega\hat{a}_-$$

(2) Based on your result above, explain what the lowering operator, $\hat{a}_-$ does to an eigenfunction of the energy operator? Hint: Apply the energy operator on $\hat{a}_-\psi(x)$, where $\psi(x)$ is an eigenfunction of the energy operator, i.e.,

$$\hat{H}\psi(x) = E\psi(x)$$

(3) Explain why the energy operator can be written either as

$$\hat{H} = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)$$

or as

$$\hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)$$

(4) Find the eigenfunction, $\psi_0(x)$, corresponding to the lowest eigenvalue,

$$E_0 = \frac{1}{2}\hbar\omega$$

of the energy operator. Hint: Solve

$$\hat{a}_-\psi(x) = 0$$

(5) We saw in recitation the following expression

$$\hat{a}_-\psi(x) = d_n\psi_{n-1}(x)$$

Find the coefficient d_n as a function of n.

Solution 4.
(1)

$$\begin{aligned}
\left[\hat{H}, \hat{a}_-\right] &= \left[\hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right), \hat{a}_-\right] \\
&= \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right) - \hbar\omega\hat{a}_-\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right) \\
&= \hbar\omega\left(\hat{a}_+\hat{a}_-\hat{a}_- + \frac{\hat{a}_-}{2} - \hat{a}_-\hat{a}_+\hat{a}_- - \frac{\hat{a}_-}{2}\right) \\
&= \hbar\omega\left(\hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+\right)\hat{a}_- \\
&= \hbar\omega\left[\hat{a}_+, \hat{a}_-\right]\hat{a}_- \\
&= -\hbar\omega\hat{a}_-
\end{aligned}$$

(2) Let E be an eigenvalue of $\hat{H}$ corresponding to the eigenfunction $\psi(x)$.

$$\begin{aligned}
\hat{H}\hat{a}_-\psi(x) &= \left(\left[\hat{H}, \hat{a}_-\right] + \hat{a}_-\hat{H}\right)\psi(x) \\
&= \left(-\hbar\omega\hat{a}_- + \hat{a}_-\hat{H}\right)\psi(x) \\
&= -\hbar\omega\hat{a}_-\psi(x) + \hat{a}_-\hat{H}\psi(x) \\
&= -\hbar\omega\hat{a}_-\psi(x) + \hat{a}_-E\psi(x) \\
&= (E - \hbar\omega)\hat{a}_-\psi(x)
\end{aligned}$$

Therefore, as $(E - \hbar\omega)$ is an eigenvalue of $\hat{H}$ corresponding to the eigenfunction $\hat{a}_-\psi(x)$, the operator lowers the eigenvalue of $\hat{H}$.

(3)

$$\begin{aligned}
\hat{H} &= \frac{1}{2m}\left(\hat{p}^2 + (m\omega\hat{x})^2\right) \\
&= \frac{1}{2m}\left((m\omega\hat{x} - i\hat{p})(m\omega\hat{x} + i\hat{p}) + m\omega\hbar\right) \\
&= \hbar\omega\left(\frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} - i\hat{p})\frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} + i\hat{p}) + \frac{1}{2}\right) \\
&= \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)
\end{aligned}$$

Similarly,

$$\hat{H} = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)$$

(4)

$$\begin{aligned}
\hat{a}_-\psi(x) &= 0 \\
\therefore \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} + i\hat{p})\psi(x) &= 0 \\
\therefore \left(m\omega\hat{x} + \hbar\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi(x) &= 0 \\
\therefore \psi(x) &= ce^{-\frac{m\omega}{2\hbar}x^2}
\end{aligned}$$

Therefore, normalizing,

$$\int_{-\infty}^{\infty} \psi(x)\psi^*(x)\,\mathrm{d}x = 1$$

Therefore, solving,

$$c = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}$$

Therefore,

$$\psi(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$$

(5) Let

$$\hat{a}_-\psi_n = d_n\psi_{n-1}$$

Let

$$\begin{aligned}
f(x) &= \hat{a}_-\psi_n \\
g(x) &= \psi_n
\end{aligned}$$

Therefore, the identity

$$\int_{-\infty}^{\infty} f^*(x)\left(\hat{a}_\pm g(x)\right)\mathrm{d}x = \int_{-\infty}^{\infty}\left(\hat{a}_\mp f(x)\right)^* g(x)\,\mathrm{d}x$$

implies

$$\begin{aligned}
\int_{-\infty}^{\infty} (\hat{a}_-\psi_n)^* (\hat{a}_-\psi_n)\,\mathrm{d}x &= \int_{-\infty}^{\infty} (\hat{a}_+\hat{a}_-\psi_n)^*\,\psi_n\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left(\left(\frac{\hat{H}}{\hbar\omega} - \frac{1}{2}\right)\psi_n\right)^* \psi_n\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left(\left(n + \frac{1}{2} - \frac{1}{2}\right)\psi_n\right)^* \psi_n\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} (n\psi_n)^*\psi_n\,\mathrm{d}x \\
&= n\int_{-\infty}^{\infty} {\psi_n}^*\psi_n\,\mathrm{d}x
\end{aligned}$$

As ψ_n is normalized, $\int_{-\infty}^{\infty} {\psi_n}^*\psi_n\,\mathrm{d}x = 1$. Therefore,

$$\begin{aligned}
\int_{-\infty}^{\infty} (d_n\psi_{n-1})^* (d_n\psi_{n-1})\,\mathrm{d}x &= (n)(1) \\
&= n \\
\therefore |d_n|^2 \int_{-\infty}^{\infty} {\psi_{n-1}}^*\psi_{n-1}\,\mathrm{d}x &= n
\end{aligned}$$

For ψ_{n-1} to be normalized,

$$\int_{-\infty}^{\infty} {\psi_{n-1}}^*\psi_{n-1}\,\mathrm{d}x = 1$$

Therefore,

$$d_n = \sqrt{n}$$