QUANTUM AND SOLID STATE PHYSICS : ASSIGNMENT 6

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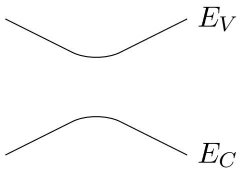
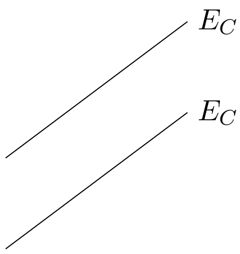
Exercise 1.

(1) The following energy diagram describes
   (a) A semiconductor which has variation in doping at the edges and centre.
   (b) A semiconductor which has an external voltage applied on both edges.
   (c) A semiconductor in which the type of material changes.



(2) The following energy diagram describes
   (a) A semiconductor made entirely of one material.
   (b) A semiconductor in which the type of material changes.
   (c) A semiconductor in which a voltage is applied across both ends.
   (d) Both a and b are correct.
   (e) Both a and c are correct.



Solution 1.

(1) The energy diagram describes a semiconductor which has variation in doping at the edges and centre.
(2) The energy diagram describes a semiconductor made of a single material, with a voltage applied across it. Therefore, both a and c are correct.

Date: Thursday 26^{th} November, 2015.

Exercise 2.
A voltage V is applied across a sample of N-type silicon at 300 K.
The electron drift velocity as a function of electric field is as shown.

(1) The silicon has the following characteristics.

$$\begin{aligned} \text{length } l &= 0.1\text{cm} \\ \text{cross-sectional area } A &= 100\,\mu\text{m}^2 \\ \text{electron mobility } \mu_n &= 1350\frac{\text{cm}^2}{\text{V s}} \\ \text{dopant concentration } N_D &= 10^{17}\frac{1}{\text{cm}^3} \end{aligned}$$

Calculate the electron drift current I_{drift_n} with $V = 10\,\text{V}$ applied across the sample. Note that the expression for drift current density is

$J_{\text{drift}_n} = qn\mu_n E$

Repeat for a Si sample which is 1 µm long.

(2) Label the direction of the electric field $\overrightarrow{E}$, and the direction of J_{drift_n}.

(3) How much time, on an average, does it take for an electron to drift 1 µm in the silicon sample at an electric field of 100 $\frac{\text{V}}{\text{cm}}$? Repeat for 10^5 $\frac{\text{V}}{\text{cm}}$.

Solution 2.
(1) If $l = 0.1$cm,

$$\begin{aligned} J_{\text{drift}_n} &= qn\mu_n E \\ &= qN_D\mu_n\frac{V}{l} \\ &= \left(1.6\times10^{-19}\,\text{C}\right)\left(10^{17}\frac{1}{\text{cm}^3}\right)\left(1350\frac{\text{cm}^2}{\text{V s}}\right)\left(\frac{10}{0.1}\frac{\text{V}}{\text{cm}}\right) \\ &= 2160\frac{\text{C}}{\text{cm}^2\,\text{s}} \\ &= 2160\frac{\text{A}}{\text{cm}^2} \end{aligned}$$

Therefore,

$$
\begin{aligned}
I_{\text{drift}_n} &= J_{\text{drift}_n} A \\
&= \left(2160 \frac{\text{A}}{\text{cm}^2}\right)\left(100\text{µm}^2\right) \\
&= \left(2160 \frac{\text{A}}{\text{cm}^2}\right)\left(10^{-6}\text{cm}^2\right) \\
&= 2.16 \times 10^{-3}\,\text{A}
\end{aligned}
$$

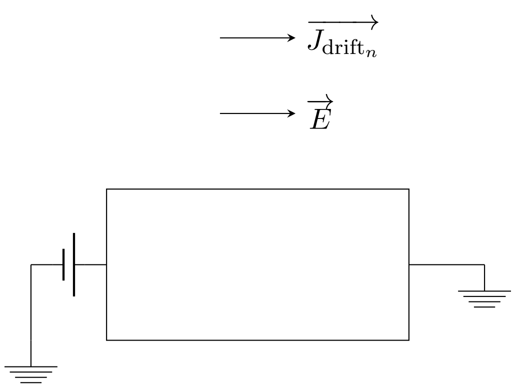
If $l = 1\text{µm} = 10^{-4}\text{cm}$,

$$
\begin{aligned}
J_{\text{drift}_n} &= q n \mu_n E \\
&= q N_D \mu_n \frac{V}{l} \\
&= \left(1.6 \times 10^{-19}\,\text{C}\right)\left(10^{17}\frac{1}{\text{cm}^3}\right)\left(1350\frac{\text{cm}^2}{\text{V s}}\right)\left(\frac{10}{0.1} \times 10^3 \frac{\text{V}}{\text{cm}}\right) \\
&= 2160 \times 10^3 \frac{\text{C}}{\text{cm}^2\,\text{s}} \\
&= 2160 \times 10^3 \frac{\text{A}}{\text{cm}^2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_{\text{drift}_n} &= J_{\text{drift}_n} A \\
&= \left(2160 \times 10^3 \frac{\text{A}}{\text{cm}^2}\right)\left(100\text{µm}^2\right) \\
&= \left(2160 \times 10^3 \frac{\text{A}}{\text{cm}^2}\right)\left(10^{-6}\text{cm}^2\right) \\
&= 2.16\,\text{A}
\end{aligned}
$$

(2)



(3) If $E = 10^2 \frac{\mathrm{V}}{\mathrm{cm}}$,

$$\begin{aligned} v_{\mathrm{drfit}_n} &= \mu_n E \\ &= \left(1350 \frac{\mathrm{cm}^2}{\mathrm{V\,s}}\right)\left(100 \frac{\mathrm{V}}{\mathrm{cm}}\right) \\ &= 1.35 \times 10^5 \frac{\mathrm{cm}}{\mathrm{s}} \end{aligned}$$

Therefore, the time required to drift 1µm is

$$\begin{aligned} t &= \frac{1\mathrm{\mu m}}{1.35 \times 10^5 \frac{\mathrm{cm}}{\mathrm{s}}} \\ t &= \frac{10^{-4}\mathrm{cm}}{1.35 \times 10^5 \frac{\mathrm{cm}}{\mathrm{s}}} \\ &= \frac{1}{1.35} \times 10^{-9}\,\mathrm{s} \\ &= 0.74 \times 10^{-9}\,\mathrm{s} \end{aligned}$$

If $E = 10^5 \frac{\mathrm{V}}{\mathrm{cm}}$,

$$\begin{aligned} v_{\mathrm{drfit}_n} &= \mu_n E \\ &= \left(1350 \frac{\mathrm{cm}^2}{\mathrm{V\,s}}\right)\left(100 \times 10^3 \frac{\mathrm{V}}{\mathrm{cm}}\right) \\ &= 1.35 \times 10^8 \frac{\mathrm{cm}}{\mathrm{s}} \end{aligned}$$

Therefore, the time required to drift 1µm is

$$\begin{aligned} t &= \frac{1\mathrm{\mu m}}{1.35 \times 10^8 \frac{\mathrm{cm}}{\mathrm{s}}} \\ t &= \frac{10^{-4}\mathrm{cm}}{1.35 \times 10^8 \frac{\mathrm{cm}}{\mathrm{s}}} \\ &= \frac{1}{1.35} \times 10^{-12}\,\mathrm{s} \\ &= 0.74 \times 10^{-12}\,\mathrm{s} \end{aligned}$$

Exercise 3.

Consider a particle at $t = 0$ described as a linear combination of two stationary states.

$$\Psi(x, 0) = c_1\psi_1(x) + c_2\psi_2(x)$$

where $\psi_1(x)$ and $\psi_2(x)$ are eigenstates of the Hamiltonian, such that

$$\begin{aligned} \hat{H}\psi_1(x) &= E_1\psi_1(x) \\ \hat{H}\psi_2(x) &= E_2\psi_2(x) \end{aligned}$$

In this question assume that the constant c_1 and c_2 are real.

(1) What are the physical meaning of ${c_1}^2$ and ${c_2}^2$? What can you say about the value of ${c_1}^2 + {c_2}^2$?

(2) What is the wave function at subsequent times, i.e. $\Psi(x,t)$? Is $\Psi(x,t)$ a stationary state? Explain your answer.
(3) Find the probability density for the wave function from the previous part.

Solution 3.

(1) ${c_1}^2$ represents the probability that the particle has energy E_1, and ${c_2}^2$ represents the probability that the particle has energy E_2.
As

$$\Psi(x,0) = c_1\psi_1(x) + c_2\psi_2(x)$$

and as the total probability must be 1,

$${c_1}^2 + {c_2}^2 = 1$$

(2) The eigenvalues of the Hamiltonian are

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

Therefore,

$$E_1 = \frac{\pi^2\hbar^2}{2ma^2}$$
$$E_2 = \frac{4\pi^2\hbar^2}{2ma^2}$$

Therefore,

$$\begin{aligned}\Psi(x,t) &= \sum_n c_n\psi_n(x)e^{-\frac{iE_nt}{\hbar}}\\ &= c_1\psi_1(x)e^{-\frac{iE_1t}{\hbar}} + c_2\psi_2(x)e^{-\frac{iE_2t}{\hbar}}\\ &= c_1\psi_1(x)e^{-\frac{i\pi^2\hbar t}{2ma^2}} + c_2\psi_2(x)e^{-\frac{4i\pi^2\hbar t}{2ma^2}}\end{aligned}$$

Therefore, as this is not in the form $\psi(x)e^{-\frac{iEt}{\hbar}}$, it is not a stationary state.

(3)

$$\left|\Psi(x,t)\right|^2 = \left|c_1\psi_1(x)e^{-\frac{i\pi^2\hbar t}{2ma^2}} + c_2\psi_2(x)e^{-\frac{4i\pi^2\hbar t}{2ma^2}}\right|$$

Exercise 4.

(1) Consider a classical particle with kinetic energy E_k trapped in an finite square well. Neglect friction and any loss of energy due to contact with the wall of the well.
   (a) Describe the motion of the particle within the well.
   (b) Draw the probability of finding the particle as a function of position in the well.
(2) Now consider a quantum particle trapped in an infinite square well. Calculate $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$, $\langle p^2\rangle$, σ_x, and σ_p for the nth stationary state.
(3) Using your results from the previous part, verify that the uncertainty principle is satisfied.

Solution 4.

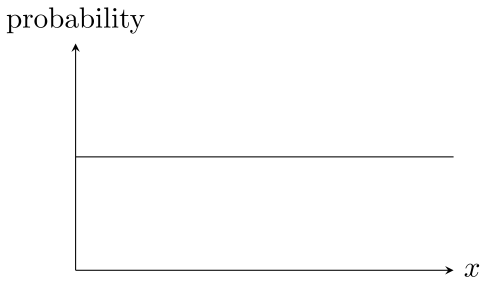
(1)

$$E_k = \frac{1}{2}mv^2$$
$$\therefore v = \sqrt{\frac{2m}{E_k}}$$

Therefore, the particle is moving with a velocity v.
Therefore, assuming the particle does not collide with any of the walls,

$$x = x_0 + vt$$

where x_0 is the original position of the particle.
Therefore,



(2)

$$\psi(x,t) = \psi(x)e^{-\frac{iE_nt}{\hbar}}$$
$$= \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)e^{-\frac{iE_nt}{\hbar}}$$

where

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

Therefore,

$$\psi(x,t) = \sqrt{\frac{2}{a}}\sin\left(\sqrt{\frac{2mE_n}{\hbar}}x\right)e^{-\frac{iE_nt}{\hbar}}$$

Therefore,

$$\langle x\rangle = \int\limits_{-\infty}^{\infty} \psi^*(x,t)x\psi(x,t)\,\mathrm{d}t$$
$$= \int\limits_{-\infty}^{\infty} \frac{2}{a}\sin^2\left(\sqrt{\frac{2mE_n}{\hbar}}\right)x\,\mathrm{d}x$$

Therefore,

$$\begin{aligned}\left\langle x^2\right\rangle &= \int_{-\infty}^{\infty} \psi^*(x,t)x^2\psi(x,t)\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \frac{2}{a}\sin^2\left(\sqrt{\frac{2mE_n}{\hbar}}\right)x^2\,\mathrm{d}x\end{aligned}$$

Therefore,

$$\begin{aligned}\langle p\rangle &= \int_{-\infty}^{\infty} \psi^*(x,t)\left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\psi(x,t)\right)\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \sqrt{\frac{2}{a}}\sin\left(\sqrt{\frac{2mE_n}{\hbar}}x\right)\left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{\frac{2}{a}}\sin\left(\sqrt{\frac{2mE_n}{\hbar}}x\right)\right)\mathrm{d}x\end{aligned}$$

Therefore,

$$\begin{aligned}\left\langle p^2\right\rangle &= \int_{-\infty}^{\infty} \psi^*(x,t)\left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\psi(x,t)\right)\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \sqrt{\frac{2}{a}}\sin\left(\sqrt{\frac{2mE_n}{\hbar}}x\right)\left(-\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sqrt{\frac{2}{a}}\sin\left(\sqrt{\frac{2mE_n}{\hbar}}x\right)\right)\mathrm{d}x\end{aligned}$$

Exercise 5.
A particle in an infinite square well is represented, at time $t = 0$ by the following wave function.

$$\Psi(x,0) = \begin{cases} Ax & ; \quad 0 \le x \le \frac{a}{2} \\ A(a-x) & ; \quad \frac{a}{2} \le x \le a \end{cases}$$

(1) Find A and sketch $\Psi(x,0)$.
(2) Write an expression for $\Psi(x,t)$.
(3) What is the probability that a measurement of the energy would yield the value E_1?
(4) Find $\langle H\rangle$.

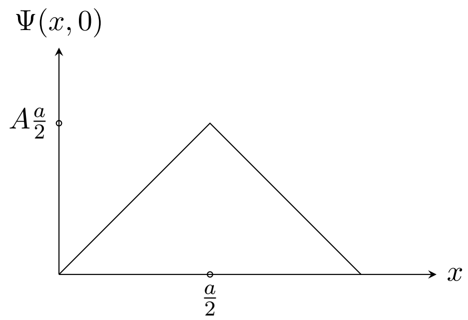
Solution 5.

(1)

$$
\begin{aligned}
1 &= \int\limits_{-\infty}^{\infty} \left|\Psi(x,0)\right|^2 \mathrm{d}x \\
&= \int\limits_{0}^{\frac{a}{2}} A^2 x^2 \,\mathrm{d}x + \int\limits_{\frac{a}{2}}^{a} A^2 \left(a^2 - 2ax + x^2\right) \mathrm{d}x \\
&= \int\limits_{0}^{\frac{a}{2}} A^2 x^2 \,\mathrm{d}x + \int\limits_{\frac{a}{2}}^{a} A^2 a^2 - 2A^2 ax + A^2 x^2 \,\mathrm{d}x \\
&= A^2 \frac{x^3}{3}\bigg|_0^{\frac{a}{2}} + A^2 a^2 x - A^2 a x^2 + A^2 \frac{x^3}{3}\bigg|_{\frac{a}{2}}^{a} \\
&= A^2 \frac{a^3}{24} + A^2 a^3 - A^2 a^3 + A^2 \frac{a^3}{3} - A^2 \frac{a^3}{2} + A^2 \frac{a^3}{4} - A^2 \frac{a^3}{24} \\
&= A^2 a^3 \left(\frac{1}{24} + 1 - 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{4} - \frac{1}{24}\right) \\
&= A^2 a^3 \frac{1}{12}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A^2 &= \frac{12}{a^3} \\
\therefore A &= \sqrt{\frac{12}{a^3}}
\end{aligned}
$$



(2)

$$
\Psi(x,0) = c_1 \psi_1(x) e^{-\frac{iE_1 t}{\hbar}} + c_2 \psi_2(x) e^{-\frac{iE_2 t}{\hbar}}
$$

(3)

$$
c_n = \int\limits_{0}^{\frac{a}{2}} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) Ax \,\mathrm{d}x + \int\limits_{\frac{a}{2}}^{a} A \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) (a - x) \,\mathrm{d}x
$$

Therefore, solving,

$$c_n = \begin{cases} 0 & ; \quad n \text{ is even} \\ \sqrt{\frac{2}{a}} \frac{3Aa^2(-1)^n}{2n^2\pi^2} & ; \quad n \text{ is odd} \end{cases}$$

Therefore,

$$c_1 = -\sqrt{\frac{2}{a}} \frac{3Aa^2}{2\pi^2}$$

Therefore,

$$\begin{aligned} |c_1|^2 &= \left(-\sqrt{\frac{2}{a}} \frac{3Aa^2}{2\pi^2} \right)^2 \\ &= \frac{54}{\pi^4} \end{aligned}$$

(4)

$$\begin{aligned} \langle H \rangle &= \sum |c_n|^2 E_n \, \mathrm{d}x \\ &= c_1 E_1 \\ &= -\sqrt{\frac{2}{a}} \frac{3Aa^2}{2\pi^2} \frac{\pi^2 \hbar^2}{2ma^2} \end{aligned}$$