QUANTUM AND SOLID STATE PHYSICS : ASSIGNMENT 5

AAKASH JOG
ID : 989323563

Exercise 2.
Consider a particle described by a wave function $\psi(x,t)$, which solves the Schrödinger equation for a potential $V(x)$.
The potential is modified such that the new potential is $V(x)+V_0$. In classical mechanics this addition doesn't have a meaning. What about quantum mechanics?

(1) Show that the new solution to the Schrödinger equation has the form $\psi(x,t)e^{-\frac{iV_0t}{\hbar}}$.
(2) How would this change in potential change the expectation values of the position operator $\hat{x}$, the momentum operator $\hat{p}$, and a general operator $\hat{Q}\left(\hat{x},\hat{p}\right)$?

Solution 2.
(1) Let

$$\Psi(x,t) = \psi(x)$$

As it a solution to the Schrödinger equation,

$$i\hbar\frac{\psi(x)\varphi'(t)}{\psi(x)\varphi(t)} = V_0$$

Therefore,

$$i\hbar\frac{\psi(x)\widetilde{\varphi}'(t)}{\psi(x)\widetilde{\varphi}(t)} = V_0 + V(x)$$

Therefore,

$$\widetilde{\varphi(t)} = e^{-\frac{iV(x)t}{\hbar}}e^{-\frac{iV_0t}{\hbar}}$$

Therefore,

$$\begin{aligned}\Psi(x,t) &= \psi(x)\widetilde{\varphi}(t)\\ &= \Psi(x,t)e^{-\frac{iV_0t}{\hbar}}\end{aligned}$$

□

Date: Thursday 19$^{\text{th}}$ November, 2015.

(2)

$$\begin{aligned}
\widetilde{\langle x \rangle} &= \int\limits_{-\infty}^{\infty} \widetilde{\Psi}^* x \widetilde{\Psi} \, \mathrm{d}x \\
&= \int\limits_{-\infty}^{\infty} \Psi^* e^{\frac{iV_0 t}{\hbar}} x \Psi e^{-\frac{iV_0 t}{\hbar}} \, \mathrm{d}x \\
&= \int\limits_{-\infty}^{\infty} \Psi^* x \Psi \, \mathrm{d}x \\
&= \langle x \rangle
\end{aligned}$$

Therefore, the change in potential <u>does not change</u> the expectation value of x.

$$\begin{aligned}
\widetilde{\langle p \rangle} &= \int\limits_{-\infty}^{\infty} \widetilde{\Psi}^* \left(-i\hbar \frac{\mathrm{d}}{\mathrm{d}x} \widetilde{\Psi} \right) \mathrm{d}x \\
&= \int\limits_{-\infty}^{\infty} \Psi^* e^{\frac{iV_0 t}{\hbar}} \left(-i\hbar \frac{\mathrm{d}}{\mathrm{d}x} \Psi e^{-\frac{iV_0 t}{\hbar}} \right) \mathrm{d}x \\
&= \int\limits_{-\infty}^{\infty} \Psi^* e^{\frac{iV_0 t}{\hbar}} e^{-\frac{iV_0 t}{\hbar}} \left(-i\hbar \frac{\mathrm{d}}{\mathrm{d}x} \Psi \right) \mathrm{d}x \\
&= \int\limits_{-\infty}^{\infty} \Psi^*(x,t) \left(-i\hbar \frac{\mathrm{d}}{\mathrm{d}x} \Psi(x,t) \right) \mathrm{d}x \\
&= \langle p \rangle
\end{aligned}$$

Therefore, the change in potential <u>does not change</u> the expectation value of p.

$$\begin{aligned}
\widetilde{\langle Q \rangle} &= \int\limits_{-\infty}^{\infty} \widetilde{\Psi}^* \hat{Q} \widetilde{\Psi} \, \mathrm{d}x \\
&= \int\limits_{-\infty}^{\infty} \Psi^* e^{\frac{iV_0 t}{\hbar}} \hat{Q} \Psi e^{-\frac{iV_0 t}{\hbar}} \, \mathrm{d}x \\
&= \int\limits_{-\infty}^{\infty} \Psi^* \hat{Q} \Psi \, \mathrm{d}x \\
&= \langle Q \rangle
\end{aligned}$$

Therefore, the change in potential <u>does not change</u> the expectation value of Q.

Exercise 3.

Prove the following commutation relations.

(1) $[\hat{p}, \hat{x}^n] = -i\hbar n \hat{x}^{n-1}$
(2) $\left[\hat{x}, \hat{p}^2\right] = 2i\hbar \hat{p}$

Exercise 4.

(1)

$$\begin{aligned}
[\hat{p}, \hat{x}^n] f(x) &= \hat{p}\hat{x}^n f(x) - \hat{x}^n \hat{p} f(x) \\
&= \hat{p} x^n f(x) - \hat{x}^n \left(-i\hbar \frac{\mathrm{d}}{\mathrm{d}x} f(x)\right) \\
&= -i\hbar \frac{\mathrm{d}}{\mathrm{d}x} \big(x^n f(x)\big) + i\hbar x^n f'(x) \\
&= -i\hbar \left(n x^{n-1} f(x) + x^n f'(x)\right) + i\hbar x^n f'(x) \\
&= -i\hbar n x^{n-1} f(x) \\
&= -i\hbar n \hat{x}^{n-1} f(x)
\end{aligned}$$

Therefore,

$$[\hat{p}, \hat{x}^n] = -i\hbar n \hat{x}^{n-1}$$

$\square$

(2)

$$\begin{aligned}
\left[\hat{x}, \hat{p}^2\right] f(x) &= \hat{x}\hat{p}^2 f(x) - \hat{p}^2 \hat{x} f(x) \\
&= \hat{x}\hat{p}\hat{p} f(x) - \hat{p}\hat{p}\hat{x} f(x) \\
&= \hat{x}\hat{p} \left(-i\hbar \frac{\mathrm{d}}{\mathrm{d}x} f(x)\right) - \hat{p}\hat{p} x f(x) \\
&= \hat{x} \left(-i\hbar \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\mathrm{d}}{\mathrm{d}x} f(x)\right)\right) - \hat{p} \left(-i\hbar \frac{\mathrm{d}}{\mathrm{d}x} x f(x)\right) \\
&= x \left(-i\hbar \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\mathrm{d}}{\mathrm{d}x} f(x)\right)\right) + i\hbar \frac{\mathrm{d}}{\mathrm{d}x} \left(-i\hbar \frac{\mathrm{d}}{\mathrm{d}x} x f(x)\right) \\
&= x \big(-i\hbar f''(x)\big) + i\hbar \frac{\mathrm{d}}{\mathrm{d}x} \big(-i\hbar x f'(x) - i\hbar f(x)\big) \\
&= -i\hbar x f''(x) + i\hbar \big(-i\hbar x f''(x) - i\hbar f'(x) - i\hbar f'(x)\big) \\
&= i\hbar \big(-x f''(x) - i\hbar x f''(x) - i\hbar f'(x) - i\hbar f'(x)\big) \\
&= -i\hbar \big(x f''(x) + i\hbar x f''(x) + i\hbar f'(x) + i\hbar f'(x)\big) \\
&= 2i\hbar \hat{p} f(x)
\end{aligned}$$

Therefore,

$$\left[\hat{x}, \hat{p}^2\right] = 2i\hbar \hat{p}$$

$\square$