QUANTUM AND SOLID STATE PHYSICS : ASSIGNMENT 8

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Exercise 1.

(1) The hole concentration of Si at $T = 300\,\mathrm{K}$ increases linearly from $x = 0\mathrm{cm}$ to $x = 0.01\mathrm{cm}$. At $x = 0\mathrm{cm}$, the hole concentration is $p = 4 \times 10^{17} \frac{1}{\mathrm{cm}^3}$. The hole diffusion coefficient is $D_p = 10 \frac{\mathrm{cm}^2}{\mathrm{s}}$. The magnitude of hole diffusion current density is $J_{\mathrm{diffusion}_p} = 20 \frac{\mathrm{A}}{\mathrm{cm}^2}$. Determine the hole concentration at $x = 0.01\mathrm{cm}$.

(2) The electron concentration in a sample of silicon is given by

$$n(x) = 10^{15} e^{-\frac{x}{L_n}} \frac{1}{\mathrm{cm}^3}$$

where $L_n = 10^{-4}\mathrm{cm}$, and $0 \leq x < \infty$. The electron diffusion coefficient is $D_n = 25 \frac{\mathrm{cm}^2}{\mathrm{s}}$. Draw the electron concentration profile as a function of distance, and indicate the direction of electron flow, and of electron current density $J_{\mathrm{diffusion}_n}$ on the figure. Determine the electron diffusion current density $J_{\mathrm{diffusion}_n}$ at $x = 0$, $x = L_n$, and as $x \to \infty$. Mathematically, what is the significance of L_n? Answer this by comparing electron concentration at $x = L_n$ to its initial value at $x = 0$. At $x = L_n$, $n(x)$ has decayed to what fraction of its initial value?

Solution 1.

(1)

$$J_{\mathrm{diffusion}_p} = q D_p \frac{\mathrm{d}p}{\mathrm{d}x}$$

$$\therefore 20 \frac{\mathrm{A}}{\mathrm{cm}^2} = (q) \left(10 \frac{\mathrm{cm}^2}{\mathrm{s}}\right) \frac{\mathrm{d}p}{\mathrm{d}x}$$

$$\therefore \frac{2}{q} \mathrm{A\,s} = \frac{\mathrm{d}p}{\mathrm{d}x}$$

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Therefore,

$$\begin{aligned}
\frac{p_0 - p_{0.01}}{0 - 0.01} &= \frac{2}{q} \\
\therefore \frac{p_0 - p_{0.01}}{0.01} &= -\frac{2}{1.6} \times 10^{19} \\
\therefore p_0 - p_{0.01} &= -1.25 \times 10^{17} \\
\therefore p_{0.01} &= 4 \times 10^{17} + 1.25 \times 10^{17} \\
&= 5.25 \times 10^{17}
\end{aligned}$$

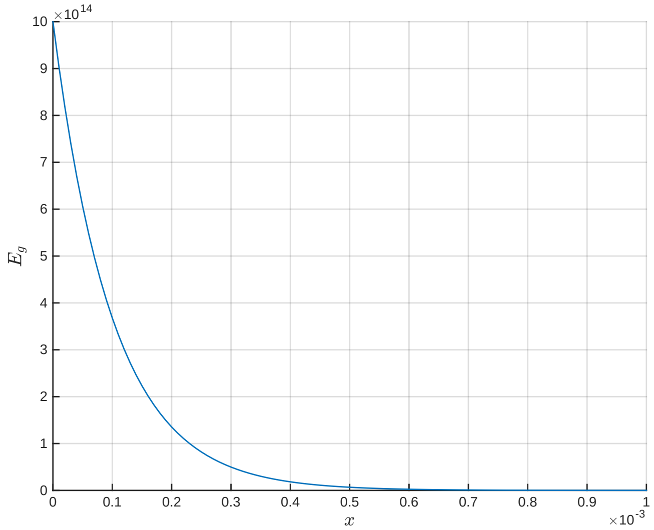
(2)

$$n(x) = 10^{15} e^{-\frac{x}{L_n}}$$

Therefore,

$$\frac{\mathrm{d}n}{\mathrm{d}x} = -\frac{10^{15}}{L_n} e^{-\frac{x}{L_n}}$$

Therefore,



The electron flow is directed from the left to the right.

$$\begin{aligned}
J_{\text{diffusion}_n} &= qD_p \frac{\mathrm{d}n}{\mathrm{d}x} \\
&= \left(1.6 \times 10^{-19}\mathrm{C}\right) \left(25 \frac{\mathrm{cm}^2}{\mathrm{s}}\right) \left(-\frac{10^{15}}{10^{-4}} e^{-\frac{x}{10^{-4}}} \frac{1}{\mathrm{cm}^4}\right) \\
&= -40 e^{-10^4 x} \frac{\mathrm{A}}{\mathrm{cm}^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
J_{\text{diffusion}_n}(x=0) &= -40e^0 \\
&= -40\frac{\text{A}}{\text{cm}^2} \\
J_{\text{diffusion}_n}(x=L_n=10^{-4}) &= -40e^{-1} \\
&= -14.72\frac{\text{A}}{\text{cm}^2} \\
J_{\text{diffusion}_n}(x\to\infty) &= -40\cdot 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
n(x) &= 10^{15}e^{-\frac{x}{L_n}}\frac{1}{\text{cm}^3} \\
\therefore n(0) &= 10^{15}e^0 \\
&= 10^{15}\frac{1}{\text{cm}^3} \\
\therefore n(L_n) &= 10^{15}e^{-1} \\
&= \frac{10^{15}}{e}\frac{1}{\text{cm}^3}
\end{aligned}$$

Therefore, at $x = L_n$, $n(x)$ is $\frac{1}{e}$ of its original value.

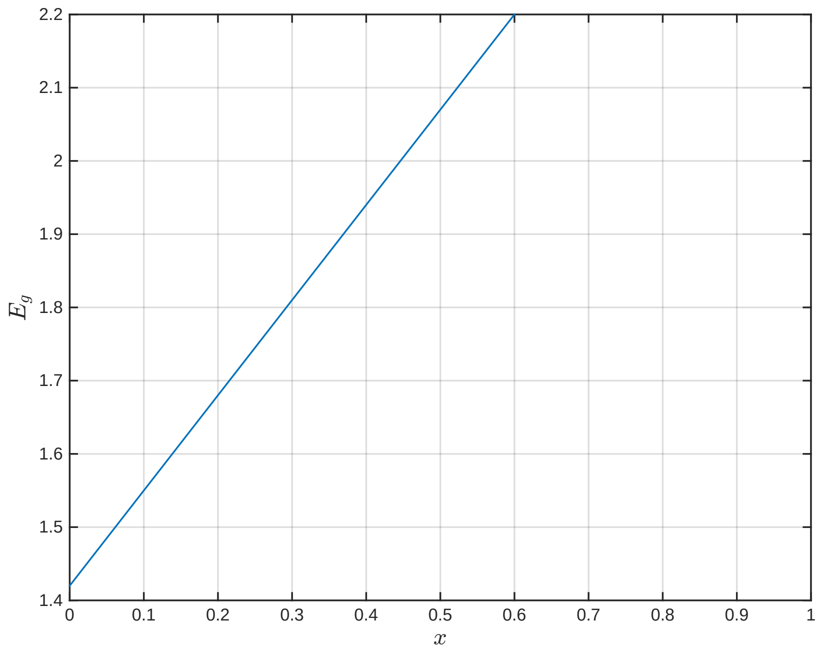
Exercise 2.

Suppose we use optical absorption and emission as a technique for determining the Al composition of a sample of $\text{Al}_x\text{Ga}_{1-x}\text{As}$. A feature of AlGaAs is that it is a material which demonstrates radiative recombination, i.e. it emits photons as a result of carrier recombination.

In our experiment, we illuminate a sample of AlGaAs with a 509 nm laser, and the wavelength of photons emitted from the material is 689 nm.

Answer the following.

(1) Is this laser a suitable choice for the experiment?
(2) Given the plot below, of E_{gap} as a function of aluminium composition x in $\text{Al}_x\text{Ga}_{1-x}\text{As}$, what is the value of x in this particular material?



Solution 2.

(1) The wavelength of the laser is less than the wavelength of the photons emitted. Therefore, the energy of the photons emitted by the laser is greater than the energy band gap of the sample. Hence, the laser is a suitable choice, but a laser which emits photons of lesser energy can also be used, as long as the wavelength of the laser is less than 689 nm.

(2)

$$
\begin{aligned}
E &= \frac{hc}{\lambda} \\
&= \frac{1.24}{689} \times 10^3 \\
&= 1.8\,\mathrm{eV}
\end{aligned}
$$

Therefore, the value of x in this particular material is approximately 0.3.

Exercise 3.

Consider a particle of mass m and the following potential $V(x)$.

$$V(x) = -\alpha\left(\delta(x+a) + \delta(x-a)\right)$$

We solved in class for the even wave functions. Repeat what we did for the odd solutions, i.e. write down the odd wave function, and find the allowed values for k. Is there an allowed energy for all values of α?

Solution 3.

$$V(x) = -\alpha\left(\delta(x+a) + \delta(x-a)\right)$$

Therefore,

$$\psi(x) = \begin{cases} Ae^{kx} & ; \quad x < -a \\ Be^{kx} + Ce^{-kx} & \quad -a < x < a \\ De^{-kx} & ; \quad a < x \end{cases}$$

where

$$k = \sqrt{-\frac{2mE}{\hbar^2}}$$

Therefore,

$$\begin{aligned} \psi(-a+\varepsilon) &= \psi(-a-\varepsilon) \\ \psi(a+\varepsilon) &= \psi(a-\varepsilon) \\ \psi'(-a+\varepsilon) - \psi'(-a-\varepsilon) &= -\frac{2m\alpha}{\hbar^2}\psi(-a) \\ \psi'(a+\varepsilon) - \psi'(a-\varepsilon) &= -\frac{2m\alpha}{\hbar^2}\psi(a) \end{aligned}$$

As $V(x)$ is even, the wave function $\psi(x)$ can be split into odd and even parts, i.e. $\psi_{\text{even}}(x)$ and $\psi_{\text{odd}}(x)$.
Therefore, for $\psi_{\text{odd}}(x)$,

$$\begin{aligned} A &= -D \\ B &= -C \end{aligned}$$

Therefore,

$$\psi_{\text{odd}}(x) = \begin{cases} Ae^{kx} & ; \quad x < -a \\ B\left(e^{kx} - e^{-kx}\right) & ; \quad -a < x < a \\ -Ae^{-kx} & ; \quad a < x \end{cases}$$

Therefore,

$$\psi_{\text{odd}}{}'(x) = \begin{cases} kAe^{kx} & ; \quad x < -a \\ kB\left(e^{kx} + e^{-kx}\right) & ; \quad -a < x < a \\ kAe^{-kx} & ; \quad a < x \end{cases}$$

Therefore,

$$\begin{aligned} \psi(-a+\varepsilon) &= \psi(-a-\varepsilon) \\ \therefore Ae^{-ka} &= B\left(e^{-ka} - e^{ka}\right) \\ \psi'(-a+\varepsilon) - \psi'(-a-\varepsilon) &= -\frac{2m\alpha}{\hbar^2}\psi(-a) \\ \therefore kB\left(e^{-ka} + e^{ka}\right) - kAe^{ka} &= -\frac{2m\alpha}{\hbar^2}Ae^{-ka} \end{aligned}$$

Therefore,

$$\begin{aligned} A &= B\left(1 - e^{2ka}\right) \\ B\left(1 + e^{2ka}\right) &= A\left(1 - \frac{2m\alpha}{\hbar^2 k}\right) \end{aligned}$$

Therefore,

$$\begin{aligned} 1 + e^{2ka} &= \left(1 - e^{2ka}\right)\left(1 - \frac{2m\alpha}{\hbar^2 k}\right) \\ \therefore 1 + e^{2ka} &= 1 - \frac{2m\alpha}{\hbar^2 k} - e^{2ka} + e^{2ka}\frac{2m\alpha}{\hbar^2 k} \\ \therefore \frac{m\alpha}{\hbar^2 k} &= e^{2ka}\left(\frac{m\alpha}{\hbar^2 k} - 1\right) \\ \therefore 1 &= e^{2ka}\left(1 - \frac{\hbar^2 k}{m\alpha}\right) \\ \therefore e^{-2ka} &= 1 - \frac{\hbar^2 k}{m\alpha} \end{aligned}$$

Let

$$\begin{aligned} z &= 2ka \\ c &= \frac{\hbar^2}{2am\alpha} \end{aligned}$$

Therefore,

$$e^{-z} = 1 - cz$$

Therefore, there exists at least one solution.

Exercise 4.
Consider a particle of mass m, and the following potential.

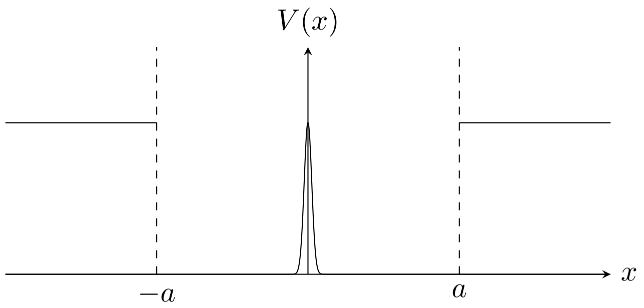
$$V(x) = \begin{cases} \infty & ; \quad x < -a \\ \alpha\delta(x) & ; \quad -a < x < a \\ \infty & ; \quad a < x \end{cases}$$

(1) Sketch the potential.
(2) Write down the general from of the wave function for all values of position x. Indicate the number of unknown parameters and list them. Hint: Use $\sin x$ and $\cos x$ functions as for the infinite potential well.
(3) Write down the boundary conditions.
(4) Find the allowed energies for the particle corresponding to the even wave functions. Hint: Even solutions satisfy $\psi(x) = \psi(-x)$.
(5) Find the allowed energies for the particle corresponding to the odd wave functions. Hint: Even solutions satisfy $\psi(x) = -\psi(-x)$.
(6) Compare your answer to the allowed energy states of an infinite well, without a delta function in the middle.

Solution 4.

(1)

$$V(x) = \begin{cases} \infty & ; \quad x < -a \\ \alpha\delta(x) & ; \quad -a < x < a \\ \infty & ; \quad a < x \end{cases}$$



(2) Therefore, solving the time independent Schrödinger equation,

$$\psi(x) = \begin{cases} 0 & ; \quad x < -a \\ Ae^{ikx} & ; \quad -a < x < 0 \\ Ae^{-ikx} & ; \quad 0 < x < a \\ 0 & ; \quad a < x \end{cases}$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

Therefore,

$$\psi(x) = \begin{cases} 0 & ; \quad x < -a \\ A\cos(kx) + B\sin(kx) & ; \quad -a < x < 0 \\ A\cos(-kx) + B\sin(-kx) & ; \quad 0 < x < a \\ 0 & ; \quad a < x \end{cases}$$

$$= \begin{cases} 0 & ; \quad x < -a \\ A\cos(kx) + B\sin(kx) & ; \quad -a < x < 0 \\ A\cos(kx) - B\sin(kx) & ; \quad 0 < x < a \\ 0 & ; \quad a < x \end{cases}$$

Therefore, the unknowns are A and B.

(3) As $\psi(x)$ is always continuous,

$$0 = A\cos(-ka) + B\sin(-ka)$$

$$0 = A\cos(ka) - B\sin(ka)$$

(4) For $\psi_{\text{even}}(x)$,

$$B = 0$$

Therefore,

$$\psi_{\text{even}}(x) = \begin{cases} 0 & ; \quad x < -a \\ A\cos(kx) + B\sin(kx) & ; \quad -a < x < 0 \\ A\cos(kx) - B\sin(kx) & ; \quad 0 < x < a \\ 0 & ; \quad a < x \end{cases}$$

Substituting the boundary condition, for $x = -a$,

$$0 = A\cos(ka) - B\sin(ka)$$

Substituting the boundary condition, for $x = a$,

$$0 = A\cos(ka) - B\sin(ka)$$

Therefore,

$$A\cos(ka) = B\sin(ka)$$

$$\therefore \tan ka = \frac{A}{B}$$

Therefore, solving the boundary conditions for ψ',

$$-2kB = \frac{2m\alpha}{\hbar^2}A$$

Therefore, solving the conditions simultaneously, as for infinite quantum well,

$$-\frac{\hbar^2}{m\alpha a}z = \tan z$$

(5) For $\psi_{\text{odd}}(x)$,

$$A = 0$$

Therefore,

$$\psi_{\text{even}}(x) = \begin{cases} 0 & ; \quad x < -a \\ A\cos(kx) + B\sin(kx) & ; \quad -a < x < 0 \\ -A\cos(kx) + B\sin(kx) & ; \quad 0 < x < a \\ 0 & ; \quad a < x \end{cases}$$

For ψ to be continuous at $x = 0$,

$$\begin{aligned} B &= -B \\ &= 0 \end{aligned}$$

Therefore,

$$\psi_{\text{even}}(x) = \begin{cases} 0 & ; \quad x < -a \\ B\sin(kx) & ; \quad -a < x < 0 \\ B\sin(kx) & ; \quad 0 < x < a \\ 0 & ; \quad a < x \end{cases}$$

Substituting the boundary condition, for $x = -a$,

$$B\sin(ka) = 0$$

Substituting the boundary condition, for $x = a$,

$B\sin(ka) = 0$

Therefore,

$ka = n\pi$

Therefore,

$ka = n\pi$

Therefore,

$$k_n = \frac{n\pi}{a}$$
$$\therefore \sqrt{\frac{2mE}{\hbar^2}} = \frac{n\pi}{a}$$
$$\therefore \frac{2mE}{\hbar^2} = \frac{n^2\pi^2}{a^2}$$
$$\therefore E = \frac{\pi^2 n^2 \hbar^2}{2ma^2}$$

(6) For the odd functions, the result is the same as the result for an infinite square well, without a delta function. For the even function, the energy levels are slightly different than that for an infinite well without a delta function.