Homework 5

STA-360

Due at 5:00 PM EDT on Friday September 24 at 5 PM EDT

Total points: 10 (reproducibility) + 15 (Q1) + 25 (Q2) = 50 points.

General instructions for homeworks: Please follow the uploading file instructions according to the syllabus. You will give the commands to answer each question in its own code block, which will also produce plots that will be automatically embedded in the output file. Each answer must be supported by written statements as well as any code used. Your code must be completely reproducible and must compile.

Advice: Start early on the homeworks and it is advised that you not wait until the day of. While the professor and the TA's check emails, they will be answered in the order they are received and last minute help will not be given unless we happen to be free.

Commenting code Code should be commented. See the Google style guide for questions regarding commenting or how to write code https://google.github.io/styleguide/Rguide.xml. No late homework's will be accepted.

1. (15 points, 5 points each) Hoff, 3.12 (Jeffrey's prior).

```
knitr::include_graphics("hoff_3_12.jpg")
```

Hoff 3.12

a $p_J(\theta)$, $y \sim \text{binomial}(n, \theta)$

$$p_J(\theta) \propto \sqrt{-E\left(\frac{\partial^2 \log p(y|\theta)}{\partial\theta^2} \,\middle|\, \theta\right)}$$

$$p(y|\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$$

$$\log p(y|\theta) = \log\binom{n}{y} + \log\theta^y + \log(1-\theta)^{n-y}$$

$$= \log\binom{n}{y} + y\log\theta + (n-y)\log(1-\theta)$$

$$\frac{\partial \log p(y|\theta)}{\partial\theta} = 0 + \frac{y}{\theta} + \frac{n-y}{1-\theta}\cdot -1 = y\theta^{-1} + (y-n)(1-\theta)^{-1}$$

$$\frac{\partial^2 \log p(y|\theta)}{\partial\theta^2} = -y\theta^{-2} + (y-n)\cdot -(1-\theta)^{-2}\cdot -1 = -\frac{y}{\theta^2} + \frac{y-n}{(1-\theta)^2}$$

$$p_J(\theta) \propto \sqrt{-E\left(-\frac{y}{\theta^2} + \frac{y+n}{(1-\theta)^2}\,\middle|\,\theta\right)} \propto \sqrt{\frac{1}{\theta^2}E(y) + \frac{1}{(1-\theta)^2}E(n-y)}$$

$$\propto \sqrt{\frac{n\theta}{\theta^2} + \frac{n-n\theta}{(1-\theta)^2}} \propto \sqrt{\frac{n\theta}{\theta\cdot\theta} + \frac{n(1-\theta)}{(1-\theta)^2}} \propto \sqrt{\frac{n}{\theta}+\frac{n}{(1-\theta)}}$$

$$\propto \sqrt{\frac{n(1-\theta)+n(\theta)}{\theta(1-\theta)}} \propto \sqrt{\frac{n}{\theta(1-\theta)}} \propto n^{1/2}\theta^{-1/2}(1-\theta)^{-1/2}$$

$\propto \theta^{-1/2}(1-\theta)^{-1/2} \leftarrow$ (Kernel for beta(0.5, 0.5))

$\propto$ Beta(0.5, 0.5)

b $$p_J(\theta) \propto \sqrt{-E\left(\frac{\partial^2 \log p(y|\psi)}{\partial\theta^2}\,\middle|\,\psi\right)}$$

$$p(y|\psi) = \binom{n}{y} e^{\psi y}(1+e^\psi)^{-n}$$

$$\log p(y|\psi) = \log\binom{n}{y} + \log e^{\psi y} + \log(1+e^\psi)^{-n}$$

$$= \log\binom{n}{y} + \psi y - n\log(1+e^\psi)$$

$$\partial \log(p(y|\psi))/\partial\psi = 0 + y - n/(1+e^\psi)\cdot e^\psi = y - ne^\psi/(e^\psi+1)$$

$$\partial^2 \log p(y|\psi)/\partial\psi^2 = 0 - n((e^\psi)(e^\psi+1) - (e^\psi)(e^\psi))/(e^\psi+1)^2 = -ne^\psi/(e^\psi+1)^2$$

$$p_J(\psi) \propto \sqrt{-E\left(-\frac{ne^\psi}{(e^\psi+1)^2}\,\middle|\,\psi\right)} \propto \sqrt{\frac{ne^\psi}{(e^\psi+1)^2}} \propto \boxed{\frac{\sqrt{ne^\psi}}{e^\psi+1}}$$

c $p_\psi(\psi) \propto p_\theta(h(\psi)) \times \left|\frac{dh}{d\psi}\right|$ (Formula from 3.10) $\frac{dh}{d\psi} = \left(\frac{e^\psi}{(1+e^\psi)}\right)' = \frac{e^\psi}{(1+e^\psi)^2}$

$$\propto \sqrt{\frac{n}{\frac{e^\psi}{1+e^\psi}\left(1-\frac{e^\psi}{1+e^\psi}\right)}}\cdot\frac{e^\psi}{(1+e^\psi)^2}$$

$$\propto \sqrt{\frac{n}{e^\psi/(1+e^\psi)^2}}\cdot\frac{e^\psi}{(1+e^\psi)^2} \propto \frac{\sqrt{n}(1+e^\psi)}{\sqrt{e^\psi}}\cdot\frac{e^\psi}{(1+e^\psi)^2} \propto \boxed{\frac{\sqrt{ne^\psi}}{e^\psi+1}}$$

Same as b ✓

2. *Lab component* (25 points total) Please refer to lab 5 and complete tasks 4—5.

(a) (10) Task 4

```r
x <- seq(0, 1, 10^-2)
fx <- function(x) sin(pi * x)^2
plot(fx, xlim = c(0,1), ylim = c(0,1.5),
     ylab = "f(x)", lwd = 2)
curve(dunif, add = TRUE, col = "blue", lwd = 2)
```

```r
curve(dbeta(x,2,2), add = TRUE, col = "red", lwd = 2)
legend("bottom", legend =
         c(expression(paste("sin(",pi,"x)"^"2")),"Unif(0,1)",
           "Beta(2,2)"), col = c("black", "blue", "red"),
       lty = c(1,1,1), bty = "n", cex = 1.1, lwd = 2)
```

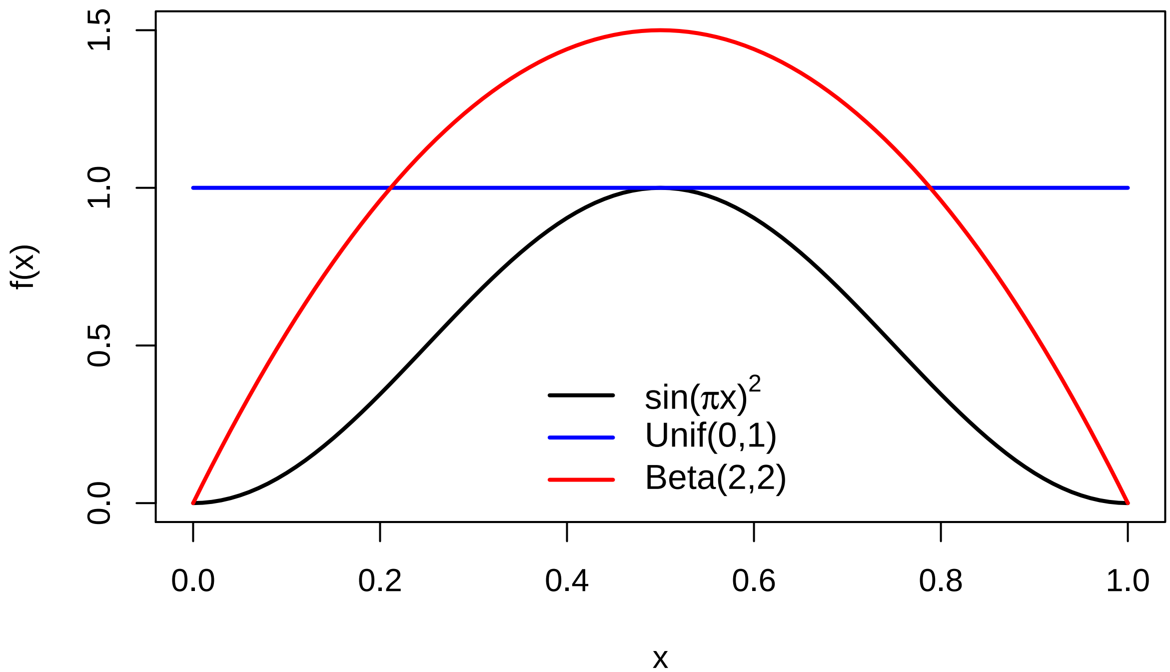

Figure 1: Comparision of the target function and the Beta(2,2) density on the same plot.

```r
sim_fun <- function(f, envelope = "unif", par1 = 0,
                    par2 = 1, n = 10^2, plot = TRUE){

  r_envelope <- match.fun(paste0("r", envelope))
  d_envelope <- match.fun(paste0("d", envelope))
  proposal <- r_envelope(n, par1, par2)
  density_ratio <- f(proposal) / d_envelope(proposal, par1, par2)
  samples <- proposal[runif(n) < density_ratio]
  acceptance_ratio <- length(samples) / n
  if (plot) {
    hist(samples, probability = TRUE,
         main = paste0("Histogram of ",
                       n, " samples from ",
                       envelope, "(", par1, ",", par2,
                       ").\n Acceptance ratio: ",
                       round(acceptance_ratio,2)),
                       cex.main = 0.75)
  }
  list(x = samples, acceptance_ratio = acceptance_ratio)
}
```

```r
# reproducibility
set.seed(1)
par(mfrow = c(2,2), mar = rep(4, 4))
unif_1 <- sim_fun(fx, envelope = "unif", par1 = 0, par2 = 1, n = 10^2)
unif_2 <- sim_fun(fx, envelope = "unif", par1 = 0, par2 = 1, n = 10^5)
```

```r
# additional code for task 4, using beta(2, 2) density
beta_1 <- sim_fun(fx, envelope = "beta", par1 = 2, par2 = 2, n = 10^2)
beta_2 <- sim_fun(fx, envelope = "beta", par1 = 2, par2 = 2, n = 10^5)
```

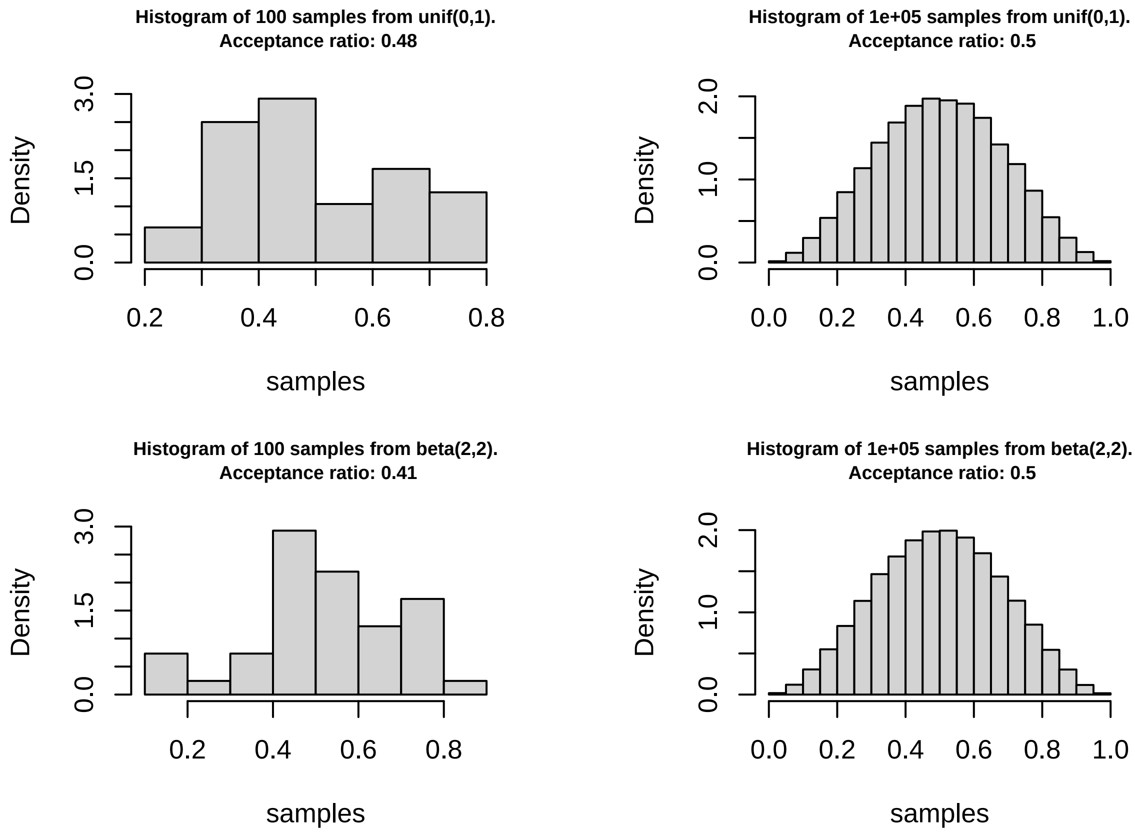

Figure 2: Comparision of the output of the rejection sampling for 100 versus 100,000 simulations with uniform and beta distributions as envelope functions.

First of all, we notice that the results for the beta(2, 2) distribution somewhat resemble the results for the unif(0, 1) distribution. The acceptance ratio in the 100 simulation case for beta is slightly lower, at 0.41, than for unif, at 0.48, but specifically with the 100,000 simulation case, we see an identical acceptance ratio of 0.5 for unif and beta. We can also see the acceptance ratio with 100,000 simulations better resembles a normal distribution than the acceptance ratio with 100 simulations for both beta and unif.

(b) (15) Task 5

I would not have a strong recommendation between the unif(0, 1) and beta(2, 2) distributions, as when examining a situation with a large number of situations, they are about the same for acceptance ratio (at 0.5 with a normal distribution for both). I would say they are too similar for me to recommend one over the other for this specific situation; they are about the same.

I would try a normal distribution if I wanted to maximize the acceptance ratio. The function f(x) we are considering better resembles a normal distribution, with something like mean 0.5 and standard deviation 0.4. This would minimize the region that produces rejections in comparison to the unif(0, 1) and beta(2, 2) examples, so I think the acceptance ratio would be higher for a high number of samples.