$\underline{\underline{Q4}}$

To prove :-

$$H\left(\frac{Y}{X}\right) + H(X) = H(X,Y)$$

Proof :-

$$\text{As } H(X) = -\sum_{x \in X} P(x) \log P(x) \quad \text{--- (1)}$$

$$H\left(\frac{Y}{X}\right) = -\sum_{x \in X}\sum_{y \in Y} P(x,y) \log P\left(\frac{Y}{X}\right) \quad \text{--- (2)}$$

$$H(X,Y) = -\sum_{x \in X}\sum_{y \in Y} P(x,y) \log P(x,y) \quad \text{---- (3)}$$

Now Taking eq 2

$$H\left(\frac{Y}{X}\right) = -\sum_{x \in X}\sum_{y \in Y} P(x,y) \log P\left(\frac{Y}{X}\right) \quad \text{----}$$

$$\text{As } P\left(\frac{Y}{X}\right) = \frac{P(Y \cap X)}{P(X)}$$

$$\text{So } H\left(\frac{Y}{X}\right) = -\sum_{x \in X}\sum_{y \in Y} P(x,y) \log \frac{P(Y \cap X)}{P(X)}$$

$$H\left(\frac{Y}{X}\right) = -\sum_{x \in X} \sum_{y \in Y} P(x,y) \log P(x,y)$$

$$\cancel{+\log} + \sum_{x \in X} \sum_{y \in Y} P(x,y) \log P(X)$$

$$H\left(\frac{Y}{X}\right) = H(X,Y) + \sum_{x \in X} \sum_{y \in Y} P(x,y) \log P(X)$$

... using eq^n 3

$$\sum_{x \in X} \sum_{y \in Y} P(x,y) \log P(X) = \sum_{x \in X} P(x) \log P(X) = -H(X)$$

$$H\left(\frac{Y}{X}\right) = H(X,Y) - H(X)$$

$$\boxed{H(X,Y) = H\left(\frac{Y}{X}\right) + H(X)}$$

Joint Entropy

Conditional Entropy

Q5

To prove :-

$$I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y/X)$$

Where. $I(X;Y) = E_{(x,y)\sim P(x,y)} \log \frac{P(X,Y)}{P(X)\,P(Y)}$

$$= \sum_{x\in X} \sum_{y\in Y} P(x,y) \log \frac{P(x,y)}{P(x)\,P(y)}$$

Proof

$$H(X) = -\sum_{x\in X} P(x) \log P(x)$$

$$H\left(\frac{X}{Y}\right) = -\sum_{x\in X}\sum_{y\in Y} P(x,y) \log P\left(\frac{x}{y}\right)$$

$$P\left(\frac{x}{y}\right) = \frac{P(x\cap y)}{P(y)} = \frac{P(x,y)}{P(y)}$$

$$H\left(\frac{X}{Y}\right) = -\sum_{x\in X}\sum_{y\in Y} P(x,y) \log P(x,y)$$

$H(X,Y)$

$$+ \sum_{x\in X}\sum_{y\in Y} P(x,y) \log P(y)$$

$\hookrightarrow -H(y)$

(as proved in problem 4)

$$H\left(\frac{X}{Y}\right) = H(X,Y) - H(Y) \quad \text{---} \quad (4)$$

$$H(X) - H(X/Y) = H(X) + H(Y) - H(X,Y)$$

$$H(X) - H(X/Y) = +\sum_{x \in X} \sum_{y \in Y} P(x,y) \log P(x,y)$$

$$- \sum_{x \in X} P(x) \log P(x) - \sum_{y \in Y} P(y) \log P(y)$$

$$\left[\begin{array}{l} \because \sum_{x \in X} P(x) \log P(x) = \sum_{x \in X} \sum_{y \in Y} P(x,y) \log P(x) \\ \sum_{y \in Y} P(y) \log P(y) = \sum_{y \in Y} \sum_{x \in X} P(x,y) \log P(y) \end{array} \right.$$

On putting values

$$H(X) - H(X/Y) = \sum_{x \in X} \sum_{y \in Y} P(x,y) \begin{bmatrix} \log P(x,y) \\ - \log P(x) \\ - \log P(y) \end{bmatrix}$$

$$= \sum_{x \in Y} \sum_{y \in Y} P(x,y) \left[\log \left(\frac{P(x,y)}{P(x)\, P(y)} \right) \right] \quad \text{equation } (5)$$

$$\text{So } H(X) - H(X/Y) = H(X) + H(Y) - H(X,Y) = I(X,Y)$$

$$H(X) - H(X/Y) = I(X;Y)$$

Now we will use the previous results to prove $I(X;Y) = H(Y) - H(Y/X)$

So as in previous part we proved

$$H\left(\frac{X}{Y}\right) = H(X,Y) - H(Y) \quad (\text{equation } 4)$$

Similarly $H\left(\frac{Y}{X}\right) = H(X,Y) - H(X)$

So $H(Y) - H(Y/X) = H(Y) + H(X) - H(X,Y)$

So using equation 5 from previous part

$$H(Y) - H(Y/X) = H(Y) + H(X) - H(X,Y) = I(X;Y)$$

So $\boxed{I(X;Y) = H(Y) - H\left(\frac{Y}{X}\right)}$

So $\boxed{I(X,Y) = H(X) - H(X/Y) = H(Y) - H\left(\frac{Y}{X}\right)}$

Hence proved

$\underline{6}$ Numerical exercise

$p(x=0, y=0) = p(x=0, y=1) = p(x=1, y=1)$

$= \frac{1}{3}$

and $p(x=1, y=0) = 0$

X, Y are random variables taking values 0 or 1

x

So

y	0	1	y Total
0	1/3	0	1/3
1	1/3	1/3	2/3
x total	2/3	1/3	

To find

a) $H(X), H(Y)$

$$H(X) = -\sum_{x \in X} p(x) \log_2 p(x)$$

$$= -\frac{1}{3}\log_2 \frac{1}{3} - \frac{2}{3}\log_2 \frac{2}{3}$$

$$= -\frac{1}{3}\left[\log\frac{1}{3} + \log\frac{4}{9}\right]$$

$$= -\frac{1}{3}\log_2\frac{4}{27}$$

$$= -\frac{1}{3}\times\left[2 - \log_2 27\right]$$

$$= \frac{-1}{3}\left[2 - 4.75\right]$$

$$\approx +\frac{2.75}{3}$$

$$\approx 0.92$$

$$\boxed{H(X) \approx 0.92}$$

$$H(Y) = -\sum_{y \in Y} P(y)\log_2 P(y)$$

$$= -\frac{1}{3}\log_2\frac{1}{3} - \frac{2}{3}\log_2\frac{2}{3}$$

$$\boxed{H(Y) \approx 0.92}$$

b) $H(X/Y)$, $H(Y/X)$

$$H\left(\frac{X}{Y}\right) = -\sum_{x \in X}\sum_{y \in Y} P(x,y) \log P\left(\frac{x}{y}\right)$$

$$H\left(\frac{X}{Y}\right) = -\sum_{x \in X}\sum_{y \in Y} P(x,y) \log \frac{P(x,y)}{P(y)}$$

$$= -\left[\frac{1}{3}\log\frac{1/3}{1/3} + \frac{1}{3}\log\frac{1/3}{2/3} + \frac{1}{3}\log\frac{1/3}{2/3}\right] = +\frac{2}{3} \approx 0.66$$

↳ can also it in this way

$$H\left(\frac{X}{Y}\right) = -\sum_{x \in Y}\sum_{y \in Y} P(x,y) \log P(x,y) \rightarrow H(X,Y)$$

$$+ \sum_{x \in X}\sum_{y \in Y} P(x,y) \log P(y)$$

$$= H(X,Y) - H(Y)$$

$$H\left(\frac{X}{Y}\right) = -\left[\frac{1}{3}\log\frac{1}{3} + \frac{1}{3}\log\frac{1}{3} + \frac{1}{3}\log\frac{1}{3}\right]$$

$$- H(Y)$$

$$H\left(\frac{X}{Y}\right) = -\left[\log \frac{1}{3}\right] - 0.92$$

$$H\left(\frac{X}{Y}\right) = + \log_2 3 - 0.92$$

$H\left(\frac{X}{Y}\right) \simeq 1.58 - 0.92$

$[\because H(X,Y) = 1.58]$

$$\boxed{H\left(\frac{X}{Y}\right) \simeq 0.66}$$

similarly

$H\left(\frac{Y}{X}\right) = H(X,Y) - H(X)$

So $H\left(\frac{Y}{X}\right) \simeq 1.58 - 0.92$

$$\boxed{H\left(\frac{Y}{X}\right) \simeq 0.66}$$

c) $H(X,Y) = -\sum_{x \in X} \sum_{y \in Y} P(x,y) \log P(x,y)$

$= -\left[\frac{1}{3}\log\frac{1}{3} + \frac{1}{3}\log\frac{1}{3} + \frac{1}{3}\log\frac{1}{3} \right]$

$= \log_2 3$

$$\boxed{H(X,Y) \simeq 1.58}$$

d) $H(Y) - H(Y/X)$

Using the previous results

$H(Y) \approx 0.92$

$H(Y/X) \approx 0.66$

So $H(Y) - H\left(\frac{Y}{X}\right) \approx 0.92 - 0.66$

$$\boxed{H(Y) - H\left(\frac{Y}{X}\right) \approx 0.26}$$

e) $$I(X;Y) = \sum_{x \in X} \sum_{y \in Y} P(x,y) \log \frac{P(x,y)}{P(x)P(y)}$$

$$= \frac{1}{3} \log \frac{1/3}{\frac{1}{3}\frac{2}{3}} + 0 + \frac{1}{3} \log \frac{1/3}{\frac{1}{3}\frac{2}{3}} + \frac{1}{3} \log \frac{1/3}{\frac{2}{3}\frac{2}{3}}$$

$$= \frac{2}{3} \log_2 \frac{3}{2} + \frac{1}{3} \log_2 \frac{3}{4}$$

$$= \frac{2}{3}\left[\log_2 3 - 1\right] + \frac{1}{3}\left[\log_2 3 - 2\right]$$

$$\frac{2}{3}\left[0.58\right] + \frac{1}{3}\left[1.58 - 2\right]$$

$$\frac{0.74}{3} \approx 0.25$$

$$\boxed{I(X;Y) \approx 0.25 \approx 0.26 = H(Y) - H\left(\frac{Y}{X}\right)}$$

↳ Little bit decimal error.