Module 6

Aakash Shah
Team 4: Pause&Ponder*

March 27, 2018

1 Linearly independent vectors

1.1 In-Class Exercise 1

How does one address the question for the above example?

The premise of the above question is as follows :
The above equation can be written as a set of 3 equations in 2 variables. We multiply α with **u** and β with **v** to get 3 equations. If we solve these equations we get a solution. The solution shows that the corresponding lines intersect at the particular position of alpha and beta (here it is 3,2).

$w = 3u + 4v$

Thus, the above question says that **w** can be expressed as a linear combination of **u** and **v** by adding a bunch of scalars. These scalars help to scale the basic structure(vectors) of the equation in a linear fashion. The word linear means that the grid lines are parallel to each other and the standard basis vectors start at the origin.
There will always exist a trivial solution when we make all of them 0.

1.2 In-Class Exercise 2

Can u be expressed in terms of v and w?

Yes **u** can be expressed in terms of **v** and **w** just like the above equation. We have a different set of scalars that will make the equation true :

$u = \frac{-4}{3}\mathbf{v} + \frac{1}{3}\mathbf{w}$

1.3 In-Class Exercise 3

Show that this is the case.

$\mathbf{z} = (6, 1, 8)$ We can prove that z can not depend on u and v in a linear fashion. This is because it will not have any solution. The vectors can not possibly add each other in a fashion that it can form the resultant vector **z**. And hence it will cause a contradiction if we solve it using substitution.
The contradiction in this case will be $7 = 6$ which is absurd and not true and hence no solution or z can not be expressed as a linear combination of u and v.

*Team Member: Rohan Shetty

1.4 In-Class Exercise 4

Can u above be expressed as a linear combination of v and z? Or v as a linear combination of u and z??

$u = \alpha\mathbf{v} + \beta\mathbf{z}$
$=> 6\beta = 2$
$=> \alpha + \beta = -1$
$=> \alpha + 8\beta = 1$

This can lead to many contradictions and hence it can not be expressed as a linear combination. It means there are no scalars alpha or beta that can be used to express u in terms of a linear combination of v and z.

$v = \alpha\mathbf{u} + \beta\mathbf{z}$
$=> 2\alpha + 6\beta = 0$
$=> -\alpha + \beta = 1$
$=> \alpha + 8\beta = 1$

This can lead to many contradictions and hence it can not be expressed as a linear combination. It means there are no scalars alpha or beta that can be used to express v in terms of a linear combination of u and z.

2 RREF and independent vectors

2.1 In-Class Exercise 5

Use the definition of linear independence to show both parts of Proposition 6.1 are true for this example, and then generalize to prove 6.1 for all RREFs.

The definition of linear independence states that :

""In the theory of vector spaces, a set of vectors is said to be linearly independent if none of the vectors in the set can be defined as a linear combination of the others;""

Thus in the above example the pivot rows are linearly independent. That is the rows can not be expressed as a linear combination of the other. The other two non-pivot rows are dependent vectors and hence can be expressed as a linear combination of the other vectors. In this case these are 0 and definitely dependent. Since we can multiply it by a scalar of 0 to reach the null vector.

If $r_1, r_2, ..., r_n$ are linearly independent $c_1r_1 + c_2r_2 + ... + c_nr_n = 0$ iff $c_1 = c_2 = ... = c_n = 0$. Considering $r_n = 0$, we can get $c_1r_1 + c_2r_2 + ... + c_nr_n = 0$ by setting $c_1 = c_2 = \cdots = c_{n-1} = 0$ and taking any $c_n \neq 0$. So by definition, any set of vectors that contain the zero vector is linearly dependent.

Since the RREF's will naturally be in REF form and thus the REF's will have the pivot rows defined in such a way that the pivot rows on the top will be independent and the ones below the final pivot row will be dependent. If the matrix has n rows and n pivot rows, in that case it does not have any dependent row vectors.

2.2 In-Class Exercise 6

Use the definition of linear independence to show both parts of Proposition 6.2 are true for this example, and then generalize to prove 6.2 for all RREFs

The proposition 6.2 says that there are no independent vectors beyond the last pivot column.

This can be proved with a small example :

$c_3 = \alpha c_1 + \beta c_2$
$=> \alpha + 0\beta = -1$
$=> 0\alpha + \beta = 2$

And hence it is proven that c_3 is linearly dependent on c_1 and c_2. Similarly we can prove it for c_4 and c_5.

2.3 In-Class Exercise 7

Show that the above is true given what was shown earlier for v_2 and v_4

It is to be understood that the rank of a collection of vectors is the size of the largest subset of independent vectors and hence v_2 and v_4 are independent and all other vectors can be expressed as a linear combination of these two vectors.

2.4 In-Class Exercise 8

Prove the following: if the vectors $v_1, v_2, \ldots, v_n$ are independent, then so is any subset of these vectors.

Let the set $v_1, v_2, v_3, , ...v_m$ be any subset of the set $v_1, v_2, \ldots, v_n$

Let, us assume that the subset $v_1, v_2, ..., v_m$ is linearly dependent. So, there exists a vector in the subset $v_!, v_2, , ..., v_m$ which is the linear combination of the remaining vectors.

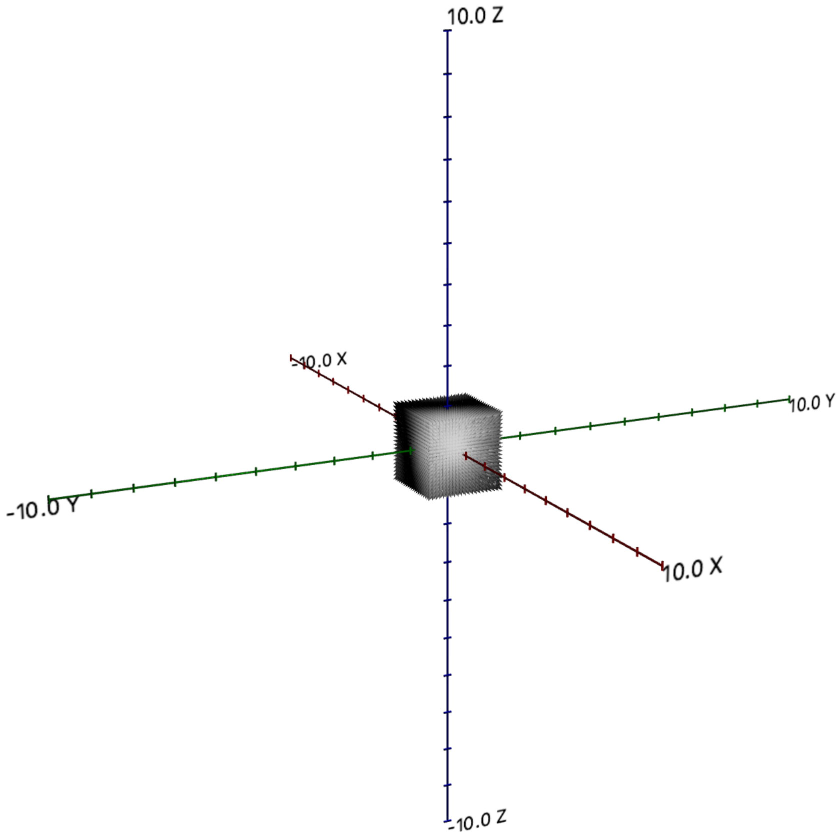
So let, v_k, be such a vector. Also, the subset $v_1, v_2, v_3, , ...v_m$ is contained in the set $v_1, v_2, \ldots, v_n$. So, the vector v_k, is also contained in the set $v_1, v_2, \ldots, v_n$ which is the linear combination of the remaining vectors. $= v_1, v_2, \ldots, v_n$ is linearly dependent set.

This is a contradiction to the given hypothesis. Thus, our assumption was wrong.
Hence, any subset of linearly independent set is also linearly independent.

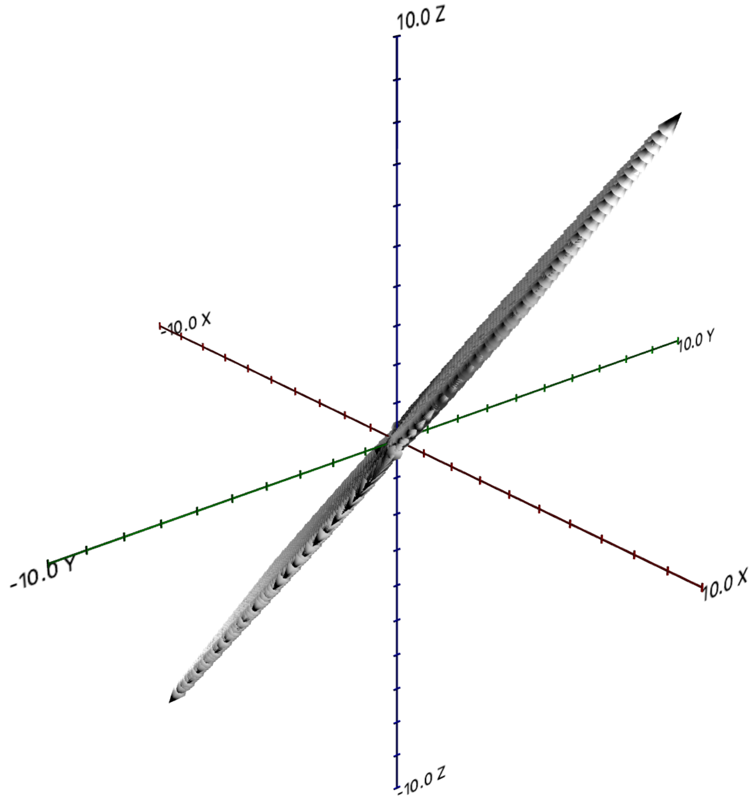
2.5 In-Class Exercise 9

In 3D, explore the span of $u = (1, 1, 2), v = (2, 1, 3), w = (3, 1, 4)$ in Span3DExample.java. You will need MatrixTool.java from earlier, or you can use your implementation of LinTool to compute scalar multiplication and addition of vectors.

We are exploring the Span of the three vectors. The span allows us to see the correlation between the vectors. We can see that the basis vectors can span from -1 to 1 in all directions of the grid. The picture below shows that the span is nearly equal to the entire 3D Space.



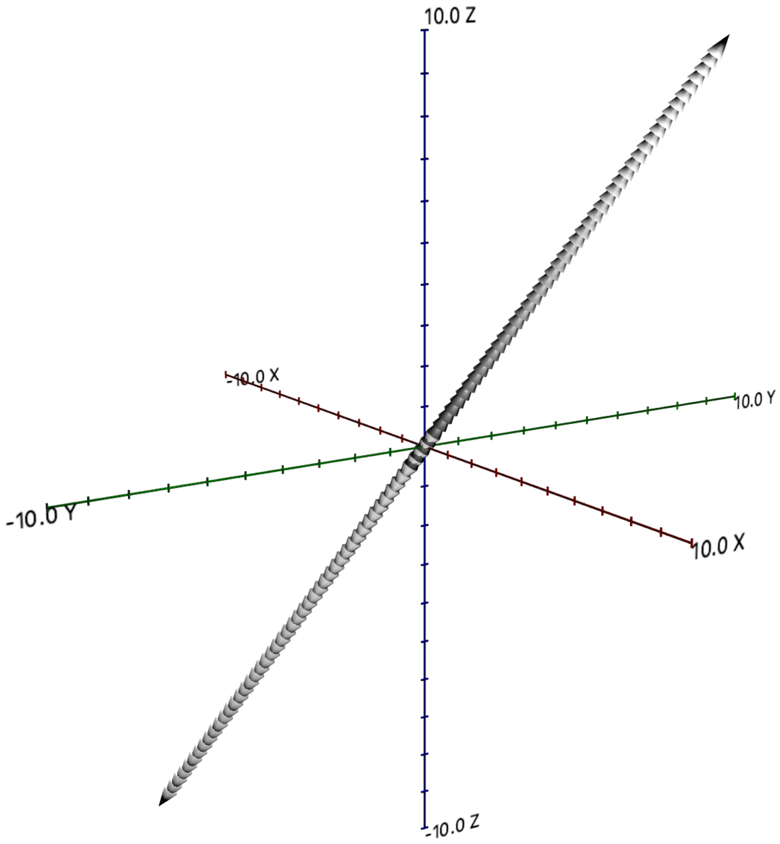
The given vectors : $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent vectors, which means that one vector can be composed of the other two vectors. In that case the span of the vectors reduces down from the whole 3D Space to a 2D plane. This can be seen in the picture below that the span is limited to the plane because the vectors are proportional and hence can be expressed as a linear combination of other two vectors.



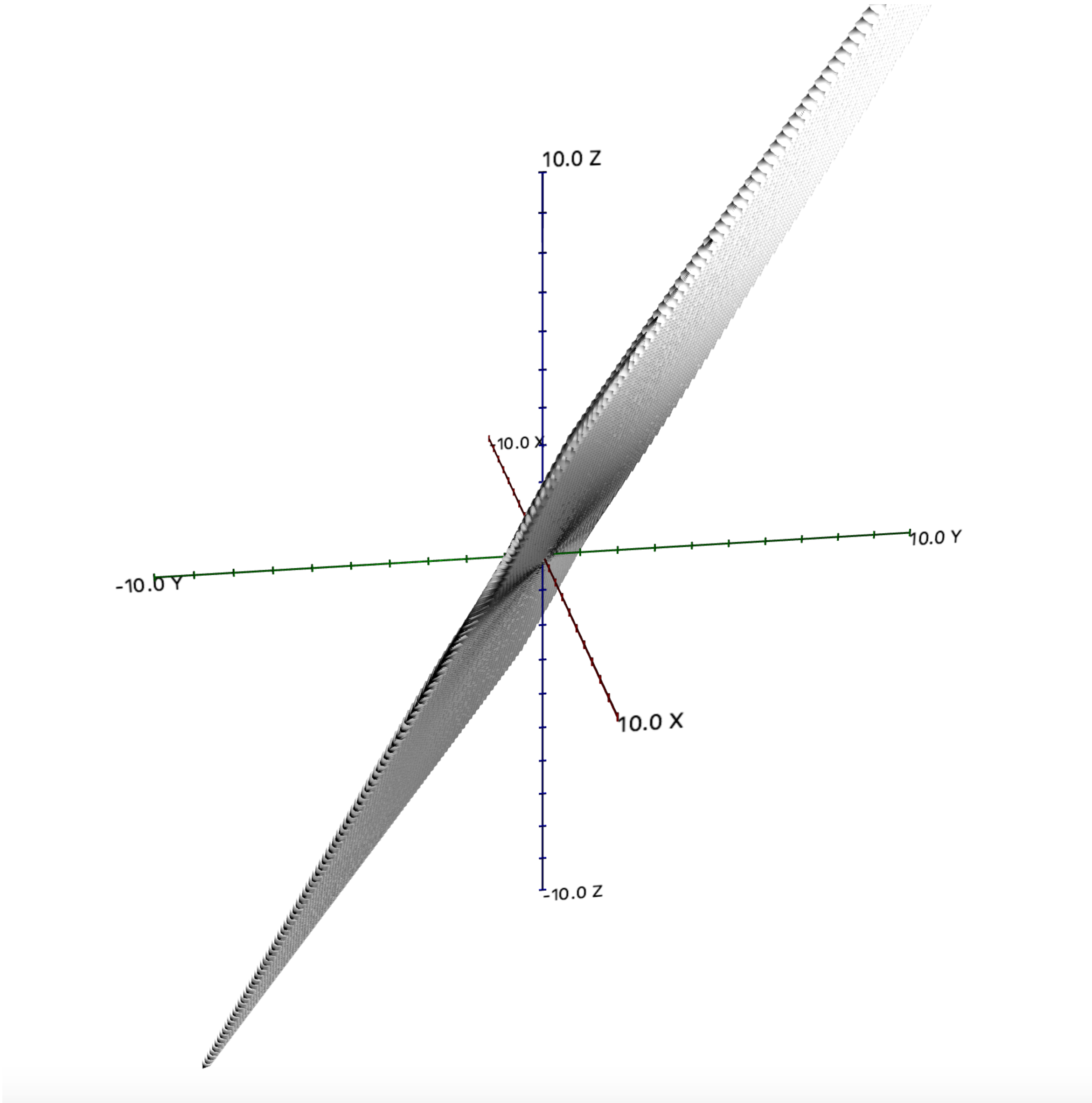
2.6 In-Class Exercise 10

We'll now explore the span of just two of the above vectors: $u = (1, 1, 2), v = (2, 1, 3)$ **in Span3DExample2.java. Is the span of the two the same as the span of the three? Try different bounds for the scalars. What is the span of just u=(1,1,2) all by itself, and is that the same as any of the other spans?**

When we just try the vector **u** and try to find its span with α ranging from -5 to 5, It draws a line and spans from -5 to 5 in the 3D space.



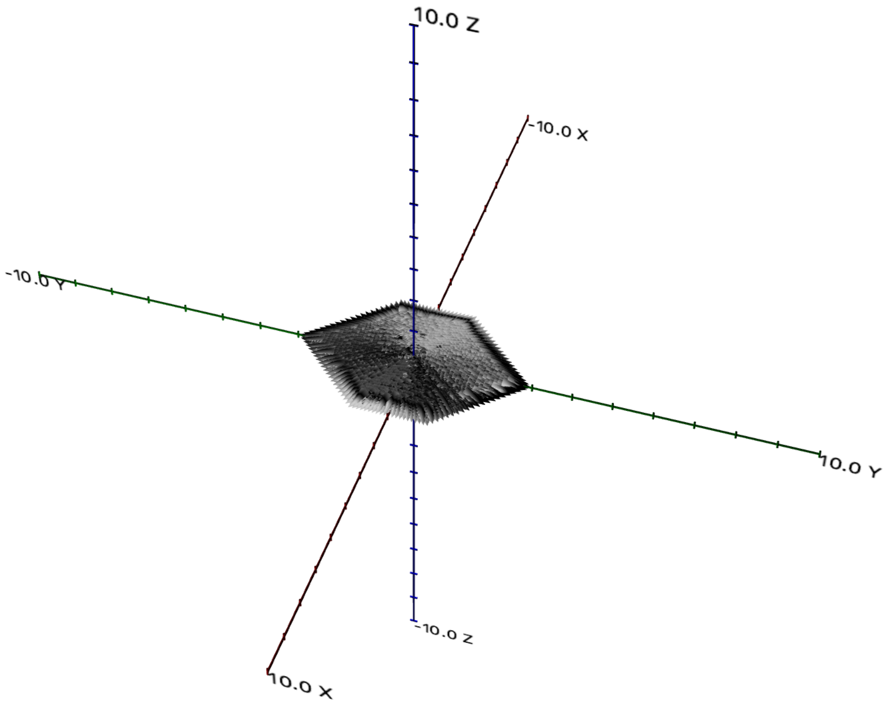
Exploring the span of two vectors **u** and **v** we see that the span is similar to the span of 3 vectors. This means that the third vector can be expressed as a linear combination of the first two vectors. And thus in such cases the span of the vectors is reduced from the whole 3d space to a 2d plane because these vectors will not be able to span outside this. You can also imagine this as the end-points of the vector arrows for every value of alpha and beta.



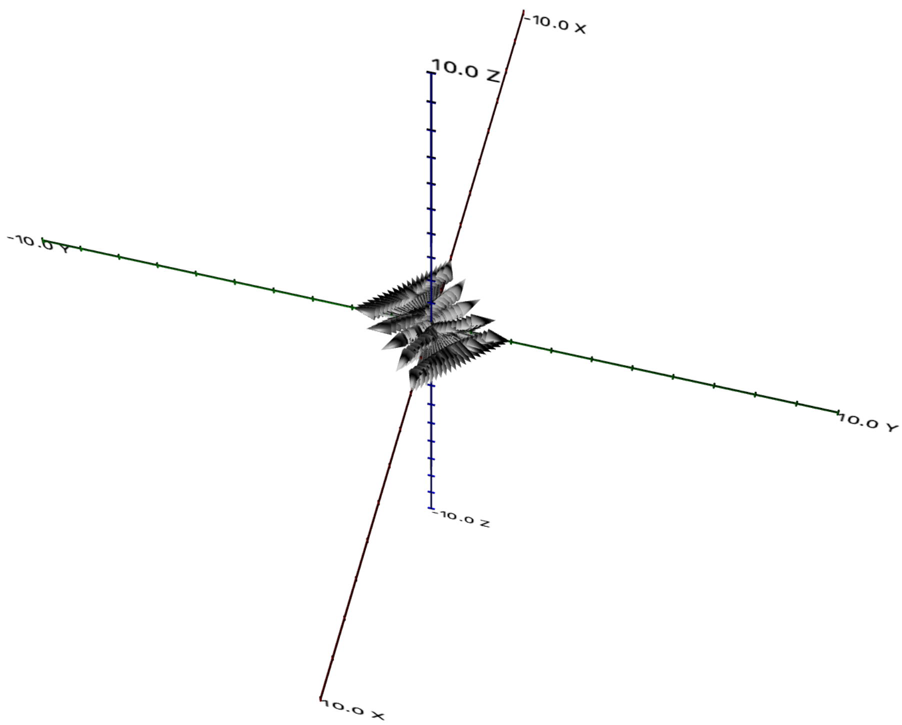
2.7 In-Class Exercise 11

On paper, draw the vectors $u = (1, 1, 0), v = (-1, 1, 0), w = (0, -1, 0)$. What is the set of vectors spanned by these three vectors? What two obvious vectors should one use to span the same set?

The span of the above vectors looks like this :



The two obvious vectors that span the same set should be the ones that are proportional to the first two vectors. $u = (1, 1, 0), v = (-1, 1, 0)$ are enough to generate the subspace. The vector w is merely the negative y axis itself. The vector w can be expressed as a linear combination of u and v.



2.8 In-Class Exercise 12

Why?

No single vector by itself can generate the (x,y)-plane. To generate the span or subspace of the vector, The vector is added with other vectors to see where does the resultant vector lie. In the case of a single vector, the scalar values will scale the vector in its own direction. The vector scaled in its own direction will always be a straight line. It can never generate a plane.

2.9 In-Class Exercise 13

Show that u,v are independent and that w is a linear combination of u and v.

The vectors u and v are independent because they can not be defined in terms of each other which is the very basic definition of linear independence. The vector w can be expressed as a linear combination of u and v and hence vector w is linearly dependent on u and v.

$\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}$
$=> \alpha - \beta = 0$
$=> \alpha + \beta = -1$
$=> 0 + 0 = 0 \textit{trivial}$

therefore $\alpha = \frac{-1}{2}; \beta = \frac{-1}{2}$ Thus w is a linear combination and not counted while calculating dimension of the space.

If $\alpha_1 v_1 + \alpha_2 v_2 = 0$, then u and v are linearly independent. We can see that $\alpha_1 - \alpha_2 = 0$

This means that α = 0 is the only solution to the equation. Hence, vectors u and v are linearly independent.

2.10 In-Class Exercise 14

Prove Proposition 6.5. Hint: start by assuming that one vector among $v_1, v_2, ..., v_n$ is dependent on the others. Can the span be generated by the others?

Initial Hypothesis: Let the set $v_1, v_2, \ldots, v_n$ be bunch of n vectors and the span$(v_1, v_2, \ldots, v_n)$ with dimension = n.

Assumption: Let, us assume one of these vectors can be expressed as a linear combination of the other vectors or essentially one of the vectors is linearly dependent on the remaining vectors. So let, v_k, be such a vector.

Since, the vector v_k, can be expressed in terms of other remaining vectors the span of this vector should not affect the span of the rest of the vectors or we can say that the span can be generated by the remaining n-1 vectors. Therefore, span$(v_1, v_2, \ldots, v_n)$ has dimension = n-1.

This is a contradiction to the given hypothesis. Thus, our assumption was wrong.
Hence, if there is a set of vectors that span a space and if there's dependence between them, you need fewer of them (the independent ones) to achieve the same span.

3 Row space and column space of a matrix

3.1 In-Class Exercise 15

Show that the rowspace of A′ is the same as the rowspace of A. Do this for each of three types of row operations.

The three row operations in creating an RREF are:
- Swapping two rows
- Scaling a row by a number
- Adding to a row a scalar multiple of another row.

Swapping two rows

Matrix A has rows r1,. . . ,rm and its rowspace: span(r1,. . . ,rm). A rowspace is defined as the span of the row vectors. Thus the span of these vectors is represented as span(r1,. . . ,rm). Thus writing the system of equations or a set of vectors in Ax = b form. We can say that

$\mathbf{b} = x_1.\mathbf{r_1} + x_2.\mathbf{r_2} + x_3.\mathbf{r_3} + ... + x_n\mathbf{r_n}$

After swapping a row the above equation will become :
$\mathbf{b} = x_1.\mathbf{r_2} + x_2.\mathbf{r_1} + x_3.\mathbf{r_3} + ... + x_n\mathbf{r_n}$

Thus the rowspace of **A'** = span(r2,r1,. . . ,rm) = span(r1,r2,. . . ,rm).

Scaling a row by a number

$\mathbf{b} = x_1.\mathbf{r_1} + x_2.\mathbf{r_2} + x_3.\mathbf{r_3} + ... + x_n\mathbf{r_n}$

$1 = \frac{\mathbf{w} - (x_2.\mathbf{r_2} + x_3.\mathbf{r_3} + ... + x_n\mathbf{r_n})}{x_1\mathbf{r_1}}$

After scaling a row by a scalar θ the equation will become :

$\mathbf{b} = x_1.\theta\mathbf{r_2} + x_2.\mathbf{r_1} + x_3.\mathbf{r_3} + ... + x_n\mathbf{r_n}$

$\theta = \frac{\mathbf{w}-(x_2.\mathbf{r_2}+x_3.\mathbf{r_3}+...+x_n\mathbf{r_n})}{x_1\mathbf{r_1}}$

$\therefore \quad \theta = 1$

Thus the rowspace of **A'** = span(r1,r2,. . . ,rm).

Adding to a row a scalar multiple of another row.

$\mathbf{b} = x_1.\mathbf{r_1} + x_2.\mathbf{r_2} + x_3.\mathbf{r_3} + ... + x_n\mathbf{r_n}$

$1 = \frac{\mathbf{w}-(x_2.\mathbf{r_2}+x_3.\mathbf{r_3}+...+x_n\mathbf{r_n})}{x_1\mathbf{r_1}}$

After scaling a row by a scalar θ and adding with another row the equation will become :
$\mathbf{b} = x_1.(\mathbf{r_1} + \theta\mathbf{r_2}) + x_2.\mathbf{r_2} + x_3.\mathbf{r_3} + ... + x_n\mathbf{r_n}$

$x_1\theta\mathbf{r_2} = \mathbf{w} - (x_1.\mathbf{r_1} + x_2.\mathbf{r_2} + x_3.\mathbf{r_3} + ... + x_n\mathbf{r_n})$

$\therefore \quad x_1\theta\mathbf{r_2} = 1$

Thus the rowspace of **A'** = span(r1,r2,. . . ,rm).

3.2 In-Class Exercise 16

Why are the pivot rows independent?

A matrix is in Row Reduced Echelon Form (RREF) if: All the pivot rows are bunched together at the top. The remaining non-pivot rows are all zero.

Suppose a non-pivot row has a non-pivot element $c \neq 0$ in column k, When it was column k's turn to look for a pivot, element c was ignored as a pivot. This means there was a pivot d above it. But that means, the pivot d would have eliminated c through row reduction. Therefore we have a contradiction. Entries in a pivot row can be non-zero in the non-pivot columns but No non pivot row will have a non-zero element. Which means that it will not be linearly dependent.

Other way to look at it is that the number of pivot rows is basically equal to the rank of the matrix. The rank of the matrix is defined as the maximum number of linearly independent row vectors or column vectors in the matrix which makes it a self fulfilling prophecy.

Therefore the pivot rows will have non zero elements in its rows which will be linearly independent.

3.3 In-Class Exercise 17

Complete the proof of Proposition 6.4.

Proposition 6.4 states that: The rank of the collection $v_1, v_2, \ldots, v_n$ is the number of pivot columns of RREF(A).

In particular A and A^T share any information that is carried by the diagonal part, and the (column) rank is one of them. This essential point of this argument is that elementary row operations, which by construction don't alter the row rank, also no not alter the column rank (and similarly for column operations). That this is so is because doing an elementary row operation just amounts to expressing all columns in a different basis.

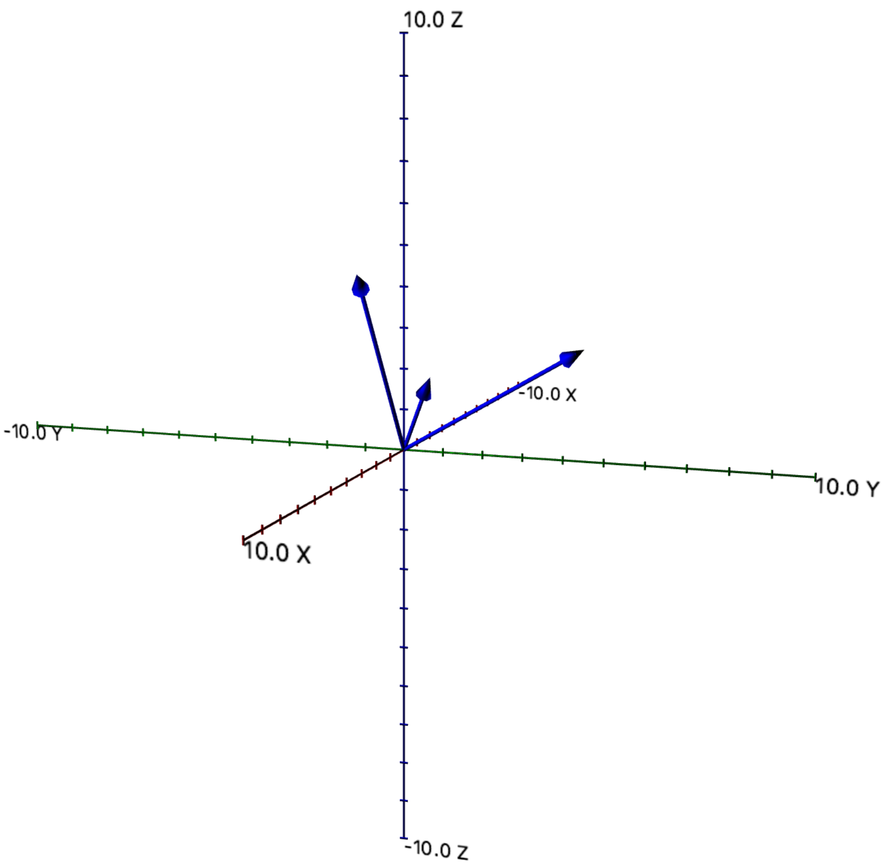
The rank of the matrix does not change when the matrix is transposed. Which means that if the lines are intersecting or if they have a solution the slope of the lines will not be equal and hence the determinant will not be 0. This means that when a set of vectors is transposed, the number of pivot columns becomes the pivot rows or we can say that the column rank becomes the row rank of the matrix.

The rank of the collection $v_1, v_2, \ldots, v_n$ is the number of pivot columns of RREF(A) = number of pivot rows of RREF(A').

3.4 In-Class Exercise 18

Prove or disprove the following. Suppose $v_1, v_2, \ldots, v_n$ is a collection of m-dimensional linearly independent vectors and u $\in span(v_1, v_2, \ldots, v_n)$. Then there is exactly one linear combination of $v_1, v_2, \ldots, v_n$ that will produce u. Consider both cases, $m = n and m \neq n$.

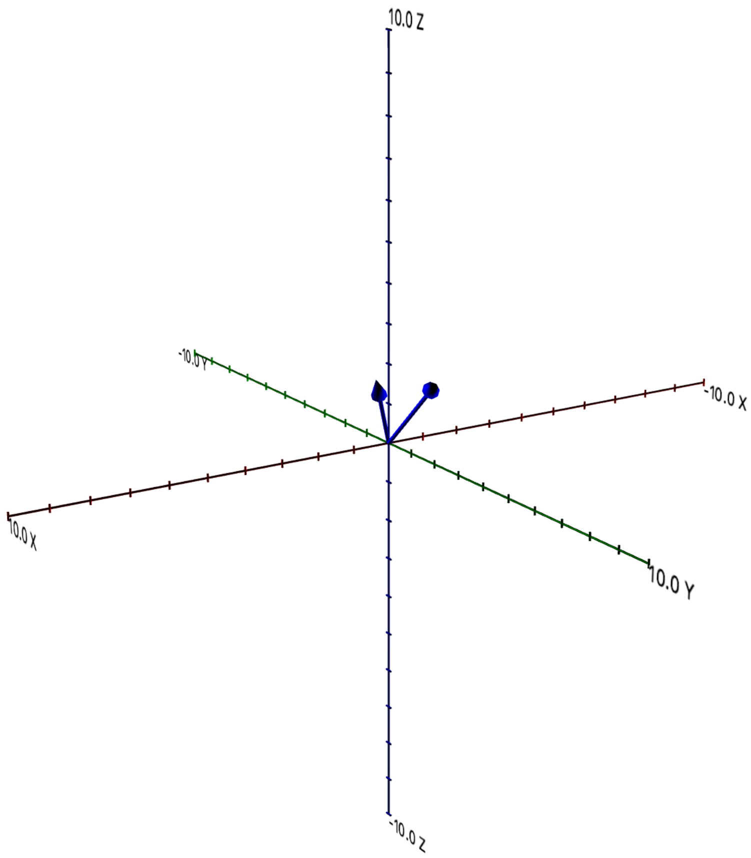
For m=n



In the above picture there are 3 vectors in 3 dimensions. We have to prove that there will exist a u such that there will be just one linear combination of v_1, v_2, v_3 that will produce u or a set of 3 linearly independent vectors will have only one solution to form the resultant vector. We can also say that the 3 independent lines will only intersect at one point.

$\therefore$ $\mathbf{u} = x_1.\mathbf{v_1} + x_2.\mathbf{v_2} + x_3.\mathbf{v_3}$
The above equation will have one solution only when $x_1...x_n = x_1...x_m$.

For m ≠ n



In the above picture there are 2 vectors in 3 dimensions. We have to prove that there will exist a u such that there will be just one linear combination of v_1, v_2 that will produce u or alternatively a set of 2 linearly independent vectors in 3 dimensions will have only one solution to form the resultant vector. We can also say that the 2 independent lines will only intersect at one point.

$\therefore$ $\mathbf{u} = x_1.\mathbf{v_1} + x_2.\mathbf{v_2}$
We require a minimum of independent vectors and thus only one linear combination is required.

4 Basis

4.1 In-Class Exercise 19

What is the more natural basis (3D vectors) for the (x,y) plane?

The question can be elaborated by thinking of 2 vectors in 3 dimensional space, that are not pointing in the same direction what does it mean to take their span their span is a collection of all possible combinations of the two vectors. Turning two different knobs That tip will trace out the fal

The basis of a vector space is a set of linearly independent vectors that span that full space. We can imagine the Euclidean Space R^n (here n = 3 dimensions) which is also called the real vector space. In this space, one of the most common set of linearly independent vectors would be orthogonal to each other that is $\mathbf{u}[1,0,0], \mathbf{v}[0,1,0], \mathbf{w}[0,0,1]$. These vectors span the full 3D space.

The linear combination of the following vectors will span the 3D space. $\therefore\ x_1.\mathbf{u} + x_2.\mathbf{v} + x_3.\mathbf{w}$

These specific set of vectors also have an alternative naming convention : $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ which corresponds to the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ respectively.

Now for the (X,y) plane in this 3D space, we can ignore the third vector then the linear combination of the other two vectors will span into a xy plane.

5 Orthogonality and linear independence

5.1 In-Class Exercise 20

Prove this result:
1) Apply the definition of linear independence.
2) Then compute the dot product with v1 on both sides.
3) What does this imply for the value of α_1?
4) Complete the proof.

If $v_1, v_2, \ldots, v_n$ are orthogonal, they are linearly independent.

The definition of linear independence says that ""In the theory of vector spaces, a set of vectors is said to be linearly independent if none of the vectors in the set can be defined as a linear combination of the others;""

Mathematically linear independence can be shown as :
if $\alpha_1.\mathbf{v_1} + \alpha_2.\mathbf{v_2} = 0$ then $\alpha_1 = \alpha_2 = 0$

Let us multiply the first vector on both sides. Or try to translate the equations by v_1

$(\alpha_1.\mathbf{v_1} + \alpha_2\mathbf{v_2}).\mathbf{v_1} = 0$
$(\alpha_1.\mathbf{v_1})\mathbf{v_1} + (\alpha_2\mathbf{v_2}).\mathbf{v_1} = 0$
$(\alpha_1.\mathbf{v_1})\mathbf{v_1} + 0 = 0$
$\alpha_1.\mathbf{v_1}^2 = 0$
$\alpha_1 = 0$

This happened because the premise was that the vectors are orthogonal. If the vectors are orthogonal their dot product is 0 because their dot product will essentially just rotate the vector.

And hence the value of $\alpha_1 = 0$

Similar explanation can be given for $\alpha_2 = 0$ by multiplying with $\mathbf{v_1}$ and hence we can prove that the orthogonal vectors will always be linearly independent.

5.2 In-Class Exercise 21

Prove this result:
Does the same proof work for complex vectors? Recall that the dot-product for complex vectors is a bit different. If the proof fails, why does it fail? If the proof works, explain all the steps in terms of complex vectors

The definition of linear independence persists from the above equation. Two vectors u, v in a complex inner product space V are called orthogonal if $(u, v) = 0$.

When both u, v $\in$ V are nonzero, the angle θ between u and v is defined by

$\cos\theta = \frac{(u,v)}{||u||.||v||}$ and $\theta = 90$, and $\cos(90') = 0$

The complex vectors can be defined as

$(a\mathbf{u} + b\mathbf{v}) = 0$

Multiplying with the complex vector on both sides :

$(a\mathbf{u} + b\mathbf{v}, \mathbf{u}) = a(\mathbf{u}, \mathbf{u}) + b(\mathbf{v}, \mathbf{u}) = 0$

Since we have proved above that $(u, v) = 0$. if u and v are orthogonal. Thus the product becomes

$a(\mathbf{u}, \mathbf{u}) + 0 = 0$
$a = 0$

Thus we can prove this for complex number b. Hence proved.