Module 5

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1 Matrices that transform matrices

1.1 In-Class Exercise 1

Design a matrix to swap rows 2 and 3 of a 3×3 matrix, and show how it works on an example. Then design a matrix to swap rows 3 and 2. What happens when you multiply the matrix that does the 2-3 swap with the one that does the 3-2 swap? What is the matrix that swaps rows i and j of an n×n matrix? What happens when you multiply this matrix by itself?

$BA = C$

$$A = \begin{bmatrix} Row1 \\ Row2 \\ Row3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 1 & 10 \end{bmatrix}$$

We have to design a matrix B which transforms the matrix A into C where $C = \begin{bmatrix} Row1 \\ Row2 \\ Row3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 10 \\ 3 & 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 1 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 10 \\ 3 & 1 & 1 \end{bmatrix}$$

Hence, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$B'A' = C'$

$$A' = \begin{bmatrix} Row1 \\ Row2 \\ Row3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 1 & 10 \end{bmatrix}$$

We have to design a matrix B which transforms the matrix A' into C' where $C' = \begin{bmatrix} Row1 \\ Row2 \\ Row3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 10 \\ 3 & 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 1 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 10 \\ 3 & 1 & 1 \end{bmatrix}$$

Hence, $B' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

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$$\text{Thus, } B * B' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Thus, Generic Matrix to swap rows i and j from a n x n matrix is.

$$\begin{bmatrix} Row & i \\ Row & . \\ Row & j \\ Row & . \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 & \cdots \\ 3 & 1 & 1 & \cdots \\ 2 & 1 & 10 & \cdots \\ \vdots & \vdots & & \vdots \end{bmatrix}_{n*n}$$

Thus over here i=1 and j=3,

$$\begin{bmatrix} 0 & 0 & 1 & \cdots \\ 0 & 1 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & & \vdots \end{bmatrix}_{n*n} * \begin{bmatrix} Row & i \\ Row & . \\ Row & j \\ Row & . \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 & \cdots \\ 3 & 1 & 1 & \cdots \\ 2 & 1 & 10 & \cdots \\ \vdots & \vdots & & \vdots \end{bmatrix}_{n*n} = \begin{bmatrix} Row & j \\ Row & . \\ Row & i \\ Row & . \end{bmatrix} \begin{bmatrix} 2 & 1 & 10 & \cdots \\ 3 & 1 & 1 & \cdots \\ 1 & 2 & 4 & \cdots \\ \vdots & \vdots & & \vdots \end{bmatrix}_{n*n}$$

Let's consider
$i_{ij} = 1$; $i_{ji} = 1$
$i_{ij} = 1$ when $i = j$ where i is the row number and j is the column but $(i \neq i, j)$

$$\begin{bmatrix} 0_{11} & 0_{12} & 1_{13} & \cdots \\ 0_{21} & 1_{22} & 0_{23} & \cdots \\ 1_{31} & 0_{32} & 0_{33} & \cdots \\ \vdots & \vdots & \vdots & 1_{44} \end{bmatrix}_{n*n} * \begin{bmatrix} 0_{11} & 0_{12} & 1_{13} & \cdots \\ 0_{21} & 1_{22} & 0_{23} & \cdots \\ 1_{31} & 0_{32} & 0_{33} & \cdots \\ \vdots & \vdots & \vdots & 1_{44} \end{bmatrix}_{n*n} = \begin{bmatrix} 1_{11} & 0_{12} & 0_{13} & \cdots \\ 0_{21} & 1_{22} & 0_{23} & \cdots \\ 0_{31} & 0_{32} & 1_{33} & \cdots \\ \vdots & \vdots & \vdots & 1_{44} \end{bmatrix}_{n*n}$$

1.2 In-Class Exercise 2

Design a matrix to divide row 3 of a 3×3 matrix by 2 and show how it works on an example. What is the matrix that divides row i of an n×n matrix by the number β? What is the inverse of this matrix?

$BA = C$

$$A = \begin{bmatrix} Row1 \\ Row2 \\ Row3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 1 & 10 \end{bmatrix}$$

We have to design a matrix B which transforms the matrix A into C where $C = \begin{bmatrix} Row1 \\ Row2 \\ Row3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 4/3 \\ 3 & 1 & 1 \\ 2 & 1 & 10 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 1 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 1 \\ 2/3 & 1/3 & 10/3 \end{bmatrix}$$

Hence, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

This means that in a diagonal matrix of n x n the $i_{ii} = \frac{i_{ii}}{\beta}$. The value is divided by β, Therefore,

$$\begin{bmatrix} 1_{11} & 0_{12} & 0_{13} & \cdots \\ 0_{21} & 1_{22} & 0_{23} & \cdots \\ 0_{31} & 0_{32} & 1/\beta_{33} & \cdots \\ \vdots & \vdots & \vdots & 1_{44} \end{bmatrix}_{n*n}$$

1.3 In-Class Exercise 3

What is $r_A(2) + r_A(3)$ in the above example?

$r_A(2) + r_A(3) = (11, 13, 15)$

1.4 In-Class Exercise 4

What is the transformation matrix that achieves $r_A(1) \leftarrow r_A(1) - 3r_A(2)$ for a 2×2 matrix? What is the effect when applied to $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$?

$$BA = C = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1.5 In-Class Exercise 5

What is the transformation matrix that achieves $r_A(i) \leftarrow r_A(i) + \alpha r_A(j)$ for an n x n matrix? What is the inverse of this matrix?

Lets take an example of : $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$

$\alpha\ r_A(2) = (\alpha\ \ 3\alpha)$
$r_A(1) = (2\ \ 4)$

$r_A(1) = (2 - \alpha \quad 4 + 3\alpha)$

The transformed matrix is : $\begin{bmatrix} 2-\alpha & 4+3\alpha \\ 1 & 3 \end{bmatrix}$

$$BA = C = \begin{bmatrix} 1 & \alpha \\ 0 & \alpha \end{bmatrix} * \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2-\alpha & 4+3\alpha \\ 1 & 3 \end{bmatrix}$$

We can say that multiplying the ith row with alpha and adding row is equivalent to multiplying alpha with replacing the jth column in an identity matrix.

1.6 In-Class Exercise 6

Prove that I as defined above is indeed the multiplicative identity for matrices for any size n×n. What is the geometric intuition for why this must be so?

I is called the Identity matrix. We can prove that the multiplicative identity is equivalent to multiplying the matrix with the basis vectors in the standard plane. Matrix transformation or Matrix Multiplication is basically transforming a matrix into another Cartesian plane with a different basis vectors. The identity matrix is a n x n matrix which means it is a n dimensional orthogonal basis vector in the standard plane. It is a reference to $\hat{\imath}, \hat{\jmath}, \hat{k}$... Geometrically the parallelogram that is built by the vectors when multiplied by the identity matrix lies on the origin and the scaling or stretching is reflected on its respective axes.

1.7 In-Class Exercise 7

What is the reason for this notation? Why does it make sense?

An inverse matrix is also called an undo matrix. The resultant work done when a matrix is multiplied by its multiplicative inverse(or undo matrix) is 0 and so the vectors stay put at their original location and that is the reason it leads to **I**. The x^{-1} notation also denotes that this is the same vectors of x, with the same magnitude, but in an opposite direction that is clockwise in most cases.

1.8 In-Class Exercise 8

Confirm the above by doing the calculations by hand.

$$\begin{bmatrix} -0.2 & 0.3 \\ 0.4 & -0.1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
Matrix (2x2):
 -0.200  0.300
  0.400 -0.100
Matrix (2x2):
  1.000  3.000
  4.000  2.000
Matrix (2x2):
  1.000 -0.000
  0.000  1.000
```

2 Solving by hand: some insights

2.1 In-Class Exercise 9

Write the above equations in matrix form.

$\mathbf{x_1 + 3x_2 = 7.5 \ \ (1)}$
$\mathbf{4x_1 + 2x_2 = 10 \ \ (2)}$

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7.5 \\ 10 \end{bmatrix}$$

2.2 In-Class Exercise 10

Why? What is the precise reasoning?

Although we have changed the vectors the solution of the equations remain the same.

2.3 In-Class Exercise 11

Write the new set in matrix form.

$\mathbf{4x_1 + 12x_2 = 30 \ \ (1)}$
$\mathbf{4x_1 + 2x_2 = 10 \ \ (2)}$

$$\begin{bmatrix} 4 & 12 \\ 4 & 2 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \end{bmatrix}$$

3 Properties of matrix multiplication

3.1 In-Class Exercise 12

Write the new set (3), (4) in matrix form.

$\mathbf{4x_1 + 12x_2 = 30 \ \ (1)}$
$\mathbf{10x_2 = 20 \ \ (2)}$

$$\begin{bmatrix} 4 & 12 \\ 0 & 10 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 30 \\ 20 \end{bmatrix}$$

3.2 In-class Exercise 13

Write the new set (1), (5) in matrix form.

$\mathbf{1x_1 + 3x_2 = 7.5 \ \ (1)}$
$\mathbf{x_2 = 2 \ \ (2)}$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7.5 \\ 2 \end{bmatrix}$$

3.3 In-class Exercise 14

Write the new set (7), (5) in matrix form.

$\mathbf{x_1 + 0 = 1.5 \ \ (1)}$
$\mathbf{0 + x_2 = 2 \ \ (2)}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 2 \end{bmatrix}$$

3.4 In-class Exercise 15

Write down the corresponding five matrices in sequence: the starting matrix, and then the resulting matrices after each row operation.

$$\begin{bmatrix} 2 & -3 & 2 \\ -2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3/2 & 1 \\ -2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3/2 & 1 \\ 0 & -2 & 2 \\ 1 & 1 & -1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3/2 & 1 \\ 0 & -2 & 2 \\ 0 & 5/2 & -2 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 4 \\ -5/2 \end{bmatrix}$$

3.5 In-class Exercise 16

Write down the corresponding three matrices in sequence after applying the above three row operations.

$$\begin{bmatrix} 1 & -3/2 & 1 \\ 0 & 1 & -1 \\ 0 & 5/2 & -2 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -2 \\ -5/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3/2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1/2 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -2 \\ 5/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3/2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -2 \\ 5 \end{bmatrix}$$

3.6 In-class Exercise 17

Write down the corresponding three matrices in sequence after applying the above three row operations.

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

4 Solving by hand using an augmented matrix

4.1 In-class Exercise 18

Apply the same operations in the same order to the 4×4 identity matrix to compute the inverse of the coefficient matrix. Then enter the inverse matrix in FourVariableExample.java and check that $A^{-1}A = I$.

$$\mathbf{A^{-1}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 <-R_2 + R_1$
$R_3 <-R_3 - 3R_1$
$R_4 <-R_4 + 2R_1$

$$\mathbf{A^{-1}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 <-R_3 - R_2$
$R_4 <-R_4 - 2R_2$

$$\mathbf{A^{-1}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -4 & -1 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

$R_4 <-R_4 - 2R_3$

$$\mathbf{A^{-1}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -4 & -1 & 1 & 0 \\ 8 & 0 & -2 & 1 \end{bmatrix}$$

$R_1 <-R_1 - 2R_2$

$$\mathbf{A^{-1}} = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -4 & -1 & 1 & 0 \\ 8 & 0 & -2 & 1 \end{bmatrix}$$

$R_1 <-R_1 + 2R_3$
$R_2 <-R_2 - R_3$

$$\mathbf{A^{-1}} = \begin{bmatrix} -9 & -4 & 2 & 0 \\ 5 & 2 & -1 & 0 \\ -4 & -1 & 1 & 0 \\ 8 & 0 & -2 & 1 \end{bmatrix}$$

$R_1 < -R_1 - R_4$
$R_2 < -R_2 + 3R_4$
$R_3 < -R_3 - 2R_4$

$$\mathbf{A^{-1}} = \begin{bmatrix} -17 & -4 & 4 & -1 \\ 29 & 2 & -7 & 3 \\ -20 & -1 & 5 & -2 \\ 8 & 0 & -2 & 1 \end{bmatrix}$$

```java
        double[][] A = {
            {1,2,0,-5},
            {-1,-1,1,4},
            {3,7,2,-14},
            {-2,-2,4,13}
        };

        // Right hand side:
        double[] b = {-5,6,-9,23};

        // Solution found in RREF:
        double[] x = {2,-1,3,1};

        // Inverse of A:
        double[][] Ainv = {
            {-17 , -4, 4, -1},
            { 29, 2, -7, 3},
            {-20, -1,  5, -2},
            {8 ,  0, -2, 1}
        };

        // b2 should be equal to b.
        double[] b2 = MatrixTool.matrixVectorMult (A, x);
        MatrixTool.print (b2);

        // This should give us the identity matrix.
        double[][] I = MatrixTool.matrixMult (Ainv, A);
        MatrixTool.print (I);
        }

    }

```

Problems @ Javadoc Declaration Console

```
<terminated> FourVariableExample [Java Application] /Library/Java/JavaVirtualMachines/jdk1.8.0_151.jdk/
Vector: -5.000  6.000 -9.000 23.000
Matrix (4x4):
  1.000  0.000  0.000  0.000
  0.000  1.000  0.000  0.000
  0.000  0.000  1.000  0.000
  0.000  0.000  0.000  1.000
```

4.2 In-class Exercise 19

Apply the method above, step-by-step, to solve

$$\begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ -3 \\ 1 \end{bmatrix}$$

Enter your solution in FourVariableExample2.java and check that $Ax = b$. How many pivots were identified?

```java
public class FourVariableExample2 {

    public static void main (String[] argv)
    {
    // The coefficient matrix A:
    double[][] A = {
        {2, 1, 1, 3},
        {1, -1, 1, 0},
        {0, 0, 2, 1},
        {1, 0, 1, 0}
        };

    // Right hand side:
    double[] b = {6,-2,-2,1};

    // Enter solution:
    double[] x = {2, 3, -1, 0};

    // b2 should be equal to b.
    double[] b2 = MatrixTool.matrixVectorMult (A, x);
    MatrixTool.print (b2);
    }

}
```

```
Problems  @ Javadoc  Declaration  Console
minated> FourVariableExample2 [Java Application] /Library/Java/JavaVirtualMach
cor:   6.000 -2.000 -2.000  1.000
```

5 Handling special cases

5.1 In-class Exercise 20

How do we know this? Could it be possible that some row swaps applied first could result in a normal sequence of row operations that produce a solution?

In the above matrix the last row is 0 hence all of the x3 coeffecients of the linear combination will be 0. Hence we have only 2 equations but 3 unknowns. Hence no solution.

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} x1 \\ y2 \\ x3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 6 \end{bmatrix}$$

Switching r2 and r3 we get

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} . \begin{bmatrix} x1 \\ y2 \\ x3 \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ 13 \end{bmatrix}$$

R2 $<$ - R2/ 2

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} . \begin{bmatrix} x1 \\ y2 \\ x3 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 13 \end{bmatrix}$$

Which is absurd and hence no solution. It can be seen through substitution.

5.2 In-class Exercise 21

Apply row operations and identify the pivot.

$$\begin{bmatrix} 1 & 2 & -1 & | & 5 \\ 0 & 1 & 2 & | & -9 \\ 0 & 1 & 2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3/2 & 1 & | & 5/2 \\ -3 & 7 & -5 & | & -9 \\ 1 & 1 & -1 & | & 0 \end{bmatrix}$$

$R_2 <-R_2 + 3R_1$
$R_3 <-R_3 - R_1$

$$\begin{bmatrix} 1 & -3/2 & 1 & | & 5/2 \\ 0 & 5/2 & -4/5 & | & -3/2 \\ 0 & 5/2 & -2 & | & 0 \end{bmatrix}$$

$R_2 <-R_2(2/5)$

$$\begin{bmatrix} 1 & 3/2 & 1 & | & 5/2 \\ 0 & 1 & -4/5 & | & -3/5 \\ 0 & 5/2 & -2 & | & 0 \end{bmatrix}$$

$R_2 <-R_2(2/5)$

$$\begin{bmatrix} 1 & -3/2 & 1 & | & 5/2 \\ 0 & 1 & -4/5 & | & -3/5 \\ 0 & 0 & 0 & | & 3/2 \end{bmatrix}$$

Hence there is no solution.

6 The affine trick (for translation)

6.1 In-class Exercise 22

Apply row operations and Identify the pivots and one possible solution.

$$\begin{bmatrix} 2 & -3 & 2 & | & 5 \\ -3 & 7 & -5 & | & -10 \\ 1 & 1 & -1 & | & 0 \end{bmatrix}$$

$R_2 <-R_1/2$

$$\begin{bmatrix} 1 & -3/2 & 1 & | & 5/2 \\ -3 & 7 & -5 & | & -10 \\ 1 & 1 & -1 & | & 0 \end{bmatrix}$$

$R_2 <-R_2 + 3R_1$
$R_3 <-R_3 - R_1$

$$\begin{bmatrix} 1 & -3/2 & 1 & | & 5/2 \\ 0 & 5/2 & -2 & | & -5/2 \\ 0 & 5/2 & -2 & | & -5/2 \end{bmatrix}$$

This will lead to row full of zeros after the first row. Thus no solution.

6.2 In-class Exercise 23

Work through the steps to compute the RREF in the example above.

$$\begin{bmatrix} 2 & 0 & | & 4 \\ 0 & 3 & | & 7 \\ 1 & 1 & | & 5 \\ 3 & 0 & | & 6 \end{bmatrix}$$

$R_1 < -R_1/2$

$$\begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 3 & | & 9 \\ 1 & 1 & | & 5 \\ 0 & 0 & | & 6 \end{bmatrix}$$

$R_3 < -R_3 - R_1$
$R_4 < -R_4 - 3R_1$

$$\begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 3 & | & 9 \\ 0 & 1 & | & 3 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$R_2 < -R_2/3$

$$\begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 3 \\ 0 & 1 & | & 3 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$R_3 < -R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 3 \\ 0 & 0 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

6.3 In-class Exercise 24

Why is true that a non-pivot row has all zeroes in that row except possibly the last (augmented) one?

If a non pivot row has non-zero elements it should have been subtracted from the corresponding pivot column. Hence if there is a non pivot row it will only contain 0s. This proof is done in the next assignment.

6.4 In-class Exercise 25

First, create example RREFs for each of the two cases. Then prove the above result.

Case I :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Case II :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix}$$

6.5 In-class Exercise 26

Fill in the steps from augmented matrix to RREF above.

$$\begin{bmatrix} 2 & 1 & 4 & -1 & | & 4 \\ 1 & 0 & -2 & 4 & | & 1 \\ 3 & 2 & 10 & -6 & | & 7 \end{bmatrix}$$

$R_1 < -R_1/2$

$$\begin{bmatrix} 1 & 1/2 & 2 & -1/2 & | & 2 \\ 1 & 0 & -2 & 4 & | & 1 \\ 3 & 2 & 10 & -6 & | & 7 \end{bmatrix}$$

$R_2 < -R_2 - R_1$
$R_3 < -R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1/2 & 2 & -1/2 & | & 2 \\ 0 & -1/2 & -4 & 9/2 & | & -1 \\ 0 & 1/2 & 4 & -9/2 & | & 1 \end{bmatrix}$$

$R_2 < -R_2(-2)$

$$\begin{bmatrix} 1 & 1/2 & 2 & -1/2 & | & 2 \\ 0 & 1 & 8 & -9 & | & 2 \\ 0 & 1/2 & 4 & -9/2 & | & 1 \end{bmatrix}$$

$R_3 < -R_3 - R_2/2$

$$\begin{bmatrix} 1 & 1/2 & 2 & -1/2 & | & 2 \\ 0 & 1 & 8 & -9 & | & 2 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Hence no solution.