Module 2

Aakash Shah
Team 4: Pause&Ponder*

February 22, 2018

1 Meet the integers

1.1 In-Class Exercise 1

Prove that the square root of an even square number is even.

It is not possible to determine an even or odd number if it is an irrational number in that case. Assuming that numbers mean : integers. and it is a perfect square. because

$2 + 6 = 8$
even integer + even integer = even integer
$4.8 + 2.2 = 7.0$
even rational + even rational $\neq$ *even* rational

Taking the smallest even square number

=> $x = 4$ //since, $x = 2p$ and $\sqrt{4} = 2$
=> $x = 16$ //since, $x = 2p$ and $\sqrt{16} = 4$

So, every even square number is divisible by 4 and thats why it is an even number.

$\sqrt{2}$ is irrational if it were not a perfect square, but irrational numbers can not have even and odd because it is difficult to determine the parity.

2 Meet the rationals

2.1 In-Class Exercise 2

Prove or disprove: given any two rational numbers x and y such that $x < y$ there is at least one rational number z in between: $x < z < y$.

Let $z = \frac{x+y}{2}$ //It complies to $x < z < y$

x and y are rational number and it is in the form of $\frac{p}{q}$ where q = 2
p = x + y

p is a rational number because sum of two rational number is a rational number. Thus given two rational numbers x and y, there is at least one rational number in between them.

*Team Member: Rohan Shetty

2.2 In-Class Exercise 3

Is there a real-world physical length corresponding to $\sqrt{2}$?

Aspect Ration of an A4 size sheet is $1 : \sqrt{2}$. This ratio of lengths of the shorter over the longer side is quite important. Also if converted into length, $\sqrt{2}/2 = 0.70710$(approx.) which is a common quantity in geometry.
Value : 1 : 1.4142

3 Meet the reals

3.1 In-Class Exercise 4

Why don't we allow the polynomial coefficients to be rational numbers in the definition above?
If we apply any of the standard arithmetic operators, $+, -, \times, \div$ to two algebraic numbers, is the result always algebraic?
Are there non-rational algebraic numbers?

We can definitely allow polynomial coefficients to be rational. When these polynomial expressions are converted into matrix like we saw in the previous module, The corresponding matrix calculations and multiplications gets complicated. That is the reason we stay limited to Integers.

An algebraic number basically has a_0 to a_n coefficients to the polynomial expression. Applying the standard arithmetic operators to such numbers would result in something like.

=> 5x + 2 = 0 algebraic number 1
=> 4y + 3 = 0 algebraic number 2
=> 5x + 2 + 4y + 3 = 0 + 0 adding two numbers
=> 5x + 4y + 5 = 0 does not result in an algebraic number in one variable, but nonetheless it can always be an algebraic number if they both have the same variable.

=> 5x + 2 = 0 algebraic number 1
=> 4x + 3 = 0 algebraic number 2
=> 5x + 2 + 4x + 3 = 0 + 0 adding two numbers
=> 9x + 5 = 0 results in an algebraic number in one variable

Assuming that there are just two classes namely rationals and irrationals, there can be an irrational algebraic number like the golden ration which is a root of the expression and also and approximation and an irrational number. There are multiple such numbers. e.g.

$x^5 - x - 1 = 0$
$x^2 - x - 1 = 0$ //The root or $x = \phi = \frac{1+\sqrt{5}}{2} = 1.6180339...$

4 Countability

4.1 In-Class Exercise 5

What is the 1-1 function that takes an integer and produces the corresponding natural number?

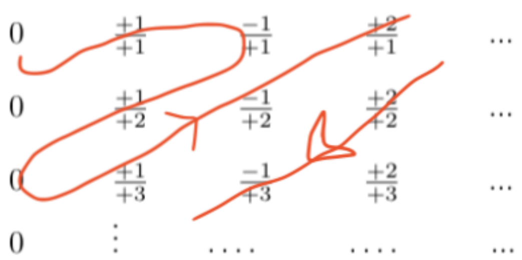
The idea is to hava a 1-1 mapping. This might seem bizarre but to think it like if we can have a way of converting all non negative numbers into even numbers (2,4,...) and the negative numbers as (1,3,...) then, it will be easy to prove that they have the same cardinality.

$$f(x) = \begin{cases} f : Z \rightarrow N \\ 2n + 2, & \text{if } n \geq 0 \\ -2(n) - 1, & \text{n} < 0 \end{cases}$$

4.2 In-Class Exercise 6

What about the size of the rationals compared to integers?
The size of rationals vs integers is the same. Taking integers (0,-1,1,-2,2,-3,3,...) and traversing the following grid of all rational numbers (below is the entirety of rational numbers belonging to the the set Q) in a special way.

0	$\frac{+1}{+1}$	$\frac{-1}{+1}$	$\frac{+2}{+1}$	...
0	$\frac{+1}{+2}$	$\frac{-1}{+2}$	$\frac{+2}{+2}$	...
0	$\frac{+1}{+3}$	$\frac{-1}{+3}$	$\frac{+2}{+3}$	...
0	$\vdots$	....	....	...



This path will always have just the integers in its numerators and it goes to countable infinity. This way we can show the bijection or the one to one mapping between rational numbers and integers. This also proves that they have the same cardinality. There will be duplicates.

4.3 In-Class Exercise 7

Show that the cardinality of $[0,1]$ is the same as that of any $[a,b]$, for example $[0,10]$.

Two sets A and B have the same cardinality if there exists a bijection, that is, an injective and surjective function, from A to B.
For example, the set A = [0,1] has the same cardinality as the set B = [a,b] , since the size of both sets = 2 : is a bijection from A to B.

5 Functions and coordinates

5.1 In-Class Exercise 8

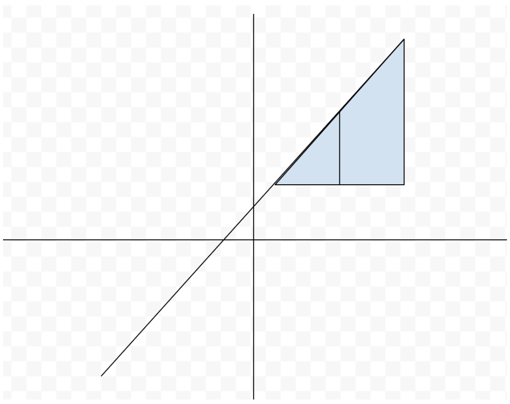
The cardinality of the plane is the same as the line. Why is this true?

Cardinal arithmetic can be used to show not only that the number of points in a real number line is equal to the number of points in any segment of that line, but that this is equal to the number of points on a plane and, indeed, in any finite-dimensional space.
The number of points on any segment or line is equal to the number of points in a plane. Thus the size is always same. Thus Cardinality of plane is same as the line

5.2 In-Class Exercise 9

Why is $ax + by + c = 0$ the equation of a line? Can you find an argument using simple geometry (similar triangles)?

Using SAS similarity in the following diagram. Since the angles are same along the line, the line is ought to be straight.



5.3 In-Class Exercise 10

Consider the two-variable function f(x,y)=(x+y)n. Show that

$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k * y^{n-k}$

=> $(x + y) * (x + y) * (x + y) * \ldots n$

=> $x^n + y^n + \binom{n}{3} x^3 * y^{n-3} + ...$

=> $\sum_{k=0}^{n} \binom{n}{k} x^k * y^{n-k}$

These will be $\binom{n}{3}$ such terms. This is a simple expansion and it is visible that it is true.

6 Polynomials

6.1 In-Class Exercise 11

Why is $b_{n,k}(x)$ a polynomial?

$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$

Taking the binomial expansion of $(1-x)^{n-k}$ will be a polynomial because of the above proof. Multiplying it with x^k will also be a polynomial.

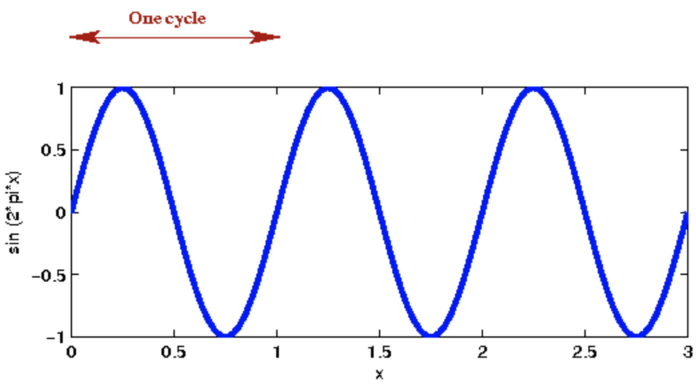
7 Trigonometric functions

7.1 In-Class Exercise 12

Note: 2π is approximately 6.29. Why is it OK to define a value for $sin(17.5)$ or $sin(-156.32)$? How are these values defined? Download *PlotSin.java*, *Function.java*, and *SimplePlotPanel.java*. Then compile and execute *PlotSin.java* to plot the function sin and cos in the range $[0, 20]$. How many times does the function repeat in this.

Answer: $sin(17.5)$ can be equal to $sin(2\pi\omega t)$. sine function has a period of 360 deg (or $2\pi rad$). We can use Taylor series to find out the value of $sin(x)$ for any real number x, where x will be in radians. And hence it is OK to define a value for $sin(17.5)$.

I want to put across an analogy with a thread. Imagine a thread of length 6.28cms, how many such threads would we need to make 17.5cms, That will give us the number of cycles required. In our case it is approximately 2.78.



These values are found out using approximate polynomial functions, and taking derivatives around known values of cos like 0, π etc. The derivatives of these polynomial functions are very easy to calculate as it has to deal with only one number, because rest of the small powers boil down to a constant and in turn 0. This is the crux of Taylor series approximation. Basically finding the area under the curve which is close to the sine or cosine function.

=> $sin(17.5) = sin(2 * \pi * 2.78)$
=> $17.5 = 2\pi\omega t$ //using the above formula
=> $17.5 = 6.29 * \omega t$ //Replacing the approximate value of π
=> $\omega t = 2.78$

* The interval is from 0 to 20 and the increment is of 0.2 which equals to **100 repetitions**. 101 in this case to display it from 0 to 20.

7.2 In-class Exercise 13

Compile and execute *PlotSin2.java* **to plot** $sin(2\pi\omega t)$ **in the range** $[0, 1]$ **with** $\omega = 1, 2, 3$. **What is the relationship between the three curves?**

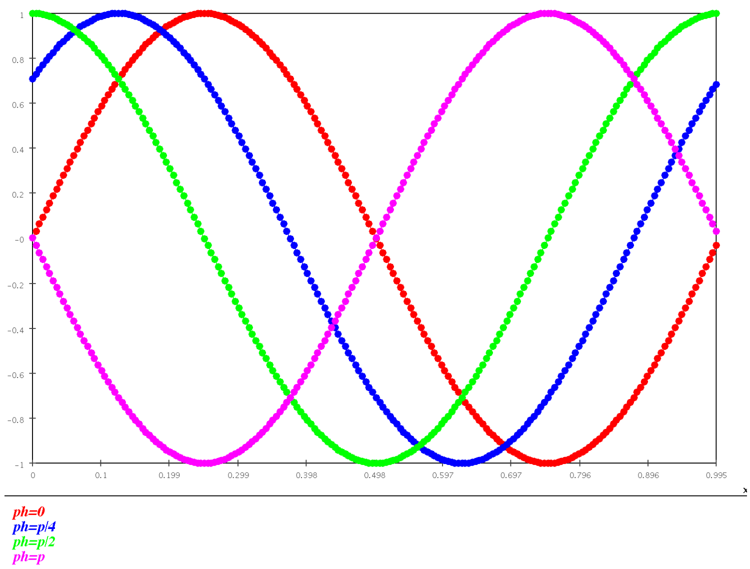
The code for *PlotSin2.java* looks like this.

```
F.add (x, Math.sin(2*Math.PI*1*x));
G.add (x, Math.sin(2*Math.PI*2*x));
H.add (x, Math.sin(2*Math.PI*3*x));
```

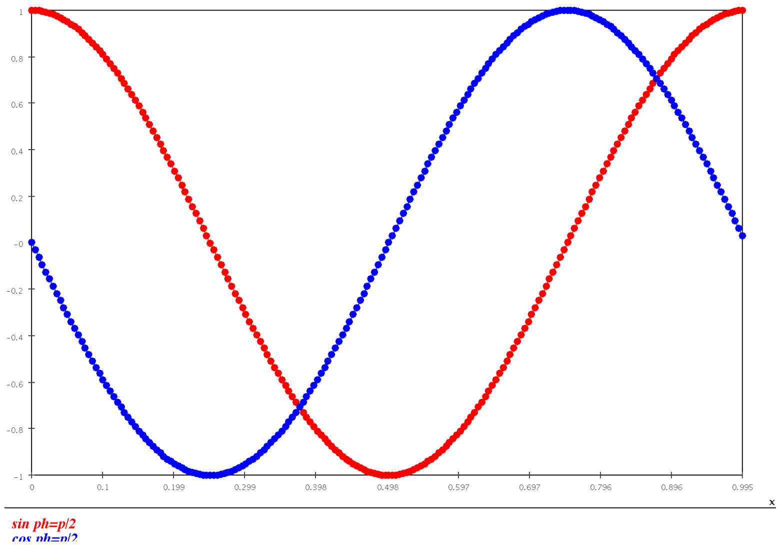
The relationship looks like the frequency of the F curve is 1 cycle in the given time frame. G curve has 2 cycles in the given time interval and H has 3 cycles in the same time. The math.sin function is computing the radians at every time step. These radians are basically number of cycles(ω) of a sine period (2π radians). It is done using Taylor series in the libraries.

7.3 In-class Exercise 14

Modify *PlotSin3.java* **to plot** $sin(2\pi t + \phi)$ **in the range** $[0, 1]$**. Plot four separate curves with** $\phi = 0, \pi/4, \pi/2, \pi$. **What can you say about** $sin(2\pi t + \pi/2)$**? Similarly, plot** $cos(2\pi t + \pi/2)$ **and comment on the function.**



In the above diagram, there are 4 plots of sine wave with different phases. Phases are just lags or delays in the sinusoidal function or for any wave function. The phase is increasing from 0 to π. You can see that sin wave with a ϕ (phase offset) of $\pi/2$ makes it look similar to a cosine wave.



To illustrate the same point the phase angles are the same for the sine and cosine wave and we can see that they are always exactly a $\pi/2$ phase apart.

7.4 In-class Exercise 15

Why?

The amplitude of a sine wave does not change the period of the wave is analogous to say that the mass of the pendulum does not matter in calculating its oscillation or frequency. It is because the maximum and minimum value of the sin(t) stays the same. If it were different then the frequency would be affected.

Also, Since the maximum value of $sin(t)$ is 1.0, the amplitude of $\alpha sin(t)$ is α.

7.5 In-class Exercise 16

Does it make sense to use a right-angled triangle to define sin(z) for a complex number z?

To define sinz using right angle triangle we need to consider z as an angle. Although the definition of sin and cos involves right angle triangle it is also defined using power series. If we are trying to express sin z in terms of power series, it makes sense for z to be a complex number, however using a right angle triangle makes it much more complex to define sin z where z is complex. To have a close look lets expand sin z
$=> sin(x + iy) = sinxcosiy + cosxsiniy$
$=> sin(x + iy) = sinxcoshy + icosxsinhy$

if we consider the above equation as the value of sin z we can use right angle triangle to define sinx and y,cosx and y,sinhx and y,coshx and y which will inturn result in the value of sin z.

8 Arrays

8.1 In-class Exercise 17

As a little practice exercise with one dimensional arrays, write some code to determine whether a string is a double-palindrome. Download ArrayExamples.java, which explains what a double-palindrome is, and write your code in Palindrome.java.

8.2 In-class Exercise 18

Write code to blur an image. Start by writing pseudocode to solve the problem. Then, download ImageBlur.java, which contains instructions, and where you will write your code. You will also need ImageTool.java. And as a test image, you can use ace.jpg.

8.3 In-class Exercise 19

Let's examine the idea of computing the "distance" between two arrays.

1. A distance measure should take two arrays and return a non-negative number.
2. Propose a "distance" calculation for 1D arrays.
3. Propose an extension of that idea to 2D arrays.
4. Do the arrays have to be of the same length?

For us to calculate the distance between two arrays we need to make sure that they are of same size. If array A contains $a1, a2, a3...$ and B contains $b1, b2, b3....$ the distance can be calculated by

$D = \sqrt{(b1 - a1)^2 + (b2 - a2)^2 + (b3 - a3)^2....}$

The same can be applied to a two dimensional array, instead of having $a1, a2, a3...$ we have $a11, a12, a13, a21.....$ As long as we have a one to one correspondence between arrays A and B the same method can be applied to any dimensional array.

8.4 In-class Exercise 20

How is a 2D array stored in memory? Consider this code snippet:

The above image is an example of how data is stored in a 2D array of size $2X3$. The first dimension(2) can be considered as rows and the second dimension(3) can be considered as columns. The first array of size 2 consists of memory address of the next array of size 3. so when we say $A[0][0]$, the pointer is taken to 0 of the first array where it will find the address of the second array and the second 0 will lead it to the desired location where the data is stored.

In the above code snippet the value of $A[1][4] is 0$. This happens because when we say $double[][]B = A$ we are not copying the contents of $AtoB$, we are copying the address stored in $AtoB$. So after that assignment operation now B is also pointing to the same address location as A, hence we we assign $B[1][4] = 0$ the data being changed will affect A and B because they both are sharing the same data.

References

https://math.stackexchange.com/questions/1366765/if-the-square-of-a-number-is-even-then-the-number-if-even-isnt-that-not-true/1366767
https://www.youtube.com/watch?v=6NKa-iwBJFc
https://math.stackexchange.com/questions/1311/are-there-more-rational-numbers-than-integers
https://math.stackexchange.com/questions/92451/can-decimal-numbers-be-considered-even-or-odd
https://math.stackexchange.com/questions/873927/how-to-show-the-integers-have-same-cardinality-as-the-natural-numbers
https://www.khanacademy.org/math/algebra-home/alg-intro-to-algebra/alg-irrational-numbers-intro/v/square-roots-and-real-numbers