Oscillatory Motion

Oscillatory motion is defined as the to and fro motion of an object from its mean position.

Examples of Oscillatory Motion

Oscillation of simple pendulum

Vibrating strings of musical instruments

Movement of spring

Alternating current

Cosmological model

Simple Harmonic Motion

Simple harmonic motion (SHM) is a type of oscillatory motion which is defined for the particle moving along the straight line with an acceleration which is moving towards a fixed point on the line such that the magnitude is proportional to the distance from the fixed point.

A simple harmonic motion can be represented by the relations

$$y = a \sin(\omega t + \Phi) \text{ ………. } 1$$

or

$$y = a \cos(\omega t + \Phi) \text{ ……… } 2$$

The system when displaced from its equilibrium position experiences a restoring force F proportional to the displacement y is known as harmonic oscillator.

This restoring force acts in the direction opposite the displacement from the equilibrium position.

From Newton's second law of motion,

F = ma

where F is restoring force of oscillator = - ky.

$$-k\,y = m\frac{d^2y}{dt^2}$$

$$\frac{d^2y}{dt^2} + \frac{ky}{m} = 0$$

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \quad \text{where } \omega^2 = \frac{k}{m}$$

This is a second order, linear differential equation.

Multiplying on both sides by $2\frac{dy}{dt}$

$$2\frac{dy}{dt}\frac{d^2y}{dt^2} + 2\omega^2 y\frac{dy}{dt} = 0$$

Integrating on both sides, we get

$$\left(\frac{dy}{dt}\right)^2 + \omega^2 y^2 = C$$

When the displacement is maximum i.e. at y = a and $\frac{dy}{dt} = 0$

$$0 + \omega^2 a^2 = C \quad \text{or} \quad C = \omega^2 a^2$$

$$\left(\frac{dy}{dt}\right)^2 + \omega^2 y^2 = \omega^2 a^2$$

$$v = \frac{dy}{dt} = \omega\sqrt{(a^2 - y^2)}$$

this is the expression for **velocity** of a particle executing simple harmonic motion at a time t.

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again $\frac{dy}{\sqrt{(a^2-y^2)}} = \omega dt$

integrating $\sin^{-1}\frac{y}{a} = \omega t + \Phi$ $\quad \frac{y}{a} = \sin(\omega t + \Phi)$

$y = a\sin(\omega t + \Phi)$ where Φ is known as **phase constant**.

Acceleration $a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}[a\omega\cos(\omega t + \Phi)] = -a\omega^2\sin(\omega t + \Phi)$

$$a = -\omega^2 y$$

Time period $T = \frac{2\pi}{\omega} = 2\pi\frac{1}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}}$

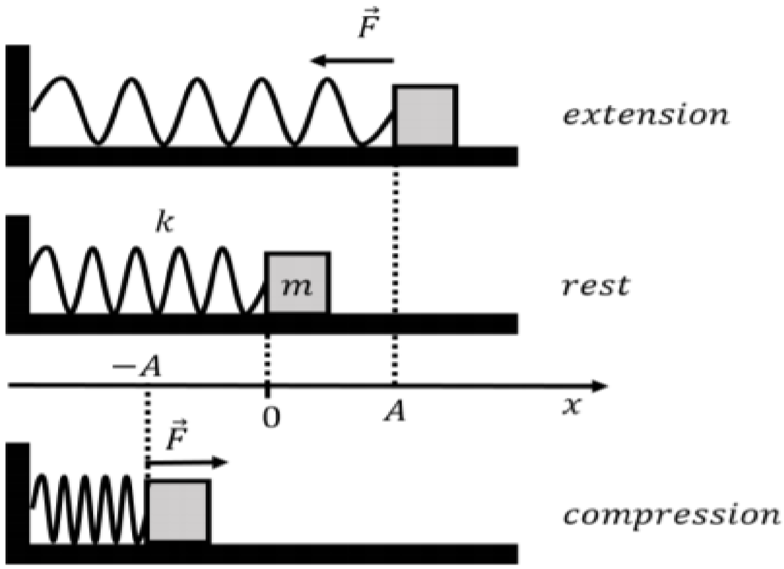
Therefore, the **frequency** of oscillator

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Mass Spring System

A spring-mass system consists of suspending or attaching a mass to the free end of the spring, while the other end of the spring is connected to a fixed point. The spring-mass system can usually be used to find the period of any object that performs simple harmonic motion. For example, the spring-mass system can be used to simulate the movement of human tendons using computer graphics, as well as the deformation of the skin of the feet.

It is assumed that the mass 'm' and the spring are on a smooth horizontal surface. When the force is applied to stretch the spring and then released, the spring regains the position of equilibrium.



Due to this force, the mass begins to vibrate back and forth. If x is displacement of mass from its equilibrium position at any instant of time t, then from Hoke's law the restoring force acting on the mass is given by

Force α extension

$$F = -k\,x \qquad \ldots\ldots\ldots\ldots \qquad 1$$

Here –ve sign shows that force and displacement are in opposite direction, and k = $\frac{F}{x}$ be the proportionality constant and called force constant or stiffness of spring

From Newtons second law of motion

$$F = \frac{dP}{dt} = \frac{d(mv)}{dt} = m\frac{dv}{dt} = m\,\frac{d^2X}{dt^2} \qquad \ldots\ldots\ldots \qquad 2$$

From equation 1 and 2

$$m\,\frac{d^2X}{dt^2} = -kx$$

$$\frac{d^2X}{dt^2} + \frac{k}{m}\,x = 0$$

$$\frac{d^2X}{dt^2} + \omega^2\,x = 0 \qquad \text{where } \omega^2 = \frac{k}{m} \qquad \ldots\ldots\ldots \qquad 3$$

This is differential equation of mass attached to the spring, as the acceleration is proportional to displacement, the motion of mass on the spring is simple harmonic and time period of oscillation

$$T = \frac{2\pi}{\omega} = 2\pi\,\frac{1}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}} \qquad \ldots\ldots\ldots \qquad 4$$

The equation shows that the period of oscillation is independent of both the amplitude and acceleration due to gravity. The above equation is also valid in the case when a constant force is being applied on the mass, i.e. a constant force cannot change the period of oscillation.

Energy in Simple Harmonic Motion

A particle executing SHM is moving under the action of restoring force so this particle possesses both potential energy due to restoring force and kinetic energy due to its motion. Total mechanical energy is the sum of kinetic energy and potential energy.

Consider a particle with mass m performing simple harmonic motion the instantaneous velocity of the particle performing S.H.M. at a distance x from the mean position is given by

$$v = \omega \sqrt{(a^2 - x^2)}$$

$$\therefore v^2 = \omega^2 (a^2 - x^2)$$

$$\therefore \text{Kinetic energy K.E.} = \frac{1}{2} mv^2 = \frac{1}{2} m\,\omega^2 (a^2 - x^2)$$

Consider a particle of mass m performing simple harmonic motion at a distance x from its mean position. Since the restoring force acting on the particle is

$F = -kx$ where k is the force constant.

Now, the particle is given further infinitesimal displacement dx against the restoring force F. Let the work done to displace the particle be dw. Therefore, the work done dw during the displacement is

$$dw = - F.\,dx = -(-k.\,x)\,dx = k.x.dx$$

Therefore, the total work done to displace the particle now from 0 to x is

$$= \int_0^x dw = \int_0^x kx\,dx = k \int_0^x x\,dx = m.\omega^2.\frac{x^2}{2} \qquad \text{where } \omega^2 = \frac{k}{m}$$

Hence Total work done $= \frac{1}{2} m.\omega^2 x^2$

The total work done here is stored in the form of potential energy.

$$\therefore \text{Potential energy P.E.} = \frac{1}{2} m\omega^2 x^2$$

Total energy = Kinetic energy + Potential energy

$$E = \frac{1}{2} m\,\omega^2 (a^2 - x^2) + \frac{1}{2} m\omega^2 x^2 = \frac{1}{2} m\,\omega^2 a^2 - \frac{1}{2} m\,\omega^2 x^2 + \frac{1}{2} m\omega^2 x^2$$

$$E = \frac{1}{2} m\omega^2 a^2$$

The law of conservation of energy states that energy can neither be created nor destroyed. Therefore, the total energy in simple harmonic motion will always be constant. However, kinetic energy and potential energy are interchangeable. Given below is the graph of kinetic and potential energy vs instantaneous displacement.

At the **mean position**, the total energy in simple harmonic motion is purely kinetic and at the **extreme position**, the total energy in simple harmonic motion is purely potential energy.

At **other positions**, kinetic and potential energies are interconvertible and their sum is equal to $\frac{1}{2}k\omega^2\ a^2$ **.**