Chapter 5

Vector Space

Exercise 5.1

1. **Solution.** $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} x \geq 0, y \geq 0 \right\}$

a. If u, v ∈ W then u and v lies in 1st quadrant i.e., u and v have non-negative entries, we know that the sum of non-negative numbers are non-negative so, u + v has non-negative entries. Thus, u + v ∈ W

b. Take, $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in W$ and $c = -1$

Then, $cu = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \notin W$ so W is not a vector space.

2. **Solution.** Given, W be the union of the 1st and 3rd quadrants in the xy-plane

i.e., $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x\,y \geq 0 \right\}$

a. Let $u = \begin{bmatrix} x \\ y \end{bmatrix} \in W$ and c be any scalar then

$cu = c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$ is in W since $xy \geq 0$, so $(cx)(cy) = c^2(xy) \geq 0$

b. Let $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in W$ and $v = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \in W$

but $u + v = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \notin W$ because u + v does not lies in 1st or 3rd quadrants.

∴ W is not a vector space.

3. **Solution.** Given, $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1 \right\}$

Let $u = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \in H$ and $c = 4$ then

$cu = 4\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \notin H$

∴ H is not closed under multiplication. So, it is not subspace of R^2.

4. **Solution.**

(a) Given, $P(t) = at^2$

So $P(t) = \text{Span}\{t^2\}$

Using theorem 1, the set span by $\{t^2\}$ is subspace of P_n.

(b) No, the polynomial of the form $P(t) = a + t^2$ is not subspace of P_n because, the set does not contains zero vectors.

(c) No, the set is not closed under multiplication by scalars which are not integers.

(d) Yes, it is subspace of P_n because the zero vector is in this set H. If p and q is H then $(p + q)(0) = p(0) + q(0) = 0 + 0 = 0$

So, $p + q \in H$

and $(cp)(0) = c.\,p(0) = c\,.\,0 = 0$ so $cp \in H$.

5. **Solution.** Given $H = \left\{ \begin{bmatrix} -2S \\ 5S \\ 3S \end{bmatrix} \right\} = \left\{ S\begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} \right\} = \{S\,V\}$ where, $v = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} \in R^3$

∴ H = span {v} and H is subspace of R^3 because H is spaned by v and v is in R^3 which is vector space (by theorem 1).

6. **Solution.** Given $H = \left\{ \begin{bmatrix} 5t \\ 0 \\ -2t \end{bmatrix} \right\} = \{tv\}$ where, $v = \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix} \in R^3$

Since, the set H = Span {v} where $v \in R^3$ so using theorem 1 H is subspace of R^3.

7. **Solution.** Given, $W = \left\{ \begin{bmatrix} 5b + 2c \\ b \\ c \end{bmatrix} \right\} = \left\{ b\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\} = \{(bv_1 + cv_2)\}$

where $v_1 = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Since, W = Span $\{v_1, v_2\}$ and $v_1, v_2 \in R^3$ so using theorem 1, W is subspace of R^3.

8. **Solution.**

Given, $W = \left\{ \begin{bmatrix} 2s + 4t \\ 3s \\ 2s - 3t \\ 3t \end{bmatrix} \right\} = \left\{ S\begin{bmatrix} 2 \\ 3 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ 0 \\ -3 \\ 3 \end{bmatrix} \right\} = \{(sv_1 + tv_2)\}$

where $v_1 = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 0 \\ -3 \\ 3 \end{bmatrix}$

Which shows that W = Span $\{v_1, v_2\}$ and since $v_1, v_2 \in R^4$. So by using theorem 1 W is subspace of R^4.

9. **Solution.**

a. The vector w is not in the set $\{v_1, v_2, v_3\}$. There are 3 vectors in the set $\{v_1, v_2, v_3\}$.

b. The set span $\{v_1, v_2, v_3\}$ contains infinitely many vectors.

c. The vector w is in subspace spanned by $\{v_1, v_2, v_3\}$. iff the equation $av_1 + bv_2 + cv_3 = w$ has a solution.

By using row reducing the augmented matrix for this system

$$[v_1\ \ v_2\ \ v_3\ \ w] = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the equation has a solution because there is free variables.

Thus, w is in subspace spanned by $\{v_1, v_2, v_3\}$.

10. **Solution.** If w in the subspace spanned by $\{v_1, v_2, v_3\}$, iff the equation $av_1 + bv_2 + cv_3 = w$ has a solution.

Since, $[v_1\ v_2\ v_3\ w]$

$$= \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 1 & 2 & 3 \\ -1 & 3 & 6 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which show that $av_1 + bv_2 + cv_3 = W$ has a solution.

∴ w is in the subspace spanned by $\{v_1, v_2, v_3\}$.

11. **Solution.**

a. Given $W = \left\{\begin{bmatrix} 3a+b \\ 4 \\ a-5b \end{bmatrix}\right\}$

This set does not contains the zero vector, so it is not vector space.

b. Similarly, it also does not contains zero vector.

c. Given, $W = \left\{\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix} \quad a, b, c, \in R\right\}$

$$= \left\{a\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

$= \{av_1 + bv_2 + cv_3,\ a, b, c \in R\}$

where, $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

which shows that $W = \text{Span}\{v_1, v_2, v_3\}$ since, $v_1, v_2, v_3 \in R^4$

So W is subspace of R^4 i.e., W is vector space.

Here Set $S = \left\{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}$

d. We have,

$$W = \left\{\begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix}\right\} = \left\{a\begin{bmatrix} 4 \\ 0 \\ 1 \\ -2 \end{bmatrix} + b\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\}$$

∴ $S = \left\{\begin{bmatrix} 4 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ is a set that spans W.

Exercise 5.2

1. **Solution.** $AW = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 3-15+12 \\ 6-6+0 \\ -8+12-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$

∴ W is in Nul A.

Similarly, $Au = 0$ So, u is also in Nul A.

2. **Solution.**

(i) 1st we find the general solution of $Ax = 0$ in term of the free variables.

Since, $[A, 0] \sim \begin{bmatrix} 1 & 0 & -2 & 4 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{bmatrix}$

Here, x_3 and x_4 are free variables so

$x_1 = 2x_3 - 4x_4$

$x_2 = -3x_3 + 2x_4$

∴ $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

∴ A spanning set for Nul A is $\left\{\begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix}\right\}$

(ii) Similarly as (i) we get $\left\{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

(iii) Similarly we get $\left\{\begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}\right\}$

3. **Solution.**

(i) We have,

$$\begin{bmatrix} 2s+t \\ r-s+2t \\ 3r+s \\ 2r-s-t \end{bmatrix} = r\begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix} + s\begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

∴ $A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \\ 2 & -1 & -1 \end{bmatrix}$

(ii) $\begin{bmatrix} b-c \\ 2b+3d \\ b+3c-3d \\ c+d \end{bmatrix} = b\begin{bmatrix}1\\2\\1\\0\end{bmatrix} + c\begin{bmatrix}-1\\0\\3\\1\end{bmatrix} + d\begin{bmatrix}0\\3\\-3\\1\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\2 & 0 & 3\\1 & 3 & -3\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}b\\c\\d\end{bmatrix}$

$\therefore$ $A = \begin{bmatrix}1 & -1 & 0\\2 & 0 & 3\\1 & 3 & -3\\0 & 1 & 1\end{bmatrix}$

4. **Solution.**

(i) The matrix A is 4×2 order thus NulA is a subspace of R^2 and col A is a subspace of R^4.

$\therefore$ For NulA, $k = 2$, for ColA, $k = 4$

Similarly for

(ii) $k = 3$ for NulA and ColA

(iii) $k = 5$ for NulA and $k = 2$ for ColA

(iv) $k = 5$ for NulA and $k = 1$ for ColA

5. **Solution.** Consider the system with agumeted matrix

$[A\ \ w] = \begin{bmatrix}-2 & 4 & 2\\-1 & 2 & 1\end{bmatrix} \sim \begin{bmatrix}1 & -2 & -1\\-1 & 2 & 1\end{bmatrix} \sim \begin{bmatrix}1 & -2 & -1\\0 & 0 & 0\end{bmatrix}$

which shows that the system is consistent so w is in ColA.

Since, $Aw = \begin{bmatrix}-2 & 4\\-1 & 2\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\therefore$ w is in NulA.

6. **Solution.** Since,

$[A, W] \sim \begin{bmatrix}1 & 0 & 0 & -1 & 1\\0 & 1 & 0 & -1 & 1\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$ which is consistent

$\therefore$ w is in ColA.

Also $Aw = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ $\quad\therefore$ w is in NulA

7. **Solution.**

For (i) we have,

$[A\ \ 0] = \begin{bmatrix}6 & -4 & 0\\-3 & 2 & 0\\-9 & 6 & 0\\9 & -6 & 0\end{bmatrix} \sim \begin{bmatrix}0 & 0 & 0\\-3 & 2 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -2/3 & 0\\0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$

$\therefore$ The general solution, $x_1 = \frac{2}{3}x_2$

x_2 free variables

$\therefore$ $x = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}\frac{2}{3}x_2\\x_2\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}$ (taking $x_2 = 3$)

$\therefore$ $\begin{bmatrix}2\\3\end{bmatrix}$ is in NulA

Any non zero vector of column of A is in ColA.

$\therefore$ $\begin{bmatrix}6\\-3\\-9\\9\end{bmatrix}$ is in ColA.

(ii) We have

$[A\ \ 0] = \begin{bmatrix}5 & -2 & 3 & 0\\-1 & 0 & -1 & 0\\0 & -2 & -2 & 0\\-5 & 7 & 2 & 0\end{bmatrix} \sim \begin{bmatrix}0 & -2 & -2 & 0\\-1 & 0 & -1 & 0\\0 & -2 & -2 & 0\\0 & 7 & 7 & 0\end{bmatrix}$

$\sim \begin{bmatrix}-1 & 0 & -1 & 0\\0 & -2 & -2 & 0\\0 & -2 & -2 & 0\\0 & 7 & 7 & 0\end{bmatrix}$

$\sim \begin{bmatrix}-1 & 0 & -1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$

Which shows that the system is consistent so the general solution is

$x_1 = -x_3$

$x_2 = -x_3$

x_3 free

$\therefore$ $x = \begin{bmatrix}-x_3\\-x_3\\x_3\end{bmatrix} = \begin{bmatrix}1\\1\\-1\end{bmatrix}$ taking $x_3 = -1$

$\therefore$ $\begin{bmatrix}1\\1\\-1\end{bmatrix}$ is in NulA.

Any column of A is a nonzero vector is ColA.

8. **Solution.**

a. Since the augmented matrix

$[B\ \ a_3] \sim \begin{bmatrix}1 & 0 & 0 & 1/3\\0 & 1 & 0 & 1/3\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}$ and $[B\ \ a_5] \sim \begin{bmatrix}1 & 0 & 0 & 10/3\\0 & 1 & 0 & -26/3\\0 & 0 & 1 & -4\\0 & 0 & 0 & 0\end{bmatrix}$

Which shows that both system is consistent

$\therefore$ a_3 and a_5 are in ColB.

b. We have using above

$A \sim \begin{bmatrix}1 & 0 & 1/3 & 0 & 10/3\\0 & 1 & 1/3 & 0 & -26/3\\0 & 0 & 0 & 1 & -4\\0 & 0 & 0 & 0 & 0\end{bmatrix}$

$\therefore$ The general solution is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1/3 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -10/3 \\ 26/3 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

∴ The spanning set for Nul A is

$$\left\{ \begin{bmatrix} -1/3 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -10/3 \\ 26/3 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

c. The reduced row echelon form of A shows that the columns of A are linearly dependent and do not span R^4.

∴ T is neither one to one nor onto.

9. **Solution.** We have,

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= Ax$$

Where $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

Since, kerT = Nul A so

$$[A \quad 0] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

∴ The general solution, $x_1 = x_3$, $x_2 = -x_3$, x_3 free

∴ $x = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

∴ $\ker T = \left\{ x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \; x_3 \in R \right\}$

For ImT we have,

Col A $= \text{span}\{a_1, a_2, a_3\}$

$$= \left\{ a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a + b \\ b + c \end{pmatrix} : a, b, c \in R \right\}$$

10. **Solution.** By rearranging the equations that describe the elements of H, we see that H is the set of all solution of the following system of homogeneous linear equations

$$a - 2b + 5c - d = 0$$
$$-a - b + c = 0$$

Thus by using theorem 2, H is a subspace of R^4.

Exercise 5.3

1. **Solution.**

(i) Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

and $A = [v_1 \; v_2 \; v_3]$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Since there is no free variables, all columns of A are pivot columns so $\{v_1 \; v_2 \; v_3\}$ is a basis for R^3.

(ii), (iii), (iv) and (v)→ Similarly to (i).

(vi) Let $v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} -7 \\ 5 \\ 4 \end{bmatrix}$

$A = [v_1 \; v_2 \; v_3]$

$$= \begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & -3 & -15 \\ 0 & 1 & 13 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 13 \\ 0 & -3 & -15 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 13 \\ 0 & 0 & 24 \end{bmatrix}$$

Which shows that $\{v_1, v_2, v_3\}$ is basis for R^3 because each columns and row of A has pivot positions.

vii. Since there is more vectors then their entries. So the set is linearly dependent, so it is not basis for R^3.

viii. Similar as vii.

2. **Solution.**

Given, $v_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 7 \\ -9 \end{bmatrix}$

Since $A = [v_1 \ \ v_2] = \begin{bmatrix} 1 & -2 \\ -2 & 7 \\ 3 & -9 \end{bmatrix}$ has at most two pivot positions which shows that $\{v_1, v_2\}$ can not span R^3.

$\therefore$ The set $\{v_1, v_2\}$ is not basis for R^3.

No, $\{v_1, v_2\}$ is not basis for R^2 because v_1 and v_2 are not from R^2.

3. Solution.

Let $A = [v_1 \ \ v_2 \ \ v_3 \ \ v_4]$

$$= \begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 20 & 4 & -20 \\ 0 & -25 & -5 & 25 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 1 & 1/5 & -1 \\ 0 & 1 & 1/5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 1 & 1/5 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Since, 1st and 2nd columns of matrix A are pivot columns.

$\therefore$ $\{v_1, v_2\}$ is basis for subspace W.

4. Solution.

Since,

$$\begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = sv_1 + sv_2$$

which shows that every vector in H is a linear combination of v_1 and v_2 so $\{v_1, v_2\}$ spans H.

Also since $\{v_1, v_2\}$ is linearly independent so it is basis for H.

5. Solution.

We find 1st general solution of $Ax = 0$ for

$$[A \ \ 0] = \begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 3 & -2 & 1 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 0 & -2 & 10 & -8 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The general solution is $x_1 = 3x_3 - 2x_4$, $x_2 = 5x_3 - 4x_4$, x_3 free, x_4 free

$$\therefore \quad x = \begin{bmatrix} 3x_3 - 2x_4 \\ 5x_3 - 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ The set $\left\{\begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix}\right\}$ is basis for Nul A.

6. Solution.

a. Given $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$

Now, if we reduced A as in echelon form then

$$A \sim \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which shows that 1st and 2nd columns are pivot columns so

$\left\{\begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix}\right\}$ is basis for col A.

To find basis for Nul A, we find the general solution of $Ax = 0$

Here, $x_1 = -6x_3 - 5x_4$

$$x_2 = -\frac{5}{2}x_3 - \frac{3}{2}x_4$$

x_3 free, x_4 free

$$\therefore \quad x = x_3\begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}$$

Hence, $\left\{\begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for Nul A.

b. Given $A = \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{bmatrix}$ doing same as above we get

$\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \\ 9 \\ 9 \end{bmatrix}\right\}$ is basis for col A.

and $\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}\right\}$ is a basis for Nul A.

7. Solution.

Given, $y = -3x$

or $3x + y = 0$

or $[3 \ \ 1]\begin{bmatrix} x \\ y \end{bmatrix} = 0$

or $AX = 0$ where $A = [3 \;\; 1]$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$

Since, $[A \;\; 0] = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

which shows that x is basic and y is free variables.

So,

The general solution

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}y \\ y \end{bmatrix} = y\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} = -3y\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$\therefore$ $\left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\}$ is a basis for Nul A

8. **Solution.** Since $4v_1 + 5v_2 - 3v_3 = 0$ which shows that each of the vectors is a linear combination of the others.
So, the sets $\{v_1, v_2\}$, $\{v_1, v_3\}$ and $\{v_2, v_3\}$ all span H
Since, none of the three vectors is a multiple of any of the others, the set $\{v_1, v_2\}$, $\{v_1, v_3\}$ and $\{v_2, v_3\}$ are linearly independent and thus each forms a basis for H.

Exercise 5.4

1. **Solution.**

(i) We have, $x = 5\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 3\begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 15 \\ -25 \end{bmatrix} + \begin{bmatrix} -12 \\ 18 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$

(ii) Similarly, we get $\begin{bmatrix} -7 \\ 4 \\ 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 8 \\ -5 \\ 1 \end{bmatrix}$

2. **Solution.** (i) Let $[x]_B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ then,

$x = c_1 b_1 + c_2 b_2$

$$\begin{pmatrix} -1 \\ -6 \end{pmatrix} = \begin{pmatrix} c_1 \\ -4c_1 \end{pmatrix} + \begin{pmatrix} 2c_2 \\ -3c_2 \end{pmatrix}$$

$\therefore$ $c_1 + 2c_2 = -1$ .... (i)

$-4c_1 - 3c_2 = -6$ .... (ii)

Solving we get

$c_1 = 3$, $c_2 = -2$

$\therefore$ $[x]_B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

(ii) We have,

$[b_1 \;\; b_2 \;\; b_3 \;\; x]$

$$= \begin{bmatrix} 1 & -3 & 2 & 8 \\ -1 & 4 & -2 & -9 \\ -3 & 9 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$\therefore$ $[x]_B = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$

(iii) Similarly, we get $[x]_B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

3. **Solution.**

(i) $P_B = [b_1 \;\; b_2] = \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix}$

(ii) $P_B = [b_1 \;\; b_2 \;\; b_3] = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & -2 \\ 6 & -4 & 3 \end{bmatrix}$

4. **Solution.**

(i) Here, $P_B = \begin{bmatrix} 1 & -3 \\ -2 & 5 \end{bmatrix}$, $x = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

So $P_B^{-1} = \frac{1}{-1}\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -3 \\ -2 & -1 \end{bmatrix}$

$\therefore$ $[x]_B = P_B^{-1} x = \begin{bmatrix} -5 & -3 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

(ii) Similarly as above we get $[x]_B = \begin{bmatrix} -8 \\ 5 \end{bmatrix}$

5. **Solution.** Given,
$1 + 2t^2$, $4 + t + 5t^2$, $3 + 2t$
So the coordinate vectors are (1, 0, 2), (4, 1, 5) and (3, 2, 0) respectively.
Now, using these vectors in columns in A, the augmented matrix

$$\begin{pmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 5 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which shows that columns of A are linearly dependent. So the corresponding polynomials are linearly dependent.

6. **Solution.** Let (c_1, c_2, c_3) be the coordinate vector of p(t)
Then,
$c_1(1 + t^2) + c_2(t + t^2) + c_3(1 + 2t + t^2) = 1 + 4t + 7t^2$

or $(c_1 + c_3) + (c_2 + 2c_3)t + (c_1 + c_2 + c_3)t^2 = 1 + 4t + 7t^2$

Comparing we get

$c_1 + c_3 = 1$ .... (i)

$c_2 + 2c_3 = 4$ .... (ii)

$c_1 + c_2 + c_3 = 7$ .... (iii)

Now, the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

So $[P]_B = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$

7. **Solution.** Solve as Q. No. 6 then we get

$[P]_B = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

8. **Solution.**

(a) The coordinate vectors of given polynomials are (1, 0, 1), (0, 1, –3) and (1, 1, –3) respectively.

Let A be a matrix using them as in columns then

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

which shows that the matrix A is invertible. So the three columns of A form a basis for R^3. So the corresponding polynomials are form a basis for P_2. (Isomorphism between R^3 and P_2).

(b) Since, $[q]_B = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ so

$q = -1 . P_1 + 1 . P_2 + 2 . P_3$

$q = -(1 + t^2) + (t - 3t^2) + 2(1 + t - 3t^2)$

$q = 1 + 3t - 10t^2$

$\therefore$ $q(t) = 1 + 3t - 10t^2$