Chapter 4

Determinants

Exercise 4.1

1.
Here,

$$A = \begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix} = (3)\begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix} - 0 + (4)\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix}$$

$= (3)(-3-10) - 0 + (4)(10-0)$
$= -39 + 40$
$= 1.$

2.
Here,

$$A = \begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix} = (0) - (5)\begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix} + (1)\begin{vmatrix} 4 & -3 \\ 2 & 4 \end{vmatrix}$$

$= 0 - (5)(4-0) + 1(16+6)$
$= -20 + 22$
$= 2.$

Q.3 - Q.8 Similar to Q.1.

9.
Here

$$A = \begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix} = 2\begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 8 \end{vmatrix}$$

[Expanded from a_{31}, so $(-1)^{3+1} = 1$]

$= (2)(5)\begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix}$

$= (2)(5)(7-6) = 10.$

10.
Here,

$$A = \begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix} = (-3)\begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix}$$

[Expanded from a_{23}, so $(-1)^{2+3} = -1$]

$= (-3)\left(5\begin{vmatrix} -2 & 2 \\ -6 & 5 \end{vmatrix} - 0 + 4\begin{vmatrix} 1 & -2 \\ 2 & -6 \end{vmatrix}\right)$

$= (-3)(5(-10+12) - 0 + 4(-6+4))$
$= (-3)(10-8) = -6.$

Q.11 - Q.14 → Similar to Q.9.
Q.15 - Q.18 → Similar to Q.1.

19.
Here,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \text{ and } \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -(ad - bc) = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

This shows if two rows are interchanged, the determinant changes its sign.

20. Here,

$$\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = (18) - (20) = -2$$

and $\begin{vmatrix} 3 & 4 \\ 5+3k & 6+4k \end{vmatrix} = (18+12k) - (20+12k) = -2$

$$\Rightarrow \begin{vmatrix} 3 & 4 \\ 5+3k & 6+4k \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 3k & 4k \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} + k\begin{vmatrix} 3 & 4 \\ 3 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} + k(0)$$

$$= \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}$$

One row times k (scalar) is added (or subtract) to another row, the determinant does not change.

Q.21 → Similar to Q.20
Q.22, 23 → Similar to Q.19

24.

Here, $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{vmatrix} = (1)(1)(1) = 1$

The triangular determinant is same as the multiple of its leading diagonal entries.

Q.25 - Q.27 Similar to Q.24

28.

Here, $\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0 - (1)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 0 = -(1)(1-0) = -1.$

29.

Here, $\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0 - 0 + (1) \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = (1)(0 - 1) = -1$

30. Since the elementary scaling matrix with k on its diagonal is an identity matrix with one non-zero entry is replaced by k. That is the elementary scaling matrix with k on its diagonal is:

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & k & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Clearly the matrix is a triangular matrix, so its determinant is the product of its diagonal elements. So, the determinant of the matrix is k.

31.

Let,

$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then,

$$EA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

Therefore,

$$\det(EA) = \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad$$

$$\det(A) = \begin{vmatrix} a & d \\ c & b \end{vmatrix} = ad - bc,$$

$$\det(E) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

Thus,

$\det(E)\det(A) = (-1)(ad - bc) = bc - ad = \det(EA)$.

Q.32 – 34 → Similar to Q.31.

35. Let

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$$

Then

$$5A = 5\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ 20 & 10 \end{bmatrix}$$

Therefore,

$$\det(A) = \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = 6 - 4 = 2$$

$$\det(5A) = \begin{vmatrix} 15 & 5 \\ 20 & 10 \end{vmatrix} = 150 - 100 = 50$$

And,

$$5\det(A) = 5(2) = 10 \neq \det(5A)$$

36. Let,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and let k is a scalar.

Then

$$kA = k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

Since,

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

And,

$$\det(kA) = \begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = k^2ad - k^2bc = k^2(ad - bc) = k^2\det(A).$$

Thus,

$\det(kA) = k^2\det(A)$ when k is 2×2 matrix.

Exercise 4.2

1. If two rows of a determinant A, are interchanged to produce a determinant B, then det (A) = – det (B).
In our problem, the first and second rows are interchanged.

2. If a row of a matrix A is multiplied by a scalar value k, produce a matrix B then

$$k\det(A) = \det(B)$$

In our problem, the first row of the determinant

$$\begin{vmatrix} 1 & -3 & 2 \\ 3 & 5 & -2 \\ 1 & 6 & 3 \end{vmatrix}$$

is multiplied by 2, produce another determinant which is given in right side.

3. If a multiple of one row of a matrix A is added to another row to produce a matrix B then det (B) = det (A).
In our problem, the first row is multiplied by –2 and add it with second row then

$$\det(A) = \begin{vmatrix} 1 & 3 & -4 \\ 2 & 0 & -3 \\ 5 & -4 & 7 \end{vmatrix}$$

produce another matrix B where

$$\det(B) = \begin{vmatrix} 1 & 3 & -4 \\ 0 & -6 & 5 \\ 5 & -4 & 7 \end{vmatrix}$$

Then det (A) = det (B).

4.

Similar to Q.3.

Here first row is multiplied by –3 and is added with third row.

5.

Here,

$$\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 3 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{vmatrix} \quad [\text{Applying } R_3 \to R_3 - 3R_2]$$

$= (1)\,(1)\,(3)$ [Multiple leading diagonal entries]

$= 3$

Q.6 – Q.9 → Similar to Q.5.

10. Here

$$\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ -2 & -6 & 2 & 3 & 9 \\ 3 & 7 & -3 & 8 & -7 \\ 3 & 5 & 5 & 2 & 7 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & -2 & 0 & 8 & -1 \\ 0 & -4 & 8 & 2 & 13 \end{vmatrix} \quad \begin{array}{l}\text{Applying} \\ R_3 \to R_3 + 2R_1 \\ R_4 \to R_4 - 3R_1 \\ R_5 \to R_5 - 3R_1\end{array}$$

$$= \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & -4 & 7 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \quad \begin{array}{l}\text{Applying} \\ R_4 \to R_4 + R_2 \\ R_5 \to R_5 + 2R_2\end{array}$$

$$= - \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & -4 & 7 & -3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \quad \begin{array}{l}\text{Interchanging third} \\ \text{and fourth rows.}\end{array}$$

$= (-1)\,(1)\,(2)\,(-4)\,(3)\,(1)$

$= 24$

11.

Here,

$$\begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 0 & 0 & 2 & 1 \end{vmatrix} \quad [\text{Applying } R_4 \to R_4 - 2R_1]$$

$$= (-5) \begin{vmatrix} 3 & 1 & -3 \\ -6 & -4 & 9 \\ 0 & 2 & 1 \end{vmatrix} \quad \left[\begin{array}{l}\text{Expanded from 2nd} \\ \text{column.}\end{array}\right]$$

$$= (-5) \begin{vmatrix} 3 & 1 & -3 \\ 0 & -2 & 3 \\ 0 & 2 & 1 \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 + 2R_1]$$

$$= (-5)\,(3) \begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix}$$

$= (-5)\,(3)\,(-2 - 6)$

$= 120.$

Q.12 – Q.14 → Similar to Q.11.

15. Let,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

Now,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix} = (5) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad \left[\begin{array}{l}\text{Taking-out the} \\ \text{common factor 5} \\ \text{from 3rd row.}\end{array}\right]$$

$= (5)\,(7) = 35.$

16. Similar to Q.15.

17. Let,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

Now,

$$\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = (-1) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad \left[\begin{array}{l}\text{Interchanging the} \\ \text{2nd and 3rd row.}\end{array}\right]$$

$= (-1)\,(7) = -7$

18. Let,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

Now,

$$\begin{vmatrix} a & b & c \\ 2d + a & 2e + b & 2f + c \\ g & h & i \end{vmatrix} = (2) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ a & b & c \\ g & h & i \end{vmatrix}$$

$$= (2)\,(7) + 0 \qquad \begin{bmatrix} \text{1st and 2nd rows have} \\ \text{same value in 2nd} \\ \text{determinant so its} \\ \text{value is 0.} \end{bmatrix}$$

$$= 14$$

19. Similar to Q.18.

20.

Here,

$$\begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 0 \\ -3 & -5 & 0 \\ 1 & 2 & 1 \end{vmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_2 \to R_2 - 4R_3 \end{bmatrix}$$

$$= (1) \begin{vmatrix} 2 & 3 \\ -3 & -5 \end{vmatrix}$$

$$= (1)\,(-10 + 9)$$

$$= -1 \neq 0$$

This means the given matrix is invertible.

Note: The matrix A is invertible if det (A) ≠ 0.

21. → Similar to Q.20.

22. Let

$$v_1 = \begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix},\ v_2 = \begin{bmatrix} -7 \\ 0 \\ 2 \end{bmatrix},\ v_3 = \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}$$

Here,

$$\det [v_1 \quad v_2 \quad v_3] = \begin{vmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ -7 & 2 & 6 \end{vmatrix}$$

$$= (-1)\,(-7) \begin{vmatrix} 6 & -5 \\ -7 & 6 \end{vmatrix} + 0 + (-1)\,(2) \begin{vmatrix} 4 & -3 \\ 6 & -5 \end{vmatrix}$$

$$= (7)\,(36 - 35) - 2(-20 + 18)$$

$$= 7 + 4$$

$$= 11 \neq 0$$

This means v_1, v_2, v_3 are linearly independent.

Note: The vectors $v_1, v_2, \ldots, v_n$ are linearly independent if det $[v_1, v_2, \ldots, v_n] \neq 0$ and linearly dependent for otherwise.

Q. 23 – 24 → Similar to Q.22.

Exercise 4.3

1.

Let the given system is in the form Ax = b then

$$A = \begin{bmatrix} 5 & 7 \\ 2 & 4 \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Here,

$$\det (A) = \begin{vmatrix} 5 & 7 \\ 2 & 4 \end{vmatrix} = 20 - 14 = 6 \neq 0$$

So, the system has unique solution. Also,

$$\det (A_1(b)) = \begin{vmatrix} 3 & 7 \\ 1 & 4 \end{vmatrix} = 12 - 7 = 5$$

$$\det (A_2(b)) = \begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix} = 5 - 6 = -1$$

Now, by Cramer's rule,

$$x_1 = \frac{\det (A_1(b))}{\det (A)} = \frac{5}{6}$$

and $$x_2 = \frac{\det (A_2(b))}{\det (A)} = \frac{-1}{6}$$

Thus, the solution of given system is $x_1 = \frac{5}{6}$, $x_2 = \frac{-1}{6}$.

Q.2 – Q.4 → Similar to Q.1.

5. Let the given system is in the form Ax = b then

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ b = \begin{bmatrix} 7 \\ -8 \\ -3 \end{bmatrix}$$

Here, $$\det (A) = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = (2) \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} - (-3) \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + 0$$

$$= 2(0 - 1) + 3(2 - 0)$$

$$= -2 + 6$$

$$= 4 \neq 0$$

So, the system has unique solution. Also,

$$\det \quad (A_1(b)) = \begin{vmatrix} 7 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{vmatrix} = -\ (1) \begin{vmatrix} -8 & 1 \\ -3 & 2 \end{vmatrix} + 0 - (1) \begin{vmatrix} 7 & 0 \\ -8 & 1 \end{vmatrix}$$

$$= -1\,(-16 + 3) + 0 - 1\,(7 - 0) = 13 - 7 = 6$$

$$\det(A_2(b)) = \begin{vmatrix} 2 & 7 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{vmatrix} = 2\begin{vmatrix} -8 & 1 \\ -3 & 2 \end{vmatrix} - (-3)\begin{vmatrix} 7 & 0 \\ -3 & 2 \end{vmatrix} + 0$$

$$= 2(-16+3) + 3(14-0) + 0$$

$$= -26 + 42$$

$$= 16$$

$$\det(A_3(b)) = \begin{vmatrix} 2 & 1 & 7 \\ -3 & 0 & -8 \\ 0 & 1 & -3 \end{vmatrix} = 2\begin{vmatrix} 0 & -8 \\ 1 & -3 \end{vmatrix} - (-3)\begin{vmatrix} 1 & 7 \\ 1 & -3 \end{vmatrix} + 0$$

$$= 2(0+8) + 3(-3-7) + 0$$

$$= 16 - 30$$

$$= -14$$

Now, by Cramer's rule,

$$x_1 = \frac{\det(A_1(b))}{\det(A)} = \frac{6}{4} = \frac{3}{2} = 1.5$$

$$x_2 = \frac{\det(A_2(b))}{\det(A)} = \frac{16}{4} = 4$$

$$x_3 = \frac{\det(A_3(b))}{\det(A)} = \frac{-14}{4} = \frac{-7}{2} = -3.5$$

Thus, the solution of given system is $x_1 = \frac{2}{3}$, $x_2 = 4$, $x_3 = \frac{-7}{2}$.

6. **Similar to Q.5.**

7. Let the given system is in $Ax = b$ then

$$A = \begin{bmatrix} 6s & 4 \\ 9 & 2s \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, b = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

Here,

$$\det(A) = \begin{bmatrix} 6s & 4 \\ 9 & 2s \end{bmatrix} = 12s^2 - 36 = 12(s^2 - 3) \neq 0 \text{ for } s \neq \pm\sqrt{3}.$$

This means the has unique solution for $s \neq \pm\sqrt{3}$.

Also,
$$\det(A_1(b)) = \begin{vmatrix} 5 & 4 \\ -2 & 2s \end{vmatrix} = 10s + 8$$

$$\det(A_2(b)) = \begin{vmatrix} 6s & 5 \\ 9 & -2 \end{vmatrix} = -12s - 45$$

Now, by Cramer's rule

$$x_1 = \frac{\det(A_1(b))}{\det(A)} = \frac{10s+8}{12(s^2-3)} = \frac{5s+4}{6(s^2-3)}$$

$$x_2 = \frac{\det(A_2(b))}{\det(A)} = \frac{-12s-45}{12(s^2-3)} = \frac{-4s-15}{4(s^2-3)}$$

Thus, the solution of the given system is

$$x_1 = \frac{5s+4}{6(s^2-3)}, x_2 = \frac{-4s-15}{4(s^2-3)} \text{ for } s \neq \pm\sqrt{3}.$$

Q.8 – Q.10 → Similar to Q.7.

11.

Let
$$A = \begin{bmatrix} 0 & -2 & -1 \\ 3 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

Here,

$$\det(A) = \begin{vmatrix} 0 & -2 & -1 \\ 3 & 0 & 0 \\ -1 & 1 & 1 \end{vmatrix} = -(3)\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} + 0 - 0$$

$$= -3(-2+1)$$

$$= 3 \neq 0$$

This means A is invertible and A^{-1} exists.

Here, the cofactors of A are,

$$C_{11} = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0 \qquad C_{12} = -\begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} = -3$$

$$C_{13} = \begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} = 3 \qquad C_{21} = -\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} = -(-2+1) = 1$$

$$C_{22} = \begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} = -1 \qquad C_{23} = -\begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix} = 2$$

$$C_{31} = \begin{vmatrix} -2 & -1 \\ 0 & 0 \end{vmatrix} = 0 \qquad C_{32} = -\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} = -3$$

$$C_{33} = \begin{vmatrix} 0 & -2 \\ 3 & 0 \end{vmatrix} = 6$$

Therefore, the cofactor matrix of A is

$$\text{Cofactor matrix of } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 0 & -3 & 3 \\ 1 & -1 & 2 \\ 0 & -3 & 6 \end{bmatrix}$$

So, the adjoint matrix of A is,

$$\text{Adj}(A) = (\text{Cofactor matrix of } A)^T$$

$$= \begin{bmatrix} 0 & -3 & 3 \\ 1 & -1 & 2 \\ 0 & -3 & 6 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & -3 \\ 3 & 2 & 6 \end{bmatrix}$$

Now, the inverse matrix of A i.e., A^{-1} is,

$$A^{-1} = \frac{1}{\det(A)} \text{Adj.}(A) = \frac{1}{3}\begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & -3 \\ 3 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 0 \\ -1 & -1/3 & -1 \\ 1 & 2/3 & 2 \end{bmatrix}$$

Q.12 – Q.16 → Similar to Q.11.

17. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Here,

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

This means A is invertible and A^{-1} exists for $ad - bc \neq 0$.
The cofactor matrix of A is

$$\text{cofactor matrix of } A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

and the adjoint matrix of A is,

$$\text{adj.}(A) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Now, the inverse matrix (A^{-1}) of A is,

$$A^{-1} = \frac{1}{\det(A)} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

18.

Let all the entries of a matrix A, are integers.
Given that det (A) = 1 ≠ 0. So, the inverse of A, exist. And,

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \text{adj}(A) \ldots \text{(i)}$$

Since the multiple, addition and subtraction between integers, is again an integer. So, all the entries of cofactor matrix, are integers. Therefore, all the entries of adjoint matrix, are integers.
So, by (i) all the entries of A^{-1}, are integers.

19.

Let the vertices of the parallelogram are A(0, 0), B(5, 2), C(6, 4), D(11, 6).

Then, $u = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (5, 2) - (0, 0) = (5, 2)$

$v = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (6, 4) - (0, 0) = (6, 4)$

Since u and v are adjacent sides of the parallelogram ABCD.
Now, the area of the paralleogram ABCD is

$$\text{area} = \det \begin{bmatrix} u \\ v \end{bmatrix} = \begin{vmatrix} 5 & 2 \\ 6 & 4 \end{vmatrix} = 20 - 12 = 8$$

Thus, the area of the parallelogram ABCD is 8 sq. units.

Q.20 – Q.22 → Similar to Q. 19.

23.

Let A is invertible. So, A^{-1} exists. And, we have

$$AA^{-1} = I$$

So, $\det(AA^{-1}) = \det(I) = 1$

$\Rightarrow \quad \det(A)\det(A^{-1}) = 1$

$\Rightarrow \quad \det(A^{-1}) = \dfrac{1}{\det(A)}.$

24. Let the one vertex of the parallelopiped is at origin and the adjacent vertices are at (1, 0, –2), (1, 2, 4) and (7, 1, 4). Then,

$$\det(A) = \begin{vmatrix} 1 & 1 & 7 \\ 0 & 2 & 1 \\ -2 & 4 & 4 \end{vmatrix}.$$

$$= 1\begin{vmatrix} 2 & 1 \\ 4 & 4 \end{vmatrix} + 0 - 2\begin{vmatrix} 1 & 7 \\ 2 & 1 \end{vmatrix}.$$

$$= 4 + 0 - 26.$$

$$= -22.$$

Thus, the volume of the parallelepiped with $|-22| = 22$.

25.

(i) Given that S is the parallelogram determined by the vectors $b_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $b_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$. So,

$$\det(S) = \begin{vmatrix} -2 & -2 \\ 3 & 5 \end{vmatrix} = -10 + 6 = -4.$$

Thus,

Area of S = |–4| = 4.

And given that

$$A = \begin{bmatrix} 6 & -2 \\ -3 & 2 \end{bmatrix}$$

Then,

$$\det(A) = \begin{vmatrix} 6 & -2 \\ -3 & 2 \end{vmatrix} = 12 - 6 = 6.$$

Therefore, the a area of S under the mapping $x \to Ax$ is,

area of image of S = Area of T(S)

= |det A| [Area of S]

= 6×4 = 24 sq. unit

(ii) Similar to (i).

26. The matrix represented to a square having vertices at O(0, 0, 1), A(1, 0, 1), B(0, 1, 1) and C(1, 1, 1) is

$$S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and given matrix is

$$A = \begin{bmatrix} 1 & 1 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, the effect of pre-multiplication of S by A is,

$$S' = A.S = \begin{bmatrix} 1 & 1 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} a & 1+a & 2+a \\ b & 1+b & 1+b \\ 1 & 1 & 1 \end{bmatrix}$$

This means the vertices of the effect of the square A are O'(a, b, 1), A'(1 +a, 1 + b, 1), B(2 +a,1 + b, 1).