Chapter 6

Vector Space Continued

Exercise 6.1

1. **Solution.**

(i) Since $H = \left\{\begin{bmatrix}2a\\-4b\\-2a\end{bmatrix}\right\} = \left\{a\begin{bmatrix}2\\0\\-2\end{bmatrix} + b\begin{bmatrix}0\\-4\\0\end{bmatrix}\right\}$

which shows that $H = \text{span}\{v_1, v_2\}$ where $v_1 = \begin{bmatrix}2\\0\\-2\end{bmatrix}, v_2 = \begin{bmatrix}0\\-4\\0\end{bmatrix}$.

Since, v_1 and v_2 are not multiple of each other so, $\{v_1, v_2\}$ is linearly independent. Thus, it is basis for H and dim H = 2.

(ii) Since, $H = \left\{\begin{bmatrix}2c\\a-b\\b-3c\\a+2b\end{bmatrix}\right\} = \left\{a\begin{bmatrix}0\\1\\0\\1\end{bmatrix} + b\begin{bmatrix}0\\-1\\1\\2\end{bmatrix} + c\begin{bmatrix}2\\0\\-3\\0\end{bmatrix}\right\}$

$= \{(av_1 + bv_2 + cv_3): a, b, c \in R\}$

which shows that $H = \text{span}\{v_1, v_2, v_3\}$

also, since $v_1 \pm 0$, v_2 is not a multiple of v_1 and v_3 is not multiple of v_1 and v_2.

∴ $\{v_1, v_2, v_3\}$ is linearly independent (or using $[v_1\ v_2\ v_3\ 0]$ show for linearly independent)

Thus, $\left\{\begin{bmatrix}0\\1\\0\\1\end{bmatrix} + \begin{bmatrix}0\\-1\\1\\2\end{bmatrix} + \begin{bmatrix}2\\0\\-3\\0\end{bmatrix}\right\}$ is basis for H and dim H = 3

Similarly do for others

(iii) $\left\{\begin{bmatrix}1\\-1\\3\\1\end{bmatrix}, \begin{bmatrix}2\\0\\-1\\1\end{bmatrix}\right\}$, dim H = 2

(iv) $\left\{\begin{bmatrix}1\\2\\0\\-3\end{bmatrix}, \begin{bmatrix}-2\\0\\-2\\0\end{bmatrix} + \begin{bmatrix}0\\5\\2\\6\end{bmatrix}\right\}$, dim H = 3

(v) We have, $H = \{(a, b, c): a - 3b + c = 0, b - 2c = 0, 2b - c = 0\}$

which shows that H = Nul A where $A = \begin{bmatrix}1 & -3 & 1\\0 & 1 & -2\\0 & 2 & -1\end{bmatrix}$

Since, $[A\ \ 0] \sim \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}$. Which shows that the all columns of A are pivot columns and there is no free variables.

Thus, H = Nul A = {0}, and dim H = 0

2. **Solution.** According to questions we have,

$H = \left\{\begin{bmatrix}a\\b\\a\end{bmatrix}: a, b \in R\right\} = \left\{a\begin{bmatrix}1\\0\\1\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix}: a, b \in R\right\}$

$= \{(av_1 + bv_2): a, b \in R\}$

$= \text{span}\{v_1, v_2\}$

Since, $\{v_1, v_2\}$ is linearly independent so, it is basis

∴ dim H = 2

3. **Solution.**

(i) Let A form a matrix using these vectors in columns

$A = \begin{bmatrix}1 & 3 & -2 & 5\\0 & 1 & -1 & 2\\2 & 1 & 1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & -1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$

There are three pivot columns so

dim = 3

(ii) Similar as (i) we get dim = 3.

4. **Solution.**

(i) Since these are three pivot columns and there are 2 not pivot columns

So, dim Col A = 3

dim Nul A = 2

(ii) $A = \begin{bmatrix}3 & 2\\-6 & 5\end{bmatrix} \sim \begin{bmatrix}3 & 2\\0 & 1\end{bmatrix}$

So dim Col A = 2

dim Nul A = 0

(iii) dim col A = 2

dim Nul A = 2

(iv) $A \sim \begin{bmatrix}1 & 0 & 6 & 5\\0 & 2 & 5 & 3\\0 & 0 & 0 & 0\end{bmatrix}$

dim Col A = 2

dim Nul A = 2

(v) $A \sim \begin{bmatrix}1 & 2 & 0 & 4 & 5\\0 & 0 & 5 & -7 & 8\\0 & 0 & 0 & 0 & -9\\0 & 0 & 0 & 0 & 0\end{bmatrix}$

dim Col A = 3

dim Nul A = 2

Exercise 6.2

1. **Solution.**

(i) Since there are two pivot position so

Rank A = 2, dim NulA = 2

For bases, 1st and 2nd columns are pivot columns so

$\left\{\begin{bmatrix}1\\-1\\5\end{bmatrix}, \begin{bmatrix}-4\\2\\-6\end{bmatrix}\right\}$ is basis for ColA

{(1, 0, −1, 5), (0, −2, 5, −6)} is basis for Row A.

For NulA, the general solution of $Ax = 0$ is

$x_1 = x_3 - 5x_4$

$x_2 = \frac{5}{2}x_3 - 3x_4$

x_3 free

x_4 free

$$\therefore \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \quad \left\{ \begin{bmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is basis for NulA.}$$

similarly do for (ii) and (iii)

(ii) Rank A = 3, dim NulA = 3

$$\text{Basis for col A} = \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ 3 \end{bmatrix} \right\}$$

Basis for row A = {(2, 6, –6, 6, 3, 6), (0, 3, 0, 3, 3, 0), (0, 0, 0, 0, 3, 0)}

$$\text{Basis for NulA} = \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(iii) Rank A = 5, dim NulA = 1

$$\text{Basis for ColA} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 6 \\ 0 \\ -1 \end{bmatrix} \right\}$$

Basis for row A = {(1, 1,–2, 0, 1,–2), (0, 1,–1, 0,–3,–1), (0, 0, 1, 1,–13, –1), (0, 0, 0, 1, –1), (0, 0, 0, 0, 0, 1)}

$$\text{Basis for NulA} = \left\{ \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

2. **Solution.** Using row-operation we have

$$A \sim \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1st and 2nd columns are pivot columns so

Rank A = 2, dim NulA = 3

$$\text{Basis for col A} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 8 \\ -5 \end{bmatrix} \right\}$$

Basis for row A = {(1, –2, –4, 3, –2), (0, 3, 9, –12, 12)}

<u>For Nul A</u>

$$A \sim \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 1 & 3 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -5 & 6 \\ 0 & 1 & 3 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So the general solution of $Ax = 0$ is

$x = -2x_3 + 5x_4 - 6x_5$

$x_2 = -3x_3 + 4x_4 - 4x_5$

x_3 free

x_4 free

x_5 free

$$\therefore \quad \left\{ \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is basis for Nul A.}$$

3. **Solution.** By rank theorem dim NulA = 7 – Rank A = 7 – 3 = 4

Since, dim Row A = Rank A = 3

Also, rank A^T = dim ColA^T = dim Row A

$\therefore$ Rank $A^T = 3$

$\therefore$ dim NulA = 4, dim Row A = 3, Rank $A^T = 3$

4. **Solution.** Yes ColA = R^4 since A has four pivot columns. dim ColA = 4. Thus ColA is a four dimensional subspace of R^4, and ColA = R^4

No, NulA ≠ R^3, it is true that dim Nul A = 3, but NulA is a subspace of R^7.

Exercise 6.3

1. **Solution.** (a) Here, $[b_1]_c = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, $[b_2]_c = \begin{bmatrix} 9 \\ -4 \end{bmatrix}$

Now, $\underset{C \leftarrow B}{P} = [[b_1]_C \, [b_2]_C] = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$

(b) Since, $[x]_B = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$[x]_c = \underset{C \leftarrow B}{P}[x]_B = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

2. **Solution.** Similar as Q. No. 1 we get

$\underset{C \leftarrow B}{P} = \begin{bmatrix} -2 & 3 \\ 4 & -6 \end{bmatrix}$

$[x]_e = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$

3. **Solution.**

a. Here, $[a_1]_B = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$, $[a_2]_B = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $[a_3]_B = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

$\therefore$ $\underset{B \leftarrow A}{P} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

b. Here, $[x]_A = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$

$\therefore$ $[x]_B = \underset{B \leftarrow A}{P}[x]_A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$

4. **Solution.** Do as Q. No. 3 we get

$\underset{D \leftarrow F}{P} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

$[x]_D = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}$

5. **Solution.**

a. We have,

$[c_1 \; c_2 \; b_1 \; b_2] \sim \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{bmatrix}$

$\therefore$ $\underset{C \leftarrow B}{P} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$

$\underset{B \leftarrow C}{P} = \left(\underset{C \leftarrow B}{P}\right)^{-1} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$

Similarly do for b, c, d

b. $\underset{C \leftarrow B}{P} = \begin{bmatrix} 9 & -8 \\ -10 & 9 \end{bmatrix}$, $\underset{B \leftarrow C}{P} = \begin{bmatrix} 9 & 8 \\ 10 & 9 \end{bmatrix}$

c. $\underset{C \leftarrow B}{P} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$, $\underset{B \leftarrow C}{P} = \begin{bmatrix} 1/2 & 3/2 \\ 0 & -1 \end{bmatrix}$

d. $\underset{C \leftarrow B}{P} = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$, $\underset{B \leftarrow C}{P} = \begin{bmatrix} 0 & -1/2 \\ 1 & 3/2 \end{bmatrix}$

6. **Solution.** Let $B = \{b_1, b_2, b_3\}$, $C = \{c_1, c_2, c_3\}$ then

$[b_1]_c = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $[b_2]_c = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$, $[b_3]_c = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

$\therefore$ $\underset{C \leftarrow B}{P} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$

since, $x = -1 + 2t$, So $[x]_c = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

$\therefore$ $[x]_c = \underset{C \leftarrow B}{P}[x]_B$

$\therefore$ The augmented matrix is

$\left[\underset{C \leftarrow B}{P} \; [x]_C\right] = \begin{bmatrix} 1 & 3 & 0 & -1 \\ -2 & -5 & 2 & 2 \\ 1 & 4 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\therefore$ $[x]_B = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$

7. **Solution.**

(i) Let $y_k = 2^k$. Then,

$y_{k+2} + 2y_{k+1} - 8y_k = 2^{k+2} + 2 \,.\, 2^{k+1} - 8.2^k$

$= 2^k\,(2^2 + 2^2 - 8)$

$= 2^k \times 0 = 0$ for all k

Since the difference equation holds for all k, 2^k is a solution.

Let $y_k = (-4)^k$, then

$y_{k+2} + 2y_{k+1} - 8y_k = (-4)^{k+2} + 2\,(-4)^{k+1} - 8\,(-4)^k$

$= (-4)^k\,\{(-4)^2 + 2\,.\,(-4) - 8\}$

$= (-4)^k \,.\, 0 = 0$ for all k

$\therefore$ For all k, $(-4)^k$ is a solution.

(ii) Similarly do for (ii)

8. **Solution.** (i) Compute and row reduce the casorati matrix for the signals 1^k, 2^k and $(-2)^k$, setting $k = 0$,

$\begin{bmatrix} 1^0 & 2^0 & (-2)^0 \\ 1^1 & 2^1 & (-2)^1 \\ 1^2 & 2^2 & (-2)^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

This casorati matrix is row equivalent to the identity matrix, thus is invertible by the IMT. Hence the signals $\{1^k, 2^k, (-2)^k\}$ is linearly independent.

(ii) Similarly as above linearly independent.

(ii) Similarly as above linearly independent.

9. **Solution.** The auxiliary equation for this difference equation is $r^2 - r + \frac{2}{9} = 0$.

$\therefore$ $r = \frac{2}{3}, \frac{1}{3}$

So two solutions of the difference equations are $\left(\frac{2}{3}\right)^k$ and $\left(\frac{1}{3}\right)^k$.

10. **Solution.** (a) Let H stand for 'Healthy' and I stand for 'ill' then

From		To
H	I	
0.95	0.45	H
0.05	0.55	I

So the stochastic matrix is $P = \begin{bmatrix} 0.95 & 0.45 \\ 0.05 & 0.55 \end{bmatrix}$

(b) Since 20% of the students are ill on Monday, the initial state vector is $x_0 = \begin{bmatrix} 0.80 \\ 0.20 \end{bmatrix}$,

For Tuesday, $x_1 = Px_0 = \begin{bmatrix} 0.85 \\ 0.15 \end{bmatrix}$

$$x_2 = Px_1 = \begin{bmatrix} 0.875 \\ 0.125 \end{bmatrix}$$

Thus 15% of the students are ill on Tuesday, and 12.5% on Wednesday.

(c) Since the student is well today, the initial state vector is $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then

$x_1 = Px_0 = \begin{bmatrix} 0.95 \\ 0.05 \end{bmatrix}$ $\quad x_2 = Px_1 = \begin{bmatrix} 0.925 \\ 0.075 \end{bmatrix}$

Thus, the probability that the student is well two days from now is 0.925.