Chapter 2
Transformation
Exercise 2

1. $T(u) = Au = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

   $T(v) = Av = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

2. Hint: See questions no. 1.

3. Consider $Ax = b$, where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

   It's augmented matrix is

   $$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ -2 & 1 & 6 & 7 \\ 3 & -2 & -5 & -3 \end{array}\right]$$

   $R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - 3R_1$

   $$\sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & 1 & 0 \end{array}\right]$$

   $R_3 \to R_3 + 2R_2$

   $$\sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 5 & 10 \end{array}\right]$$

   $R_3 \to \frac{1}{5}R_3$

   $$\sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

   Corresponding system is,

   $x_1 - 2x_3 = -1$

   $x_2 + 2x_3 = 5$

   $x_3 = 2$

   Using back substitution

   $x_1 = 3, x_2 = 1, x_3 = 2$

4. Hints: See Q. No. 3
5. Hints: Given
6. Consider $Ax = b$, where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

   It's augmented matrix is,

   $$\left[\begin{array}{cccc|c} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 2 & -6 & 6 & -4 & 0 \end{array}\right]$$

   $R_3 \to R_3 - 2R_1$

   $$\sim \left[\begin{array}{cccc|c} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 2 & -8 & 6 & 2 \end{array}\right]$$

   $R_3 \to \frac{1}{2}R_3$

   $$\sim \left[\begin{array}{cccc|c} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 1 & -4 & 3 & 1 \end{array}\right]$$

   $R_3 \to R_3 - R_2$

   $$\sim \left[\begin{array}{cccc|c} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

   Since there is no row of the form $[0 \ \ 0 \ \ 0 \ \ 0 \,|\, b]$, $b \neq 0$ so the system $Ax = b$ is consistent. Thus, b is in the range of linear transformation $x \to Ax$.

7.

(i) Here $T(x_1, x_2, x_3, x_4) = \begin{bmatrix} 0 \\ x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{bmatrix}$

$$= \begin{bmatrix} 0x_1 + 0x_2 + 0x_3 + 0x_4 \\ 1x_1 + 1x_2 + 0x_3 + 0x_4 \\ 0x_1 + 1x_2 + 1x_3 + 0x_4 \\ 0x_1 + 0x_2 + 1x_3 + 1x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$= Ax$

Where, $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ is standard matrix for T and hence T is linear transformation.

(ii) Hint: see 7(i)

8.

(i) The standard matrix for T is

$A = [T(e_1) \quad T(e_2)]$

$$= \begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$$

(ii) Hint: See Q. No. 8(i).

9.

$$T(x_1, x_2) = \begin{bmatrix} x_1 + x_2 \\ 4x_1 + 5x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= Ax$$

So, $T(x) = (3, 8)$

$\Rightarrow$ $Ax = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

It's augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 4 & 5 & 8 \end{array}\right]$$

$R_2 \to R_2 - 4R_1$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -4 \end{array}\right]$$

$\Rightarrow$ $x_1 + x_2 = 3$

$x_2 = -4$

Using back substitution,

$x_1 = 7, x_2 = -4$

$\therefore$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$

10. Hint: Given in the question.
11. Hint: Given in the question.

12. See solution of Q. No. 7(i)

We have,

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Using row operations, we get

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Showing that x_4 is free variable, so columns of A are linearly dependent. Hence, T is not one to one.

Also,

Since, $Ax = b$ is inconsistent for all $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \in \mathbb{R}^4$, so A is not onto.

13. Hint: See solution. Q. No. 12