Chapter 7

Eigenvalue and Eigenvector

Exercise 7.1

1. **Solution.**

Given, $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$

Now, $Au = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6-30 \\ 30-10 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4u$

Hence, u is eigenvector of A.

2. **Solution.**

Let $A = \begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$Au = \begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -3+4 \\ -3+32 \end{bmatrix} = \begin{bmatrix} 1 \\ 29 \end{bmatrix} \neq \lambda u$

So, u is not eigenvector of A.

3. Let $A = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$, $u = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Now, $Au = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6+4 \\ 4-4 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix} \neq \lambda u$

So, u is not eigenvector of A.

4. Let $A = \begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}$, $u = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$

Now, $Au = \begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 12-21+9 \\ -16+15+1 \\ 8-12+4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0u$

Hence, u is eigenvector of A and 0 is eigen value.

5. Let $A = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$, $\lambda = 1$

Since, $[A - \lambda I \quad 0] = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix}$ Since there is no free variables.

which shows that $Ax = 1x$ has trivial solution.

So, 1 is not eigen value of $\begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$

6. Given, $\lambda = 5$, $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$

we have, $[A - 5I \quad 0]$

$= \begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ There is free variables so

$Ax = 5x$ has non trivial solution. So 5 is eigen value of A. So,

$2x_1 - 4x_2 = 0$

x_2 free

$\therefore \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\therefore \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is eigen vector of A.

7. Given, $\lambda = 3$, $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Here, $[A - 3I \quad 0]$

$= \begin{bmatrix} -2 & 2 & 2 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$ $\quad R_1 \to -\frac{1}{2}R_1$

$\sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$ $\quad R_2 \to R_2 - 3R_1$

$\sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$ $\quad R_2 \to -\frac{1}{2}R_2$

$\sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

which shows that x_3 is free variables so

$Ax = 3x$ has no trivial solution. So 3 is an eigen value of A

and $\quad x_1 - x_2 - x_3 = 0$

$x_2 - 2x_3 = 0$

x_3 free

$\therefore \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 + x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$\therefore \quad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is eigen vector of A.

8. Given, $\lambda = 2, A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$

Here, $[A - 2I \quad 0]$

$$= \begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which shows that x_2 and x_3 are free variables so $Ax = 2x$ has non trivial solution.

So, 2 is eigen value of A.

9. Given, hint is book

10.

a. Given, $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}, \lambda = 1$

Now, $[A - \lambda I \quad 0]$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here x_2 is free variable so

$x_1 = 0$

x_2 free

So $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore$ $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is basis for the eigen space.

b. Given, $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}, \lambda = 4$

Now, $[A - \lambda I \quad 0]$

$$= \begin{bmatrix} 6 & -9 & 0 \\ 4 & -6 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 6 & -9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_2 \to R_2 - \frac{4}{5}R_1$$

$6x_1 - 9x_2 = 0$

x_2 free

$\therefore$ $x = \begin{bmatrix} \frac{9}{6}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} = \frac{x_2}{2} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\therefore$ $\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ is basis for the eigen space.

c. Given, $A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}, \lambda = 10$

Now, $[A - \lambda I \quad 0]$

$$= \begin{bmatrix} -6 & -2 & 0 \\ -3 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 1 & 0 \\ -3 & -1 & 0 \end{bmatrix} \qquad R_1 \to -\frac{1}{2}R_1$$

$$\sim \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $x = \begin{bmatrix} -1/3\,x_2 \\ x_2 \end{bmatrix} = \frac{1}{3}x_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$\therefore$ $\left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$ is basis for eigen space.

d. Given, $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}, \lambda = 2$

Now, $[A - \lambda I \quad 0]$

$$= \begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_2 and x_3 are free

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2}x_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is basis for eigen space.

e. Given, $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}, \lambda = 1, \lambda = 2$

Now, for $\lambda = 1$.

$[A - \lambda I \quad 0]$

$$= \begin{bmatrix} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ x_3 free variables. So,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

∴ $\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}$ is basis for eigen space for $\lambda = 1$.

For $\lambda = 2$,

$$[A - \lambda I \quad 0]$$

$$= \begin{bmatrix} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad R_1 \to -\frac{1}{2}R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$$

x_2 and x_3 is free variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

∴ $\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ is basis for eigen space corresponding to $\lambda = 2$.

11.

a. Given, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{bmatrix}$

Here, $\lambda = 0, 2, -1$ are eigen value.

b. Given, $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

Here, $\lambda = 4, 0, 3$ are eigen value.

c. $\lambda = 3, 2, 1$

12. We have, $Ax = \lambda x$

So, $A^2x = A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x$

Similarly, $A^3x = \lambda^3 x$

13. We have, $A^3x = \lambda^3 x$

$$= (-4)^3 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -64 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = 64 \begin{bmatrix} -6 \\ 5 \end{bmatrix} = \begin{bmatrix} -384 \\ 320 \end{bmatrix}$$

Exercise 7.2

1.

a. Given, $A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

the characteristic polynomial is

$|A - \lambda I|$

$$\begin{vmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{vmatrix}$$

$= 4 + \lambda^2 - 4\lambda - 49$

$= \lambda^2 - 4\lambda - 45$

For eigen value, $|A - \lambda I| = 0$

$\lambda^2 - 4\lambda - 45 = 0$

$\lambda^2 - 9\lambda + 5\lambda - 45 = 0$

$\lambda(\lambda - 9) + 5(\lambda - 9) = 0$

$(\lambda - 9)(\lambda + 5) = 0$

∴ $\lambda = -5, 9$

(b), (c), (d) do similar as (a).

e. Given, $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$

The characteristic polynomial is

$|A - \lambda I|$

$$\begin{vmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -2 & 2-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 2-\lambda & 2 & -1 \\ -1+\lambda & 3-\lambda & 0 \\ -1 & -2 & 2-\lambda \end{vmatrix} \qquad R_2 \to R_2 - R_1$$

$$= 1-\lambda \begin{vmatrix} 2-\lambda & 2 & -1 \\ -1 & 1 & 0 \\ -1 & -2 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 4-\lambda & 2 & -1 \\ 0 & 1 & 0 \\ -3 & -2 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda) \times 1 \times \begin{vmatrix} 4-\pi & -1 \\ -3 & 2-\lambda \end{vmatrix}$$

$= (1-\lambda)(8 + \lambda^2 - 6\lambda - 3)$

$= (1-\lambda)(\lambda^2 - 6\lambda + 5)$

$= (1-\lambda)(\lambda - 1)(\lambda - 5)$

$= (\lambda - 1)^2(\lambda - 5)$

For eigen value, $|A - \lambda I| = 0$

$(\lambda - 1)^2(\lambda - 5) = 0$

∴ $\lambda = 1, 5$

f. Given, $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

For characteristic polynomial

$|A - \lambda I|$

$$= \begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} -1-\lambda & 4 & -2 \\ 0 & 3-\lambda & -3+\lambda \\ -3 & 1 & 3-\lambda \end{vmatrix} \quad R_2 \to R_2 - R_3$$

$$= (3-\lambda) \begin{vmatrix} -1-\lambda & 4 & -2 \\ 0 & 1 & -1 \\ -3 & 1 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda) \begin{vmatrix} -1-\lambda & 4 & 2 \\ 0 & 1 & 0 \\ -3 & 1 & 4-\lambda \end{vmatrix} \quad C_3 \to C_3 + C_2$$

$$= (3-\lambda) \times 1 \times \begin{vmatrix} -1-\lambda & 2 \\ -3 & 4-\lambda \end{vmatrix}$$

$= (3-\lambda)(\lambda^2 - 3\lambda - 4 + 6)$

$= (3-\lambda)(\lambda^2 - 3\lambda + 2)$

$= (3-\lambda)(\lambda - 1)(\lambda - 2)$

For eigen value: $(3-\lambda)(\lambda - 1)(\lambda - 2) = 0$

$\therefore \quad \lambda = 1, 2, 3$

g. Given, $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$

So, characteristic polynomial is

$|A - \lambda I|$

$$= \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ -3 & 5 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda) \begin{vmatrix} -1-\lambda & 0 \\ -3 & 4-\lambda \end{vmatrix}$$

$= (2-\lambda)(-1-\lambda)(4-\lambda)$

For eigen value, $|A - \lambda I| = 0$

$\lambda = -1, 2, 4$

Similarly, for (i) and (j) they are triangular matrix.

4. See in book similar as examples.

5. **Solution.**

Statement: Every square matrix A satisfies its characteristic equation i.e.,

$|A - \lambda I| = 0$

Given,

$A = \begin{bmatrix} 6 & 2 & -1 \\ -6 & -1 & 2 \\ 7 & 2 & -2 \end{bmatrix}$

For characteristic polynomial is $|A - \lambda I|$

$$= \begin{vmatrix} 6-\lambda & 2 & -1 \\ -6 & -1-\lambda & 2 \\ 7 & 2 & -2-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 6-\lambda & 2 & -1 \\ -6 & -1-\lambda & 2 \\ 1+\lambda & 0 & -1-\lambda \end{vmatrix} \quad R_3 \to R_3 - R_1$$

$$= \begin{vmatrix} 5-\lambda & 2 & -1 \\ -4 & -1-\lambda & 2 \\ 0 & 0 & -1-\lambda \end{vmatrix}$$

$$= -1-\lambda \begin{vmatrix} 5-\lambda & 2 \\ -4 & -1-\lambda \end{vmatrix}$$

$= (-1-\lambda)(\lambda^2 - 4\lambda + 3)$

$= -\lambda^3 + 3\lambda^2 + \lambda - 3$

Now, replacing λ by A and show,

$-A^3 + 3A^2 + A - 3I = 0$

Exercise 7.3

1. Given, $P = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

We have, $A^5 = P D^5 P^{-1}$

$$= \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^5 \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 32 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 96 & 4 \\ 32 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -96+4 & 384-12 \\ -32+1 & 128-3 \end{bmatrix}$$

$$= \begin{bmatrix} -92 & 372 \\ -31 & 125 \end{bmatrix}$$

2. Given, $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$

For eigen value, $|A - \lambda I| = 0$

$$\begin{vmatrix} 7-\lambda & 2 \\ -4 & 1-\lambda \end{vmatrix} = 0$$

$\lambda^2 - 8\lambda + 7 + 8 = 0$

$\lambda^2 - 8\lambda + 15 = 0$

$\lambda^2 - 5\lambda - 3\lambda + 15 = 0$

$$\lambda(\lambda - 5) - 3(\lambda - 5) = 0$$
$$(\lambda - 5)(\lambda - 3) = 0$$
$$\therefore \quad \lambda = 3, 5$$

For $\lambda = 3$, corresponding to eigen vector $v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

For $\lambda = 5$, corresponding to eigen vector $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So, $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

So, $A^4 = P D^4 P^{-1}$

$$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^4 & 0 \\ 0 & 3^4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5^4 & 3^4 \\ -5^4 & -2 \times 3^4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 5^4 - 3^4 & 5^4 - 3^4 \\ -2 \times 5^4 + 2 \times 3^4 & -5^4 + 2 \times 3^4 \end{bmatrix}$$

$$= \begin{bmatrix} 1169 & 544 \\ -1088 & -463 \end{bmatrix}$$

3. **See in answer**

4.

a. Given, $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

For eigen value, $|A - \lambda I| = 0$

$$\begin{vmatrix} 1 - \lambda & 0 \\ 6 & -1 - \lambda \end{vmatrix} = 0$$

$$(1 - \lambda)(-1 - \lambda) = 0$$
$$\therefore \quad \lambda = 1, -1$$

For $\lambda = 1$, the eigen vector is

$$[A - \lambda I \quad 0]$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 6 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here x_2 is free variables so

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} x_2 \\ x_2 \end{bmatrix} = \frac{1}{3} x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\therefore \quad v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

For $\lambda = -1$, $\begin{bmatrix} 2 & 0 & 0 \\ 6 & 0 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So, $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

So, $AP = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$

$PD = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$

Since, AP = PD. So A is diagonalizable.

b. Given, $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

For eigen values, $|A - \lambda I| = 0$

$$\begin{vmatrix} 2 - \lambda & 3 \\ 4 & 1 - \lambda \end{vmatrix} = 0$$

$$\lambda^2 - 3\lambda + 2 - 12 = 0$$
$$\lambda^2 - 3\lambda - 10 = 0$$
$$\lambda^2 - 5\lambda + 2\lambda - 10 = 0$$
$$\lambda(\lambda - 5) + 2(\lambda - 5) = 0$$
$$(\lambda - 5)(\lambda + 2) = 0$$
$$\therefore \quad \lambda = -2, 5$$

Since, A is 2×2 matrix and there are two distinct eigen values. So A is diagonalizable.

c. Given, $A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3 - \lambda & -1 \\ 1 & 5 - \lambda \end{vmatrix} = 0$$

$$\lambda^2 - 8\lambda + 15 + 1 = 0$$
$$\lambda^2 - 8\lambda + 16 = 0$$
$$(\lambda - 4)^2 = 0$$
$$\therefore \quad \lambda = 4$$

For eigen vector, $[A - 4I \quad 0]$

$$\begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\therefore$ $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Since, A is 2×2 matrix so we need two linearly independent eigen vectors but here is only one eigen vector.

$\therefore$ A is not diagonalizable.

d. Given, $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

For eigen value, $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda) \begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(\lambda^2 - 3\lambda + 2) = 0$$

$$(2-\lambda)(\lambda - 2)(\lambda - 1) = 0$$

$\therefore$ $\lambda = 1, 2$

For $\lambda = 1$, the eigen vector,

$[A - I \quad 0]$

$$= \begin{bmatrix} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here x_3 is free variable so

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$\therefore$ $v_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = 2$, the eigen vector

$[A - 2I \quad 0]$

$$= \begin{bmatrix} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

So $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ $P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and PD = AP. So A is diagonalizable.

e. Given, $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$

For eigen value,

Since, the matrix is triangular matrix. So

$\lambda = 1, 2, 2$

For eigen vector,

$\lambda = 1$, $[A - I \quad 0]$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & 5 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 8 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 free, $8x_2 = -x_3$ $x_1 + x_2 = 0$

$x_2 = -\frac{1}{8}x_3$ $x_1 = -x_2$, $x_1 = \frac{1}{8}x_3$

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{8}x_3 \\ \frac{-1}{8}x_3 \\ x_3 \end{bmatrix} = \frac{1}{8}x_3 \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}$

$\therefore$ $v_1 = \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}$

For $\lambda = 2$, $[A - 2I \quad 0]$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_3 is free variable,

$$x = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Since, A is 3×3 matrix and there are only two linearly independent eigen vector. So A is not diagonalizable.

f, g, h, i, j do similar as above.

5. Given, $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Here, $Av_1 = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3v_1.$

$\therefore$ $\lambda = 3$ is eigen value corresponding to v_1.

Similarly, $Av_2 = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1\, v_2$

$\lambda = -1$ is eigen value corresponding to v_2.

$\therefore$ $P = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

$$AP = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 6 & -1 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 6 & -1 \end{bmatrix}$$

Since, AP = PD

$\therefore$ A is diagonalizable.

Exercise 7.4

1. Given, $T(b_1) = 3c_1 - 2c_2 + 5c_3$

$\therefore$ $[T(b_1)]_c = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$

Similarly, $[T(b_2)]_c = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$

$\therefore$ $[T]_{B.C} = [[T(b_1)]_c\ [T(b_2)]_c]$

$$= \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$$

2. Similar as 1.

3. Given, $T(x, y) = (y, -5x + 13y, -7x + 16y)$

Let $T(b_1) = a_1 c_1 + a_2c_2 + a_3 c_3$ where, $[T(b_1)]_c = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$T(31) = a_1 (1, 0, -1) + a_2 (-1, 2, 2) + a_3 (0, 1, 2)$

$\therefore$ $(1, -5 \times 3 + 13 \times 1, -7 \times 3 + 16 \times 1) = (a_1 - a_2, 2a_2 + a_3, -a_1 + 2a_2 + 2a_3)$

$\therefore$ $a_1 - a_2 = 1$ .... (1)

$2a_2 + a_3 = -2$ .... (2)

$-a_1 + 2a_2 + 2a_3 = -5$ .... (3)

Solving (1) and (3), we get

$a_2 + 2a_3 = -4$ .... (4)

Using equation (4) and (2)

$$\begin{aligned} 2a_2 + a_3 &= -2 \\ 2a_2 + 4a_3 &= -8 \\ - \quad - \quad & \quad + \\ \hline -3a_3 &= 6 \end{aligned}$$

$a_3 = -2$

$\therefore$ $2a_2 + a_3 = -2$

$2a_2 - 2 = -2$

$a_2 = 0$

$\therefore$ $a_1 = 1$

Hence, $[T(b_1)]_c = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

Similarly we get $[T(b_2)]_c = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

$\therefore$ $[T]_{B.C} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$

4. Given, $T(x, y) = \begin{pmatrix} x - y \\ x + y \end{pmatrix}$

Let $T(b_1) = c_1b_1 + c_2b_2$

$T(1, 1) = c_1 (1, 1) + c_2 (-1, 0)$

$(0, 2) = (c_1 - c_2, c_1)$

$\therefore$ $c_1 = 2, \quad c_1 - c_2 = 0$

$c_2 = 2.$

So, $[T(b_1)]_B = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

Similarly, $[T(b_2)]_B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\therefore$ $[T]_B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$

5. Given, $T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$

So $T(1) = T(1 + 0 . t + 0 . t^2) = 0$

$T(t) = T(0 + 1 . t + 0 . t^2) = 1 + 0 = 1$

$T(t^2) = T(0 + 0 . t + 1 . t^2) = 0 + 2 . 1 . t = 2t$

$\therefore$ $[T]_B = [[T(1)]_B\ [T(t)]_B\ [T(t^2)]_B]$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

6. Given, $T(P(t)) = P(t) + t^2 P(t)$

So, $T(2 - t + t^2) = 2 - t + t^2 + t^2(2 - t + t^2)$

$= 2 - t + t^2 + 2t^2 - t^3 + t^4$

$= 2 - t + 3t^2 - t^3 + t^4$

For next,

Let $B = \{1, t, t^2\}$, $C = \{1, t, t^2, t^3, t^4\}$

Then,

$T(1) = 1 + t^2 . 1 = 1 + t^2 = 1 . 1 + 0 . t + 1 . t^2 + 0 . t^3 + 0 . t^4$

$$\therefore\ [T(1)]_c = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$T(t) = t + t^2 . t = 0 . 1 + 1 . t + 0 . t^2 + 1 . t^3 + 0 . t^4$

$$\therefore\ [T(t)]_c = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$T(t^2) = t^2 + t^2 . t^2$

$= t^2 + t^4$

$= 0 . 1 + 0 . t + 1 . t^2 = 0 . t^3 + 1 . t^4$

$$\therefore\ [T(t^2)]_c = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore\ [T]_{B.C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7. Given, $T(a_0 + a_1 t + a_2 t^2) = 3a_0 + (5a_0 - 2a_1) t + (4a_1 + a_2) t^2$

$T(1) = 3 . 1 + (5 . 1 - 2 . 0) t + (4 . 0 + 0) t^2$

$= 3 + 5t$

$= 3 + 5t + 0 t^2$

$$\therefore\ [T(1)]_B = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$$

$T(t) = 3 . 0 + (5 . 0 - 2 . 1) t + (4 . 1 + 0) t^2$

$= - 2t + 4t^2$

$= 0 - 2t + 4t^2$

$$\therefore\ [T(t)]_B = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}$$

$T(t^2) = 3 . 0 + (5 . 0 - 2 . 0) t + (4 . 0 + 1) t^2$

$= t^2$

$= 0 + 0 . t + 1 . t^2$

$$\therefore\ [T(t^2)]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ $[T]_B = [\{T(1)]_B\ [T(t)]_B\ [T(t^2)]_B]$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 5 & -2 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

8.

i. Given, $A = \begin{bmatrix} 5 & -3 \\ -7 & 1 \end{bmatrix}$

We know, $D = [T]_B = P^{-1} AP$, where $B = [b_1\ \ b_2]$ is basis for R^2.

So $P = [b_1\ \ b_2]$

The characteristic equation

$|A - \lambda I| = 0$

$$\begin{vmatrix} 5 - \lambda & -3 \\ -7 & 1 - \lambda \end{vmatrix} = 0$$

$\lambda^2 - 6\lambda - 16 = 0$

$\lambda^2 - 8\lambda + 2\lambda - 16 = 0$

$\lambda(\lambda - 8) + 2(\lambda - 8) = 0$

$(\lambda - 8)(\lambda + 2) = 0$

$\therefore$ $\lambda = -2, 8$

For $\lambda = -2$,

$[A + 2I\ \ 0]$

$$= \begin{bmatrix} 7 & -3 & 0 \\ -7 & 3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 7 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $7x_1 - 3x_2 = 0 \Rightarrow x_1 = \frac{3x_2}{7}$, x_2 free

$$\therefore\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} x_2 \\ x_2 \end{bmatrix} = \frac{x_2}{7} \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

So, $b_1 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

Similarly, for $\lambda = 8$

$[A - 8I\ \ 0]$

$$= \begin{bmatrix} -3 & -3 & 0 \\ -7 & -7 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -3 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $x_1 = -x_2$, x_2 free

$\therefore$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\therefore$ $B = \left\{ \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is basis for R^2.

ii. Similar as (i).

9. Given, $A = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}$, $b_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

We have, $P = [b_1 \ \ b_2] = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

$\therefore$ We have,

B-matrix = D = P^{-1} AP

$$= \frac{1}{5} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 11 \\ -1 & -3 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 5 & 25 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

10. **Similar as 9.**

Exercise 7.5

1.

i. Given, $A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$\lambda^2 - 4\lambda + 3 + 2 = 0$

$\lambda^2 - 4\lambda + 5 = 0$

$\therefore$ $\lambda = 2 \pm i$

For $\lambda = 2 - i$

$(A - \lambda I)x = 0$

$$\begin{pmatrix} -1+i & -2 \\ 1 & 1+i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$(-1 + i)x_1 - 2x_2 = 0$ .... (i)

$x_1 + (1 + i)x_2 = 0$ .... (ii)

Since both equations are identical. So from equation (ii)

$x_1 = -(1 + i)x_2$

Let $x_2 = -1$, then $x_1 = 1 + i$

$\therefore$ $x = \begin{bmatrix} 1+i \\ -1 \end{bmatrix}$ similarly for $\lambda = 2 + i$, we get $x = \begin{bmatrix} 1-i \\ -1 \end{bmatrix}$

Hence, $\lambda = 2 \pm i$, $x = \begin{bmatrix} 1 \pm i \\ -1 \end{bmatrix}$

Similarly solve for (ii) and iii)

(ii) $\lambda = 2 \pm 3i$, $\begin{bmatrix} 1 \pm 3i \\ 2 \end{bmatrix}$

(iii) $\lambda = 2 \pm 2i$, $\begin{bmatrix} 1 \\ 2 \pm 2i \end{bmatrix}$

2.

i. Given, $A = \begin{pmatrix} 5 & -2 \\ 1 & 3 \end{pmatrix}$

For eigen value,

$|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$\lambda^2 - 8\lambda + 15 + 2 = 0$

$\lambda^2 - 8\lambda + 17 = 0$

$\therefore$ $\lambda = \dfrac{8 \pm \sqrt{(-8)^2 - 4.1.17}}{2 \times 1}$

$\lambda = \dfrac{8 \pm \sqrt{-4}}{2}$

$\therefore$ $\lambda = 4 \pm i$

For basis, $\lambda = 4 + i$

$(A - \lambda I)x = 0$

$$\begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(1 - i)x_1 - 2x_2 = 0$ .... (i)

$x_1 - (1 + i)x_2 = 0$ .... (ii)

both are identical equation so

$x_1 = (1 + i)x_2$

Let $x_2 = 1$ then $x_1 = 1 + i$

$\therefore$ $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ is basis corresponding to $\lambda = 4 + i$.

Similarly, we get $\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$ fro $\lambda = 4 - i$.

ii. Given, $A = \begin{pmatrix} 1.52 & -0.7 \\ 0.56 & 0.4 \end{pmatrix}$

For eigen value,

$|A - \lambda I| = 0$

$\begin{bmatrix} 1.52-\lambda & -0.7 \\ 0.56 & 0.4-\lambda \end{bmatrix} = 0$

$\lambda^2 - 1.92\lambda + 0.608 + 0.392 = 0$

$\lambda^2 - 1.92\lambda + 1 = 0$

$\therefore \quad \lambda = \dfrac{1.92 \pm \sqrt{(-1.92)^2 - 4.1.1}}{2.1}$

$\lambda = \dfrac{1.92 \pm 0.56\,i}{2}$

$\therefore \quad \lambda = 0.96 \pm 0.28\,i$

For $\lambda = 0.96 + 0.28\,i$, $(A - \lambda I)\,x = 0$

$\begin{pmatrix} 0.56-0.28\,i & -0.7 \\ 0.56 & -0.56-0.28\,i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$(0.56 - 0.28i)\,x_1 - 0.7\,x_2 = 0$ .... (i)

$0.56x_1 - (0.56 + 0.28i)\,x_2 = 0$ .... (ii)

Both are identical equation so

$0.56\,x_1 = (0.56 + 0.28i)\,x_2$

Let $x_2 = 2$ then $x_1 = 2 + i$

$\therefore \begin{bmatrix} 2+i \\ 2 \end{bmatrix}$ is basis for $\lambda = 0.96 + 0.28i$

Similarly $\begin{bmatrix} 2-i \\ 2 \end{bmatrix}$ is basis for $\lambda = 0.96 - 0.28i$

iii. Similar as (ii)

$\lambda = -0.6 + 0.8i$, $\begin{bmatrix} 2+i \\ 5 \end{bmatrix}$, $\lambda = -0.6 - 0.8\,i$, $\begin{bmatrix} 2-i \\ 5 \end{bmatrix}$

3.

i. Given, $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$

For eigen value, $|A - \lambda I| = 0$

$\begin{vmatrix} 5-\lambda & -5 \\ 1 & 1-\lambda \end{vmatrix} = 0$

or $\quad \lambda^2 - 6\lambda + 5 + 5 = 0$

$\lambda^2 - 6\lambda + 10 = 0$

$\lambda = \dfrac{6 \pm \sqrt{(-6)^2 - 4.1.10}}{2}$

$\lambda = 3 \pm i$

For $\quad \lambda = 3 - i$

We have, $\quad a = 3, b = 1$

$C = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

For eigen vector corresponding to $\lambda = 3 - i$.

$(A - \lambda I)\,x = 0$

$\begin{pmatrix} 2+i & -5 \\ 1 & -2+i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$(2 + i)\,x_1 - 5x_2 = 0$

$x_1 + (-2 + i)\,x_2 = 0$

both are identical equation so

$x_1 = -(-2 + i)\,x_2$

Let $x_2 = 1$ then $x_1 = 2 - i$

$\therefore \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$

$\therefore \quad P = [\text{Re } x \quad \text{Im } x] = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

For check

$AP = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 3 & -1 \end{bmatrix}$

$PC = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 3 & -1 \end{bmatrix}$

ii. Given, $A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$

For eigen value, $|A - \lambda I| = 0$

$\begin{vmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$\lambda^2 - 8\lambda + 15 + 2 = 0$

$\lambda^2 - 8\lambda + 17 = 0$

$\therefore \quad \lambda = 4 \pm i$

For $\lambda = 4 - i$, $C = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$

For $\lambda = 4 - i$ we get basis

$x = \begin{bmatrix} 1-i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$

$\therefore \quad P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

So $\quad AP = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & -1 \end{bmatrix}$

$PC = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & -1 \end{bmatrix}$

iii. Similar as above we get

$P = \begin{bmatrix} 2 & -1 \\ 5 & 0 \end{bmatrix}$, $C = \begin{bmatrix} -0.6 & -0.8 \\ 0.8 & -0.6 \end{bmatrix}$

iv. Similarly as above we get

$$P = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix},\ C = \begin{bmatrix} 0.96 & -0.28 \\ 0.28 & 0.96 \end{bmatrix}$$

Exercise 7.6

1. Given,

$$A = \begin{bmatrix} 0.5 & 0.4 \\ -p & 1.1 \end{bmatrix}, \text{ where } p = 0.2$$

For eigen values, $|A - \lambda I| = 0$

$$\begin{vmatrix} 0.5-\lambda & 0.4 \\ -0.2 & 1.1-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $\lambda^2 - 1.6\lambda + 0.55 + 0.08 = 0$

$\Rightarrow$ $\lambda^2 - 1.6\lambda + 0.63 = 0$

$\lambda^2 - 0.9\lambda - 0.7\lambda + 0.63 = 0$

$\Rightarrow$ $(\lambda - 0.9)(\lambda - 0.7) = 0$

$\therefore$ $\lambda = 0.9, 0.7$

For $\lambda_1 = 0.9$ $[A - \lambda I \quad 0]$

$$= \begin{bmatrix} -0.4 & 0.4 & 0 \\ -0.2 & 0.2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -0.4 & 0.4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $x = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$ $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding $\lambda_1 = 0.9$.

For $\lambda_2 = 0.7$, we have, $[A - \lambda I \quad 0]$

$$= \begin{bmatrix} -0.2 & 0.4 & 0 \\ -0.2 & 0.4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -0.2 & 0.4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $x = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\therefore$ $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is eigen vector corresponding to $\lambda_2 = 0.7$.

Hence, we have

$x_k = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2$

$$x_k = c_1 (0.9)^k \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 (0.7)^k \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

as $k \to 0$ then $(0.9)^k \to 0$ and $(0.7)^k \to 0$. So,

$x_k \to 0$ so owl population decline.

ii. The rat population is also decline?

2. Similar as 1.

3. **Solution.** The initial $x_0 = c_1 v_1 + c_2 v_2 + c_3 v_3$

So,

$$\begin{bmatrix} 1 \\ 11 \\ -2 \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Comparing and solving we get $c_1 = 2$, $c_2 = 1$, $c_3 = 3$ and we have,

$x_k = c_1 (\lambda_1)^k v_1 + c_2 (\lambda_2)^k v_2 + c_3 (\lambda_3)^k v_3$

$$x_k = 2 \cdot (1)^k \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + 1 \cdot \left(\frac{2}{3}\right)^k \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 3 \cdot \left(\frac{1}{3}\right)^k \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$\therefore$ The general solution is

$$x_k = 2 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + \left(\frac{2}{3}\right)^k \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 3 \cdot \left(\frac{1}{3}\right)^k \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

as $k \to \infty$, then we get $\left(\frac{2}{3}\right)^k \to 0$, $\left(\frac{1}{3}\right)^k \to 0$. So $x_k = 2 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$.

4. **Solution.** Given, $A = \begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix}$

For eigen value, $|A - \lambda I| = 0$

$$= \begin{vmatrix} 4-\lambda & -5 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$\lambda^2 - 5\lambda - 6 = 0$ $\therefore$ $\lambda = 6, -1$

The eigen vector corresponding to $\lambda_1 = 6$ is $v_1 = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$

The eigen vector corresponding to $\lambda_2 = -1$ is $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Let $x_1(t) = v_1 e^{\lambda_1 t}$ and $x_2(t) = v_2 e^{\lambda_2 t}$ be eigen functions satisfy the differential equations $x' = Ax$, so their linear combination.

$\therefore$ $x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$

$$x(t) = c_1 \begin{pmatrix} -5 \\ 2 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

Since, $x_0 = \begin{bmatrix} 2.9 \\ 2.6 \end{bmatrix}$, i.e., $x(0) = x_0$. So,

$$\begin{bmatrix} 2.9 \\ 2.6 \end{bmatrix} = c_1 \begin{pmatrix} -5 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

or $-5c_1 + c_2 = 2.9 \quad \ldots. (1)$

$2c_1 + c_2 = 2.6 \quad \ldots. (2)$

Solving (1) and (2) we get

$$c_1 = \frac{-3}{70}, \; c_2 = \frac{188}{70}$$

$$x(t) = \frac{-3}{70} \begin{pmatrix} -5 \\ 2 \end{pmatrix} e^{6t} + \frac{188}{70} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

or

$$\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} \frac{15}{70} e^{6t} + \frac{188}{70} e^{-t} \\ \frac{-6}{70} e^{6t} + \frac{188}{70} e^{-t} \end{pmatrix}$$