Chapter 9

Group and Subgroups

Exercise 9.1

What three things must we check to determine whether a function $\phi : S \to S'$ is an isomorphism of a binary structure?

olution. To determine whether a function ϕ: $S \to S'$ $<S, *>$ with $<S', *'>$, is an isomorphism: we have to check the following three conditions

(i) Does ϕ is one-to-one function? That is, suppose that $\phi(x) = \phi(y)$ in S' and deduce from this $x = y$ in s.

(ii) Does ϕ is onto S'? That is suppose that $s' \in S'$ is given and show that there does exist $s \in S$ such that $\phi\begin{pmatrix} s \\ x \end{pmatrix} = s'$.

(iii) Show that if $\phi(x * y) = \phi(x) * \phi(y)$ for all $x, y \in S$. This is just a question of computation. Compute both sides of the equation and see whether they are the same.

$<\mathbb{Z}, +>$ with $<\mathbb{Z}, +>$ where $\phi(x) = -n$ for $n \in \mathbb{Z}$.

Step 1: ϕ is one to one: for any $x, y \in \mathbb{Z}$ with $\phi(x) = \phi(y)$

$\Rightarrow -x = -y \Rightarrow x = y$: Hence ϕ is one to one.

Step 2: ϕ is on to: for any $y \in \mathbb{Z}$ such that $\phi(-y) = -(-y) = y$

$\therefore$ ϕ is on to [for any $y \in \mathbb{Z}$ there is $-y \in \mathbb{Z}$. $\phi(-y) = -((-y)) = y$.

Step 3: Now $\phi(x + y) = -(x + y) = (-x) + (-y) = \phi(x) + \phi(y)$

Then, ϕ is isomorphism.

$<\mathbb{Z}, +>$ with $<\mathbb{Z}, +>$ where $\phi(x) = 2x$ for $x \in \mathbb{Z}$.

ϕ is one-to-one: Let $x \in \mathbb{Z}$, then

$\phi(x) = 2x$ and $y \in \mathbb{Z}$. $\phi(y) = 2y$

Such that $\phi(x) = \phi(y)$

$\Rightarrow 2x = 2y$

$\Rightarrow x = y$

$\therefore$ $\phi(x) = \phi(y) \Rightarrow x = y$

Thus ϕ is one to one.

ii. ϕ is one to : $\forall x = n \in \mathbb{Z}$ the number $\frac{n}{2}$ may not be integer $\phi\left(\frac{n}{2}\right) = 2 \times \frac{n}{2} = n$.

Thus ϕ is not on to and hence ϕ is not isomorphism.

c. $<\mathbb{Z}, +>$ and $<\mathbb{Z}, +>$ where $\phi(n) = n + 1$

i. $\phi(x) = \phi(y)$ for $x, y \in \mathbb{Z}$

$\Rightarrow x + 1 = y + 1$

$\Rightarrow x = y$ $\therefore$ ϕ is one to one

ii. ϕ is onto: for every $x \in \mathbb{Z}$ there exists

$x - 1 \in \mathbb{Z}$ such that $\phi(x - 1) = x - 1 + 1 = x$

$\therefore \phi$ is onto

iii. For $x, y \in \mathbb{Z}$ and $\phi(x + y) = x + y - 1 \neq x - 1 + y - 1 = \phi(x) + \phi(y)$

Then, $\phi(x + y) \neq \phi(x) + \phi(y)$

$\therefore$ ϕ is not homomorphism and ϕ is not isomorphism.

d. $<Q, +>$ when $<Q, +>$ when $\phi(x) = \frac{x}{2}$ for $x \in Q$.

i. For $x, y, \in Q$ with $\phi(x) = \phi(y)$

$\Rightarrow \frac{x}{2} = \frac{y}{2}$

$\Rightarrow x = y$

$\therefore$ ϕ is one to one.

ii. For $y \in Q$ there exists $2y \in Q$ such that $\phi(2y) = \frac{2y}{2} = y$. Then ϕ is on to.

iii. Let $\phi(x + y) = \frac{x + y}{2} = \frac{x}{2} + \frac{y}{2} = \phi(x) + \phi(y)$

Hence ϕ is isomorphism.

e. $<Q, .>$ and $<Q, .>$ where $\phi(x) = x^2$ for $x \in Q$.

i. For $x, y \in Q$ with $\phi(x) = \phi(y)$

$\Rightarrow$ $x^2 = y^2$ may not be $x = y$, Since $x^2 = y^2$ is true,

also for $-x = -y, -x = y, x = -y$.

Then ϕ is not one to one and is not isomorphism.

f. $<\mathbb{R}, .>$ with $<\mathbb{R}, .>$ where $\phi(x) = x^3$ for $x \in \mathbb{R}$.

i. For $x, y \in \mathbb{R}$ with $\phi(x) = \phi(y)$

$x^3 = y^3$ and $x, y \in \mathbb{R}$

$\Rightarrow$ $x = y$ or $x^2 - xy + y^2 = 0$

$\because$ ϕ is not isomorphism, but not possible in general.

Then ϕ is one to one.

ii. $\forall$ real number $x \in \mathbb{R}$ there is $x^{\frac{1}{3}} \in \mathbb{R}$. Such that $\phi(x^{1/3}) = (x^{1/3})^3 = x$

Then ϕ is on to.

iii. For $x, y \in \mathbb{R}$. Then $\phi(xy) = (xy)^3$

$$= x^3 y^3$$

$$= \phi(x)\, \phi(y)$$

Then ϕ is an isomorphism

g. $<M_2(\mathbb{R}), .>$ with $<\mathbb{R}, .>$ where $\phi(A)$ is the determinant of matrix A.

i. For A, B $\in M_2(R)$ with $\phi(A) = \phi(B) \Rightarrow |A| = |B|$

It is not necessary to the matrix are equal for equal determinant. We can find different matrices with same determinant. Hence ϕ is not one to one and ϕ is not isomorphism.

h. $<M_1(R), .>$ where $\phi(A)$ is the determinant of matrix A.

. For one to one, let A, B $\in M_1(R)$ with $\phi(A) = \phi(B) \Rightarrow |A| = |B|$, A = B, if the determinant of $|X|$ matrices are equal then the matrices must be equal.

i. For even $x \in R$ there exists a matrix. $A = [x]$ such that $\phi(A) = |x| = x$. Then ϕ is on to.

ii. For A, B $\in M_1(R)$ with $\phi(AB) = |AB| = AB = |A|\,|B| = \phi(A)\,\phi(B)$

Here, both A and B are 1×1 matrices and AB is also 1×1 matrix.

Then ϕ is isomorphism.

i). $<\mathbb{R}, +>$ with $<\mathbb{R}^+, .>$ where $\phi(r) = \left(\frac{1}{2}\right)^r$ for $r \in \mathbb{R}$

. Let $x, y \in \mathbb{R}$

Then $\phi(x) = \phi(y)$

$\Rightarrow$ $\left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^y$

$\Rightarrow$ $x = y$ and $\phi(x) = \phi(y) \Rightarrow x = y$

i. For $y \in \mathbb{R}^+$ there is $x \in \mathbb{R}$ such that $\phi(x) = y$, $\left(\frac{1}{2}\right)^x = y$ is true only for $x = 0 \in R$, $y \in^+ \mathbb{R}$. Hence ϕ is not isomorphism

3. Determine whether the binary operation * gives a group structure on the given set. If no group results, give the first axioms in order G_0, G_1, G_2, G_3 from definition that does not hold.

a. Let $*$ be defined on $\mathbb{Z}$ by letting $a * b = ab$.

Closure G_0: For any integers a and b the product ab is again an integer hence $a * b \in G_0$.

Associative G_1: For any integers $a, b, c \in \mathbb{Z}$.

$(a * b) * c = a * (b * c)$ is true; since multiplication of integers are always associative.

Existence of identity G_2: For any integers $a \in \mathbb{Z}$ there exists unique integer $1 \in \mathbb{Z}$ such that $a * 1 = a = 1 * a$.

Existence of inverse G_3: For any integers are $a \in \mathbb{Z}$ the integer $\frac{1}{a}$ in some cases may not be integer or not defined particularly for $a = 0$. Hence G_3 is not hold. Hence $(G, *)$ is not group.

b. Let $*$ be defined on $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ by letting $a * b = a + b$.

Solution.

G_0: For any integers n the integer 2n is always even then for any $a = 2n_1$ and $b = 2n_2$: $a * b = 2n_1 + 2n_2 = 2(n_1 + n_2)$ is again even integer. Thus $a * b \in$ G.

For G_1: Now for any three even integers $a = 2n_1$, $b = 2n_2$, $c = 2n_3$

Then, $(a * b) * c = (2n_1 * 2n_2) * 2n_3$

$$= (2n_1 + 2n_2) * 2n_3 = 2n_1 + 2n_2 + 2n_3 = 2(n_1 + n_2 + n_3)$$

Again, $a * (b * c) = (2n_1) * (2n_2 * 2n_3) = 2n_1 * (2n_2 + 2n_3)$

$$= 2n_1 + 2n_2 + 2n_3$$

$$= 2(n_1 + n_2 + n_3)$$

$\therefore$ $(a * b) * c = a * (b * c)$. Thus, G_1 is hold.

G_2 : For any integer $a = 2n_1 \in G$ there exists $e = 0 = 2.0 \in G$ such that $a * e = a + e = 2n_1 + 0 = 2n_1 = a$. Thus, G_2 hold.

G_3 : For any integer $a = 2n_1 \in G$ there exists unique integer $a' = -2n_1$ such that $a * a' = a + a' = 2n_1 - 2n_1 = 0$. Thus, G_3 is also hold. $\therefore$ G is group with the binary operation $a * b = a + b$.

c. Let $*$ be defined on $\mathbb{R}^+$ by letting $a * b = \sqrt{ab}$.

Solution.

G_0: For any positive real numbers $a, b \in \mathbb{R}^+$, $a*b \in \mathbb{R}^+$.

Now,

G_1: For any $a, b, c \in \mathbb{R}^+$ then

$$a * (b * c) = a * \sqrt{bc} = \sqrt{a\sqrt{bc}}$$

and $(a * b) * c = \sqrt{ab} * c = \sqrt{\sqrt{ab}\, c}$

Since, $\sqrt{a\sqrt{bc}} \neq \sqrt{\sqrt{ab}\, c}$. Thus, elements of $\mathbb{R}^+$ are not associative under the binary operation '$*$'.

Thus, $\mathbb{R}^+$ is not group with respect to binary operation $*$.

d. Let $*$ be defined on Q by letting $a * b = ab$.

Solution.

G_0: For any rational number a, b

$a * b = ab$ is also rational i.e. $a * b \in Q$.

G_1: For any rational numbers $a, b, c \in Q$. Then,

$$(a * b) * c = (ab) * c = abc$$

$$a * (b * c) = a * (bc) = abc$$

which shows that elements of Q are associative.

G_2: For any $a \in Q$ there exists unique rational numbers $1 \in Q$, such that, $a * 1 = 1 * a = a\,1 = a$

G_3: For any $a \in Q$ there exists unique element $b = \frac{1}{a} \in Q$ for $a \neq 0$, Such that

$$a * b = a * \frac{1}{a} = 1$$

But for $a = 0$, $b = \frac{1}{a}$ is not defined, so multiplicative inverse for zero is not defined.

Hence, G is not a group.

Let $*$ be defined on the set $\mathbb{R}^+$ of non-zero real numbers by letting $a*b = \frac{a}{b}$.

olution.

G_0: For any two non-zero real numbers $a, b : a * b = \frac{a}{b}$ is also non zero positive real numbers.

$\therefore \quad a * b \in \mathbb{R}$

G_1: For any three non-zero positive real numbers $a, b, c \in \mathbb{R}$: $(a * b) * c = \left(\frac{a}{b}\right) * c = \frac{\frac{a}{b}}{c} = \frac{a}{bc}$

Again, $a * (b * c) = a * \left(\frac{b}{c}\right) = \frac{a}{\frac{b}{c}} = \frac{a\,c}{b}$

$\therefore \quad (a * b) * c \neq a * (b * c) \qquad \therefore$ Elements of $\mathbb{R}^*$ are not associative under binary operation $*$.

$\therefore \quad \mathbb{R}^*$ is not a group with binary operation $*$.

f. Let $*$ be defined on $\mathbb{C}$ by letting $a * b = |ab|$

Solution. G_0: For any two complex number a, b. Then,

$a * b = |a\,b| \qquad |ab|$ is also complex number. (Every real number is a complex number)

G_1: For any complex numbers a, b, c. Then,

$$(a * b) * c = |ab| * c = ||ab|c| = |a|\,|b|\,|c|$$

Again,

$$a * (b * c) = a * |bc| = |a\,|bc|\,| = |a|\,|b|\,|c|$$

Thus, elements of $\mathbb{C}$ are associative.

G_2: For any $a \in \mathbb{C}$ there is an element $1 \in \mathbb{C}$. Such that

$a * 1 = |a| = a$ is not in general.

Hence existence of identity element is not true. Then G is a group.

Exercise 9.2

Determine whether the given subset of the complex numbers is a subgroup of the group $\mathbb{C}$ of complex numbers under addition.

1. $\mathbb{R}$: Here $\mathbb{R}$ is a non-empty subset of the group $\mathbb{C}$.
   Closure: Sum of any two real numbers is again a real number.
   Existence of Additive Identity: For any real number $x \in R$ there exists $0 \in R$ such that $x + 0 = 0 + x = x$.
   Existence of additive inverse: For any real number $x \in R$ there exists $-x \in R$ such that $x + (-x) = 0 = (-x) + x$.
   Hence R is subgroup of $\mathbb{C}$.

2. Q^+

Here Q^+ is set of all positive rational numbers is non-empty: sum of two positive rational number is positive rational number.

Existence of additive identity: For any $x \in Q^+$

$0 \notin Q^+$ such that $0 + x = x + 0 = x$

$\therefore$ Q^+ is not subgroup of $\mathbb{C}$.

3. $7\mathbb{Z} = \{0, 7, 14, ...\} \cup \{-7, -14, ...\}$

$= \{...., -14, -7, 0, 7, 14, ...\}$

Closure: For each elements $x = 7n_1$ and $y = 7n_2$, where $n_1, n_2 \in \mathbb{Z}$. The sum $x + y = 7n_1 + 7n_2 = 7(n_1 + n_2) \in 7\mathbb{Z}$

Existence of identity element: For each $x = 7n \in 7\mathbb{Z}$ there exists an element $0 = 7.0 \in 7\mathbb{Z}$ such that $0 + x = 0 + 7n = 7n = x + 0$

Existence of inverse: For each element $x = 7n \in 7\mathbb{Z}$ there exists an element $-7n = -x \in 7\mathbb{Z}$ such that $x + (-x) = 7n - 7n = 0$

$$-x + x = -7n + 7n = 0$$

Thus, $7\mathbb{Z}$ form a sub-group under addition.

4. The set $i\mathbb{R}$ of pure imaginary numbers including 0.

Solution. $H = \{ix, x \in \mathbb{R}\} \cup \{0\}$

Closure: Sum of two pure imaginary numbers including 0 are either 0 or pure imaginary. Hence elements of $i\mathbb{R}$ are closed under addition.

Existence of additive identity: For each element $x \in H\ \{0\}$. There exists $0 \in H\ \{0\}$. Such that, $x + 0 = 0 + x = x$.

Existence of additive inverse: For each $x \in H$ there exists $-x \in H$. Such that $x + (-x) = (-x) + x = 0$. Thus, H is a subgroup of $\mathbb{C}$.

5. The set πQ of rational multiples of π.

Solution. Let $H = \{\pi x : x$ is rational numbers$\}$

Since $0 \in Q$. Hence $0 \in H$.

i. Closure: For any elements $x, y \in H$

Then $x = \pi n_1$ and $y = \pi n_2$ where $n_1, n_2 \in Q$. Then $x + y = \pi(n_1 + n_2) \in \pi Q = H$.

ii. Existence of additive identity: For each $x \in \pi Q$ there is $0 \in \pi Q$. Such that $x + 0 = 0 + x = x$.

iii. Existence of additive inverse: For each $x \in \pi Q$ say $x = \pi n$, $n \in Q$ there exists $y = -\pi n$, $-n \in Q$ such that

$x + y = 0 = y + x$

$\therefore$ πQ is a group under addition.

6. The set $G = \{\pi^n \mid n \in \mathbb{Z}\}$

Here for each $x = \pi^{n_1}$

$y = \pi^{n_2}$ where $n_1, n_2 \in \mathbb{Z}$

$x + y = \pi^{n_1} + \pi^{n_2}$ may not in the form of π^n. Hence, G is not a sub group under addition.

7. Which of the sets from 1 to 6 are subgroup of $\mathbb{C}^*$ (of non-zero complex number) under multiplication.

Solution. $\mathbb{Q}^+$ is subgroup of $\mathbb{C}^*$ under multiplication and all other are not.

Determine whether the given set of invertible n×n matrices with real entries is a subgroup of GL(n, R).

8. The n×n matrices with determinant 2.

Solution. Here $G = \{A : A$ is n×n invertible matrix with real entries$\}$

$H = \{B : B$ is n×n invertible matrix with real entries of determinant $2\}$

Closure: For B_1 and $B_2 \in H$ then $|B_1| = 2$, $|B_2| = 2$

Then $|B_1 B_2| = |B_1||B_2| = 2 \times 2 = 4$

Thus, element of H are not closed under multiplication of matrices.

9. The diagonal n×n matrices with no zeros on the diagonal.

Solution. (i) Let n = 3 and $H = \{A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$; $a \neq 0, b \neq 0, c \neq 0$.

Let A, B, $\in$ H then,

$$A + B = \begin{bmatrix} a + x & 0 & 0 \\ 0 & b + y & 0 \\ 0 & 0 & c + z \end{bmatrix}$$

Where $B = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ $x \neq 0, y \neq 0, z \neq 0$, here A + B may have zero in its main diagonal.

$a + x, b + y, c + z$ may be zero.

Hence, H is not closed under addition and hence is not subgroup of G (n, $\mathbb{R}$).

(ii) If we consider the binary operation is multiplication of matrices then

Closure, $AB = \begin{bmatrix} ax & 0 & 0 \\ 0 & by & 0 \\ 0 & 0 & cz \end{bmatrix} \in H$. Since $ax, by, cz \neq 0$.

Existence of multiplicative identity: There exists a matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Such that $AI = IA = A$.

Existence of multiplicative inverse: For any $A \in H$ there exists,

$$A^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix} \in H$$

Such that $AA^{-1} = A^{-1}A = I$

Thus H is subgroup under multiplication.

10. The upper-triangular n×n matrices with no zeros on the diagonal.

Solution. $H = \{A : A$ is upper triangular with no zeros in diagonal$\}$

Existence of identity: Here, the identity matrix of n×n order is an upper triangular with no zeros in it's diagonal is an element of H. Thus, H has identity element.

Closure: Let A, B $\in$ H. Then both A and B are upper triangular with no zeros in their diagonal. Now, AB is being product of two upper triangular matrices is again upper triangular with no zero in it's diagonal.

$\therefore$ AB $\in$ H

Existence of Inverse: Since, the determinant of upper triangular matrix is just product of main diagonal elements; here each diagonal elements are non-zero. The matrix are invertible and hence each of them has multiplicative inverse in H.

Thus, H is subgroup of G.

11. The n×n matrices with determinant –1.

$$H = \left[A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = -1 \right]$$

we consider 2×2 matrices.

Let $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} : eh - gf = -1$

Then $|AB| = |A|\ |B| = 1$

The elements of H are not closed under multiplication. Thus H is not subgroup.

12. $H = \{A: |A| \text{ is} - 1 \text{ or } 1\}$

Solution. Here, H is non empty. Since the identity matrix of order n×n is in H with determinant 1, which play the role of identity element in H.

Closure: For any A, B ∈ H then both A and B are non singular with their determinant 1 or –1.

$$|AB| = |A|\,|B| = \begin{cases} -1 \times 1 = -1 \text{ if } |A| = -1 \text{ and } |B| = 1 \\ -1 \times -1 = 1 \text{ if } |A| = -1 \text{ and } |B| = -1 \\ 1 \times -1 = -1 \text{ if } |A| = 1 \text{ and } |B| = -1 \\ 1 \times 1 = 1 \text{ if } |A| = 1,\ |B| = 1 \end{cases}$$

In all cases $|AB| = 1$ or -1 hence $AB \in H$

Existence of Inverse: Since the determinant of all $A \in H$ is either –1 or 1 in any cases A is non singular and hence A^{-1} exists.

Now, $AA^{-1} = I \Rightarrow |AA^{-1}| = |I| \Rightarrow |A|\ |A^{-1}| = 1$. if $|A| = 1$ then $|A^{-1}| = 1$, if $|A| = -1$ then $|A^{-1}| = -1$

∴ $A^{-1} \in H$. Hence H is subgroup of G.

13–20. Let F be the set of all real valued functions with domain ℝ and let $\tilde{F}$ be the subset of F consisting of those functions that have a non-zero value at every point in ℝ. In exercises 14 through 19, determine whether the given subset of F with the induced operations is (a) a subgroup of group F under addition, (b) a subgroup of the group $\tilde{F}$ under multiplication.

13. F

Solution. F = {f : f is real valued function whose domain is ℝ}

$\tilde{F}$ = {f : f is real valued : $f(x) \neq 0, x \in \mathbb{R}$}

Does $\tilde{F}$ is subgroup of F.

Closure: Let $f_1, f_2 \in \tilde{F}$. Then $\forall x \in \mathbb{R} : f_1(x) \neq 0, f_2(x) \neq 0$

Now,

$(f_1 + f_2)(x) = f_1(x) + f_2(x)$

Here if $f_1(x) = 2$ and $f_2(x) = -2$. Then

$f_1(x) + f_2(x) = 2 - 2 = 0$ i.e. $f_1 + f_2 \notin \tilde{F}$

∴ $\tilde{F}$ is not subgroup under addition.

14. The subset of all f ∈ F, such that f(1) = 0.

Solution. Let H = {f ∈ F: f(1) = 0}

Here H is not empty, since 0 ∈ H since 0(1) = 0

Closure: Let $f_1, f_2 \in H$. Then $f_1(1) = 0$, $f_2(1) = 0$

Then $(f_1 + f_2)(1) = f_1(1) + f_2(1) = 0 + 0 = 0$

∴ $f_1 + f_2 \in H$

ii. Elements of H are associative under addition i.e.

$[(f_1 + f_2) + f_3](1)$

$= (f_1 + f_2) + f_3(1)$

$= (f_1 + f_2)(1) + f_3(1) = 0 + 0 + 0 = 0$

$= f_1(0) + f_2(0)$

Again, $(f_1 + (f_2 + f_3))(1)$

$= f_1(1) + (f_2 + f_3)(1)$

$= f_1(1) + 0 + 0$

$= 0 + 0 + 0$

$= 0$

iii. For every f ∈ H

There exists 0 ∈ H. Such that

$(f + 0)(1) = f(1) + 0(1) = 0 + 0 = 0 = f(1)$

Again, $\forall f \in H$, $(0 + f)(1) = 0(1) + f(1) = 0 + 0 = 0 = f(1)$

∴ $0 + f = f$

∴ 0 ∈H is an additive identity of H.

Existence of additive inverse: $\forall f \in H$. There exists $-f \in H$. Such that

$[f + (-f)](1) = f(1) - f(1) = 0 + 0 = 0 = 0(1)$

∴ – f is inverse of f.

∴ H is a subgroup of F.

15. The subset of all $f \in \tilde{F}$ such that f(1) = 1

Solution. Here, H = {f ∈ $\tilde{F}$: f(1) = 1}

Since the identity map is in H. So, H is non empty.

Closure: For $f_1, f_2 \in H : (f_1 . f_2)(1) = f_1(1) . f_2(1) = 1 . 1 = 2$

∴ Elements of H are closed under multiplication

ii. Existence of identity is obvious.

i.e. $\forall f \in H$, there exists I ∈ H such that (f . I) (1) = f(1) I(1) = 1 . 1 = 1 = f(1)

Similarly, (I . f) (1) = f(1)

(iii) Existence of inverse: Here 0 ∉ H, every other f∈ H has their inverse themselves and form a subgroup under multiplication.

16. H = {f ∈ $\tilde{F}$: f(0) = 1}

Solution.

Closure: For every f, g ∈ H, (f . g) (0) = f(0) . g(0) = 1 . 1 = 1

∴ f . g ∈ H

Existence of Identity: For all f ∈ H the identity map I(x) = x ∉ H. Such that (f . I) (0) = f(0) . I(0) = 1 . 0 = 0 ≠ f(0)

Thus, H is not subgroup of $\tilde{F}$ under multiplication.

17. The subset of all f∈ $\tilde{F}$ such that f(0) = –1

Solution. Here for all f, g ∈ H : (f . g) (0) = f(0) . g(0) = –1 . –1 = 1

∴ f.g ∉ H

Hence, H is not subgroup of $\tilde{F}$ under multiplication.

18. The subset of all constant function in F.

Solution. $H = \{f \in \tilde{F} : f(x) = c \; \forall \in \mathbb{R}\}$

Here $0 \in H$. Since $0(x) = 0 \; \forall \; x \in \mathbb{R}$ but the inverse of 0 mapping does not exists since 0 mapping is not one to one. Thus, H is not a subgroup under multiplication.

19. Write at least 5 elements of each of the following cyclic groups.

a. $25\mathbb{Z}$ under addition

$25\mathbb{Z} = \{-100, -75, -25, 0, 25, 50, 75, \ldots.\}$

b. $\left\{\left(\frac{1}{2}\right)^n \mid n \in \mathbb{Z}\right\}$ under multiplication}

$= \left\{\frac{1}{2}, \frac{1}{4}, 2, 4, \frac{1}{8}, \ldots.\right\}$

c. $\{\pi^n \mid n \in \mathbb{Z}\}$ under multiplication

$= \left\{\pi, \frac{1}{\pi}, \pi^2, \frac{1}{\pi^2}, \ldots..\right\}$

20. Which of the following groups are cyclic? For each cyclic group, list all the generators of the group. $G_1 = \langle \mathbb{Z}, + \rangle$, $G_2 = \langle Q, + \rangle$, $G_3 = \langle Q^+, . \rangle$, $G_4 = \langle 6\mathbb{Z}, + \rangle$

Solution. $G_1 = \langle \mathbb{Z}, + \rangle$

21. Find the order of the cyclic subgroup of $\mathbb{Z}_4$ generated by 3.

Solution. Here $\mathbb{Z}_4$ is a group under addition. The cyclic subgroup generated by 3 is itself $\mathbb{Z}_4$. Since, $3 + 3 = 2$, $3 + 3 + 3 = 1$, $3 + 3 + 3 + 3 = 0$, $3 + 3 + 3 + 3 + 3 = 3$.

Hence order of cyclic subgroup of $\mathbb{Z}_4$ generated by 3 is 4 and order of the generator 3 is 3.