Chapter 10

Ring and Field

Exercise 10

1.

(a) $(12)(6) = 22$

(b) Since, $-8 = 18$; so $(20)(-8) = (20)(18) = 22$

(c) Since In Z_4, $-3 = 1$ and In Z_{11}, $-4 = 7$

So, $(-3, 5)(2, -4) = (1, 5)(2, 7) = (2, 2)$

2.

(a) $nZ = \{nm : \forall m \in Z\}$(supple $n \neq 0, 1$), otherwise is ring)

The addition is defined as $nm_1 + nm_2 = n(m_1 + m_2)$ ..... (1)

$\forall m, m_2 \in Z$

and multiplication is defined as $nm_1 . nm_2 = n^2(m_1m_2)$ ..... (2)

R_1: Here nZ is abelian group under b.0 addition

Because it is closed by (1) and associative, since for nm_1, nm_2 and $nm_3 \in nZ$.

$nm_1 + (nm_2 + nm_3) = n[m_1 + m_2 + m_3]$ and

$(nm_1 + nm_2) + nm_3 = n[m_1 + m_2 + m_3]$

Here, $0 = n . 0 \in nZ$ act as additive inverse and for every $nm \in nZ \; \exists -n\,m \in n\,Z$ st $nm + (-nm) = 0$.

So existence of inverse. Also, $nm_1 + nm_2 = n(m_1 + m_2) = n(m_2 + m_1) = nm_2 + nm_1$. So Abellian.

R_2: From (2) it is closed under binary operation.

R_3: Since, $(nm_1 . nm_2)(nm_3) = n^3(m_1 m_2 m_3)$ and

$(nm_1(nm_2 nm_3) = n_3(m_1 m_2 m_3)$, so it is associative under b . 0 Multiplication.

R_4: Here $nm_1 . (nm_2 + nm_3) = nm_1[n(m_2 + m_3)]$

$= (nm_1) n(m_2 + m_3)$

$= n^2 m_1 (m_2 + m_3)$

$= n^2 (m_1m_2 + m_1m_3)$

(b) $Z{\times}Z = \{(n, m) : \forall n_1 m \; Z\}$

The + binary operation defined as

$(n_1, m_1) + (n_2, m_2) = (n_1 + n_2, m_1 + m_2)$ .... (1)

$(n_1, m_1) . (n_2, m_2) = (n_1n_2, m_1m_2)$ .... (2)

R1: Here Z×Z is abelian group under b . o t.

It is closed from (1). It is associative also, because

$[(n_1, m_1) + (n_2, m_2)] + (n_3, m_3) = (n_1 + n_2 + n_3, m_1 + m_2 + m_3)$ and

$(n_1m_1) + [(n_2, m_2) + (n_3 + m_3)] = (n_1 + n_2 + n_3, m_1 + m_2 + m_3)$

Here $(0, 0) \in Z \times Z$ act as identity element.

For all $(n_1m) \in Z{\times}Z \; \exists (-n, -m)\,(Z \times Z)$ is inverse element so that $(n_1m) + (-n_1 -m) = (0, 0)$

Gain, $(n_1, m_1) + (n_2, m_2) = (n_1 + n_2, m_1 + m_2)$

$= (n_2 + n_1, m_2 + m_1)$

$= (n_2, m_2) + (n_2, m_1)$

$\therefore$ Z × Z is abelian.

R2: From (2), it is closed under b . 0.

R3: Here $[(n_1, m_1)(n_2, m_2)](n_3, m_3)] = (n_1 n_2 n_3, m_1 m_2 m_3)$ and

$(n_1, m_1)[(n_2, m_2)(n_3, m_3)] = (n_1 n_2 n_3, m_1 m_2 m_3)$. So, it is associative.

R4: Since $(n_1, m_1)[(n_2, m_2) + (n_3, m_3)] = (n_1, m_1)[(n_2 + n_3, m_2 + m_3)]$

$= [n_1(n_2 + n_3, m_1(m_2 + m_3)]$

$= (n_1n_2 + n_1n_3, m_1 m_2 + m_1m_3)$

and $[(n_1, m_1)(n_2, m_2) + (n_1, m_1)(n_3, m_3)] = (n_1n_2, m_1m_2) + (n_1n_3, m_1m_3)$

$= (n_1n_2 + n_1n_3, m_1m_2 + m_1m_3)$

Hence, left distributive hold similarly for right distributive also T.

This is abelian group with identity because $(n_1, m_1)(n_2, m_2) = (n_2, m_1)$, (n_1, m_1) and $(1, 1) \in Z{\times} Z$. But it is not field because for $(2, 3) \in Z \times Z$ has no multiplication inverse.

(c) Z^+ is not group because $5 \in Z^+ - 5 \notin Z^+$ i.e. has no additive inverse of 5 in Z^+. Even though it satisfy all properties of ring.

(d) $G = \{a + b\sqrt{2}, : a, b \in Q\}$

In example (3) of book prove $G = \{a + b\sqrt{2}, : a, b \in Z\}$ is commutative ring. It has identity element also because $1 + 0\sqrt{2} \in G$ act as multiplicative identity and is field also because for every non zero element $a + b\sqrt{2} \in G \; \exists \dfrac{1}{a + b\sqrt{Z}}$

$$= \frac{a - b\sqrt{Z}}{a^2 - 2b^2}$$

$$= \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2} \text{ in Z}$$

Such that $(a + b\sqrt{Z}) . \left(\dfrac{a}{a^2 - 2b^2} - \dfrac{b}{a^2 - 2b^2}\sqrt{Z}\right) = 1$

(e) $M_2(R) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R\right\}$

addition b . 0 is defined as for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in M_2(R)$

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix} \quad .... (1)$$

and multiplication b . 0 is defined as

$$AB = \begin{pmatrix} a_{11} b_{11} + a_{12} b_{21} & a_{11} b_{12} + a_{12} b_{22} \\ a_{21} + b_{11} + a_{22} b_{21} & a_{21} b_{12} + a_{22} b_{22} \end{pmatrix}$$

R1: Here $M_2(R)$ is abelian group under b. 0 +.

Because it is closed under b. 0 + from (1).

Again, matrix addition is associative also.

i.e. $A + (B + C) = (A + B) + C$.

Here $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(R)$ act as additive inverse and for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R) \; \exists \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ s.t. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, So, existence of inverse also. And matrix addition is commutative.

Also, because $A + B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$

and

$$B + A = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11}+a_{11} & b_{12}+a_{12} \\ b_{21}+a_{21} & b_{22}+a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

$$= A + B$$

R2: Here $M_2(R)$ is closed under b . 0 multiplication, shown from (2).

R3: Matrix multiplication is associative. Also,

Because $(AB)C = A(BC)$

Here, $(AB)C = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}+b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$

Similarly, we find A(BC) then we get $(AB)C = A(BC)$

R4: Here, $A(B + C) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11}+a_{11} & b_{12}+a_{12} \\ b_{21}+a_{21} & b_{22}+a_{22} \end{pmatrix}$

and $AB + AC = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$

3. For additive inverse of $(3, 2) \in Z_4 \times Z_7$

$(3, 2)^i = (3^i, 2^i)$

For $3 \in Z_4$, $3^i = 1$

and $2 \in Z_7$, $2^i = 5$

Thus, $(3, 2)^i = (3^i, 2^i) = (1, 5)$

For multiplicative inverse of $(3, 2) \in Z_4\ Z_7$

For $3 \in Z_4$ $3^i = 3$

$2 \in Z_7$ $2^i = 4$

$\therefore$ $(3, 2)^i = (3^i, 2^i) = (3, 4)$

4. Given equation is

$x^2 - 5x + 6 = 0$

$x^2 - 3x - 2x + 6 = 0$

$x(x - 3) - 2(x - 3) = 0$

$(x - 3)(x - 2) = 0$

$\therefore$ $x = 2, 3$ also solution in Z_{12}. For other solution.

When $x = 4$; $x^2 - 5x + 6 = (x - 2)(x - 3) = (2)(1) \neq 0$

$x = 5$, $= (3)(2) \neq 0$

$x = 6$, $= (4)(3) \neq 0$

$x = 7$, $= (5)(4) \neq 0$

$x = 8$, $= (6)(5) \neq 0$

$x = 9$, $= (7)(6) \neq 0$

$x = 10$, $= (8)(7) \neq 0$

$x = 11$, $= (9)(8) \neq 0$

$\therefore$ $x = 6$ and $x = 11$ also solution of given equation is Z_{12}

5. $3x = 2$, in Z_7

Here, $3x - 2 = 0$

When $x = 1$, $3x-2 \neq 0$

$x = 2$, $3x - 2 \neq 0$

$x = 3$, $3x - 2 = 0$

$x = 4$, $3x - 2 \neq 0$

$x = 5$, $3x - 2 \neq 0$

$x = 6$, $3x - 2 \neq 0$

$\therefore$ $x = 3$ is solution in Z_7.

6.

(a) Z_{20}

Here, 2, 4, 5, 6, 8, 10, 12, 14, 15, 16, 18 are number whose ged with 20 are not 1. Hence these are zero divisor of Z_{20} because (2) (10) = (4) (5) = (6) (5) = (12) (5) = (8) (5) = (18) (10) = (16) (5) = (14) (10) = 0

(b) Z_{16}

Here, 2, 4, 6, 8, 10, 12, 14 are number whose ged with 16 are not 1. So, these are zero divisor of Z_{16} because

(4) (4) = (2) (8) = (10) (8) = (6) (8) = 0

(c) Z_{10}

Here, 2, 4, 5, 6, and 8 are zero division because

(2) (5) = (4) (5) = (6) (5) = (8) (5) = 0

(d) In Z_{11} has no zero divisor i.e. no $a, b \in Z_{11}$ St $ab = 0$

7. Z_7 is integral domain.

Since, it has commutative with identity and has no zero divisor so it integral domain. Here no two $a, b \in Z_7$ st $ab = 0$.

8. Z_{12} is not integral domain because it has zero divisor since, $(4)(3) = 0$.

9. Z_{11} is field because it is ring (given) and commutative with identity and every non-zero elements has multiplicative inverse. Here, inverse of 1 is 1. Inver of 2 is 6. Inverse of 3 is 4. Inverse of 5 is 9, Inverse of 7 is 8, inverse of 10 is itself and inverse of 11 also itself.