Chapter 1

Linear Equations in Linear Algebra

Exercise 1.1

1.

(i) $\begin{bmatrix}2 & 3 & h\\4 & 6 & 7\end{bmatrix} \sim \begin{bmatrix}2 & 3 & h\\0 & 0 & 7-2h\end{bmatrix}$ $R_2 \to R_2 - 2R_1$

For consistency,

$7 - 2h = 0 \quad \Rightarrow h = 7/2$

(ii) $\begin{bmatrix}1 & -2 & 3\\3 & h & -2\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 3\\3 & h+6 & -11\end{bmatrix}$ $R_2 \to R_2 - 3R_1$

For consistency,

$h + 6 \neq 0 \quad \Rightarrow h \neq -6$

2.

(i) $\begin{bmatrix}1 & h & 2\\4 & 8 & k\end{bmatrix} \sim \begin{bmatrix}1 & h & 2\\0 & 8-4h & k-8\end{bmatrix}$ $R_2 \to R_2 - 4R_1$

(a) For no. solution,

$8 - 4h = 0 \qquad k - 8 \neq 0$

$\Rightarrow h = 2$ and $k \neq 8$

(b) For unique solution,

$8 - 4h \neq 0$

$\Rightarrow h \neq 2$

(c) For infinitely many solutions,

$8 - 4h = 0$ and $k - 8 = 0$

$\Rightarrow h = 2$ and $k = 8$

(ii) $\begin{bmatrix}1 & 3 & 2\\3 & h & k\end{bmatrix} \sim \begin{bmatrix}1 & 3 & 2\\0 & h-9 & k-6\end{bmatrix}$ $R_2 \to R_2 - 3R_1$

(a) For No solution, $h - 9 = 0 \Rightarrow h = 9$ and $k - 6 \neq 0 \Rightarrow k \neq 6$.

(b) For unique solution,

$h - 9 \neq 0; \qquad k - 6 = 0$

$\Rightarrow h = 9, \qquad k = 6$

3.

(i) $\begin{bmatrix}1 & 3 & 4\\3 & 9 & 7\end{bmatrix} \sim \begin{bmatrix}1 & 3 & 4 & 7\\0 & 0 & -5 & -15\end{bmatrix}$ $R_2 \to R_2 - 3R_1$

$\sim \begin{bmatrix}1 & 3 & 4 & 7\\0 & 0 & 1 & 3\end{bmatrix}$ $R_2 \to -\frac{1}{5}R_2$

$\sim \begin{bmatrix}1 & 3 & 0 & -5\\0 & 0 & 1 & 3\end{bmatrix}$ $R_1 \to R_1 - 4R_2$

Corresponding system is

$x_1 + 3x_2 = -5$

x_2 is free variable.

$x_3 = 3$

Using back substitution, we get

$x_3 = -5 - 3x_2$

x_2 is free variable.

(ii) $\begin{bmatrix}0 & 1 & -6 & 5\\1 & -2 & 7 & -6\end{bmatrix} \sim \begin{bmatrix}1 & 1 & -6 & 5\\0 & -2 & 7 & -6\end{bmatrix}$ $R_1 \to R_2$ -

$\sim \begin{bmatrix}1 & 0 & -5 & 4\\0 & 1 & -6 & 5\end{bmatrix}$ $R_1 \to R_1 + 2R_2$

Corresponding system is

$x_1 - 5x_3 = 4$

$x_2 - 6x_3 = 5$

x_3 is free variable

$\therefore\ x_1 = 4 + 5x_3$

$x_2 = 5 + 6x_3$

x_3 is free variable.

(iii) $\begin{bmatrix}3 & -4 & 2 & 0\\-9 & 12 & -6 & 0\\-6 & 8 & -4 & 0\end{bmatrix} \sim \begin{bmatrix}3 & -4 & 2 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$ $R_2 \to R_2 + 3R_1,\ R_3 \to R_3 + 2R_1$

Corresponding system is $3x_1 - 4x_2 + 2x_3 = 0$

x_2, x_3 are free variables

$\therefore\ x_1 = \frac{4}{3}x_2 - \frac{2}{3}x_3$

x_2, x_3 are free variables.

4.

(i) $\begin{bmatrix}2 & 3 & 4 & 20\\3 & 4 & 5 & 26\\3 & 5 & 6 & 31\end{bmatrix} \sim \begin{bmatrix}2 & 3 & 4 & 20\\0 & -1/2 & -1 & -4\\0 & 1/2 & 0 & 1\end{bmatrix}$ $R_2 \to R_2 - \frac{3}{2}R_1,\ R_3 \to R_3 - \frac{3}{2}R_1$

$\sim \begin{bmatrix}2 & 3 & 4 & 20\\0 & -1/2 & -1 & -4\\0 & 0 & -1 & -3\end{bmatrix}$ $R_3 \to R_3 + R_2$

$\sim \begin{bmatrix}2 & 3 & 4 & 20\\0 & -1 & -2 & -8\\0 & 0 & -1 & -3\end{bmatrix}$ $R_2 \to 2R_2$

Corresponding system is

$2x + 3y + 4z = 20$

$-y - 2z = -8$

$-z = -3$

Using back substitution,

$x = 1, y = 2, z = 3$

(ii) $\begin{bmatrix}1 & 0 & -3 & 8\\2 & 2 & 9 & 7\\0 & 1 & 5 & -2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -3 & 8\\0 & 2 & 15 & -9\\0 & 1 & 5 & -2\end{bmatrix}$ $R_2 \to R_2 - 2R_1$

$\sim \begin{bmatrix}1 & 0 & -3 & 8\\0 & 1 & 5 & -2\\0 & 2 & 15 & -9\end{bmatrix}$ $R_2 \to R_3$

$$\sim\begin{bmatrix}1 & 0 & -3 & 8\\ 0 & 1 & 5 & -2\\ 0 & 0 & 5 & -5\end{bmatrix} R_3 \to R_3 - 2R_2$$

$$\sim\begin{bmatrix}1 & 0 & -3 & 8\\ 0 & 1 & 5 & -2\\ 0 & 0 & 0 & -3\end{bmatrix} R_3 \to R_3 - R_2$$

Since last row is off the form [0 0 0 b], b ≠ 0. So, the given system is inconsistent.

5.

(i) $$\begin{bmatrix}1 & 0 & 3 & 0 & 2\\ 0 & 1 & 0 & -3 & 3\\ 0 & -2 & 3 & 2 & 1\\ 3 & 0 & 0 & 7 & -5\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 3 & 0 & 2\\ 0 & 1 & -3 & 3 & 3\\ 0 & -2 & -3 & 8 & 7\\ 0 & 0 & -3 & 7 & -8\end{bmatrix} R_4 \to R_4 - 3R_1$$

$$\sim\begin{bmatrix}1 & 0 & 3 & 0 & 2\\ 0 & 1 & -3 & 3 & 3\\ 0 & 0 & -3 & 8 & 7\\ 0 & 0 & 0 & -1 & -15\end{bmatrix} R_4 \to R_4 - R_3$$

Since, last column is not pivot column, so the system is consistent.

(ii) $$\begin{bmatrix}0 & 1 & -8 & 8\\ 2 & -3 & 2 & 1\\ 5 & -8 & 7 & 1\end{bmatrix} \sim \begin{bmatrix}2 & -3 & 2 & 1\\ 0 & 1 & -8 & 8\\ 5 & -8 & 7 & 1\end{bmatrix} R_1 \to R_2$$

$$\sim\begin{bmatrix}2 & -3 & 2 & 1\\ 0 & 1 & -8 & 8\\ 0 & -1/2 & 2 & -3/2\end{bmatrix} R_3 \to R_3 - \frac{5}{2}R_1$$

$$\sim\begin{bmatrix}2 & -3 & 2 & 1\\ 0 & 1 & -8 & 8\\ 0 & 0 & -2 & 5/2\end{bmatrix} R_3 \to R_3 + \frac{1}{2}R_2$$

Since last column is not pivot column, so the system is consistent.

6: Let $A = \begin{bmatrix}1 & 3 & 4\\ -4 & 2 & -6\\ -3 & -2 & -7\end{bmatrix}$ **and** $b = \begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}$. **Is the equation $Ax = b$ consistent for all possible b_1, b_2, b_3?**

Solution: The augmented matrix of $Ax = b$ is

$$\begin{bmatrix}1 & 3 & 4 & b_1\\ -4 & 2 & -6 & b_2\\ -3 & -2 & -7 & b_3\end{bmatrix}.$$

$$\sim\begin{bmatrix}1 & 3 & 4 & b_1\\ 0 & 14 & 10 & b_2+4b_1\\ 0 & 7 & 5 & b_3+3b_1\end{bmatrix} \quad \because \text{applying } \begin{matrix}R_2 \to R_2 + 4R_1\\ R_3 \to R_3 + 3R_1\end{matrix}$$

$$\sim\begin{bmatrix}1 & 3 & 4 & b_1\\ 0 & 14 & 10 & b_2+4b_1\\ 0 & 0 & 0 & b_1 - (1/2)b_2 + b_3\end{bmatrix} \quad [\text{applying } R_3 \to R_3 - (1/2)R_2]$$

This shows that the equation $Ax = b$ is inconsistent for every b, because some choice of b can make $b_1 - \frac{1}{2}b_2 + b_3$ is non-zero.

Exercise 1.2

1. (i) $$u + v = \begin{bmatrix}6\\ -1\\ 5\end{bmatrix} + \begin{bmatrix}-3\\ 4\\ 6\end{bmatrix} = \begin{bmatrix}3\\ 3\\ 5\end{bmatrix}$$

and $$3u - 2v = 3\begin{bmatrix}6\\ -1\\ 5\end{bmatrix} - 2\begin{bmatrix}-3\\ 4\\ 6\end{bmatrix} = \begin{bmatrix}18+6\\ -3-8\\ 15-12\end{bmatrix} = \begin{bmatrix}24\\ -11\\ 3\end{bmatrix}.$$

(ii) Similar to (i).

2. (i) $$[a_1 \; a_2 \; a_3 \; b] = \begin{bmatrix}1 & 0 & 5 & 2\\ -2 & 1 & -6 & -1\\ 0 & 2 & 8 & 6\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 0 & 5 & 2\\ 0 & 1 & 4 & 3\\ 0 & 2 & 8 & 6\end{bmatrix} R_2 \to R_2 + 2R_1$$

$$\begin{bmatrix}1 & 0 & 5 & 2\\ 0 & 1 & 4 & 3\\ 0 & 0 & 0 & 0\end{bmatrix} R_3 \to R_3 - 2R_2$$

Thus system $x_1 a_1 + x_2 a_2 + x_3 a_3 = b$ is consistent. Hence, b is linear combination of a_1, a_2, a_3.

(ii) Similar to (i).

3.

(i) $$\begin{bmatrix}1 & -4 & 2 & 3\\ 0 & 3 & 5 & -7\\ -2 & 8 & -4 & -3\end{bmatrix} \sim \begin{bmatrix}1 & -4 & 2 & 3\\ 0 & 3 & 5 & -7\\ 0 & 0 & 0 & 3\end{bmatrix} R=3 \to R_3 + 2R_1$$

This system $Ax = b$ is inconsistent.

Hence, b is not linear combination of columns of A.

(ii) Similar to (i).

4.

(i) $$\begin{bmatrix}1 & -2 & 4\\ 4 & -3 & 1\\ -2 & 7 & h\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 4\\ 0 & 5 & -15\\ 0 & 3 & h+8\end{bmatrix} R_2 \to R_2 - 4R_1, R_3 \to R_3 + 2R_1$$

$$\sim\begin{bmatrix}1 & -2 & 4\\ 0 & 1 & -3\\ 0 & 3 & h+8\end{bmatrix} R_2 \to \frac{1}{5}R_1$$

$$\sim\begin{bmatrix}1 & -2 & 4\\ 0 & 1 & -3\\ 0 & 0 & h+17\end{bmatrix} R_3 \to R_3 - 3R_2$$

If b is in span {u, v}

Then $h + 17 = 0 \Rightarrow h = -17$.

(ii) Similar to (i).

5. Solved in book.

6. The augmented matrix of $Ax = b$ is

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix}.$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2+4b_1 \\ 0 & 7 & 5 & b_3+3b_1 \end{bmatrix} \text{[applying } \begin{matrix} R_2 \to R_2 + 4R_1 \\ R_3 \to R_3 + 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2+4b_1 \\ 0 & 0 & 0 & b_1 - (1/2)b_2 + b_3 \end{bmatrix} \quad \text{[applying } R_3 \to R_3 - (1/2)R_2]$$

This shows that the equation $Ax = b$ is inconsistent for every b, because some choice of b can make $b_1 - \frac{1}{2} b_2 + b_3$ is non-zero.

Exercise 1.3

A.

1. $\begin{bmatrix} 6 & 4 & 2 \\ 3 & -5 & -34 \end{bmatrix} \sim \begin{bmatrix} 6 & 4 & 2 \\ 0 & -7 & -35 \end{bmatrix} \quad R_2 \to R_2 - \frac{1}{2} R_1$

$\sim \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 5 \end{bmatrix} \quad R_1 \to \frac{1}{2} R_1,\ R_2 \to -\frac{1}{7} R_2$

Corresponding system is

$3x + 2y = 1$

$y = 5$

Using back substitution,

$x = -3,\ y = 5$

2. $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 5 \\ 3 & 1 & 1 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -1 \\ 0 & -2 & -2 & -10 \end{bmatrix} \quad R_2 \to R_2 - R_1,\ R_3 \to R_3 - 3R_1$

$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & -2 & -9 \end{bmatrix} \quad R_3 \to R_3 - R_2$

Corresponding system is,

$x + y + z = 6$

$-2y = -1$

$-2z = -9$

Using back substituting

$x = 1,\ y = 1/2,\ z = 9/2$

3-6 → Similar to Q. 1.

B.

(i) $\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \quad R_2 \to R_2 + R_1,\ R_3 \to R_3 - 2R_1$

$\sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 3R_2$

Here x_3 is free variable so system has non trivial solution.

Corresponding system is

$3x_1 + 5x_2 - 4x_3 = 0$

$3x_2 = 0$

x_3 is free variable.

Using back substitution,

$x_1 = \frac{4}{3} x_3;\ x_2 = 0;\ x_3$ is favourable.

$\therefore$ Solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$

(ii) Similar to (i).

Exercise 1.4

1. Consider homogeneous system $x_1v_1 + x_2v_2 + x_3v_3 = 0$

(i) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

It is in echelon (reduced) form and x_1, x_2, x_3 are all basic variables, so homogeneous system has only trivial solution $x_1 = 0$, $x_2 = 0$, $x_3 = 0$. Hence, vectors are linearly independent.

(ii) Vectors are not multiples of each other. So, they are linearly independent.

(iii) Vectors are not multiples of each other. So, they are linearly independent.

(v) Vector the given vectors one vector is zero vector, so vectors are linearly dependent.

(vi) Vectors are not multiples of each other. So, they are linearly independent.

2.

(i) $\begin{bmatrix} 0 & -8 & 5 & 0 \\ 3 & -2 & 4 & 0 \\ -1 & 5 & -4 & 0 \\ 1 & -3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 3 & -2 & 4 & 0 \\ -1 & 5 & -4 & 0 \\ 0 & -8 & 5 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_4$

$\sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 7 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -8 & 5 & 0 \end{bmatrix} \quad R_2 \to R_2 - 3R_1,\ R_2 \to R_2 + R_1$

$\sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 7 & -2 & 0 \\ 0 & 0 & -10/7 & 0 \\ 0 & 0 & 51/7 & 0 \end{bmatrix} \quad R_3 \to R_3 - \frac{2}{7} R_2,\ R_4 \to R_4 - \frac{8}{7} R_2$

$\sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 7 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad R_3 \to -\frac{7}{10} R_3,\ R_4 \to \frac{7}{51} R_4$

$$\sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 7 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 \to R_4 - R_3$$

No, free variables, hence columns of given matrix are linearly independent.

(ii) Similar to (i).

3.

(i) Consider homogeneous system

$x_1v_1 + x_2v_2 + x_3v_3 = 0$

It's augmented matrix is

$$\begin{bmatrix} 1 & -3 & 5 & 0 \\ -3 & 9 & -7 & 0 \\ 2 & -6 & h & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & h-10 & 0 \end{bmatrix} R_2 \to R_2 + 3R_1,\ R_3 \to R_3 - 2R_1$$

Here, x_2 is free variable for all values of h.

So, vectors are linearly independent for all values of h.

(ii) $$\begin{bmatrix} 1 & 3 & -1 & 0 \\ -1 & -5 & 5 & 0 \\ 4 & 7 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & -5 & h+4 & 0 \end{bmatrix} R_2 \to R_2 + R_1,\ R_3 \to R_3 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & h-6 & 0 \end{bmatrix} R_2 \to -\frac{1}{2}R_2$$

For vectors to be linearly dependent. x_3 must be free variable i.e. if

$h - 6 = 0 \Rightarrow h = 6.$

(iii) Similar to (ii).