Chapter 3

Matrix Algebra

Exercise 3.1

1. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}$.

(i) Here,

$$BA = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 26 & -25 & 13 \\ 14 & -20 & 7 \end{bmatrix}$$

(ii) Here, A has more entries in a row than B has in a column. So, AB is not possible.

2. Let, $A = \begin{bmatrix} -9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix}$.

Then,

$$A - 5I = \begin{bmatrix} -9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -14 & -1 & 3 \\ -8 & 2 & -6 \\ -4 & 1 & 3 \end{bmatrix}$$

3. Let $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$.

Here,

$$AB = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$

And $$AC = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$

Thus, $$AB = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix} = AC.$$

4.

(i) $|A| = 2 \neq 0$, so A is nonsingular.

(iii) $|A| = 1 \neq 0$, so A is nonsingular.

5. (i) $\begin{vmatrix} 3 & 2 \\ 7 & 4 \end{vmatrix} = 12 - 14 = -2 \neq 0.$

So A is non singular.

6. Let, $A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$

$|A| = 32 - 30 = 2$

Cofactor of $a_{11} = 4$Cofactor of $a_{12} = -5$

Cofactor of $a_{21} = -6$ Cofactor of $a_{22} = 8$

$$\text{adj. }(A) = \begin{bmatrix} 4 & -5 \\ -6 & 8 \end{bmatrix}^t = \begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj. }A}{|A|} = \frac{\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix}}{2} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}$$

$\therefore$ Since A^{-1} exists,

$X = A^{-1}C$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad = \begin{bmatrix} 7 \\ -9 \end{bmatrix}$$

7.

(i) $\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$

$$\sim \begin{bmatrix} 5 & 7 \\ 0 & -9/5 \end{bmatrix} \quad R_2 \to R_2 + \frac{3}{5}R_1$$

$\therefore$ No. of pivot = 2

Size of A = 2

Hence, by investible matrix theorem, A is invertible ($\because$ no. of pivot = size of A).

(ii) Similar to (i).

(iii) $$\begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ -4 & -9 & 7 \end{bmatrix} \quad R_1 \to R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & -9 & 15 \end{bmatrix} \quad R_3 \to R_3 + 4R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 3R_2$$

No. of pivot = 2,

Size of A = 3

Hence, A is not invertible.

8. **Note:** An algorithm to fund A^{-1}, if A^{-1} exists

$[A : I] \sim [I : A^{-1}]$

$[A^{-1} : I] \sim [I : A]$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ -3 & 3 & -1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Using element row operations, we shall arrive

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 2 & 4 & 1 \\ 0 & 0 & 1 & 3 & 5 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}.$$

9. $A^T = \begin{bmatrix} 1 & 3 & -5 \\ 0 & 1 & -1 \\ -2 & -2 & 9 \end{bmatrix}$

$R_3 \to R_3 + 2R_1$

$$\sim \begin{bmatrix} 1 & 3 & -5 \\ 0 & 1 & -1 \\ 0 & 4 & -1 \end{bmatrix}$$

$R_3 \to R_3 - 4R_2$

$$\sim \begin{bmatrix} 1 & 3 & -5 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

Since, number of pivot in $A^T = 3 =$ size of A^T

$\therefore$ A^T is invertible.

10. See Q. No. 6.

Exercise 3.2

1. Not possible since $A_{11} B_{21}$ is not defined.

2. $\text{Col}_1(A)\,\text{row}_1(B) = \begin{bmatrix} -3 \\ 1 \end{bmatrix} [a \;\; b] = \begin{bmatrix} -3a & -3b \\ a & b \end{bmatrix}$

$\text{Col}_2(A) \,.\, \text{row}_2(B) = \begin{bmatrix} 1 \\ -4 \end{bmatrix} [c \;\; d] = \begin{bmatrix} c & d \\ -4c & -4d \end{bmatrix}$

$\text{Col}_3(A) \,.\, \text{row}_3(B) = \begin{bmatrix} 2 \\ 5 \end{bmatrix} [e \;\; f] = \begin{bmatrix} 2e & 2f \\ 5e & 5f \end{bmatrix}$

$$\therefore \quad AB = \sum_{k=1}^{3} \text{col}_k(A) \,.\, \text{row}_k(B) = \begin{bmatrix} -3a & -3b \\ a & b \end{bmatrix} + \begin{bmatrix} c & d \\ -4c & -4d \end{bmatrix} + \begin{bmatrix} 2e & 2f \\ 5e & 5f \end{bmatrix}$$

$$= \begin{bmatrix} -3a + c + 2c & -3b + d + 2f \\ a - 4c + 5e & b - 4d + 5f \end{bmatrix}$$

3. Similar to Q. No. 2

4. Here,

$$A_{11}^{-1} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}; \quad A_{22}^{-1} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$$

and $\quad -A_{11}^{-1} A_{12} A_{22}^{-1} = \begin{bmatrix} 13 & 39 \\ 8 & -23 \end{bmatrix}$

Using form...

Exercise 3.3

1. $I - C = \begin{bmatrix} 0.9 & -0.6 \\ -0.5 & 0.8 \end{bmatrix}$ Find $(I - C)^{-1}$ and use, $x =$

2. $I - C = \begin{bmatrix} 0.8 & -0.2 & 0.0 \\ -0.3 & 0.9 & -0.3 \\ -0.1 & 0.0 & 0.8 \end{bmatrix}$

The augmented matrix of $(I - C)x = d$

$$\left[\begin{array}{ccc|c} 0.8 & -0.2 & 0.0 & 40 \\ -0.3 & 0.9 & -0.3 & 60 \\ -0.1 & 0.0 & 0.8 & 80 \end{array}\right]$$

using elementary row operations,

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 82.8 \\ 0 & 1 & 0 & 131 \\ 0 & 0 & 1 & 110.3 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 82.8 \\ 131 \\ 110 \end{bmatrix} \approx \begin{bmatrix} 83 \\ 131 \\ 110 \end{bmatrix}$$

3. Translation matrix $= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$(x, y) \to (3, 1)$

$$\text{Rotation matrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \xrightarrow{T} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\xrightarrow{R} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

Note: Translation transformation matrix $= \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$

8. $$A = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & -3 & -1 & 6 \\ 0 & 6 & 2 & -7 \\ 0 & -9 & -3 & 13 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

and $$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 1 & 0 \\ -3 & -3 & 2 & 0 & 1 \end{bmatrix}$$