j) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$

Solution: Here,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$$

Let $u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$.

Let $v_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{\frac{1}{2}}}$

Now, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+\sqrt{n+1}}}{\sqrt{n}}$

$$= \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}(1+\frac{\sqrt{n+1}}{\sqrt{n}})}.}{\sqrt{n}}$$

$$= \lim_{n\to\infty} (1 + \sqrt{1 + \frac{1}{n}}) = 2 > 0$$

Also, $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n}$ and it is diverges by p-test.

Hence the given $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$ series is diverges by Limit Comparison Test

K) $\frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7}+\ \ldots$

Solution: Here, given series is

$$\frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7}+\ \ldots$$

n^{th} term of the series $(a_n)=\frac{n}{(2n-1)(2n+1)}$

Then the given series can be written as

$$\frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7}+\ \ldots = \sum_{n=1}^{\infty}\frac{n}{(2n-1)(2n+1)}$$

Let $u_n = \frac{n}{(2n-1)(2n+1)}$.

Let $v_n = \frac{n}{n^2} = \frac{1}{n}$

Now, $$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\frac{n}{(2n-1)(2n+1)}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\ \frac{n}{n^2(2-\frac{1}{n})(2+\frac{1}{n})}\times n$$

$$= \lim_{n\to\infty}\frac{1}{(2-\frac{1}{n})(2+\frac{1}{n})} = \frac{1}{4} > 0$$

Also, $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty}\frac{1}{n}$ and it is diverges by p-test.

Hence the given series $\sum_{n=1}^{\infty}\frac{n}{(2n-1)(2n+1)}$ is diverges by Limit Comparison Test.

n) $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ \ldots$

Solution: Here, given series is

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ \ldots$$

n^{th} term of the series $(a_n)=\frac{n}{n+1}$

Then the given series can be written as

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ \ldots\ =\ \sum_{n=1}^{\infty}\frac{n}{n+1}$$

Also, $\sum_{n=1}^{\infty}\frac{n}{n+1} > \sum_{n=1}^{\infty}\frac{1}{n}$

Also, $\sum_{n=1}^{\infty}\frac{1}{n}$ and it is diverges by p-test.

Hence the given series $\sum_{n=1}^{\infty}\frac{n}{n+1}$ is diverges by Comparison Test.

o) $\sum_{n=1}^{\infty} \left(\frac{\sqrt{n+2}}{2n^2+n+1}\right)$

Solution: Here,

$$\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{\sqrt{n+2}}{2n^2+n+1}\right)$$

Let $u_n = \sum_{n=1}^{\infty} \left(\frac{\sqrt{n+2}}{2n^2+n+1}\right)$.

Let $v_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{2-\frac{1}{2}}} = \frac{1}{n^{\frac{3}{2}}}$

Now, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\left(\frac{\sqrt{n+2}}{2n^2+n+1}\right)}{\frac{1}{n^{\frac{3}{2}}}}$

$$= \lim_{n\to\infty} \frac{\left(\frac{\sqrt{n}(\sqrt{1+\frac{2}{n}})}{n^2(2+\frac{1}{n}+\frac{1}{n^2})}\right)}{n^{\frac{3}{2}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+\frac{2}{n}}}{2+\frac{1}{n}+\frac{1}{n^2}} = \frac{1}{2} > 0$$

Also, $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ and it is converges by p-test.

Hence the given series is converges by Limit Comparison Test.

5. Investigate the convergence of the following series.

a) $\sum_{n=0}^{\infty} \left(\frac{2^n+5}{3^n}\right)$

Theorem:(D' Alembert Ratio Test)

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = \ell.$$

Then,

a. If $\ell<1$ then the series converges

b. If $\ell>1$ then the series diverges

c. If $\ell=1$ then the test is inclusive and further test is needed.

a) $\sum_{n=0}^{\infty}\left(\frac{2^n+5}{3^n}\right)$

Solution: Here, given series is $\sum_{n=0}^{\infty}\left(\frac{2^n+5}{3^n}\right)$

Let $a_n = \frac{2^n+5}{3^n}$

Now, $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}}\right)$

$$= \lim_{n\to\infty}\left(\frac{\frac{2^n\times 2+5}{3^n\times 3}}{\frac{2^n+5}{3^n}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\frac{2^n\left(2+\frac{5}{2^n}\right)}{3^n\times 3}}{\frac{2^n\left(1+\frac{5}{2^n}\right)}{3^n}}\right)$$

$$= \left(\frac{\frac{2+\frac{5}{2^\infty}}{3}}{1+\frac{5}{2^\infty}}\right) = \frac{2}{3} < 1$$

Thus, by D' Alembert Ratio test, the given series is convergent.

a) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$

Solution: Here, given series is $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$.

Let $a_n = \frac{(2n)!}{n!n!}$

Now, $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{(2n+2)!}{(n+1)!(n+1)!}}{\frac{(2n)!}{n!n!}}\right)$

$$= \lim_{n\to\infty}\left(\frac{\frac{(2n+2)(2n+1)(2n)!}{(n+1)\,n!\,(n+1)\,n!}}{\frac{(2n)!}{n!n!}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(2n+2)(2n+1)}{(n+1)(n+1)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2\,(2n+1)}{n+1}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2\ n\left(2+\frac{1}{n}\right)}{n\left(1+\frac{1}{n}\right)}\right) = \frac{2}{1} = 2 > 1$$

Thus, by D' Alembert Ratio test, the given series is divergent.

d) $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\, n! 3^n}$

Solution: Here, given series is $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\, n! 3^n}$.

Let $a_n = \frac{(n+3)!}{3!\, n! 3^n}$

Now, $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{(n+1+3)!}{3!(n+1)! 3^{n+1}}}{\frac{(n+3)!}{3!\, n! 3^n}}\right)$

$$= \lim_{n\to\infty}\left(\frac{\frac{(n+4)(n+3)!}{3!\,(n+1)! 3^n \times 3}}{\frac{(n+3)!}{3!\, n! 3^n}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\frac{n+4}{(n+1)! \times 3}}{\frac{1}{n!}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n+4}{3(n+1)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n\left(1+\frac{4}{n}\right)}{3\, n\left(1+\frac{1}{n}\right)}\right) = \frac{1}{3} < 1$$

Thus, by D' Alembert Ratio test, the given series is convergent.

h) $\sum_{n=1}^{\infty} \sqrt{\frac{2^n-1}{3^n-1}}$

Solution: Here, given series is $\sum_{n=1}^{\infty} \sqrt{\frac{2^n-1}{3^n-1}}$.

Let $a_n = \frac{2^n-1}{3^n-1}$

Now, $\lim\limits_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim\limits_{n\to\infty} \frac{\sqrt{\frac{2^{n+1}-1}{3^{n+1}-1}}}{\sqrt{\frac{2^n-1}{3^n-1}}}$

$$= \lim_{n\to\infty} \sqrt{\frac{\frac{2^n\times2-1}{3^n\times3-1}}{\frac{2^n-1}{3^n-1}}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{\frac{2^n\left(2-\frac{1}{2^n}\right)}{3^n\left(3-\frac{1}{3^n}\right)}}{\frac{2^n\left(1-\frac{1}{2^n}\right)}{3^n\left(1-\frac{1}{3^n}\right)}}}$$

$$= \frac{2}{3} < 1$$

Thus, by D'Alembert Ratio test, the given series is convergent.

6. Investigate the convergence of the following series.

a) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$

Theorem: (The n^{th} Root Test)(Cauchy's Radical Test)

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \ell.$$

Then,

a. If $\ell<1$ then the series converges

b. If $\ell>1$ then the series diverges

c. If $\ell=1$ then the test is inclusive and further test is needed.

6. a) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$

Solution: Here, given series is $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$.

Let $a_n = \left(1+\frac{1}{n}\right)^{-n^2}$

Now,

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^{-n^2}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$= \frac{1}{e} \approx 0.37 < 1 \quad \left[\because \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} = e\right]$$

Thus, the series is converges by Cauchy's Radical Test.

6. e) $\sum_{n=1}^{\infty} \left(\frac{n^n}{(2^n)^2}\right)$

Solution: Here, given series is $\sum_{n=1}^{\infty} \left(\frac{n^n}{(2^n)^2}\right)$.

Let $a_n = \sum_{n=1}^{\infty} \left(\frac{n^n}{(2^n)^2}\right)$

Now,

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n^n}{(2^n)^2}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left(\frac{n}{2^2}\right)$$

$$= \lim_{n\to\infty} \frac{n}{4}$$

$$= \infty$$

Thus, the series is converges by **Cauchy's Radical Test.**

Alternative Series:

An infinite series $\sum(-1)^n a_n$ is known as an alternative series.

Leibnitz's Theorem (The Alternative Series Test)

The series,

$$\sum_{n=1}^{\infty}(-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \ \dots$$

Converges if it satisfies the following conditions:

a) $b_n > 0$, for all n

b) $b_n \geq b_{n+1}$ for all n$\geq N$ for some integer N.

C) $\lim_{n\to\infty} b_n = 0.$

Exercise 8.4

1. **Test the convergency of the following alternative series.**

a) $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^2}$

Solution: Here, given series is

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^2}$$

Here, comparing this series with $\sum_{n=1}^{\infty}(-1)^{n+1}\, b_n$

$$b_n = \frac{1}{n^2}$$

i) $b_n \geq 0$, for all n.

ii) $b_{n+1} - b_n = \frac{1}{(n+1)^2} - \frac{1}{n^2}$

$$= \frac{n^2 - n^2 - 2n - 1}{n^2(n+1)^2}$$

$$= \frac{-(2n+1)}{n^2(n+1)^2} < 0$$

$b_{n+1} < b_n$

iii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n^2} = 0.$

Thus, the series is converges by **Leibnitz's Theorem (The Alternative Series Test).**

b) $\sum_{n=1}^{\infty} (-1)^n \frac{\ell n\,(n)}{n}$

Solution: Here, given series is

$$\sum_{n=1}^{\infty} (-1)^n \frac{\ell n\,(n)}{n}$$

Here, comparing this series with $\sum_{n=1}^{\infty}(-1)^{n+1}\, b_n$

$$b_n = \frac{\ell n\,(n)}{n}$$

i) $b_n \geq 0$, for all n> 0.

ii) $$b_{n+1} - b_n = \frac{\ell n\,(n+1)}{n+1} - \frac{\ell n\,(n)}{n}$$

$$= \frac{n\ell n\,(n) - (n+1)\ell n\,(n)}{n\,(n+1)}$$

$$= \frac{\ell n\left(\frac{n^n}{n^{n+1}}\right)}{n\,(n+1)} = \frac{\ell n\left(\frac{1}{n}\right)}{n\,(n+1)} \leq 0 \text{ for } n \geq 1$$

$b_{n+1} \leq b_n$

iii) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{\ell n\,(n)}{n} = \lim\limits_{n\to\infty} \frac{1}{n}$ (using L Hospital rule) = 0.

Thus, the series is converges by Leibnitz's Theorem (The Alternative Series Test).

d) $\sum_{n=2}^{\infty} (-1)^n \frac{\ell n\,(n)}{\ell n\,(n^2)}$

Solution: Here, given series is

$$\sum_{n=2}^{\infty} (-1)^n \frac{\ell n\,(n)}{\ell n\,(n^2)}$$

Here, comparing this series with $\sum_{n=1}^{\infty}(-1)^{n+1}\, b_n$

$$b_n = \frac{\ell n\,(n)}{\ell n\,(n^2)} = \frac{\ell n\,(n)}{2\ell n\,(n)} = \frac{1}{2}$$

i) $b_n \geq 0$, for all n> 2.

iii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} \neq 0.$

Thus, the series is diverges by Leibnitz's Theorem

f) $\sum_{n=1}^{\infty} (-1)^n \ell n\left(1+\frac{1}{n}\right)$

Solution: Here, given series is

$$\sum_{n=1}^{\infty} (-1)^n \ell n\left(1+\frac{1}{n}\right)$$

Here, comparing this series with $\sum_{n=1}^{\infty}(-1)^{n+1} b_n$

$$b_n = \ell n\left(1+\frac{1}{n}\right)$$

i) $b_n \geq 0$ for $n \geq 1$

iii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \ell n\left(1+\frac{1}{n}\right) = 0.$

Thus, the series is converges by Leibnitz's Theorem

Absolute Convergence

A series $\sum \boldsymbol{a_n}$ converges absolutely if the corresponding series $\sum |\boldsymbol{a_n}|$ converges.

Theorem (The Absolute Convergence Test)

If $\sum |\boldsymbol{a_n}|$ converges then series $\sum \boldsymbol{a_n}$ converges.

Normally , Absolute Convergence implies the general convergence.

Conditional Convergence

A series $\sum \boldsymbol{a_n}$ converges but $\sum |\boldsymbol{a_n}|$ does not converges, then the series is called convergent conditionally.

2. Test the absolute convergence and conditional convergence of the following series.

a) $\sum_{\mathbf{n=1}}^{\infty} (-\mathbf{1})^{\mathbf{n+1}} \frac{\mathbf{1}}{\mathbf{n^2}}$

Solution: Here, given series is

$$\sum_{\mathbf{n=1}}^{\infty} (-\mathbf{1})^{\mathbf{n+1}} \frac{\mathbf{1}}{\mathbf{n^2}}$$

Let $a_n = (-\mathbf{1})^{\mathbf{n+1}} \frac{\mathbf{1}}{\mathbf{n^2}}$

Here, $|a_n| = |(-\mathbf{1})^{\mathbf{n+1}} \frac{\mathbf{1}}{\mathbf{n^2}}| = \frac{\mathbf{1}}{\boldsymbol{n^2}}$

$$\sum_{\mathbf{n=1}}^{\infty} |a_n| = \sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{\boldsymbol{n^2}}$$

Clearly the $\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{\boldsymbol{n^2}}$ is convergent by p-Test with p=2.

Hence, the given series convergent absolutely.

d) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{a+nb}$

Solution: Here, given series is

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{a+nb}$$

Let $a_n = (-1)^{n+1} \frac{1}{a+nb}$

Also, $(-1)^{n+1} \frac{1}{a+nb} < (-1)^{n+1} \frac{1}{n}$

And the series, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ is convergent being alternative harmonic series.

Again,

Here, $|a_n| = \left|(-1)^{n+1} \frac{1}{a+nb}\right| = \frac{1}{a+nb}$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{a+nb}$$

$$= \lim_{k\to\infty} \int_1^k \frac{1}{a+xb}\, dx$$

$$= \lim_{k\to\infty} \left[\frac{\ell n\,(a+bx)}{b}\right]_1^k$$

$$= \lim_{k\to\infty} \left[\frac{\ell n\,(a+kb)}{b} - \frac{\ell n\,(a+b}{b}\right]$$

$$= \infty$$

Clearly the series $\sum_{n=1}^{\infty} \frac{1}{a+nb}$ is diverges.

Hence, the given series is conditionally convergent .

e) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ell n\,(n+1)}$

Solution: Here, given series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ell n\,(n+1)}$

Let $a_n = (-1)^{n+1} \frac{1}{\ell n\,(n+1)}$

Here, $|a_n| = \left| \frac{1}{\ell n\,(n+1)} \right| = \frac{1}{\ell n\,(n+1)}$.

Now,

$$\frac{1}{\ell n\,(n+1)} > \frac{1}{n+1}$$

$$\Rightarrow \frac{1}{\ell n\,(n)} > \frac{1}{n}$$

Since the series $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)$ is diverges by p-test.

Thus $\sum |a_n| = \sum \frac{1}{\ell n\,(n+1)}$ diverges and hence the given series is divergent.

g) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{3^{n-2}}$

Solution: Here, given series is $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{3^{n-2}}$

Let $a_n = (-1)^{n-1} \frac{1}{3^{n-2}}$

Here, $|a_n| = \left| (-1)^{n-1} \frac{1}{3^{n-2}} \right| = \frac{1}{3^{n-2}} = \frac{9}{3^n}$

$\sum_{n=1}^{\infty} \frac{9}{3^n}$ is geometric series with common ratio $\frac{1}{3} < 1$, and hence it is convergent by geometric ratio test.

Thus, the given series is convergent absolutely.

8.3 Power Series, Taylor and Maclaurin's series:

Power Series:

A power series about x = 0 is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$$

A power series about x = a is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots + c_n (x-a)^n + \ldots$$

in which 'a' is center and the coefficients $c_0, c_1, c_2, \ldots + c_n + \ldots$ all are constants.

Interval, Center, and Radius of Convergence of a Power Series.

Consider the power series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1\mathrm{x} + c_2\mathrm{x}^2 + \ldots + c_n x^n + \ldots$$

If there exists a positive number r such that the series converges for |x|<r and diverges for |x|>r, then

(-r, r) is called the interval,

$\frac{-r+r}{2} = 0$ is called center

r is called radius of convergence.

NOTE:- If there exists a positive number r such that the series converges for |x|≤ r and diverges for |x|>r, then

[-r, r] is called the interval,

$\frac{-r+r}{2} = 0$ is called center

r is called radius of convergence.

Example: For what value of x does the following series converges?

$$\sum_{n=0}^{\infty}(-1)^{n-1}\left(\frac{x^n}{n}\right)$$

Solution: Let $a_n = (-1)^{n-1}\frac{x^n}{n}$

Then, $\left|\frac{a_{n+1}}{a_n}\right| = \left|(-1)^{n+1-1}\frac{x^{n+1}}{n+1}\times\frac{n}{(-1)^{n-1}\,x^n}\right|$

$$= \left|\frac{nx}{n+1}\right|$$

$$= \left|\frac{x}{1+\frac{1}{n}}\right|$$

So, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x}{1+\frac{1}{n}}\right| = |x|$

By ratio test, the given series is converges for |x|<1, i.e. for -< x <1.

At x = -1, the given series becomes

$$\sum_{n=0}^{\infty}(-1)^{n-1}\left(\frac{(-1)^n}{n}\right) = \sum_{n=0}^{\infty}(-1)^{2n-1}\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty}\frac{1}{n}$$

Which is divergent by p-test with p=1.

At x = 1, the given series becomes

$$\sum_{n=0}^{\infty}(-1)^{n-1}\left(\frac{(1)^n}{n}\right)$$

$$=\sum_{n=0}^{\infty}(-1)^{n-1}\left(\frac{1}{n}\right)$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n}\right)$$

Which is divergent being an alternative harmonic series .

Thus, the given series is convergent for any value of x in $-1 < x \leq 1$.

The Convergence Theorem for Power Series:

If the series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1\text{x}+c_2\text{x}^2 + \dots + c_n x^n + \dots$$

Converges for x = a$\neq$ 0 the it converges absolutely for all |x|<|a|. If the series diverges for x = b the it diverges for all |x|>|b|.

Development of Taylor Series

Consider the series,

$$f(x) = c_0 + c_1(x-a)+c_2(x-a)^2 + c_3(x-a)^3 \ldots + c_n(x-a)^n + \ldots \text{ (i)}$$

Differentiating f(x) term by term, we get

$$f'(x) = c_1+2c_2(x-a) + 3c_3(x-a)^2+ \ldots + n\, c_n(x-a)^{n-1} + \ldots$$

$$f''(x) = 2c_2 + 6c_3(x-a)+ \ldots + n(n\text{-}1)\, c_n(x-a)^{n-2} + \ldots$$

$$f'''(x) = 6c_3+ 24c_4(x-a)+ \ldots + n(n\text{-}1)(n\text{-}2)\, c_n(x-a)^{n-3} + \ldots$$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots. \quad \ldots \quad \ldots.$

$$f^n(x) = n(n\text{-}1)(n\text{-}2) \ldots.. 2\times 1\, c_n + \ldots$$

$$= n!\, c_n$$

At x= a,

$$f(a) = c_0$$

$$f'(a) = c_1$$

$$f''(a) = 2c_2$$

Or, $c_2 = \dfrac{f''(a)}{2!}$

$$f'''(a) = 6c_3 \Rightarrow c_3 = \frac{f'''(a)}{3!}$$

$\ldots \quad \ldots \quad \ldots$

$$f^n(a) = n!\, c_n \Rightarrow c_n = \frac{f^n(a)}{n!}$$

Replacing the values of $c_0, c_1, c_2, c_3, \ldots, c_n$ in (i), we get

$$f(x) = f(a)+ f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \ldots+ \frac{f^n(a)}{n!}(x-a)^n +\ldots$$

This is known as taylor's Series.

Thus, the Taylor's Series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \ldots + \frac{f^n(a)}{n!}(x-a)^n + \ldots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

Also, when a = 0

The Maclaurin's series generated by f is

$$f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 \ldots + \frac{f^{(n)}(0)}{n!}(x)^n + \ldots$$

Thus, the Maclurin's Series can be written as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n.$$

Define Taylor's polynomial of order n.

If f has its first n-derivatives are continuous on [a, b] or [b, a] and $f^{(n)}$ is differentiable on (a, b) or (b, a) for each positive integer n for each x in (a, b)

$$P_n(x) = f(a) + f'(x-a)(x) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(x-a)}{n!}(x)^n$$

is called Taylor's polynomial of order n.

Taylor's Formula:

If f has derivatives of all orders in an open interval I containing a then for each positive integers n and for each x in I

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \ldots + \frac{f^n(a)}{n!}(x-a)^n + \ldots + R_n(x)$$

Where the remainder $R_n(x)$ is,

$$R_n(x) = \frac{f^{(n+1)}(a)(c)}{(n+1)!}(x)^{n+1}$$

for some c between a and x.

Note1:The remainder value $R_n(x)$ is also known as error term of Taylor's Series.

Note 2: If $R_n(x) \to 0 \ as\ n \to \infty$ for all x in I then the Taylor's series reduces to

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

In this condition, we say that the Taylor's series converges to f on I.

Theorem: (Taylor's Inequality)

Is there are positive constant M and t such that $|f^{(n+1)}(t)| \leq M$ for all t between a and x then the remainder term $|R_n(x)| \leq \frac{|(x-a)^{n+1}|}{(n+1)!}$.

If these condition hold for all the other conditions of Taylor's theorem Are satisfied by f then the series converges to f(x).

Example: Find the Maclaurin's series for cos x and show that it represent cos x for all x.(TU 2077)

Solution: Here,

Let f(x) = cos x

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{iv}(x) = \cos x$

$f(0) = \cos 0 = 1$

$f'(0) = -\sin 0 = 0$

$f''(0) = -\cos 0 = -1$

$f'''(0) = \sin 0 = 0$

$f^{iv}(0) = \cos 0 = 1$

The Maclaurin's series generated by f is

$$f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \ldots + \frac{f^n(0)}{n!}(x)^n + R_n(x)$$

$$\text{Cosx} = 1 + 0\,(x) + \frac{-1}{2!}(x)^2 + \frac{0}{3!}(x)^3 + \frac{1}{4!}(x)^4 + \ldots + (-1)^{n+1}\frac{x^{2n}}{(2n)!} + R_{2n+1}(x)$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots + (-1)^{n+1}\frac{x^{2n}}{(2n)!} + R_{2n+1}(x) \quad \ldots.\text{(i)}$$

We know that cosine function and all the derivatives of cosine function have absolute value less than or equal to 1. So, with M = 1.

$$|R_{2n+1}(x)| \leq M \cdot \frac{|(x)^{2n+1}|}{(2n+1)!}.$$

$$\Rightarrow \qquad |R_{2n+1}(x)| \leq 1 \cdot \frac{|x|^{2n+1}|}{(2n+1)!}.$$

Since, $\frac{|x|^{2n+1}|}{(2n+1)!} \to 0$ as n $\to \infty$ for all values of x.

This shows that the series on the right of (i) , converges to cosx for every value of x. Hence, Maclaurin's series for cosx it represent cosx for all values of x.

Exercise 8.5

1. Find for what value of x, the following series convergent.

a) $\sum_{n=1}^{\infty} n^2 x^{n-1}$

Solution: Here, given series is

$$\sum_{n=1}^{\infty} n^2 x^{n-1}$$

Let $a_n = n^2 x^{n-1}$ each term of a_n is positive for $n \geq 0$.

$$\text{Also, } \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2\, x^{n+1-1}}{n^2 x^{n-1}}$$

$$= \lim_{n\to\infty} \frac{(n^2+2n+1)\, x^{n-n+1}}{n^2}$$

$$= \lim_{n\to\infty} \left(\frac{n^2}{n^2} + \frac{2n}{n^2} + \frac{1}{n^2}\right)x$$

$$= \lim_{n\to\infty} \left(1 + \frac{2}{n^2} + \frac{1}{n^2}\right)x$$

$$= x$$

Hence by the D'Alembert Ratio Test, series is convergent if |x|<1 .

Thus, the value of x is |x|<1. i.e. -1<x<1.

1. Find for what value of x, the following series convergent.

b) $\sum_{n=1}^{\infty}\left(\frac{2^n-2}{2^n+1}\right)x^{n-1}$

Solution: Here, given series is

$$\sum_{n=1}^{\infty}\left(\frac{2^n-2}{2^n+1}\right)x^{n-1}$$

Let $a_n=\left(\frac{2^n-2}{2^n+1}\right)x^{n-1}$ each term of a_n is positive for $n\geq 1$.

Also, $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\frac{\left(\frac{2^{n+1}-2}{2^{n+1}+1}\right)x^{n+1-1}}{\left(\frac{2^n-2}{2^n+1}\right)x^{n-1}}$

$$=\lim_{n\to\infty}\frac{\left(\frac{2^n\times2-2}{2^n\times2+1}\right)x^{n-n+1}}{\left(\frac{2^n-2}{2^n+1}\right)}$$

$$=\lim_{n\to\infty}\frac{\left(\frac{2^n(2-\frac{2}{2^n}}{2^n(2+\frac{1}{2^n}}\right)x^1}{\left(\frac{2^n(1-\frac{2}{2^n}}{2^n(1+\frac{1}{2^n}}\right)}$$

$$=\lim_{n\to\infty}\frac{\left(\frac{(2+-\frac{2}{2^n}}{(2+\frac{1}{2^n}}\right)x}{\left(\frac{(1-\frac{2}{2^n}}{(1+\frac{1}{2^n}}\right)}$$

$= x$

Hence by the D'Alembert Ratio Test, series is convergent if |x|<1 .

js sir i.e. -1<x<1.

1. Find for what value of x, the following series convergent.

c) $\sum_{n=1}^{\infty}\left(\frac{x^n}{3^n n^2}\right)$ for x>0

Solution: Here, given series is

$$\sum_{n=1}^{\infty}\left(\frac{x^n}{3^n n^2}\right) \text{ for } x>0$$

Let $a_n = \frac{x^n}{3^n n^2}$ each term of a_n is positive for $n \geq 1$.

Also, $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\left(\frac{x^{n+1}}{3^{n+1}(n+1)^2}\right)}{\left(\frac{x^n}{3^n n^2}\right)}$

$$= \lim_{n\to\infty}\frac{x^{n+1}}{3^{n+1}(n+1)^2}\times\frac{3^n n^2}{x^n}$$

$$= \lim_{n\to\infty}\frac{x\, n^2}{3\,(n^2+2n+1)}$$

$$= \lim_{n\to\infty}\frac{x\, n^2}{3\, n^2\,(1+\frac{2n}{n^2}+\frac{1}{n^2})}$$

$$= \frac{x}{3}$$

Hence by the D'Alembert Ratio Test, series is convergent if

$$\left|\frac{x}{3}\right| < 1$$

$$\Rightarrow |x| < 3.$$

$$\Rightarrow -3 < x < 3$$

And diverges for $\left|\frac{x}{3}\right| > 1.$

$\Rightarrow |x| > 3$

And at $\left|\frac{x}{3}\right| = 1$ further test is needed.

Since x > 0. So, the series is converges for 0<x< 3 and diverges for x>3 and further test is needed at x = 3.

At x = 3

$$a_n = \frac{x^n}{3^n n^2} = \frac{3^n}{3^n n^2} = \frac{1}{n^2}$$

Here, $= \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right)$ is convergent by p-test with p=2.

Thus, the given series is convergent for 0<x≤ 3 and divergent for x>3.

Hence the interval of convergence of the given series is (0,3].

2. Find the interval, centre and radius of convergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$

Solution: Here, given series is

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

Let $a_n = \frac{x^{n-1}}{(n-1)!}$ **each term of** a_n **is positive for** $n \geq 1$.

Also, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{x^{n+1-1}}{(n+1-1)!}}{\frac{x^{n-1}}{(n-1)!}}$

$$= \lim_{n\to\infty} \frac{x^n}{n!} \times \frac{(n-1)!}{x^{n-1}}$$

$$= \lim_{n\to\infty} \frac{x^{n-n+1}}{n\,(n-1)!} \times \frac{(n-1)!}{1}$$

$$= \lim_{n\to\infty} \frac{x}{n}$$

$$= 0$$

Hence by the D'Alembert Ratio Test, series is convergent for all values of x $(-\infty, \infty)$.

2\. b) $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

Solution: Here, given series is

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

n^{th} term $(a_n) = (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)}$

$$|a_n| = |(-1)^{n+1} \frac{x^{2n-1}}{(2n-1)}| = \frac{x^{2n-1}}{(2n-1)}$$

Also, $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \left| \frac{\frac{x^{2(n+1)-1}}{(2(n+1)-1)}}{\frac{x^{2n-1}}{(2n-1)}} \right|$

$$= \lim_{n\to\infty} \left| \frac{x^{2n+1}}{2n+1} \times \frac{2n-1}{x^{2n-1}} \right|$$

$$= \lim_{n\to\infty} |x^2| \left| \frac{2n-1}{2n+1} \right|$$

$$= |x^2|$$

Hence by the D'Alembert Ratio Test, series is convergent if $|x^2| < 1$.

$$|x^2| < 1.$$

$\Rightarrow$ $\quad |x| < 1.$

i.e. -1<x<1.

And the series is divergent if |x|>1 and further test is required for |x |= 1.

At x = 1

$$a_n = (-1)^{n+1}\frac{x^{2n-1}}{(2n-1)}$$

$$= (-1)^{n+1}\frac{1^{2n-1}}{(2n-1)}$$

$$= (-1)^{n+1}\frac{1}{(2n-1)} < (-1)^{n+1}\frac{1}{n}$$

Here, the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is alternative harmonic series. So, it is convergent.

Again, at x = -1

$$a_n = (-1)^{n+1}\frac{(-1)^{2n-1}}{(2n-1)}$$

$$= (-1)^{n+2}\frac{1}{2n-1}$$

$$= (-1)^{n+2}\frac{1}{(2n-1)} < (-1)^{n+2}\frac{1}{n}$$

Here, the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is alternative harmonic series. So, it is convergent.

Hence the interval of convergence is $-1 \leq x \leq 1$

i.e. [-1, 1]

Center of convergence $= \frac{1+(-1)}{2} = 0$

Radius of convergence $= \frac{1-(-1)}{2} = 1$

3. Using Maclaurin series expansion, show that

a) $\text{Cos x} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots + (-1)^{n+1}\frac{x^{2n}}{(2n)!} + \cdots$

Solution: Here,

Let $f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{iv}(x) = \cos x$

$f(0) = \cos 0 = 1$

$f'(0) = -\sin 0 = 0$

$f''(0) = -\cos 0 = -1$

$f'''(0) = \sin 0 = 0$

$f^{iv}(0) = \cos 0 = 1$

The Maclaurin's series generated by f is

$$f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \ldots + \frac{f^n(0)}{n!}(x)^n + \ldots$$

$R_n(x)$

$$\text{Cos}x = 1 + 0(x) + \frac{-1}{2!}(x)^2 + \frac{0}{3!}(x)^3 + \frac{1}{4!}(x)^4 + \ldots + (-1)^{n+1}\frac{x^{2n}}{(2n)!} + \ldots$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots + (-1)^{n+1}\frac{x^{2n}}{(2n)!} + \ldots$$

Example: (Q.6)

Show that the Maclaurin's series for sin x converges to sin x for all x.(TU 2077)

Solution: Here,

Let $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{iv}(x) = \sin x$

$f(0) = \sin 0 = 0$

$f'(0) = \cos 0 = 1$

$f''(0) = -\sin 0 = 0$

$f'''(0) = -\cos 0 = -1$

$f^{iv}(0) = \sin 0 = 0$

The Maclaurin's series generated by f is

$$f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \ldots + \frac{f^n(0)}{n!}(x)^n + R_{2n+1}(x)$$

$$\sin x = 0 + 1.(x) + \frac{0}{2!}(x)^2 - \frac{1}{3!}(x)^3 + \frac{0}{4!}(x)^4 - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + R_{2n+1}(x)$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots + (-1)^{n+1}\frac{x^{2n+1}}{(2n+1`)!} + R_n(x) \quad \ldots.(i)$$

We know that sine function and all the derivatives of sine function have absolute value less than or equal to 1. So, with $M = 1$.

$$|R_{2n+1}(x)| \leq M \, . \frac{|(x)^{3n+1}|}{(3n+1)!}.$$

$$\Rightarrow \qquad |R_{2n+1}(x)| \leq 1 \, . \frac{|x|^{3n+1}|}{(3n+1)!}.$$

Since, $\frac{|x|^{3n+1}|}{(3n+1)!} \to 0$ **as n** $\to \infty$ **for all values of x.**

This shows that the series on the right of (i) , converges to sin x for every value of x. Hence, Maclaurin's series for sinx it represent sinx for all values of x.

Q. Find the *Maclaurin* series for e^x and show that it represents e^x for all x .(TU)

Solution: See Example 15 (book, page 272)