Chapter -8

Infinite Sequence and Series

Infinite sequence:

A sequence is a list of numbers which are written as some definite order.

For example: $\{a_n\} = a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$

Here,

first term = a_1

second term = a_2

…… ….. ….. …..

n^{th} **term/general term** = a_n **or** u_n

Bounded And unbounded Sequence:

- A sequence $\{a_n\}$ is ***bounded*** if there are two fixed value k, K $\in \mathbb{R}$ such that $k \le a_n \le K$, for all n.

- A sequence $\{a_n\}$ is **bounded above** if is real number

k $\in \mathbb{R}$ such that $a_n \le k$, for all n. In such case, the sequence is called unbounded below.

- A sequence $\{a_n\}$ is **bounded below** if is real number

k $\in \mathbb{R}$ such that k $\le a_n$, for all n. In such case, the sequence is called unbounded above.

- A sequence $\{a_n\}$ is called **unbounded** if it is no bounded above and below.

Example:

Show that the sequence $a_n = \frac{1}{n}$ for n $\in \mathbb{N}$ is bounded.

Solution: Here given sequence is

$$a_n = \frac{1}{n} \text{ for n} \in \mathbb{N}.$$

Therefore,

$$\{a_n\} = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} .$$

We see that $u_n \leq 1$ for all n.

Also,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

Thus,

$$0 \leq a_n \leq 1 \qquad \text{for all n.}$$

Hence, this shows that $\{a_n\}$ is bounded below by 0 and bounded above by 1.

3. Determine whether the sequence converges or diverges. If it converges, find the limit.

a) $a_n = 1 - (0.2)^n$

Solution: Here,

$$a_n = 1 - (0.2)^n$$

$$= 1 - \left(\frac{2}{10}\right)^n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left\{1 - \left(\frac{2}{10}\right)^n\right\}$$

Here, $\lim_{n\to\infty}\left(\frac{2}{10}\right)^n = 0$

Thus, $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left\{1 - \left(\frac{2}{10}\right)^n\right\}$

$$= \lim_{n\to\infty} a_n\, 1 - \lim_{n\to\infty}\left(\frac{2}{10}\right)^n$$

$$= 1 - 0$$

$$= 1$$

Hence, the sequence $a_n = 1 - (0.2)^n$ converges and the limit is 1.

b) $a_n = \frac{3+5n^2}{n+n^2}$

Solution: Here,

$$a_n = \frac{3+5n^2}{n+n^2}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left\{\frac{3+5n^2}{n+n^2}\right\} \quad [\because \tfrac{\infty}{\infty}]$$

$$= \lim_{n\to\infty}\left\{\frac{\frac{3+5n^2}{n^2}}{\frac{n+n^2}{n^2}}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{\frac{3}{n^2}+5}{\frac{1}{n}+1}\right\}$$

$$= \left\{\frac{\frac{3}{\infty^2}+5}{\frac{1}{\infty}+1}\right\}$$

$$= 5$$

Hence, the sequence a_n converges and the limit is 5.

c) $a_n = \frac{n^3}{n+1}$

Solution: Here,

$$a_n = \frac{n^3}{n+1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left\{\frac{n^3}{1+n}\right\} \quad [\because \tfrac{\infty}{\infty}]$$

$$= \lim_{n\to\infty}\left\{\frac{\frac{n^3}{n^3}}{\frac{1+n}{n^3}}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{1}{\frac{1}{n^3}+\frac{1}{n^2}}\right\}$$

$$=\left\{\frac{1}{\frac{1}{\infty}+\frac{1}{\infty}}\right\}$$

$$= \frac{1}{0}$$

$$= \infty$$

Hence, the sequence $a_n = \frac{n^3}{n+1}$ diverges .

e) $a_n = \frac{3^{n+2}}{5^n}$

Solution: Here

$$a_n = \frac{3^{n+2}}{5^n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left\{\frac{3^{n+2}}{5^n}\right\} \quad [\because \frac{\infty}{\infty}]$$

$$= \lim_{n\to\infty}\left\{\frac{3^n \times 9}{5^n}\right\}$$

$$= \lim_{n\to\infty} 9\left\{\frac{3}{5}\right\}^n$$

$$= 9 \times 0$$

$$= 0$$

Hence, the sequence a_n converges and the limit is 0.

f) $a_n = tan\left(\frac{2n\pi}{1+8n)}\right)$

Solution: Here,

$$a_n = tan\left(\frac{2n\pi}{1+8n)}\right)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left\{tan\left(\frac{2n\pi}{1+8n)}\right)\right\}$$

$$= tan\left(\lim_{n\to\infty}(\frac{2n\pi}{1+8n)})\right) \lim_{n\to\infty}\left\{\frac{3^n\times 9}{5^n}\right\}$$

$$= \tan\left(\lim_{n\to\infty}(\frac{\frac{2n\pi}{n}}{\frac{1+8n}{n}})\right)$$

$$= tan\left(\lim_{n\to\infty}\frac{2\pi}{\frac{1}{n}+8}\right)$$

$$= \tan\frac{2\pi}{8}$$

$$= 1$$

Hence, the sequence a_n converges and the limit is 1.

i) $a_n = \frac{(2n-1)!}{(2n+1)!}$

Solution: Here,

$$\begin{aligned} a_n &= \frac{(2n-1)!}{(2n+1)!} \\ &= \frac{(2n-1)!}{(2n+1)(2n)(2n-1)!} \\ &= \frac{1}{2n(2n+1)} \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty}\left\{\frac{1}{2n(2n+1)}\right\} \\ &= \lim_{n\to\infty}\left\{\frac{1}{4n^2+2n}\right\} \\ &= \frac{1}{\infty} \\ &= 0 \end{aligned}$$

Hence, the sequence a_n converges and the limit is 0.

4.a) Determine whether the sequence defined as follows is convergent or divergent:

$$a_1 = 1, a_{n+1} = 4 - a_n \text{ for } n \geq 1$$

Solution: Here,

$$a_1 = 1,$$

$$a_{n+1} = 4 - a_n$$

So,

$$a_{1+1} = 4 - a_1 = 4 - 1 = 3$$

$$a_2 = 3$$

$$a_3 = 4 - a_2 = 4 - 3 = 1$$

$$a_4 = 4 - a_3 = 4 - 1 = 3$$

And so on.

Thus, $\{a_n\} = \{a_1, a_2, a_3, a_4 \ldots.\}$

$= \{3, 1, 3, 1, 3, \ldots\}$

We see that , this sequence is divergent.

5. Determine if the sequence is non-decreasing and if it is bounded above or below.

a) $a_n = \frac{3n+1}{n+1}$

Solution: Here,

$$a_n = \frac{3n+1}{n+1}$$

$$a_{n+1} = \frac{3(n+1)+1}{(n+1)+1} = \frac{3n+4}{n+2}$$

$$a_n - a_{n+1} = \frac{3n+1}{n+1} - \frac{3n+4}{n+2}$$

$$= \frac{(3n+1)(n+2)-(3n+4)(n+1)}{(n+1)(n+2)}$$

$$= \frac{3n^2+6n+n+2-3n^2-3n-4n-4}{(n+1)(n+2)}$$

$$= \frac{-2}{(n+1)(n+2)} < 0$$

$$\Rightarrow a_n - a_{n+1} < 0$$

$$\Rightarrow a_n < a_{n+1}$$

This shows that the sequence is increasing(non-decreasing).

Again, for bounded:

$$a_n = \frac{3n+1}{n+1}$$

$$a_1 = \frac{3\times1+1}{1+1} = 2$$

$$a_2 = \frac{3\times2+1}{2+1} = \frac{7}{3} = 2.333$$

$$a_3 = \frac{3\times3+1}{3+1} = \frac{10}{4} = 2.5$$

$$a_4 = \frac{3\times4+1}{4+1} = \frac{13}{5} = 2.6 \text{ and so on.}$$

We see that a_n is bounded below by 2 (as $2 \leq a_n$).

Also,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left\{\frac{3n+1}{n+1}\right\} = \lim_{n\to\infty}\left\{\frac{\frac{3n+1}{n}}{\frac{n+1}{n}}\right\} = 3$$

Hence, the sequence a_n bounded above by 3.

Thus, $2 \leq a_n \leq 3$ and hence the sequence is bounded.

6. Determine whether the sequence is increasing or decreasing or not monotonic.

a) $\boldsymbol{a_n = (-2)^{n+1}}$

Solution: **Here,**

$$\boldsymbol{a_n = (-2)^{n+1}}$$

$$\boldsymbol{a_1 = (-2)^{1+1} = 4}$$

$$\boldsymbol{a_2 = (-2)^{2+1} = -8}$$

$$\boldsymbol{a_3 = (-2)^{3+1}} = 16 \text{ and so on.}$$

We see that, the sequence is not monotonic and not bounded.

c) $a_n = \frac{2n-3}{3n+4}$

Solution: **Here,**

$$a_n = \frac{2n-3}{3n+4}$$

$$a_1 = \frac{2\times1-3}{3\times1+4} = \frac{-1}{7}$$

$$a_2 = \frac{2\times2-3}{3\times2+4} = \frac{1}{10}$$

$$a_3 = \frac{2\times3-3}{3\times3+4} = \frac{3}{13}$$

$$a_4 = \frac{2\times4-3}{3\times4+4} = \frac{5}{16}$$

We see that the sequence is increasing as $a_n < a_{n+1}$ and bounded below by $\frac{-1}{7}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left\{\frac{2n-3}{3n+4}\right\}$$

$$= \frac{2}{3}$$

Hence the sequence is bounded above by $\frac{2}{3}$.

Thus, $-\frac{1}{7} \leq a_n \leq \frac{2}{3}$

And therefor the sequence is bounded.

e) $a_n = \mathrm{n} + \frac{1}{n}$

Solution: Here,

$$a_n = \mathrm{n} + \frac{1}{n}$$

$$a_1 = 1 + \frac{1}{1} = 2$$

$$a_2 = 2 + \frac{1}{2} = 2.5$$

$$a_3 = 3 + \frac{1}{3} = 3.33$$

$$a_4 = 4 + \frac{1}{4} = 4.25$$

We see that the sequence is increasing and bounded below by 2.

Also,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left\{n + \frac{1}{n}\right\} = \infty + 0 = \infty$$

Hence the sequence is not bounded above.

Infinite Series:

A sequence of numbers $\{a_n\}$, an expression

$a_1 + a_2 + a_3 + a_4 + \ldots = \sum_{n=1}^{\infty} a_n$

Is called an infinite series.

Partial sum:

If we take only finite number of terms from the infinite series, then series of such finite terms is called partial sum of the series.

For a given series $\sum_{n=1}^{\infty} a_n$, the finite k-terms in the form

$$\sum_{n=1}^{k} a_n$$

is called k^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ and denoted by S_k.

i.e. $S_k = \sum_{n=1}^{k} a_n$.

Convergence and Divergence of an infinite series:

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series. If there a finite value L such that

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i = \mathrm{L} \qquad \text{i.e. } \lim_{n\to\infty} S_n = \mathrm{L},$$

then we say that the given series is convergent to L

Otherwise the series is divergent.

Telescoping series:

A series in which on expression of n^{th} partial sum, every term except first and last term are cancelled out is called telescopic series.

Example: Examine the convergency of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Solution: Given series is

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Partial sum , S_k $= \sum_{n=1}^{k} \frac{1}{n(n+1)}$

$$= \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots. + \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= 1 - \frac{1}{k+1}$$

[first and last terms are remaining.

So, it is telescopic series]

Then,

$$\lim_{n\to\infty} S_k = \lim_{n\to\infty} \left(1 - \frac{1}{k+1}\right) = 1.$$

Hence the series is convergent and its limit is 1.

Geometric Series:

A series of the form a + ar + ar^2 + … + ar^{n-1} + … = $\sum_{n=1}^{\infty} ar^{n-1}$ where a is non-zero first term and r is fixed ratio, is called infinite geometric series.

Theorem:(Geometric Ration Test)

The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ = a + ar + ar^2 + … + ar^{n-1} + … converges to $\frac{a}{1-r}$ if $|r|<1$ and diverges if $r \geq 1$.

i.e. Sum of the infinite geometric series = $\frac{a}{1-r}$ for $|r|<1$.

Theorem: The necessary condition for the convergence of an infinite series $\sum_{n=1}^{\infty} a_n$ is $\lim\limits_{n\to\infty} a_n = 0$. But the condition is not sufficient.

Proof: Let $\sum_{n=1}^{\infty} a_n$ be an infinite series.

Let $S_n = a_1 + a_2 + a_3 \ldots + a_n$

Suppose that the series $\lim\limits_{n\to\infty} \sum_{n=1}^{\infty} a_n$ is convergent to a finite value S.

So, $\lim\limits_{n\to\infty} S_n = S$

Also, $\lim\limits_{n\to\infty} S_{n-1} = S$

Now,

$a_n = S_n - S_{n-1}$

Thus,

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (S_n - S_{n-1}) = S - S = 0$

Hence, if the series is converges then necessarily $\lim\limits_{n\to\infty} a_n = 0$.

But the condition is not sufficient. That is $\lim_{n\to\infty} a_n = 0$ may not imply that the series converges.

For example: Take a series $\sum a_n = \sum \left(\frac{1}{n}\right)$

Here, $\lim_{n\to\infty} a_n = 0$ but the series is divergent.

(see book page-236 , example-8)

n^{th} term test for divergent:

If $\lim_{n\to\infty} a_n$ does not exist of $\lim_{n\to\infty} a_n \neq 0$ then the series is divergent.

Example: Prove that the series $\frac{1}{3}+\frac{2}{4}+\frac{3}{5}+\frac{4}{6}+\ \dots$ is divergent.

Solution: Given series is

$$\frac{1}{3}+\frac{2}{4}+\frac{3}{5}+\frac{4}{6}+\ \dots$$

Here, n^{th} term $a_n = \frac{n}{n+2}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+2} = \frac{1}{1+0} = 1 \neq 0$$

Therefore, by n^{th} term test for divergence, the series is divergent.

Exercise 8.2

Test the convergency of the following infinite series. If convergent, find its sum.

1. **Apply geometric series test**

a) $\sum_{n=0}^{\infty}\left(\frac{(-1)^n\,2}{3^n\;5}\right)$.

Solution: Here, given series is

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n\,2}{3^n\;5}\right)$$

which is a geometric series with first term (a) $=\frac{2}{5}$ and common ratio (r) $= -\frac{1}{3}$. So, $|r| = \left|-\frac{1}{3}\right| = \frac{1}{3} < 1$

Thus, the series is convergent by geometric ratio test.

Sum of the series $= \frac{a}{1-r} = \frac{\frac{2}{5}}{1+\frac{1}{3}} = \frac{3}{10}$.

c) $\sum_{n=2}^{\infty}\left(\sqrt{3}\right)^{n}$

Solution: Here, given series is

$$\sum_{n=2}^{\infty}\left(\sqrt{3}\right)^{n} = 3 + 3\sqrt{3} + 9 + 9\sqrt{3} + \ldots$$

which is a geometric series with first term (a) = 3

and common ratio (r) = $-\sqrt{3}$.

Here, $|r| = |\sqrt{3}| > 1$

Thus, the series is divergent by geometric ratio test.

d. $\sum_{n=0}^{\infty}\left(\frac{2^n+5}{3^n}\right)$

Solution: **Here, given series is**

$$\sum_{n=0}^{\infty}\left(\frac{2^n+5}{3^n}\right)=\sum_{n=0}^{\infty}\left(\frac{2^n}{3^n}\right)+\sum_{n=0}^{\infty}\left(\frac{5}{3^n}\right)$$

Here, 1st series on the right hand side is a geometric series with first term (a) = 1 and common ratio (r) = $\frac{2}{3}<1$.

Thus, the series $\sum_{n=0}^{\infty}\left(\frac{2^n}{3^n}\right)$ **is convergent by geometric ratio test.**

Sum of 1st series $=\frac{a}{1-r}=\frac{1}{1-\frac{2}{3}}=3.$

Again,

Here, 2nd series on the right hand side is a geometric series with first term (a) = 5 and common ratio (r) = $\frac{1}{3}<1$.

Thus, the series $\sum_{n=0}^{\infty}\left(\frac{5}{3^n}\right)$ **is convergent by geometric ratio test.**

Sum of 2nd series $=\frac{a}{1-r}=\frac{5}{1-\frac{1}{3}}=\frac{15}{2}$

Sum of the given series $\sum_{n=0}^{\infty}\left(\frac{2^n+5}{3^n}\right)=3+\frac{15}{2}=\frac{21}{2}$

g) $\sum_{n=0}^{\infty} cos(n\pi)$

Solution: **Here, given series is**

$$\sum_{n=0}^{\infty} cos(n\pi) = (-1)^n$$

$$= 1 - 1 + 1 - 1 + 1 \ ...$$

which is a series with alternating terms.

So, this series does not converges.

2. a) $\sum_{n=1}^{\infty} \frac{4}{(4n+1)(4n-3))}$.

Solution: Given series is

$$\sum_{n=1}^{\infty} \frac{4}{(4n+1)(4n-3))}$$

Partial sum , $S_k = \sum_{n=1}^{k} \frac{4}{(4n+1)(4n-3))}$.

$$= \sum_{n=1}^{k} \left(\frac{1}{4n-3} - \frac{1}{4n+1}\right)$$

$$= \left(\frac{1}{1} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{13}\right) + \dots + \left(\frac{1}{4k-3} - \frac{1}{4k+1}\right)$$

$$= 1 - \frac{1}{4k+1}$$

First and last terms are remaining. So, it is telescopic series.

Then,

$$\lim_{n\to\infty} S_k = \lim_{n\to\infty}\left(1 - \frac{1}{4k+1}\right) = 1.$$

Hence the series is convergent and its limit is 1.

3. Show that the following series are divergent.

a) $\sum_{n=1}^{\infty} n^2$

Solution : Here, the given series is

$$\sum_{n=1}^{\infty} n^2$$

$$n^{th} \text{ term} = n^2$$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \infty$ **[limit does not exists.]**

Therefore, by n^{th} term test for divergence, the series $\sum_{n=1}^{\infty} n^2$ is divergent.

n^{th} term test for divergent:

If $\lim_{n\to\infty} a_n$ does not exist of $\lim_{n\to\infty} a_n \neq 0$ then the series is divergent.

Integral Test:

Suppose that f is continuous, positive, decreasing function on$[1,\infty)$ and $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent.

In other words,

i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent .

ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent .

Note: When we use the integral test, it is not necessary to start the series or the integral at n= 1. Also, it is not necessary that f be always decreasing.

Q. Show that the series $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$ converges. (T.U)

Solution:

Here, $f(n) = \frac{1}{1+n^2}$

Now,

$$\int_0^{\infty} f(x)dx = \lim_{n\to\infty} \int_0^{n} \frac{1}{1+x^2} dx$$

$$= \lim_{n\to\infty} [\frac{1}{1} tan^{-1}(\frac{x}{1})]_1^n$$

$$= \lim_{n\to\infty} [\tan^{-1}(n) - \tan^{-1}(1)]$$

$$= tan^{-1}(\infty) - \frac{\pi}{4}$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{3\pi}{4}$$

This shows that $\int_0^{\infty} f(x)dx$ is convergent and hence, by Integral Test, the series $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$ is convergent.

Divergence of harmonic series:(important)

- **Show that a harmonic series $\sum_{n=0}^{\infty}\left(\frac{1}{n}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \ \ldots + \frac{1}{n} + \ \ldots$, is diverges.**

Solution: Here,

$$\int_1^{\infty} f\left(\frac{1}{x}\right) dx = \lim_{k\to\infty} \int_1^{k} \frac{1}{x}\, dx$$

$$= \lim_{k\to\infty} [\text{In} x]_1^k$$

$$= \lim_{k\to\infty} [Ink - \text{In}1]$$

$$= In\infty - In1$$

$$= \infty - 0$$

$$= \infty$$

Thus, the integral $\int_1^{\infty} f\left(\frac{1}{x}\right) dx$ has no fixed value. So, the integral is divergent. Then, by integral test, the series $\sum_{n=0}^{\infty}\left(\frac{1}{n}\right)$ diverges .

Does the series $\sum_{n=0}^{\infty}\left(\frac{1}{n^2}\right)$ convergent?

- Test the convergency of a p-series $\sum_{n=1}^{\infty}\left(\frac{1}{n^p}\right)$ by integral test. (VVI)

OR

Show that the series $\sum_{n=1}^{\infty}\left(\frac{1}{n^p}\right)$ is converges for p> 2 and diverges for p$\leq$ 1.

Solution: Since f(n) $= \frac{1}{n^p}$.

Case I: For P = 1, the series reduces to $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$, which is a harmonic series. Clearly, this series is divergent by divergence of harmonic test.

Case II: If p<1, then (1- p)> 0.

$$
\begin{aligned}
\int_1^{\infty} f\left(\frac{1}{x^p}\right) dx &= \lim_{k\to\infty} \int_1^k x^{-p}\, dx \\
&= \lim_{k\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^k \\
&= \left(\frac{1}{1-p}\right) \lim_{k\to\infty} \left[x^{1-p}\right]_1^k \\
&= \left(\frac{1}{1-p}\right) \lim_{k\to\infty} \left[k^{1-p}-1\right] \\
&= \left(\frac{1}{1-p}\right) [\infty - 1] \\
&= \infty
\end{aligned}
$$

Thus, the integral $\int_1^{\infty} f\left(\frac{1}{x^p}\right) dx$ has no fixed value. So, the integral is divergent. Then, by integral test, the series $\sum_{n=0}^{\infty}\left(\frac{1}{n^p}\right)$ diverges for p<1.

Case III: If p>1, then (p- 1)> 0.

$$\int_1^\infty f\left(\frac{1}{x^p}\right) dx = \lim_{k\to\infty} \int_1^k x^{-p}\ dx$$

$$= \lim_{k\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^k$$

$$= \left(\frac{1}{1-p}\right) \lim_{k\to\infty} \left[x^{-(p-1)}\right]_1^k$$

$$= \left(\frac{1}{1-p}\right) \lim_{k\to\infty} \left[k^{-(p-1)} - 1\right]$$

$$= \left(\frac{1}{1-p}\right) \lim_{k\to\infty} \left[\frac{1}{k^{(p-1)}} - 1\right]$$

$$= \left(\frac{1}{1-p}\right) [0-1] \text{ , as (p-1) >0}$$

$$= \frac{1}{p-1}$$

Thus, the integral $\int_1^\infty f\left(\frac{1}{x^p}\right) dx$ has fixed value. So, the integral is convergent. Then, by integral test, the series $\sum_{n=0}^{\infty}\left(\frac{1}{n^p}\right)$ converges for p>1.

Theorem: (Comparison Test)

Let $\sum a_n$ be a series of non-negative terms.

a) If there is a convergent series $\sum b_n$ with $a_n \leq b_n$ for all n$\geq N$, then $\sum a_n$ is also convergent.

b) If there is a divergent series $\sum c_n$ with $a_n \geq c_n$ for all n$\geq N$, then $\sum a_n$ is also divergent.

Example: Apply the Comparison Test for $\sum_{n=1}^{\infty}\left(\frac{7}{7n-2}\right)$.

Solution: Here,

$$\text{let } \sum a_n = \sum_{n=1}^{\infty}\left(\frac{7}{7n-2}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{n-\frac{2}{7}}\right)$$

We know, $\left(n-\frac{2}{7}\right) < \mathbf{n}$ **for all n.**

$$\frac{1}{n-\frac{2}{7}} > \frac{1}{n}$$

Suppose $\sum c_n = \frac{1}{n}$

Here, $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$ diverges by p-test. Thus the series $\sum_{n=1}^{\infty}\left(\frac{7}{7n-2}\right)$ also diverges by Comparison Test.

Theorem: (The Limit Comparison Test)

Suppose $\sum a_n$ and $\sum b_n$ are series of positive terms. If,

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \ell$$

a) With ℓ is finite number $\ell > 0$ then either both $\sum a_n$ and $\sum b_n$ converges or both diverges.

b) With $\ell = 0$ or $\ell = \infty$, then the series $\sum a_n$ diverges.

Theorem: (D' Alembert Ratio Test)

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)=\ell.$$

Then,

a. If $\ell<1$ then the series converges

b. If $\ell>1$ then the series diverges

c. If $\ell=1$ then the test is inclusive and further test is needed.

Example: Test the convergency of the series $\sum_{n=1}^{\infty}\left(\frac{n^n}{n!}\right)$.

Solution: **Here, given series is**

$$\sum_{n=1}^{\infty}\left(\frac{n^n}{n!}\right)$$

General term of the series is

$$a_n = \frac{n^n}{n!}$$

Here, $\lim\limits_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim\limits_{n\to\infty}\left(\frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n}\right)$

$$= \lim_{n\to\infty}\left(\frac{(n+1)^{n+1}}{(n+1)\, n!} \times \frac{n!}{n^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

$$= e \approx 2.71 > 1$$

So, the given series is divergent by (D' Alembert Ration Test)

Theorem: (The n^{th} Root Test)(Cauchy's Radical Test)

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \ell.$$

Then,

a. If $\ell < 1$ then the series converges

b. If $\ell > 1$ then the series diverges

c. If $\ell = 1$ then the test is inclusive and further test is needed.

Alternative Series:

An infinite series $\sum(-1)^n a_n$ is known as an alternative series.

Leibnitz's Theorem (The Alternative Series Test)

The series,

$$\sum_{n=1}^{\infty}(-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \ \ldots$$

Converges if it satisfies the following conditions:

a) $b_n > 0$, for all n

b) $b_n \geq b_{n+1}$ for all n$\geq N$ for some integer N.

C) $\lim_{n\to\infty} b_n = 0.$

Convergence of an Alternative Harmonic Series:

Show that the alternative harmonic series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ \ldots\ \boldsymbol{converges}$.

Solution: Here, given series is $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ \ldots$

Let $b_n = \frac{1}{n}$

Here, a) $b_n>0$ for n≥ 1

b)$b_{n+1} - b_n = \frac{1}{n+1} - \frac{1}{n} = \frac{n-n-1}{n(n+1)} = \frac{-1}{n(n+1)} < 0$

$\Rightarrow b_{n+1} - b_n < 0$

$\Rightarrow b_{n+1} < b_n$ for all n.

c) $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0.$

Hence by Leibnitz's Theorem , the given series is convergent.

Absolute and Conditional Convergence

A series $\sum \boldsymbol{a_n}$ converges absolutely if the corresponding series $\sum |\boldsymbol{a_n}|$ converges.

Theorem (The Absolute Convergence Test)

If $\sum |\boldsymbol{a_n}|$ converges then series $\sum \boldsymbol{a_n}$ converges.

Strategy for Testing Series:

1) If the series of the form $\sum_{n=1}^{\infty}\left(\frac{1}{n^p}\right)$, it is p-series, which is convergent if p>1 and divergent if $p \leq 1$.

2) If the series is of the form $\sum ar^{n-1}$ or $\sum ar^n$, it is geometric, which is convergent if $|r|<1$ and divergent if $|r| \geq 1$.

3) If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. If a_n is a rational function or an algebraic function of n (involving roots of polynomial) the series should be compared with a p-series. The comparison tests apply only two series with positive terms. But if $\sum a_n$ has negative terms, then we can apply the comparison test to $\sum |a_n|$ and test for absolute convergence.

4) If we can see that $\lim_{n\to\infty} a_n \neq 0$ then the test of divergence should be used.

5) The series that involve factorial notation or other products (including a constant raised to the n^{th} power) are often conveniently tested using the ratio test.

6) If a_n is of the form $(b_n)^n$, then the Root Test may be useful.

7) If a_n= f(n), where $\int_1^{\infty} f(x)dx$ is easily evaluated, then the Integral Test is effective.

Exercise 8.3

1. Test the convergence of the series by Integral Test:

a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Integral Test:

Suppose that f is continuous, positive, decreasing function on$[1,\infty)$ and a_n = f(n). Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent.

In other words,

i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent .

ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent .

Note: When we use the integral test, it is not necessary to start the series or the integral at n= 1. Also, it is not necessary that f be always decreasing.

a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Solution:

Here, $f(n) = \frac{1}{n^3}$

Now,

$$\int_1^{\infty} f(x)dx = \lim_{n\to\infty} \int_1^n \frac{1}{x^3}\, dx$$

$$= \lim_{n\to\infty} \int_1^n x^{-3}\, dx$$

$$= \lim_{n\to\infty} \left[\frac{x^{-3+1}}{-3+1}\right]_1^n$$

$$= \lim_{n\to\infty} \left[-\frac{1}{2x^2}\right]_1^n$$

$$= \lim_{n\to\infty} \left[-\frac{1}{2n^2} + \frac{1}{2}\right]$$

$$= 0+\frac{1}{2}$$

$$= \frac{1}{2}$$

This shows that $\int_1^{\infty} f(x)dx$ is convergent and hence, by Integral Test, the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent.

b) $\sum_{n=2}^{\infty} \frac{\ell n\, n}{n}$

Solution:

Here, $f(n) = \frac{\ell n\, n}{n}$

Now,

$$\int_2^{\infty} f(x)dx = \int_2^{\infty} \frac{\ell n\, x}{x}\, dx$$

Let $\ell n\, x = y$

$\frac{1}{x}\, dx = dy$ and as $x = 2$; $y = \ell n\, 2$ and as $x = \infty$; $y = \infty$

Now,
$$\int_2^{\infty} f(x)dx = \int_{\ell n 2}^{\infty} \frac{\ell n\, x}{x}\, dx$$
$$= \lim_{n\to\infty} \int_{\ell n 2}^{n} y]\ dy$$
$$= \lim_{n\to\infty} [\frac{y^2}{2}]_{\ell n 2}^{n}$$
$$= \lim_{n\to\infty} [\frac{(n)^2}{2} - \frac{(\ell n\, 2)^2}{2}]$$
$$= \infty - \frac{(\ell n\, 2)^2}{2}$$
$$= \infty$$

This shows that $\int_2^{\infty} f(x)dx$ is divergent and hence, by Integral Test, the series $\sum_{n=2}^{\infty} \frac{\ell n\, n}{n}$ is convergent.

c) $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

Solution:

Here, $f(n) = \frac{e^n}{1+e^{2n}}$

Now,

$$\int_1^\infty f(x)dx = \lim_{n\to\infty} \int_1^n \frac{e^x}{1+e^{2x}}\, dx$$

Let $e^x = y$

$e^x dx = dy$. As x = 1 then y = e and as x= n ,then y = e^n

$$= \lim_{n\to\infty} \int_e^{e^n} \frac{1}{1+y^2}\, dy$$

$$= \lim_{n\to\infty} [\tan^{-1} y]_e^{e^n}$$

$$= \lim_{n\to\infty} [\tan^{-1}(e^n) - tan^{-1}(e)]$$

$$= tan^{-1}(e^\infty) - tan^{-1}(e)$$

$$=\frac{\pi}{2} - tan^{-1} e$$

This shows that $\int_1^\infty f(x)dx$ is convergent and hence, by Integral Test, the series $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$ is converges.

d) $\sum_{n=1}^{\infty} \frac{8\tan^{-1}n}{1+n^2}$

Solution:

Here, $f(n) = \frac{8\tan^{-1}n}{1+n^2}$

Now,

$$\int_1^{\infty} f(x)dx = \lim_{n\to\infty} \int_1^n \frac{8\tan^{-1}x}{1+x^2}\,dx$$

Let $\tan^{-1}x = y$

$\frac{1}{1+x^2}dx = dy$. As x = 1 then $y = \frac{\pi}{4}$ and as x= n ,then $y = \tan^{-1}n$

$$\text{So, } \lim_{n\to\infty} \int_1^n \frac{8\tan^{-1}x}{1+x^2}\,dx = \lim_{n\to\infty} \int_{\frac{\pi}{4}}^{\tan^{-1}n} 8y\,dy$$

$$= 8\lim_{n\to\infty} \left[\frac{y^2}{2}\right]_{\frac{\pi}{4}}^{\tan^{-1}n}$$

$$= 8\lim_{n\to\infty} \left[\frac{(\tan^{-1}n)^2}{2} - \frac{1}{2}\left(\frac{\pi}{4}\right)^2\right]$$

$$= \frac{1}{2} \times 8\left[(\tan^{-1}(\infty))^2 - \frac{\pi^2}{32}\right]$$

$$= 4\left[\frac{\pi}{2} - \frac{\pi^2}{32}\right]$$

This shows that , $\int_1^{\infty} f(x)dx$ is convergent and hence, by Integral Test, the series is converges.

g) $\sum_{n=1}^{\infty} \frac{1}{n(\ell n\, n)^2}$

Solution: Here, f(n) = $\frac{1}{n(\ell n\, n)^2}$

Now,

$$\int_1^{\infty} f(x)dx = \lim_{n\to\infty} \int_1^{n} \frac{1}{x(\ell n\, x)^2}\, dx$$

Let $\ell n\, x = y$

$\frac{1}{x} dx = dy$. **As x = 1 then y = 0 and as x= n ,then y =** $\ell n\, n$

So, $\lim_{n\to\infty} \int_1^{n} \frac{1}{x(\ell n\, x)^2}\, dx = \lim_{n\to\infty} \int_0^{\ell n\, n} y^{-2}\, dy$

$$= \lim_{n\to\infty} \left[\frac{y^{-2+1}}{-2+1}\right]_0^{\ell n\, n}$$

$$= \lim_{n\to\infty} \left[-\frac{1}{y}\right]_0^{\ell n\, n}$$

$$= \lim_{n\to\infty} \left[-\frac{1}{\ell n\, n} + 0\right]$$

$$= \infty$$

This shows that , $\int_1^{\infty} f(x)dx$ **is divergent and hence, by Integral Test, the series is diverges.**

2. Explain why the integral test can not be used to determine whether the series is divergent.

a) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$ b) $\sum_{n=1}^{\infty} \cos^2 n\, e^{-n^3}$

Integral Test:

Suppose that f is continuous, positive, decreasing function on$[1,\infty)$ and a_n = f(n). Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent.

In other words,

i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent .

ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent .

3. Test the convergence of series by Comparison Test

Theorem: (Comparison Test)

Let $\sum a_n$ be a series of non-negative terms.

i) If there is a convergent series $\sum b_n$ with $a_n \le b_n$ for all n$\geq N$, then $\sum a_n$ is also convergent.

ii) If there is a divergent series $\sum c_n$ with $a_n \ge c_n$ for all n$\geq N$, then $\sum a_n$ is also divergent.

a) $\sum_{n=1}^{\infty} \frac{3}{3\sqrt{n}-2}$

Solution: Here,

$$\sum_{n=1}^{\infty} \frac{3}{3\sqrt{n}-2} > \sum_{n=1}^{\infty} \frac{3}{3\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is diverges by p-series test.

Thus, by Comparison Test the given series

$\sum a_n = \sum_{n=1}^{\infty} \frac{3}{3\sqrt{n}-2}$ also diverges.

c) $\sum_{n=1}^{\infty} \frac{1}{2^n+\sqrt{n}}$

Solution: Here,

$\sum_{n=1}^{\infty} \frac{1}{2^n+\sqrt{n}} < \sum_{n=1}^{\infty} \frac{1}{2^n}$ Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with radius $|r| = \frac{1}{2} < 1$ is converges by Geometric Ratio test.

Thus, by Comparison Test the given series

$\sum_{n=1}^{\infty} \frac{1}{2^n+\sqrt{n}}$ also converges.

e) $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$

Solution: Here,

$$\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3} < \sum_{n=1}^{\infty} \frac{5}{n^2} = 5 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is converges by p-series test.

Thus, by Comparison Test the given series

$\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ also converges.

f) $\sum_{k=1}^{\infty} \frac{\ell n\, k}{k}$

Solution: **Here,**

$$\sum_{k=1}^{\infty} \frac{\ell n\, k}{k} > \sum_{k=1}^{\infty} \frac{1}{k}$$

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ **is diverges by p-series test.**

Thus, by Comparison Test the given series

$\sum_{n=1}^{\infty} \frac{\ell n\, k}{k}$ **also diverges.**

4. Test the convergence of the series.

a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Ans: Converges by p-series test.

b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Ans: Diverges by p-series test

a) $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$

Ans: Converges by p-series test

d) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$

Solution: Here,

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$$

Let $u_n = \frac{2n+1}{n^2+2n+1}$.

Let $= v_n = \frac{n}{n^2} = \frac{1}{n}$

Now, $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{\frac{2n+1}{n^2+2n+1}}{\frac{1}{n}}$

$$= \lim_{n\to\infty} \frac{\frac{n(2+\frac{1}{n})}{n^2(1+\frac{2}{n}+\frac{1}{n^2})}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{n(2+\frac{1}{n})}{n^2(1+\frac{2}{n}+\frac{1}{n^2})} \times n = 2 > 0$$

Also, $\lim\limits_{n\to\infty} v_n = \lim\limits_{n\to\infty} \frac{1}{n}$ and it is diverges by p-test.

Hence the given $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$ series is diverges by Limit Comparison Test.

e) $\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$

Solution: Here,

$$\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$$

Let $u_n = \frac{10n+1}{n(n+1)(n+2)}$.

Let $v_n = \frac{n}{n^3} = \frac{1}{n^2}$

Now, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{10n+1}{n(n+1)(n+2)}}{\frac{1}{n^2}}$

$$= \lim_{n\to\infty} \frac{\frac{n(10+\frac{1}{n})}{n^3(1+\frac{1}{n}+\frac{2}{n})}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{n(10+\frac{1}{n})}{n^3(1+\frac{1}{n}+\frac{2}{n})} \times n^2 = 10 > 0$$

Also, $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ and it is converges by p-test.

Hence the given $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$ series is converges by Limit Comparison Test.

j) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$

Solution: Here,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$$

Let $u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$.

Let $v_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{\frac{1}{2}}}$

Now, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+\sqrt{n+1}}}{\sqrt{n}}$

$$= \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}\left(1+\frac{\sqrt{n+1}}{\sqrt{n}}\right)}.}{\sqrt{n}}$$

$$= \lim_{n\to\infty}\left(1 + \sqrt{1+\frac{1}{n}}\right) = 2 > 0$$

Also, $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n}$ **and it is diverges by p-test.**

Hence the given $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$ **series is diverges by Limit Comparison Test**

j) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$

Solution: Here,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$$

Let $u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$.

Let $v_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{\frac{1}{2}}}$

Now, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+\sqrt{n+1}}}{\sqrt{n}}$

$$= \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}(1+\frac{\sqrt{n+1}}{\sqrt{n}})}.}{\sqrt{n}}$$

$$= \lim_{n\to\infty} (1 + \sqrt{1 + \frac{1}{n}}) = 2 > 0$$

Also, $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n}$ and it is diverges by p-test.

Hence the given $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$ series is diverges by Limit Comparison Test

K) $\frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7}+\ \dots$

Solution: Here, given series is

$$\frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7}+\ \dots$$

n^{th} term of the series $(a_n)=\frac{n}{(2n-1)(2n+1)}$

Then the given series can be written as

$$\frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7}+\ \dots = \sum_{n=1}^{\infty}\frac{n}{(2n-1)(2n+1)}$$

Let $u_n = \frac{n}{(2n-1)(2n+1)}$.

Let $v_n = \frac{n}{n^2} = \frac{1}{n}$

Now, $$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\frac{n}{(2n-1)(2n+1)}.}{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\ \frac{n}{n^2(2-\frac{1}{n})(2+\frac{1}{n})}\times n$$

$$= \lim_{n\to\infty}\frac{1}{(2-\frac{1}{n})(2+\frac{1}{n})} = \frac{1}{4} > 0$$

Also, $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty}\frac{1}{n}$ and it is diverges by p-test.

Hence the given series $\sum_{n=1}^{\infty}\frac{n}{(2n-1)(2n+1)}$ is diverges by Limit Comparison Test.

n) $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ \ldots$

Solution: Here, given series is

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ \ldots$$

n^{th} term of the series $(a_n)=\frac{n}{n+1}$

Then the given series can be written as

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ \ldots = \sum_{n=1}^{\infty}\frac{n}{n+1}$$

Also, $\sum_{n=1}^{\infty}\frac{n}{n+1} > \sum_{n=1}^{\infty}\frac{1}{n}$

Also, $\sum_{n=1}^{\infty}\frac{1}{n}$ and it is diverges by p-test.

Hence the given series $\sum_{n=1}^{\infty}\frac{n}{n+1}$ is diverges by Comparison Test.

o) $\sum_{n=1}^{\infty} \left(\frac{\sqrt{n+2}}{2n^2+n+1}\right)$

Solution: Here,

$$\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{\sqrt{n+2}}{2n^2+n+1}\right)$$

Let $u_n = \sum_{n=1}^{\infty} \left(\frac{\sqrt{n+2}}{2n^2+n+1}\right)$.

Let $v_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{2-\frac{1}{2}}} = \frac{1}{n^{\frac{3}{2}}}$

Now, $$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\left(\frac{\sqrt{n+2}}{2n^2+n+1}\right)}{\frac{1}{n^{\frac{3}{2}}}}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{\sqrt{n}(\sqrt{1+\frac{2}{n}})}{n^2(2+\frac{1}{n}+\frac{1}{n^2})}\right)}{n^{\frac{3}{2}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+\frac{2}{n}}}{2+\frac{1}{n}+\frac{1}{n^2}} = \frac{1}{2} > 0$$

Also, $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ and it is converges by p-test.

Hence the given series is converges by Limit Comparison Test.

5. Investigate the convergence of the following series.

a) $\sum_{n=0}^{\infty} \left(\frac{2^n+5}{3^n}\right)$

Theorem:(D' Alembert Ratio Test)

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = \ell.$$

Then,

a. If $\ell<1$ then the series converges

b. If $\ell>1$ then the series diverges

c. If $\ell=1$ then the test is inclusive and further test is needed.

a) $\sum_{n=0}^{\infty}\left(\frac{2^n+5}{3^n}\right)$

Solution: Here, given series is $\sum_{n=0}^{\infty}\left(\frac{2^n+5}{3^n}\right)$

Let $a_n = \frac{2^n+5}{3^n}$

Now, $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}}\right)$

$$= \lim_{n\to\infty}\left(\frac{\frac{2^n\times 2+5}{3^n\times 3}}{\frac{2^n+5}{3^n}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\frac{2^n\left(2+\frac{5}{2^n}\right)}{3^n\times 3}}{\frac{2^n\left(1+\frac{5}{2^n}\right)}{3^n}}\right)$$

$$= \left(\frac{\frac{2+\frac{5}{2^\infty}}{3}}{1+\frac{5}{2^\infty}}\right) = \frac{2}{3} < 1$$

Thus, by D' Alembert Ratio test, the given series is convergent.

a) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$

Solution: Here, given series is $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$.

Let $a_n = \frac{(2n)!}{n!n!}$

Now, $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{(2n+2)!}{(n+1)!(n+1)!}}{\frac{(2n)!}{n!n!}}\right)$

$$= \lim_{n\to\infty}\left(\frac{\frac{(2n+2)(2n+1)(2n)!}{(n+1)\,n!\,(n+1)\,n!}}{\frac{(2n)!}{n!n!}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(2n+2)(2n+1)}{(n+1)(n+1)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2\,(2n+1)}{n+1}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2\;n\left(2+\frac{1}{n}\right)}{n\left(1+\frac{1}{n}\right)}\right) = \frac{4}{1} = 4 > 1$$

Thus, by D' Alembert Ratio test, the given series is divergent.

d) $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\,n!3^n}$

Solution: Here, given series is $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\,n!3^n}$.

Let $a_n = \frac{(n+3)!}{3!\,n!3^n}$

$$\text{Now, } \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{(n+1+3)!}{3!(n+1)!3^{n+1}}}{\frac{(n+3)!}{3!\,n!3^n}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\frac{(n+4)(n+3)!}{3!\,(n+1)!3^n\times 3}}{\frac{(n+3)!}{3!\,n!3^n}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\frac{n+4}{(n+1)!\times 3}}{\frac{1}{n!}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n+4}{3(n+1)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n\left(1+\frac{4}{n}\right)}{3\,n\left(1+\frac{1}{n}\right)}\right) = \frac{1}{3} < 1$$

Thus, by D' Alembert Ratio test, the given series is convergent.

h) $\sum_{n=1}^{\infty} \sqrt{\frac{2^n-1}{3^n-1}}$

Solution: Here, given series is $\sum_{n=1}^{\infty} \sqrt{\frac{2^n-1}{3^n-1}}$.

Let $a_n = \frac{2^n-1}{3^n-1}$

Now, $\lim\limits_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = \lim\limits_{n\to\infty} \frac{\sqrt{\frac{2^{n+1}-1}{3^{n+1}-1}}}{\sqrt{\frac{2^n-1}{3^n-1}}}$

$$= \lim_{n\to\infty} \sqrt{\frac{\frac{2^n\times 2-1}{3^n\times 3-1}}{\frac{2^n-1}{3^n-1}}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{\frac{2^n\left(2-\frac{1}{2^n}\right)}{3^n\left(3-\frac{1}{3^n}\right)}}{\frac{2^n\left(1-\frac{1}{2^n}\right)}{3^n\left(1-\frac{1}{3^n}\right)}}}$$

$$= \sqrt{\frac{2}{3}} < 1$$

Thus, by D' Alembert Ratio test, the given series is convergent.

6. Investigate the convergence of the following series.

a) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$

Theorem: (The n^{th} Root Test)(Cauchy's Radical Test)

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \ell.$$

Then,

a. If $\ell<1$ then the series converges

b. If $\ell>1$ then the series diverges

c. If $\ell=1$ then the test is inclusive and further test is needed.

6\. a) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}$

Solution: Here, given series is $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}$.

Let $a_n=\left(1+\frac{1}{n}\right)^{-n^2}$

Now,

$$\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^{-n^2}\right)^{\frac{1}{n}}$$

$$=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n}$$

$$=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$=\frac{1}{e}\approx 0.37<1 \quad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e\right]$$

Thus, the series is converges by Cauchy's Radical Test.

6. e) $\sum_{n=1}^{\infty} \left(\frac{n^n}{(2^n)^2}\right)$

Solution: Here, given series is $\sum_{n=1}^{\infty} \left(\frac{n^n}{(2^n)^2}\right)$.

Let $a_n = \sum_{n=1}^{\infty} \left(\frac{n^n}{(2^n)^2}\right)$

Now,

$$\begin{aligned}
\lim_{n\to\infty} (a_n)^{\frac{1}{n}} &= \lim_{n\to\infty} \left(\frac{n^n}{(2^n)^2}\right)^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \left(\frac{n}{2^2}\right) \\
&= \lim_{n\to\infty} \frac{n}{4} \\
&= \infty
\end{aligned}$$

Thus, the series is converges by **Cauchy's Radical Test.**

Alternative Series:

An infinite series $\sum(-1)^n a_n$ is known as an alternative series.

Leibnitz's Theorem (The Alternative Series Test)

The series,

$$\sum_{n=1}^{\infty}(-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \ \ldots$$

Converges if it satisfies the following conditions:

a) $b_n > 0$, for all n

b) $b_n \geq b_{n+1}$ for all n$\geq N$ for some integer N.

C) $\lim_{n\to\infty} b_n = 0$.

Exercise 8.4

1. Test the convergency of the following alternative series.

a) $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^2}$

Solution: Here, given series is

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^2}$$

Here, comparing this series with $\sum_{n=1}^{\infty}(-1)^{n+1} b_n$

$$b_n = \frac{1}{n^2}$$

i) $b_n \geq 0$, for all n.

ii) $b_{n+1} - b_n = \frac{1}{(n+1)^2} - \frac{1}{n^2}$

$$= \frac{n^2 - n^2 - 2n - 1}{n^2(n+1)^2}$$

$$= \frac{-(2n+1)}{n^2(n+1)^2} < 0$$

$b_{n+1} < b_n$

iii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^2} = 0.$

Thus, the series is converges by Leibnitz's Theorem (The Alternative Series Test).

b) $\sum_{n=1}^{\infty} (-1)^n \frac{\ell n\,(n)}{n}$

Solution: Here, given series is

$$\sum_{n=1}^{\infty} (-1)^n \frac{\ell n\,(n)}{n}$$

Here, comparing this series with $\sum_{n=1}^{\infty}(-1)^{n+1}\, b_n$

$$b_n = \frac{\ell n\,(n)}{n}$$

i) $b_n \geq 0$, for all n> 0.

$$\text{ii) } b_{n+1} - b_n = \frac{\ell n\,(n+1)}{n+1} - \frac{\ell n\,(n)}{n}$$

$$= \frac{n\ell n\,(n)-(n+1)\ell n\,(n)}{n\,(n+1)}$$

$$= \frac{\ell n\left(\frac{n^n}{n^{n+1}}\right)}{n\,(n+1)} = \frac{\ell n\left(\frac{1}{n}\right)}{n\,(n+1)} \leq 0 \text{ for } n \geq 1$$

$$b_{n+1} \leq b_n$$

iii) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{\ell n\,(n)}{n} = \lim\limits_{n\to\infty} \frac{1}{n}$ (using L Hospital rule) = 0.

Thus, the series is converges by Leibnitz's Theorem (The Alternative Series Test).

d) $\sum_{n=2}^{\infty} (-1)^n \frac{\ell n\,(n)}{\ell n\,(n^2)}$

Solution: Here, given series is

$$\sum_{n=2}^{\infty} (-1)^n \frac{\ell n\,(n)}{\ell n\,(n^2)}$$

Here, comparing this series with $\sum_{n=1}^{\infty}(-1)^{n+1}\, b_n$

$$b_n = \frac{\ell n\,(n)}{\ell n\,(n^2)} = \frac{\ell n\,(n)}{2\ell n\,(n)} = \frac{1}{2}$$

i) $b_n \geq 0$, for all n> 2.

iii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} \neq 0.$

Thus, the series is diverges by Leibnitz's Theorem

Absolute Convergence

A series $\sum \boldsymbol{a_n}$ converges absolutely if the corresponding series $\sum |\boldsymbol{a_n}|$ converges.

Theorem (The Absolute Convergence Test)

If $\sum |\boldsymbol{a_n}|$ converges then series $\sum \boldsymbol{a_n}$ converges.

Conditional Convergence

A series $\sum \boldsymbol{a_n}$ converges but $\sum |\boldsymbol{a_n}|$ does not converges, then the series is called convergent conditionally.

Theorem (The Absolute Convergence Test)

If $\sum |\boldsymbol{a_n}|$ converges then series $\sum \boldsymbol{a_n}$ converges.

Normally , Absolute Convergence implies the general convergence.

8.3 Power Series, Taylor and Maclaurin's series:

Power Series:

A power series about x = 0 is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 \mathrm{x} + c_2 \mathrm{x}^2 + \ldots + c_n x^n + \ldots$$

A power series about x = a is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (\mathrm{x}-a) + c_2 (x-a)^2 + \ldots + c_n (x-a)^n + \ldots$$

in which 'a' is center and the coefficients $c_0, c_1, c_2, \ldots + c_n + \ldots$ all are constants.

Interval, Center, and Radius of Convergence of a Power Series.

Consider the power series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1\text{x}+c_2\text{x}^2 + \ldots + c_n x^n + \ldots$$

If there exists a positive number r such that the series converges for |x|<r and diverges for |x|>r, then

(-r, r) is called the interval,

$\frac{-r+r}{2} = 0$ is called center

r is called radius of convergence.

NOTE:- If there exists a positive number r such that the series converges for |x|≤ r and diverges for |x|>r, then

[-r, r] is called the interval,

$\frac{-r+r}{2} = 0$ is called center

r is called radius of convergence.

Example: For what value of x does the following series converges?

$$\sum_{n=0}^{\infty}(-1)^{n-1}\left(\frac{x^n}{n}\right)$$

Solution: Let $a_n = (-1)^{n-1}\frac{x^n}{n}$

Then, $\left|\frac{a_{n+1}}{a_n}\right| = \left|(-1)^{n+1-1}\frac{x^{n+1}}{n+1}\times\frac{n}{(-1)^{n-1}x^n}\right|$

$$= \left|\frac{nx}{n+1}\right|$$

$$= \left|\frac{x}{1+\frac{1}{n}}\right|$$

So, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x}{1+\frac{1}{n}}\right| = |x|$

By ratio test, the given series is converges for |x|<1, i.e. for -< x <1.

At x = -1, the given series becomes

$$\sum_{n=0}^{\infty}(-1)^{n-1}\left(\frac{(-1)^n}{n}\right) = \sum_{n=0}^{\infty}(-1)^{2n-1}\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty}\frac{1}{n}$$

Which is divergent by p-test with p=1.

At x = 1, the given series becomes

$$\sum_{n=0}^{\infty}(-1)^{n-1}\left(\frac{(1)^n}{n}\right)$$

$$= \sum_{n=0}^{\infty}(-1)^{n-1}\left(\frac{1}{n}\right)$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n}\right)$$

Which is divergent being an alternative harmonic series .

Thus, the given series is convergent for any value of x in $-1, x \leq 1$.

The Convergence Theorem for Power Series:

If the series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$$

Converges for $x = a \neq 0$ the it converges absolutely for all $|x| < |a|$. If the series diverges for $x = b$ the it diverges for all $|x| > |b|$.

Development of Taylor Series

Consider the series,

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 \ldots + c_n(x-a)^n + \ldots \quad (i)$$

Differentiating f(x) term by term, we get

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots + n\, c_n(x-a)^{n-1} + \ldots$$

$$f''(x) = 2c_2 + 6c_3(x-a) + \ldots + n(n\text{-}1)\, c_n(x-a)^{n-2} + \ldots$$

$$f'''(x) = 6c_3 + 24c_4(x-a) + \ldots + n(n\text{-}1)(n\text{-}2)\, c_n(x-a)^{n-3} + \ldots$$

… … … … …. … ….

$$f^n(x) = n(n\text{-}1)(n\text{-}2) \ldots\ldots 2 \times 1\, c_n + \ldots$$

$$= n!\, c_n$$

At x= a,

$$f(a) = c_0$$

$$f'(a) = c_1$$

$$f''(a) = 2c_2$$

Or, $c_2 = \dfrac{f''(a)}{2!}$

$$f'''(a) = 6c_3 \Rightarrow c_3 = \frac{f'''(a)}{3!}$$

… … …

$$f^n(a) = n!\, c_n \Rightarrow c_n = \frac{f^n(a)}{n!}$$

Replacing the values of $c_0, c_1, c_2, c_3, \ldots, c_n$ in (i), we get

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \ldots + \frac{f^n(a)}{n!}(x-a)^n + \ldots$$

This is known as taylor's Series.

Thus, the Taylor's Series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \ldots + \frac{f^n(a)}{n!}(x-a)^n + \ldots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

Also, when a = 0

The Maclaurin's series generated by f is

$$f(x) = f(a) + f'(a)(x) + \frac{f''(a)}{2!}(x)^2 + \frac{f'''(a)}{3!}(x)^3 \ldots + \frac{f^{(n)}(a)}{n!}(x)^n + \ldots$$

Thus, the Maclurin's Series can be written as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x)^n.$$

Define Taylor's polynomial of order n.

If f has its first n-derivatives are continuous on [a, b] or [b, a] and $f^{(n)}$ is differentiable on (a, b) or (b, a) for each positive integer n for each x in (a, b)

$$P_n(x) = f(a) + f'(x-a)(x) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(x-a)}{n!}(x)^n$$

is called Taylor's polynomial of order n.

Taylor's Formula:

If f has derivatives of all orders in an open interval I containing a then for each positive integers n and for each x in I

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \ldots + \frac{f^n(a)}{n!}(x-a)^n + \ldots + R_n(x)$$

Where the remainder $R_n(x)$ is,

$$R_n(x) = \frac{f^{(n+1)}(a)(c)}{(n+1)!}(x)^{n+1}$$

for some c between a and x.

Note1:The remainder value $R_n(x)$ is also known as error term of Taylor's Series.

Note 2: If $R_n(x) \to 0 \text{ as } n \to \infty$ for all x in I then the Taylor's series reduces to

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

In this condition, we say that the Taylor's series converges to f on I.

Theorem: (Taylor's Inequality)

Is there are positive constant M and t such that $|f^{(n+1)}(t)| \leq M$ for all t between a and x then the remainder term $|R_n(x)| \leq \frac{|(x-a)^{n+1}|}{(n+1)!}$.

If these condition hold for all the other conditions of Taylor's theorem Are satisfied by f then the series converges to f(x).

Example: Find the Maclaurin's series and show that it represent cosx for all x.(TU 2077)

Solution: Here,

Let $f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{iv}(x) = \cos x$

$f(0) = \cos 0 = 1$
$f'(0) = -\sin 0 = 0$
$f''(0) = -\cos 0 = -1$
$f'''(0) = \sin 0 = 0$
$f^{iv}(0) = \cos 0 = 1$

The Maclaurin's series generated by f is

$$f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \ldots + \frac{f^n(0)}{n!}(x)^n + R_n(x)$$

$$\cos x = 1 + 0\,(x) + \frac{-1}{2!}(x)^2 + \frac{0}{3!}(x)^3 + \frac{1}{4!}(x)^4 + \ldots + (-1)^{n+1}\frac{x^{2n}}{(2n)!} + R_n(x)$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots + (-1)^{n+1}\frac{x^{2n}}{(2n)!} + R_n(x) \quad \ldots. \text{(i)}$$

We know that cosine function and all the derivatives of cosine function have absolute value less than or equal to 1. So, with $M = 1$.

$$|R_{2n}(x)| \leq M \, . \frac{|(x)^{2n+1}|}{(2n+1)!}.$$

$$\Rightarrow \qquad |R_{2n}(x)| \leq 1 \, . \frac{|x|^{2n+1}|}{(2n+1)!}.$$

Since, $\frac{|x|^{2n+1}|}{(2n+1)!} \to \mathbf{0}$ **as n** $\to \infty$ **for all values of x.**

This shows that the series on the right of (i) , converges to cosx for every value of x. Hence, Maclaurin's series for cosx it represent cosx for all values of x.