Design of Experiments

B.SC CSIT 3rd Semester

Analysis of Variance (ANOVA)

- The analysis of variance (ANOVA) is a powerful statistical tools for tests of significance to evaluate differences among the parameters of several groups. If we have to test the significance difference between more than three means and t-test is not useful then ANOVA is used. In other words ANOVA is a statistical techniques specially designed to test whether the means of more than two quantitative populations are equal. It provided a technique to make inference about whether all the samples are from the same normal population having the same mean. The ANOVA was developed by Ronald Fisher in 1918 for the design of agriculture method.

Design of Experiments:

- The design of experiment is the planning the experiment in such way that relevant information should be collected in systematic way for the problem under study so that efficient inference can be drown hence it is a plan, structure and strategies for decision making. It is based on the significance of variables. It is a way of getting an answer to the question which is in the experiment unit of the problem under (i) absolute and (ii) comparative study. The modern concept of experimental design introduced by R. A Fisher in 1920 A.D. It is useful in computers science and IT sector for control of variance of different output. In computer science logics are blocking implemented in different sector programming . In computer science the big data concept experimental error is control by design of experiment.

Objective of Experimental Design:

I. . To control the variance of different experimental error.
II. To minimize the standard error of estimate.
III. To estimate the effects of various treatments and to compare the difference of effects are significant or not.
IV. To estimate the interaction effects of various treatments and to compare means.
V. To measure the efficiency of random process.
VI. To measure the efficiency of design of experiment.
VII. To predict suitable allotment of block.

Terminology in Experimental Design:

- **Experiment:** It is means of getting an answer to a question that the experimenter has in mind. In planning experiment, we clearly state our objectives and formulate the hypothesis we want to test.
- **Treatment:** These are the inputs whose outcomes are to be estimated and compared. These are the different types procedures under comparison in the experiment. In agriculture experiment different types of fertilizers, different types of cultivation process, different varieties of crops are treatments.
- **Experimental unit:** The smallest division of the experimental materials in which the treatments are applied and the effects of treatments are measured.
- **Yields(effects) :** The outcome of the experiment due to the application of treatments in experimental units are called yield.
- **Blocks:** The experimental field is divided in to relatively homogeneous subgroups or strata which is homogeneous or uniform among themselves than the field as a whole are called blocks.

Basic principles of experimental design:

- According to R.A. Fisher a good experimental design must posses the following three principles namely;
- **Replication**
- **Randomization**
- **Local control (Blocking)**
- **Replication:** It is the repetition of treatments under investigation. A treatment is repeated a large number of times in order to obtain more reliable result than is possible from single observation. The most effective way to increase precision is to increase the number of replication.

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- **Randomization:** It is a process of allocating treatments to various plots in a random manner. It ensures that each treatment will have an equal chance of being assigned to an experimental unit.
- **Local control (Blocking) :** If the experimental material is heterogeneous and different treatments are allocated to various experimental units(plots) in random manner then experimental error will be increased. It is desirable to reduce the experimental error without increasing replications or without interfereing the required randomness. The experimental error can be minimized by making the relatively heterogeneous experimental material in to relatively homogeneous blocks is called local control.

Completely Randomized Design(CRD)

- It is simplest of all the design which is based upon only two principles of design namely replication and randomization. In this design treatments are assigned completely at random manner so that each and every experimental unit has equal chance of receiving any treatment. It is appropriate for the homogeneous experimental material.

Layout of CRD:

- The placement of the treatments on the experimental units along with the arrangement of experimental unit is known as the layout of an experiment. For example, consider t = 4 (A, B, C, D) and r= 3 then treatments are allocated as shown below;

A	B	A	D
A	C	C	B
B	C	D	D

Mathematical Model:

- **Mathematical Model:**

$$\boldsymbol{y_{ij} = \mu + \alpha_i + e_{ij}; (i = 1, 2, \ldots t; j = 1, 2, \ldots r)}$$

Where,

$y_{ij} = j^{th}$ replication of i^{th} $treatment$.

$\mu = general\ mean\ effect$

$\alpha_i = The\ effect\ due\ to\ i^{th}\ treatment$

$e_{ij} = error\ due\ to\ chance$

Problem to test:

- **Null hypothesis** (H_{oT}): $\mu_1 = \mu_2 = \mu_3 \ldots\ldots\ldots = \mu_k$ i.e. there is no significant difference between the treatment effects.
- **Alternative hypothesis**(H_{1T}): At least one μ_i is different. **(i= 1, 2, 3…….k)**
- **Statistical Analysis (Sum of square):**
- Total sum of square(TSS) = Sum of square due to treatment (SST)+ Sum of square due to error (SSE)
- TSS = SST + SSE

ANOVA Table:

Source of variation (S.V)	Degree of freedom (d.f)	Sum of Square (S.S)	Mean Sum of Square (M.S.S)	Fcal	Ftab α % .
Treatment	t-1	SST	MST= $\frac{SST}{t-1}$	Fcal= $\frac{MST}{MSE}$	Ftab= $F_{\alpha\{(t-1)(N-t)\}}$
Error	N-t	SSE	MSE= $\frac{SSE}{N-t}$		
Total	N-1	TSS			

Problems:

- **Analysis the given data:**

B 40	C 60	A 40	B 50
C 50	A 70	A 45	D 60
C 55	A 50	B 60	A 70

Problems:

- **Carry out ANOVA of following output of wheat per field obtained as a result of 3 varieties of wheat A, B and C.**

A 10	B 5	A 20	C 15
B 6	A 15	C 11	B 10
C 22	B 12	C 18	A 16

Problems:

- The output of 4 verities of treatment in plots are as shown below carryout analysis

D 1401	C 2536	C 2459	C 2537	C 2827	A 2069
B 2211	A 1757	D 1170	D 1516	D 2103	C 2385
B 3366	A 2103	B 2591	C 2460	D 1070	A 2544

Advantage And Disadvantage CRD:

Advantage	Disadvantage
1. The designing very simple and easily laid out. 2. It has the simplest statistical analysis. 3. It provide the maximum number of degree of freedom to the error sum of square. 4. The design is flexible i.e. any number of treatment and replication may be used.	1. The design is applicable only to small number of treatments. 2. The main demerit lies the assumption of homogeneity of the whole experimental material is not homogeneous than there may be more error.

Randomized Block Design (RBD):

- When the experimental material is not homogeneous the RBD is better than CRD. The RBD is the design where the treatments are allocated in random manner but randomization is the restricted that each treatment must occur once in each row or once in each column. Hence this design is row wise or column wise. It is based upon that all principles of design namely replication, randomization and local control.

Layout of RBD:

- Let us consider five treatments A, B, C, D, and E each replicated four times. The treatments are allocated in the blocks as shown below;

Block 1	Block 2	Block 3	Block 4
A	E	C	A
E	D	B	D
B	C	A	E
D	B	E	C
C	A	D	B

Mathematical Model:

- **Mathematical Model:**

$$\boldsymbol{y_{ij} = \mu + \alpha_i + \beta_j + e_{ij}; (i = 1, 2, \ldots t; j = 1, 2, \ldots r)}$$

Where,

$y_{ij} = the\ response\ of\ the\ j^{th}$ block and $i^{th}\ treatment.$

$\mu \quad = general\ mean\ effect$

$\alpha_i = the\ effect\ due\ to\ i^{th}\ treatment$

$\beta_j =$ the effect due to the $\ j^{th}\ block$

$e_{ij} = error\ due\ to\ chance$ i. e. $e_{ij} \sim N(0, \sigma^2{}_e)$

Problem to test:

- **Null hypothesis (H_{oT}) :** There is no significant difference between treatments.
- **Alternative hypothesis (H_{1T}) :** There is no significant difference between treatment.
- **Null hypothesis (H_{oB}):** There is no significant difference between blocks.
- **Alternative hypothesis(H_{1B}):** There is significant difference between blocks.
- **Total Sum of Square(TSS) = Sum of Square due to treatment(SST) + Sum of Square due to Block(SSB) + Sum of square due to Error(SSE).**
- **TSS = SST + SSB +SSE**

ANOVA Table:

S.V	D.f	S.S	M.S.S	Fcal	Ftab
Treatment	t-1	SST	MST	$FT=\frac{MST}{MSE}$	$F\alpha\{(t-1),(t-1)(r-1)\}$
Block	r-1	SSB	MSB	$FB=\frac{MSB}{MSE}$	$F\alpha\{(r-1),(t-1)(r-1)\}$
Error	(t-1)(r-1)	SSE	MSE		
Total	rk-1	TSS			

Problems:

- Three varieties A, B, C were tested in RBD each with six replication. Analysis the experimental yield and state your conclusion.

Blocks					
1	2	3	4	5	6
A 17	C 35	B 22	C 25	A 30	B 19
C 33	B 23	C 29	A 17	B 23	A 32
B 19	A 29	A 25	B 15	C 37	C 27

Problems:

- Let A, B, C and D are four treatment page replacement algorithms. The following table gives the running times of programs under each replacement algorithm in 5 different blocks and each block 4 different program were used.

Block I	Block II	Block III	Block IV	Block V
A 32	B 33	D 30	A 35	C 36
B 34	C 34	C 35	C 32	D 29
C 31	A 34	B 36	B 37	A 37
D 29	D 26	A 33	D 28	B 35

- Analyse the above results to test whether there is significant difference between yields of four varieties.

One Missing value In RBD:

- In such case instead of discarding the entire experiment we estimate the missing value.
- $\boldsymbol{x = \frac{rB+kT-G}{(k-1)(r-1)}}$
- Where,
- $x = \boldsymbol{Missing\ observation}$
- $\boldsymbol{r = Number\ of\ blocks}$
- $\boldsymbol{k = Number\ of\ treatment}$
- **B = Sum of remaining values in the block with missing observations.**
- **T = Sum of remaining values of the treatment and with the missing observation**
- **G = Grand Total**
- **Adjusted factor(k)** $= \frac{(B+Tt-G)^2}{t(t-1)(r-1)^2}$ **, Adjusted SST(SST_A) = SST-k**

Problems:

- The table given below are yield of 3 varieties in a 4 replicate experiment for which one observation is missing. Estimate the missing observation and then analyse the data.

P 19	R 29	P 23	Q 33
Q 26	P ?	Q 27	R 26
R 21	Q 28	R 22	P 26

Problems:

- Consider the partially completed ANOVA table below. Complete the ANOVA table and answer the following.

S.V	S.S	D.f.	M.S.S	F
Treatment	231.5	2	?	?
Blocks	?	7	?	?
Error	573.75	?	?	
Total	903.75	23		

- What design was employed?
- How many treatments were compared?
- How many observations were analyze?
- At 0.05 level of significance, can one conclude that the treatments have different effects? Why?

Advantage and Disadvantage of RBD:

Advantage	Disadvantage
1. The principle advantage of RBD is that increase the precision of the experiment. This is due to the reduction of experiment error by adoption of local control. 2. RBD provides the better result than CRD. 3. There is no restriction on the number of treatment or replication. But at least two replication is necessary.	1. R.B.D may give misleading results if blocks are not homogeneous. 2. RBD is not suitable for large number of treatments. 3. If the data on more than two plots is missing, the statistical analysis becomes quite tedious and complicated.

Latin Square Design (LSD):

- When the experimental material is divided in to rows and columns and the treatments are allocated such that each treatments occurs only once in a row and once in a column, the design is known as Latin Square Design. In LSD the number of rows and number of columns are equal. Hence, the arrangement will form a square. It follows all principle of design of experiment.

Lay out of LSD:

- In LSD the treatments are usually denoted by alphabets like A, B, C ……etc. For a Latin squares with five treatments the arrangement may be as follows

4X4

A	D	B	C
B	C	D	A
C	B	A	D
D	A	C	B

5X5

A	B	C	D	E
B	A	E	C	D
C	D	A	E	B
D	E	B	A	C
E	C	D	B	A

Mathematical Model:

- The mathematical model for LSD is given by
- $\boldsymbol{y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + e_{ijk}\ (i = j = k = 1, 2, 3 \ldots\ldots m)}$

where,

- y_{ijk} = the response from the unit in the $i^{th}\ row, j^{th} column$ $and\ receiving\ the\ k^{th}\ treatment.$
- $\mu = General\ mean\ effect$
- $\alpha_i = i^{th} row\ effect$
- $\beta_j = j^{th} column\ effect$
- $\gamma_k = k^{th}\ treatment\ effect$
- $e_{ijk} = error\ component$

Problem to test:

- **Null hypothesis** $(H_{oR}), (H_{oC})$ **and** (H_{oT})**:** There is no significant difference between rows, columns and treatments.
- **Alternative hypothesis** $(H_{1R}), (H_{1C})$ **and** (H_{1T}):There is significant difference between rows, columns and treatments.
- Statistical Analysis:
- ***Total Sum Square(TSS)*** = Sum of Square due to row **(SSR) +** Sum of Square due to column **(SSC) +** Sum of Square due to treatment **(SST) +** Sum of Square due to error **(SSE)**
- **TSS = SSR + SSC + SST + SSE**

Missing Value of Latin Square Design:

- **The missing value of LSD is calculated by**
- $x = \frac{m(R+C+T)-2G}{(m-1)(m-2)}$

Where,

$G = total\ of\ all\ known\ values$

$x = missing\ observatio$

$R = total\ of\ all\ known\ values\ of\ i^{th}\ row$

$C = total\ of\ all\ known\ values\ of\ j^{th}\ column$

$T = total\ of\ all\ known\ values\ of\ k^{th}\ treatment$

Adjustment factor (k) = $\frac{\{(m-1)T+R+C-G\}^2}{\{(m-1)(m-2)\}^2}$

Adjusted SST (SSTA) = SST – k.

Problems:

- Carry out the analysis of variance from the given LSD design.

B 15	**C 12**	**A 18**
C 15	**A 12**	**B 25**
B 19	**B 18**	**C 10**

Problems:

- The table given below represents the yields of 4 varieties in a 4 replicate experiment for which one observation is missing. Estimate the missing value and then carry out the ANOVA

A 12	C 19	B 10	D 8
C 18	B 12	D 6	A ?
B 22	D 10	A 5	C 21
D 12	A 7	C 27	B 17

Problems:

- Consider the partially completed ANOVA table below. Compute the ANOVA table below and answer the following;

S.V	S.S	D.f.	M.S.S	F-value
Column	72	?	?	2
Row	?	?	36	?
Treatment	180	3	?	?
Error	?	6	12	?
Total	?	?		

- What design was employed?
- How many treatments were compered?

Advantage and Disadvantage of LSD:

Advantage	Disadvantage
1. With two way grouping or stratification LSD controls more off the variation than CRD or RBD. 2. More than one factor can be investigated simultaneously. 3. The missing observations can be analyzed by missing plot technique.	1. The assumption of factors are independent is not always true. 2. It is suitable for treatment 5 to 10. 3. It is not easy in the field layout.

Efficiency of Design:

- **Efficiency of RBD relative to CRD:**
- $\frac{r(t-1)MSE+(r-1)MSB}{(rt-1)MSE}$
- **Efficiency of LSD relative to CRD:**
- $\frac{(m-1)\ MSE+MSR+MSC}{(m+1)\ MSE}$
- **Efficiency of LSD relative to RBD(When row is taken as block) :** $\frac{(m-1)\ MSE+MSC}{m\ MSE}$
- **Efficiency of LSD relative to RBD(When column is taken as block):** $\frac{(m-1)\ MSE+MSR}{m\ MSE}$

Problems:

- From the following ANOVA table of RBD, determine it's efficiency with respect to CRD.

S.V	d.f.	S.S	M.S.S
Between treatments	5	750	150
Between blocks	3	180	60
Error	15	200	13.33
Total	23	1130	

Problems:

- From the following ANOVA table of 4X4 LSD determine it's efficiency with respect to CRD.

S.V	d.f.	S.S	M.S.S
Rows	3	2.133	0.711
Columns	3	2.203	0.734
Treatments	3	10.663	3.554
Error	6	7.059	1.177
Total	15	22.058	

Problems:

- From the following ANOVA table of 4X4 LSD determine it's efficiency with respect to RBD.

S.V	d.f.	S.S.	M.S.S
Rows	3	2.133	0.711
Columns	3	2.203	0.734
Treatments	3	10.663	3.554
Error	6	7.059	1.177
Total	15	22.058	

F-table of Critical Values of α = 0.05 for F(df1, df2)

	DF1=1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
DF2=1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	243.91	245.95	248.01	249.05	250.10	251.14	252.20	253.25	254.31
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51

F - Distribution ($\alpha = 0.01$ in the Right Tail)

df_2 \ df_1	Numerator Degrees of Freedom								
Denominator Degrees of Freedom	1	2	3	4	5	6	7	8	9
1	4052.2	4999.5	5403.4	5624.6	5763.6	5859.0	5928.4	5981.1	6022.5
2	98.503	99.000	99.166	99.249	99.299	99.333	99.356	99.374	99.388
3	34.116	30.817	29.457	28.710	28.237	27.911	27.672	27.489	27.345
4	21.198	18.000	16.694	15.977	15.522	15.207	14.976	14.799	14.659
5	16.258	13.274	12.060	11.392	10.967	10.672	10.456	10.289	10.158
6	13.745	10.925	9.7795	9.1483	8.7459	8.4661	8.2600	8.1017	7.9761
7	12.246	9.5466	8.4513	7.8466	7.4604	7.1914	6.9928	6.8400	6.7188
8	11.259	8.6491	7.5910	7.0061	6.6318	6.3707	6.1776	6.0289	5.9106
9	10.561	8.0215	6.9919	6.4221	6.0569	5.8018	5.6129	5.4671	5.3511
10	10.044	7.5594	6.5523	5.9943	5.6363	5.3858	5.2001	5.0567	4.9424
11	9.6460	7.2057	6.2167	5.6683	5.3160	5.0692	4.8861	4.7445	4.6315
12	9.3302	6.9266	5.9525	5.4120	5.0643	4.8206	4.6395	4.4994	4.3875
13	9.0738	6.7010	5.7394	5.2053	4.8616	4.6204	4.4410	4.3021	4.1911
14	8.8616	6.5149	5.5639	5.0354	4.6950	4.4558	4.2779	4.1399	4.0297
15	8.6831	6.3589	5.4170	4.8932	4.5556	4.3183	4.1415	4.0045	3.8948
16	8.5310	6.2262	5.2922	4.7726	4.4374	4.2016	4.0259	3.8896	3.7804
17	8.3997	6.1121	5.1850	4.6690	4.3359	4.1015	3.9267	3.7910	3.6822
18	8.2854	6.0129	5.0919	4.5790	4.2479	4.0146	3.8406	3.7054	3.5971
19	8.1849	5.9259	5.0103	4.5003	4.1708	3.9386	3.7653	3.6305	3.5225
20	8.0960	5.8489	4.9382	4.4307	4.1027	3.8714	3.6987	3.5644	3.4567
21	8.0166	5.7804	4.8740	4.3688	4.0421	3.8117	3.6396	3.5056	3.3981
22	7.9454	5.7190	4.8166	4.3134	3.9880	3.7583	3.5867	3.4530	3.3458
23	7.8811	5.6637	4.7649	4.2636	3.9392	3.7102	3.5390	3.4057	3.2986
24	7.8229	5.6136	4.7181	4.2184	3.8951	3.6667	3.4959	3.3629	3.2560
25	7.7698	5.5680	4.6755	4.1774	3.8550	3.6272	3.4568	3.3239	3.2172
26	7.7213	5.5263	4.6366	4.1400	3.8183	3.5911	3.4210	3.2884	3.1818
27	7.6767	5.4881	4.6009	4.1056	3.7848	3.5580	3.3882	3.2558	3.1494
28	7.6356	5.4529	4.5681	4.0740	3.7539	3.5276	3.3581	3.2259	3.1195
29	7.5977	5.4204	4.5378	4.0449	3.7254	3.4995	3.3303	3.1982	3.0920
30	7.5625	5.3903	4.5097	4.0179	3.6990	3.4735	3.3045	3.1726	3.0665
40	7.3141	5.1785	4.3126	3.8283	3.5138	3.2910	3.1238	2.9930	2.8876
60	7.0771	4.9774	4.1259	3.6490	3.3389	3.1187	2.9530	2.8233	2.7185
120	6.8509	4.7865	3.9491	3.4795	3.1735	2.9559	2.7918	2.6629	2.5586
∞	6.6349	4.6052	3.7816	3.3192	3.0173	2.8020	2.6393	2.5113	2.4073