Modeling Rate Eq & Diff

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A.
$$\frac{dA}{dt} = -k_1 A \quad ; \quad A(t=0) = A_0 \ .$$

$$\int_{A_0}^{A(t)} \frac{dA}{A} = -k_1 \int_0^t dt$$

$$A_0 + B_0 = A(t) + B(t)$$

$$\frac{dB}{dt} = k_1 A(t)$$
$$= k_1 (A_0 + B_0 - B(t)) .$$

$$\ln\left(\frac{A(t)}{A_0}\right) = -k_1 t$$

$$\boxed{\begin{array}{ll} A(t) = A_0 e^{-k_1 t} & B(t) = B_0 + A_0(1 - e^{-k_1 t}) \\ \text{At } t = \frac{\ln 2}{k_1}, \ A = B = \frac{A_0}{2} & \end{array}}$$

B.
$$\frac{dA}{dt} = \frac{dB}{dt} = -k_2 AB$$

$$A_0 + C_0 = A(t) + C(t)$$
$$B_0 + C_0 = B(t) + C(t)$$

$$\Rightarrow \quad A_0 - B_0 = A(t) - B(t)$$

$$\Rightarrow \quad B(t) = B_0 + A(t) - A_0$$

$$\Rightarrow \quad \frac{dA}{dt} = -k_2 A (B_0 - A_0 + A)$$

$$\int \frac{dA}{A(B_0 - A_0 + A)} = \int -k_2 \, dt$$

$$\frac{x}{A} + \frac{y}{B_0 - A_0 + A} = \frac{1}{A(B_0 - A_0 + A)}$$

$$x(B_0 - A_0 + A) + Ay = 1$$

$$A = 0 \quad \Rightarrow \quad x = \frac{1}{B_0 - A_0}$$

$$A = A_0 - B_0 \Rightarrow y = \frac{1}{A_0 - B_0} = \frac{-1}{B_0 - A_0}$$

$$\therefore \quad \frac{1}{B_0 - A_0}\left[\int_{A_0}^{A} \frac{dA}{A} - \int_{A_0}^{A} \frac{dA}{B_0 - A_0 + A}\right] = -k_2 \int_0^t dt$$

$$\Rightarrow \quad \ln\left(\frac{A}{A_0}\right) - \ln\left(\frac{B_0 - A_0 + A}{B_0}\right) = -k_2 (B_0 - A_0) t$$

$$\Rightarrow \quad \frac{B_0 A}{A_0 (B_0 - A_0 + A)} = \exp\left[-k_2 (B_0 - A_0) t\right]$$

Let $\quad \gamma = B_0 - A_0$

$$\Rightarrow \quad B_0 A = A_0 (\gamma + A) \exp(-k_2 \gamma t)$$

$$\Rightarrow \quad B_0 A + A_0 \exp(-k_2 \gamma t) A = A_0 \gamma \exp(-k_2 \gamma t)$$

$$\Rightarrow \quad A(t) = \frac{A_0 (B_0 - A_0) \exp(-k_2 (B_0 - A_0) t)}{B_0 - A_0 \exp(-k_2 (B_0 - A_0) t)}$$

$$\Rightarrow \quad A(t) = \frac{A_0 (A_0 - B_0)}{A_0 - B_0 \exp(-k_2 (A_0 - B_0) t)}$$

$$A(t) = \frac{A_0 - B_0}{1 - \frac{B_0}{A_0} \exp(-k_2 (A_0 - B_0) t)}$$

$$B(t) = A(t) - (A_0 - B_0)$$

$$= (A_0 - B_0)\left[\frac{\quad}{1 - \frac{B_0}{A}\exp\left(-k_2(A_0-B_0)t\right)} - 1\right]$$

$$= (A_0 - B_0)\left[\frac{\frac{B_0}{A_0}\exp\left(-k_2(A_0-B_0)t\right)}{1 - \frac{B_0}{A_0}\exp\left(-k_2(A_0-B_0)t\right)}\right]$$

$$= \frac{A_0 - B_0}{-1 + \frac{A_0}{B_0}\exp\left(k_2(A_0-B_0)t\right)}.$$

$$B(t) = \frac{B_0 - A_0}{1 - \frac{A_0}{B_0}\exp\left(-k_2(B_0-A_0)t\right)}$$

$$C(t) = B_0 + C_0 - B(t)$$

Given $A_0 < B_0$, let $\Phi = B_0 - A_0$ be equilibrium value.

$$\boxed{\begin{aligned} A(t) &= \frac{-\Phi}{1 - \frac{B_0}{A_0}\exp\left(k_2\Phi t\right)} \\ B(t) &= \frac{\Phi}{1 - \frac{A_0}{B_0}\exp\left(-k_2\Phi t\right)} \\ C(t) &= C_0 + \frac{A_0\left[1 - \exp\left(-k_2\Phi t\right)\right]}{1 - \frac{A_0}{B_0}\exp\left(-k_2\Phi t\right)} \end{aligned}}$$

$$\boxed{\lim_{t\to\infty} \quad \begin{aligned} A(t) &= 0 \\ B(t) &= \Phi = B_0 - A_0 \\ C(t) &= C_0 + A_0 \end{aligned}}$$