Ch 6.2: Shrinkage - Ridge regression

Lecture 16 - CMSE 381

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Announcements

Last time:

- Subset selection

This time:

- Ridge regression

Announcements:

- HW #5 due tonight
- Be sure to make note of people you worked with and resources you used.

Section 1

Last time

Subset selection

Algorithm 6.1 *Best subset selection*

1. Let $\mathcal{M}_0$ denote the *null model*, which contains no predictors. This model simply predicts the sample mean for each observation.
2. For $k = 1, 2, \ldots p$:
   (a) Fit all $\binom{p}{k}$ models that contain exactly k predictors.
   (b) Pick the best among these $\binom{p}{k}$ models, and call it $\mathcal{M}_k$. Here *best* is defined as having the smallest RSS, or equivalently largest R^2.
3. Select a single best model from among $\mathcal{M}_0, \ldots, \mathcal{M}_p$ using cross-validated prediction error, C_p (AIC), BIC, or adjusted R^2.

Algorithm 6.2 *Forward stepwise selection*

1. Let $\mathcal{M}_0$ denote the *null* model, which contains no predictors.
2. For $k = 0, \ldots, p-1$:
   (a) Consider all $p - k$ models that augment the predictors in $\mathcal{M}_k$ with one additional predictor.
   (b) Choose the *best* among these $p - k$ models, and call it $\mathcal{M}_{k+1}$. Here *best* is defined as having smallest RSS or highest R^2.
3. Select a single best model from among $\mathcal{M}_0, \ldots, \mathcal{M}_p$ using cross-validated prediction error, C_p (AIC), BIC, or adjusted R^2.

Algorithm 6.3 *Backward stepwise selection*

1. Let $\mathcal{M}_p$ denote the *full* model, which contains all p predictors.
2. For $k = p, p-1, \ldots, 1$:
   (a) Consider all k models that contain all but one of the predictors in $\mathcal{M}_k$, for a total of $k - 1$ predictors.
   (b) Choose the *best* among these k models, and call it $\mathcal{M}_{k-1}$. Here *best* is defined as having smallest RSS or highest R^2.
3. Select a single best model from among $\mathcal{M}_0, \ldots, \mathcal{M}_p$ using cross-validated prediction error, C_p (AIC), BIC, or adjusted R^2.

Ways to approximate test score

$$C_p = \frac{1}{n}(\mathrm{RSS} + 2d\hat{\sigma}^2)$$

$$\mathrm{AIC} = \frac{1}{n}(\mathrm{RSS} + 2d\hat{\sigma}^2)$$

$$\mathrm{BIC} = \frac{1}{n}(\mathrm{RSS} + \log(n)d\hat{\sigma}^2)$$

$$\text{adjusted } R^2 = 1 - \frac{\mathrm{RSS}/(n-d-1)}{\mathrm{TSS}/(n-1)}$$

Cross-Validation

Section 2

Ridge Regression

Goal

- Fit model using all p predictors
- Aim to constrain (regularize) coefficient estimates
- Shrink the coefficient estimates towards 0

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4$$

Ridge regression

Before:

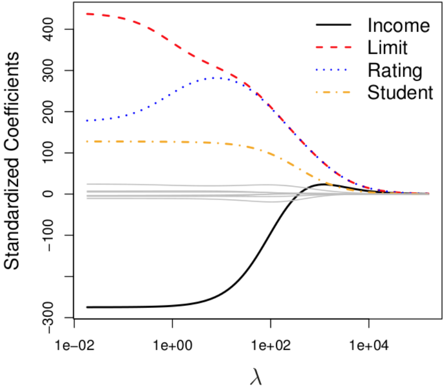
$$RSS = \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{ij} \right)^2$$

After:

$$\sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{ij} \right)^2 + \lambda \sum_{j=1}^{p} \beta_j^2 = RSS + \lambda \sum_{j=1}^{p} \beta_j^2$$

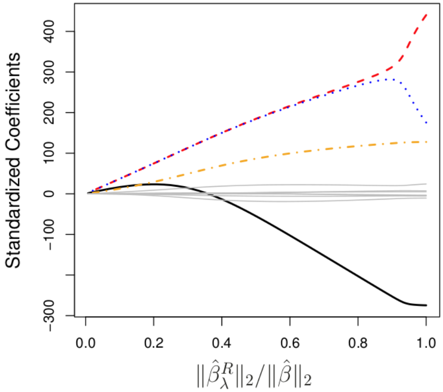
Example from the Credit data

$$RSS + \lambda \sum_{j=1}^{p} \beta_j^2$$



Same Setting, Different Plot

$$RSS + \lambda \sum_{j=1}^{p} \beta_j^2 \qquad \|\beta\|_2 = \sqrt{\sum_{j=1}^{p} \beta_j^2}$$



Scale equivavariance (or lack thereof)

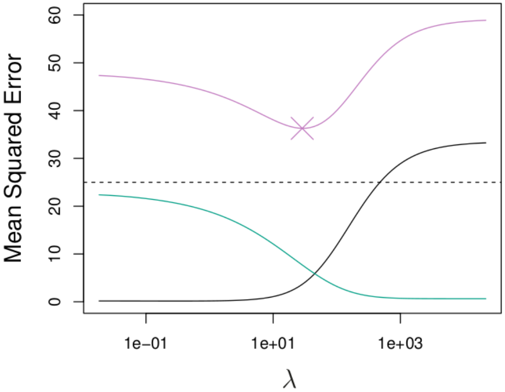
Scale equivariant: Multiplying a variable
by c (cX_i) just returns a coefficient
multiplied by $1/c$ ($1/c\beta_i$)

Solution: Standardize predictors

$$\widetilde{x}_{ij} = \frac{x_{ij}}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_{ij} - \overline{x}_j)^2}}$$

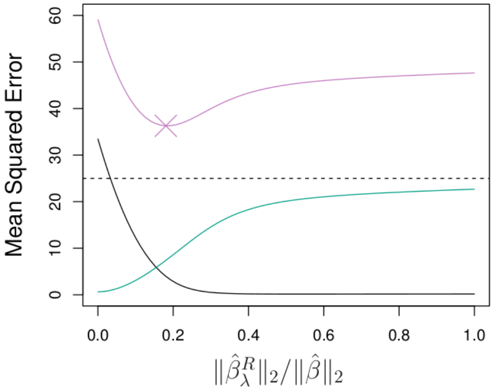
Coding

Bias-Variance tradeoff



Squared bias (black), variance (green), and test mean squared error (purple) for simulated data.

More Bias-Variance Tradeoff



Squared bias (black), variance (green), and test mean squared error (purple) for simulated data.

Advantages of Ridge

Ridge vs. Least Squares:

Ridge vs. Subset Selection:

Next time

10	M	Oct 3	Leave one out CV	5.1.1, 5.1.2	
11	W	Oct 5	k-fold CV	5.1.3	
12	F	Oct 7	More k-fold CV,	5.1.4-5	
13	M	Oct 10	k-fold CV for classification	5.1.5	HW #4 Due
14	W	Oct 12	Resampling methods: Bootstrap	5.2	
15	F	Oct 14	Subset selection	6.1	
16	M	Oct 17	Shrinkage: Ridge	6.2.1	HW #5 Due
17	W	Oct 19	Shrinkage: Lasso	6.2.2	
18	F	Oct 21	[No class, Dr Munch out of town]		
	M	Oct 24	No class - Fall break		
19	W	Oct 26	Dimension Reduction	6.3	
20	F	Oct 28	More dimension reduction; High dimensions	6.4	HW #6 Due
	M	Oct 31	Review		
	W	Nov 2	***Midterm #2***		