Fuzzy set theory

Crisp:

- A logic which demands a binary (0/1) type of handling is termed crisp in the domain of fuzzy set theory.

ex: Temperature is 37°C, The running time of a program is 30sec.

- The term fuzzy refers to things which are not clear.

Is water colorless? Crisp → Yes / → No

Is Ram honest

Fuzzy →
- Extremely honest (1)
- Very honest (0.8)
- honest at times (0.4)
- Extremely dishonest (0)

Crisp sets:

Universe of discourse → It is the set which, with reference to a particular context, contains all possible elements having the same characteristics & from which sets can be formed.

→ Universal set is denoted by E.

ex: set of all students in a university.

Set: A set is a well defined collection of objects. Here, well defined means the object either belongs to or does not belongs to the set.

ex: A = {Gandhi, Bose, Nehru}

$A = \{x \mid P(x)\}$

P(x) stands for the property P to be satisfied by the member x.

ex $A = \{x \mid x \text{ is an odd number}\}$

Venn diagram: Pictorial representation of a set.

E : universal set.

A : Set of female students.

Membership: An element x is said to a member of set A if x blongs to the set A.

$x \in A$ $\quad x \notin B$

ex: $A = \{4,5,6,7,8\}$ $\quad x = 3$ $\quad x \notin A$ $\quad x = 4.$ $x \in A$

Cardinality: The no of elements in a set called its cardinality.

ex: $A = \{4,5,6,7\}$ $\quad |A| = 4$

Family of sets: A set whose members are sets themselves.

$A = \{\{1,3,5\}, \{2,4\}\}$

Null set / Empty set: set has no elements.

ex: ϕ or $\{\ \}$ $\quad |\phi| = 0$

Singleton set: A set with a single element.

ex $A = \{a\}$ then $|A| = 1$

Subset: A is said to be a subset of B if A is fully contained in B, that is, every element of A is in B.

$A \subset B$ $\quad$ A is a proper subset of B

$A \subseteq B$ $\quad$ A is a improper subset of B.

Superset: A is said to be a superset of B. if every element of B is contained in A.

ex: $A = \{3,4\}$ $\quad B = \{3,4,5\}$ $\quad A \subset B$

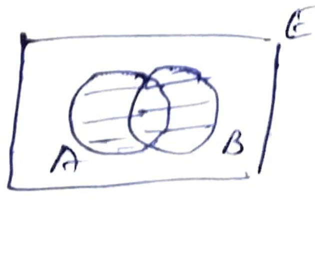
Power set: set of all possible subsets of a given set.

Let $A = \{3,4\}$ $\quad P(A) = \{\phi, \{3\}, \{4\}, \{3,4\}\}$ $\quad |P(A)| = 2^2 = 4$

Operations on crisp sets

Union $\quad A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

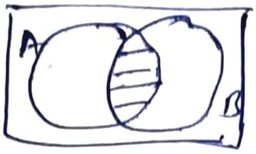
ex: $A = \{a,b,c,1,2\}$ $\quad B = \{1,2,3,a,c\}$ $\quad A \cup B = \{a,b,c,1,2,3\}$



Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

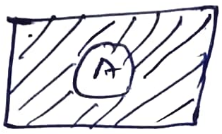
$A \cap B = \emptyset$ Disjoint sets.

$A = \{a, b, c, 1, 2\}$ $B = \{1, 2, 3, a, c\}$ $A \cap B = \{a, c, 1, 2\}$



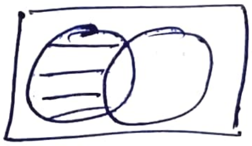
Compliment: $A^c = \{x \mid x \notin A, x \in E\}$

$E = \{1, 2, 3, 4, 5, 6, 7\}$ & $A = \{5, 4, 3\}$ $A^c = \{1, 2, 6, 7\}$



Difference: $A - B = \{x \mid x \in A \text{ \& } x \notin B\}$

$A = \{a, b, c, d, e\}$ $B = \{b, d\}$ $A - B = \{a, c, e\}$



Properties of CRISP sets

Commutativity $A \cup B = B \cup A$
$A \cap B = B \cap A$

Associativity $(A \cup B) \cup C = A \cup (B \cup C)$
$(A \cap B) \cap C = A \cap (B \cap C)$

Distributivity $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Law of absorption $A \cup (A \cap B) = A$
$A \cap (A \cup B) = A$

Law of contradiction $A \cap A^c = \emptyset$

Idempotence: $A \cup A = A$
$A \cap A = A$

Identity $A \cup \emptyset = A$ $A \cup E = E$
$A \cap E = A$ $A \cap \emptyset = \emptyset$

Transitivity $A \subseteq B, B \subseteq C, \Rightarrow A \subseteq C$

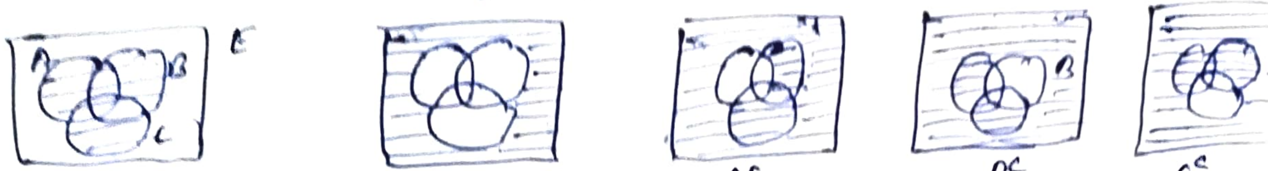
Involution $(A^c)^c = A$

Law of excluded middle $A \cup A^c = E$

Demorgan's law
$(A \cup B)^c = A^c \cap B^c$
$(A \cap B)^c = A^c \cup B^c$

Ex: Given three sets A, B, & C Prove De Morgan's Law using venn diagram? (4)

(I) $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$ (II) $(A \cap B \cap C)^c = A^c \cup B^c \cup C^c$



$A \cup B \cup C$ $(A \cup B \cup C)^c$ A^c B^c C^c

$A^c \cap B^c \cap C^c$ $(A \cap B \cap C)^c$ $A^c \cup B^c \cup C^c$

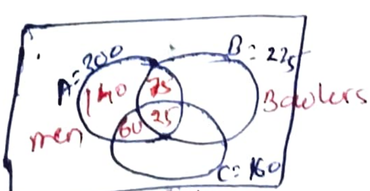
ex 2: E = all students enrolled in the university cricket club.
A = male students, B = Bowlers, C = batsmen.

Draw venn diagram for
1) female students
2) Only bowlers Not as batsmen
3) fem. can both bow & bat.

female students   Bowlers who are not batsmen   F students who can both bowl & bat.

ex 3 $|E| = 600$ $|A| = 300$ $|B| = 225$; $|C| = 160$, Also, the the no of male students who are bowlers $(A \cap B)$ be 100, 25 of whom are batsmen too $(A \cap B \cap C)$, & the total no of male students who are batsmen $(A \cap C)$ be 85.

Determine (1) female students 2) Not bowlers 3) Not batsmen
(4) Female students who can bowl.



1) No of female students = $|A^c| = |E| - |A| = 600 - 300 = 300$

2) No of students who are not bowlers $= |B^c| = |E| - |B| = 600 - 225 = 375$

3) # of students who are not batsmen $= |C^c| = |E| - |C| = 600 - 160 = 440$

4) # of female students who can bowl

$|A^c \cap B| = 125$

ex: $A = \{a, b, c, d, e\}$ $A_1 = \{a, b\}$ $A_2 = \{c, d\}$ $A_3 = \{e\}$

$|A| = 5$ & $\sum_{i=1}^{3} |A_i| = 2+2+1 = 5$

Rule of inclusion & exclusion:

Given A to be a covering of n sets $A_1, A_2, \ldots A_n$

for $n=2$ $|A| = |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

, $n=3$ $|A| = |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3|$
$- |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

ex: Given $|E| = 100$ E is a set of students who have chosen subjects from different stream in cs, it is found that 32 study subjects choosen from the CN stream, 20 from Multimedia (MM) & 45 from Software systems (SS). Also, 15 study subjects from both CN & SS 7 from both MM & SS, and 30 do not study any subjects chosen from either of three subjects.

Find the no of students who study subjects belonging to all three subjects.

$A = CC$, $MM = B$ $C = SS$ $|A \cap B \cap C| = ?$

$|A^c \cap B^c \cap C^c| = 30$, $|(A \cup B \cup C)^c| = 30$ using demorgan's law

$|E| - |A \cup B \cup C| = 30$

$|A \cup B \cup C| = E - 30 = 70$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$70 = 32 + 20 + 45 - 15 - 7 - 10 + |A \cap B \cap C|$

$70 - 65 = |A \cap B \cap C|$

$5 = |A \cap B \cap C|$

Partition & Coverings

(1) Partition:

A partition on A is defined to be a set of non-empty subsets A_i, each of which is pairwise disjoint & whose union yields the original set A.

Partition on A indicated as $\pi(A)$, is therefore

(i) $A_i \cap A_j = \phi$ for each pair $(i,j) \in I$, $i \neq j$

(ii) $\bigcup_{i \in I} A_i = A$

→ The members of A_i of partition are known as blocks.

Partition on set A

ex: $A = \{a, b, c, d, e\}$ $A_1 = \{a, b\}$ $A_2 = \{c, d\}$ $A_3 = \{e\}$

$A_1 \cap A_2 = \phi$ $A_1 \cap A_3 = \phi$; $A_2 \cap A_3 = \phi$

$A_1 \cup A_2 \cup A_3 = A = \{a, b, c, d, e\}$

Hence $\{A_1, A_2, A_3\}$ is a partition on A.

Covering

A covering on set A is defined to be a set of nonempty subsets A_i, whose union yields the original set A. The nonempty subsets needs not be disjoint.

ex: $A = \{a, b, c, d, e\}$ $A_1 = \{a, b\}$ $A_2 = \{b, c, d\}$ $A_3 = \{d, e\}$

$A_1 \cap A_2 = \{b\}$ $A_1 \cap A_3 = \phi$ $A_2 \cap A_3 = \{d\}$

$A_1 \cup A_2 \cup A_3 = \{a, b, c, d, e\} = A$ is a covering.

Rule of Addition:

Given a partition on A where A_i, $i = 1, 2, \ldots, n$ are non-empty subsets

then, $|A| = |\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i|$

Fuzzy sets

- supports a flexible sense of membership of elements to a set.
- many degree of membership (b/w 0 & 1) are allowed.
- A membership funcn $\mu_{\tilde{A}}^{(x)}$ is associated with a fuzzy set $\tilde{A}$ such that the function maps every element of the universe of discourse X (or the reference set) to the interval $[0, 1]$

Formally, the mapping is written as $\mu_{\tilde{A}}^{x} : X \rightarrow [0, 1]$

If X is a universe of discourse & x is a particular element of X, then a fuzzy set A defined on X may be written as a collection of ordered pairs.

$$A = \{ (x, \mu_{\tilde{A}}^{(x)}(x)), x \in X \}$$

where each pair $(x, \mu_{\tilde{A}}^{(x)})$ is called a singleton.

→ An alternate defn which indicates a fuzzy set as a union of all $\mu_A(x)/x$ singletons is given by:

$$A = \sum_{x_i \in X} \mu_A(x_i)/x_i \quad \text{.. in the discrete case}$$

$$A = \int_x \mu_A(x)/x \quad \text{in the continuous case.}$$

ex:

$X = \{ s_1, s_2, s_3, s_4, s_5 \}$ be the reference set of students.

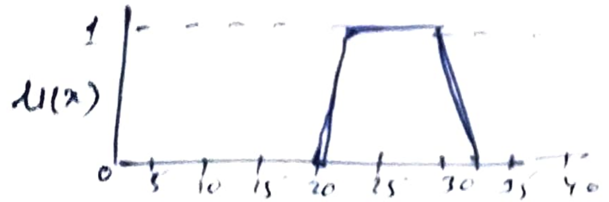
Let $\tilde{A}$ be the fuzzy set of smart students, where "smart" is a fuzzy linguistic term.

honest

$$\tilde{A} = \{ (s_1, 0.4), (s_2, 0.5), (s_3, 1), (s_4, 0.9), (s_5, 0.8) \}$$

Membership function:

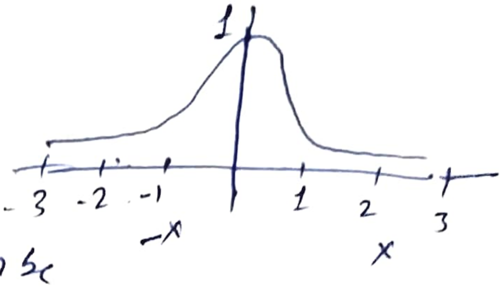
- membership function values need not be described by discrete values. Quite often, these turn out to be as described by a continuous funcn.



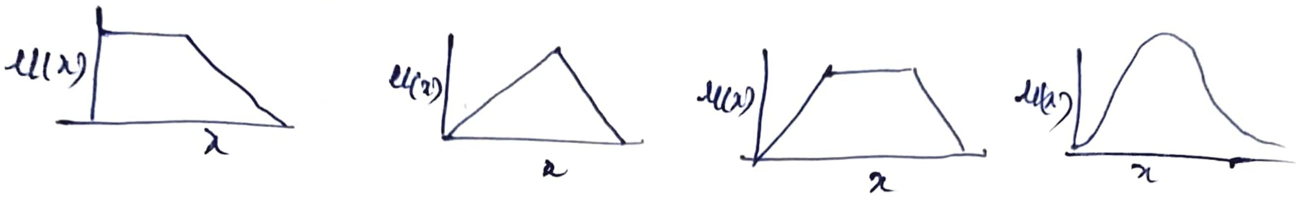
continuous membership funcn for "cool"

→ A membership function can also be given mathematically as

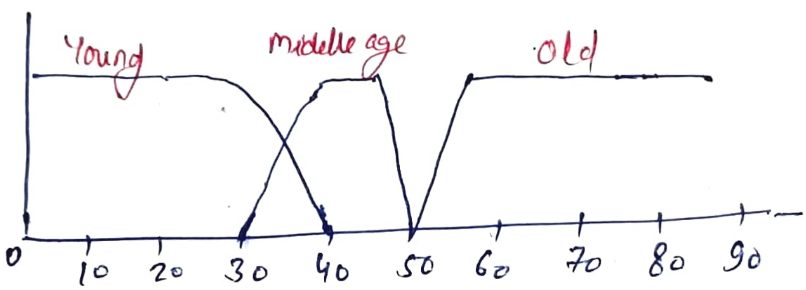
$$\mu_A(x) = \frac{1}{(1+x)^2}$$



→ Different shapes of membership functs can be



Ex: Consider the set of people in the age groups-
0-10, 10-20, 20-30, 30-40, 40-50, 50-60, 60-70, 70 & above.



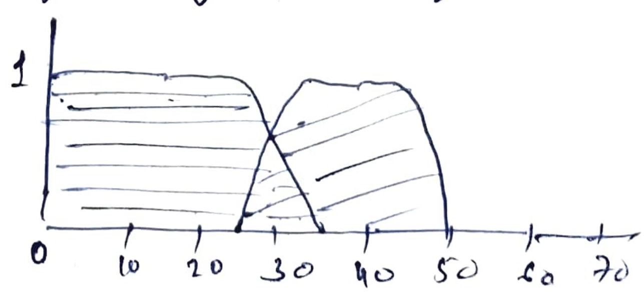
Basic fuzzy set operations:

X = universe of discourse $\tilde{A}$ & $\tilde{B}$ are fuzzy sets with $\mu_A(x)$ & $\mu_B(x)$

Union:

$$\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$$

ex: $\hat{A}$ = Set of young people. $\hat{B}$ = Set of middle aged people
$\hat{A} \cup \hat{B}$, be fuzzy set of "young or middle aged"



In its discrete form, for x_1, x_2, x_3

$\tilde{A} = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0)\}$ & $\tilde{B} = \{(x_1, 0.8), (x_2, 0.2), (x_3, 1)\}$

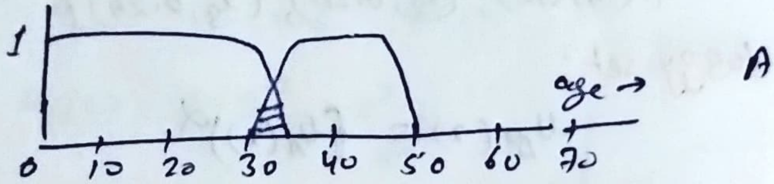
$\tilde{A} \cup \tilde{B} = \{(x_1, 0.8), (x_2, 0.7), (x_3, 1)\}$

Intersection:

$$\mu_{\tilde{A} \cap \tilde{B}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$

for $\tilde{A}$ = young $\tilde{B}$ = middle aged $\tilde{A} \cap \tilde{B}$ = young & middle aged

$\tilde{A} \cap \tilde{B}$



In the discrete form for x_1, x_2, x_3

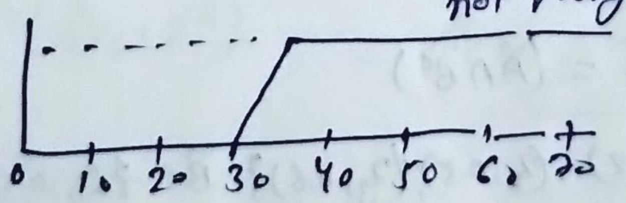
$\tilde{A} = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0)\}$ & $\tilde{B} = \{(x_1, 0.8), (x_2, 0.2), (x_3, 1)\}$

$\tilde{A} \cap \tilde{B} = \{(x_1, 0.5), (x_2, 0.2), (x_3, 0)\}$

Complement:

$$\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x)$$

$\tilde{A}^c$



$\tilde{A} = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0)\}$

$\tilde{A}^c = \{(x_1, 0.5), (x_2, 0.3), (x_3, 1)\}$

Product of two fuzzy sets:

$$\mu_{\tilde{A}.\tilde{B}}(x) = \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)$$

$\tilde{A} = \{(x_1, 0.2), (x_2, 0.8), (x_3, 0.4)\}$ $\tilde{B} = \{(x_1, 0.4), (x_2, 0), (x_3, 0.1)\}$

$\tilde{A}.\tilde{B} = \{(x_1, 0.08), (x_2, 0), (x_3, 0.04)\}$

Equality:

$$\tilde{A} = \tilde{B} \text{ if } \mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$$

ex: $A = \{(x_1, 0.2), (x_2, 0.8)\}$ $B = \{(x_1, 0.6), (x_3, 0.8)\}$ $\tilde{C} = \{(x_1, 0.2), (x_2, 0.8)\}$

$$\tilde{A} \neq \tilde{B} \qquad \tilde{A} = \tilde{C}$$

Product of a fuzzy set with a crisp number

$$\mu_{a.\tilde{A}}(x) = a.\mu_{\tilde{A}}(x)$$

ex: $\tilde{A} = \{(x_1, 0.4), (x_2, 0.6), (x_3, 0.8)\}$ $a = 0.3$

$$a.\tilde{A} = \{(x_1, 0.12), (x_2, 0.18), (x_3, 0.24)\}$$

Power of a fuzzy set:

$$\mu_{\tilde{A}^{\alpha}}(x) = \{\mu_{\tilde{A}}(x)\}^{\alpha}$$

→ Raising a fuzzy set to its second power is called concentration (con) & taking the squre root is called Dilation (DIL).

ex: $\tilde{A} = \{(x_1, 0.4), (x_2, 0.2), (x_3, 0.7)\}$, $\alpha = 2$

$$\mu_{\tilde{A}^2}(x) = \{\mu_{\tilde{A}}(x)\}^2$$

$$(\tilde{A})^2 = \{(x_1, 0.16), (x_2, 0.04), (x_3, 0.49)\}$$

Difference:

$$\tilde{A} - \tilde{B} = (\tilde{A} \cap \tilde{B}^c)$$

ex: $\tilde{A} = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.6)\}$ $\tilde{B} = \{(x_1, 0.1), (x_2, 0.4), (x_3, 0.5)\}$

$$\tilde{B}^c = \{(x_1, 0.9), (x_2, 0.6), (x_3, 0.5)\}$$

$$\tilde{A} - \tilde{B} = \tilde{A} \wedge \tilde{B}^c = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.5)\}$$

Disjunctive Sum:

$$\tilde{A} \oplus \tilde{B} = (\tilde{A}^c \wedge \tilde{B}) \cup (\tilde{A} \cap \tilde{B}^c)$$

$\tilde{A} = \{(x_1, 0.4), (x_2, 0.8), (x_3, 0.6)\}$ $\tilde{B} = \{(x_1, 0.2), (x_2, 0.6), (x_3, 0.9)\}$

$\tilde{A}^c = \{(x_1, 0.6), (x_2, 0.2) (x_3, 0.4)\}$ $\tilde{B}^c = \{(x_1, 0.8), (x_2, 0.4), (x_3, 0.1)\}$

$\tilde{A}^c \wedge \tilde{B} = \{(x_1, 0.2)(x_2, 0.2), (x_3, 0.4)\}$ $\tilde{A} \cap \tilde{B}^c = \{(x_1, 0.4), (x_2, 0.4), (x_3, 0.1)\}$

$$\tilde{A} \oplus \tilde{B} = \{(x_1, 0.4), (x_2, 0.4) (x_3, 0.4)\}$$

Properties of Fuzzy sets:

Commutativity

$\tilde{A} \cup \tilde{B} = \tilde{B} \cup \tilde{A}$

$\tilde{A} \cap \tilde{B} = \tilde{B} \cap \tilde{A}$

Associativity:

$\tilde{A} \cup (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cup \tilde{C}$

$\tilde{A} \cap (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cap \tilde{C}$

Distributivity:

$\tilde{A} \cup (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cap (\tilde{A} \cup \tilde{C})$

$\tilde{A} \cap (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{C})$

Idempotence $\tilde{A} \cup \tilde{A} = \tilde{A}$, $\tilde{A} \cap \tilde{A} = \tilde{A}$

Identity $\tilde{A} \cup \phi = \tilde{A}$, $\tilde{A} \cup X = \tilde{A}$

$\tilde{A} \cap \phi = \phi$ $\tilde{A} \cap X = \tilde{A}$

Transitivity: If $\tilde{A} \subseteq \tilde{B}$, $\tilde{B} \subseteq \tilde{C}$ then $\tilde{A} \subseteq \tilde{C}$

Involution: $(\tilde{A}^c)^c = \tilde{A}$

De morgan's law

$(\tilde{A} \cup \tilde{B})^c = (\tilde{A}^c \cap \tilde{B}^c)$

$(\tilde{A} \cap \tilde{B})^c = (\tilde{A}^c \cup \tilde{B}^c)$

exe

The task is to recognize English alphabetical characters (F, E, X, Y, I, T) in an image processing system, Define two fuzzy sets $\tilde{I}$ & $\tilde{F}$ to represents the identification of characters I & F

$\tilde{I} = \{(F,0.4),(E,0.3),(X,0.1),(Y,0.1),(I,0.9),(T,0.8)\}$

$\tilde{F} = \{(F,0.99),(E,0.8),(X,0.1),(Y,0.2),(I,0.5),(T,0.5)\}$

Find the following:

a) i) $\tilde{I} \cup \tilde{F}$ ii) $\tilde{I} - \tilde{F}$ iii) $\tilde{F} \cup \tilde{F}^c$

b) verify De morgan's Law $(\tilde{I} \cup \tilde{F})^c = \tilde{I}^c \cap \tilde{F}^c$

(a) i) $\tilde{I} \cup \tilde{F} = \{(F,0.99),(E,0.8),(X,0.1),(Y,0.2),(I,0.9),(T,0.8)\}$

ii) $\tilde{I} - \tilde{F} = \tilde{I} \cap \tilde{F}^c$

$= \{(F,0.01),(E,0.2),(X,0.1),(Y,0.1),(I,0.5),(T,0.5)\}$

iii) $\tilde{F} \cup \tilde{F}^c = \{(F,0.99),(E,0.8),(X,0.9),(Y,0.8),(I,0.5),(T,0.5)\}$

(b) $(\tilde{I} \cup \tilde{F})^c = \tilde{I}^c \cap \tilde{F}^c$

$(\tilde{I} \cup \tilde{F})^c = \{(F,0.01),(E,0.2),(X,0.9),(Y,0.8),(I,0.1),(T,0.2)\}$

$I^c = \{(F,0.6),(E,0.7),(X,0.9),(Y,0.9),(I,0.1),(T,0.2)\}$

$F^c = \{(F,0.01),(E,0.2),(X,0.9),(Y,0.8),(I,0.5),(T,0.5)\}$

$\tilde{I} \cap \tilde{F}^c = \{(F,0.01),(E,0.2),(X,0.9),(Y,0.8),(I,0.1),(T,0.2)\}$

LHS = RHS

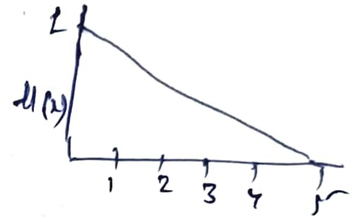
Qu) Consider the fuzzy sets $\tilde{A}$ & $\tilde{B}$ defined on the interval $X = [0, 5]$ of real numbers, by the membership grade function

$$\mu_{\tilde{A}}(x) = \frac{x}{x+1}, \quad \mu_{\tilde{B}}(x) = 2^{-x}$$

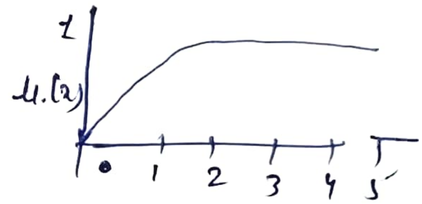
Determine the mathematical formulae & graphs of membership grade function of each of the following sets.

a) A', B^c b) $A \cup B$ c) $A \cap B$ d) $(A \cup B)^c$

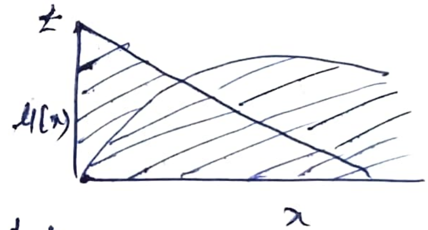
a) $$\mu_{\tilde{A}'}(x) = 1 - \mu_{\tilde{A}}(x) = 1 - \frac{x}{x+1} = \frac{1}{x+1}$$



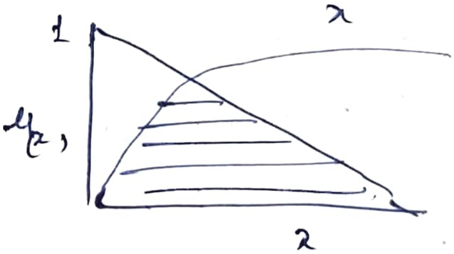
$$\mu_{\tilde{B}^c}(x) = 1 - \mu_{\tilde{B}}(x) = 1 - 2^{-x} = \frac{2^x - 1}{2^x}$$



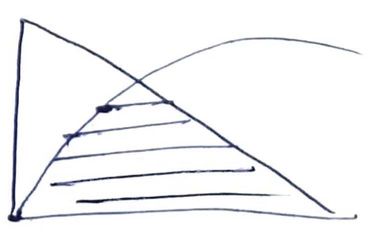
b) $$\mu_{\tilde{A} \cup \tilde{B}}(x) = \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$
$$= \max\left(\frac{x}{x+1}, 2^{-x}\right)$$



c) $$\mu_{\tilde{A} \cap \tilde{B}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$
$$= \min\left(\frac{x}{x+1}, 2^{-x}\right)$$



d) $$\mu_{(\tilde{A} \cup \tilde{B})^c}(x) = \mu_{\tilde{A}' \cap \tilde{B}^c}(x)$$
$$= \min(\mu_{\tilde{A}'}(x), \mu_{\tilde{B}^c}(x))$$
$$= \min\left(\frac{1}{x+1}, \frac{2^x - 1}{2^x}\right)$$



CRESP Relations

Cartesian Products

The Cartesian Product of two sets A & B denoted by $A \times B$ is the set of all ordered pairs such that the first element in the pair belongs to A & the second element belongs to B

i.e. $$A \times B = \{(a,b) \mid a \in A,\ b \in B\}$$

→ If $A \neq B$ & A & B are non empty sets then $A \times B \neq B \times A$

The cartesian Product could be extended to n number of sets

$$\times_{i=1}^{n} A_i = \{(a_1, a_2, a_3 \ldots, a_n) \mid a_i \in A_i \text{ for every } i = 1,2 \ldots n\}$$

observe that - $$\left|\times_{i=1}^{n} A_i\right| = \prod_{i=1}^{n} |A_i|$$

ex:

Given $A_1 = \{a, b\}$ $A_2 = \{1, 2\}$; $A_3 = \{\alpha\}$

$A_1 \times A_2 = \{(a,1),(b,1),(a,2),(b,2)\}$ $|A_1 \times A_2| = 4$ $|A_1| = |A_2| = 2$

$|A_1 \times A_2| = |A_1| \cdot |A_2|$

also

$A_1 \times A_2 \times A_3 = \{(a,1,\alpha),(a,2,\alpha),(b,1,\alpha),(b,2,\alpha)\}$

$|A_1 \times A_2 \times A_3| = 4 = |A_1| \cdot |A_2| \cdot |A_3|$

ex:2 Given $X = \{1,2,3,4\}$

$X \times X = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),$
$(4,1),(4,2),(4,3),(4,4)\}$

let the R be defined as $R = \{(x,y) \mid y = x+1,\ x,y \in X\}$

$R = \{(1,2),(2,3),(3,4)\}$

The Relation matrix R is given by

$$R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

Operations on Relation:-

Union $R \cup S$ $\qquad R \cup S(x,y) = \max\{R(x,y), S(x,y)\}$

Intersection $\qquad R \cap S(x,y) = \min(R(x,y), S(x,y))$

Complement $\qquad \bar{R}(x,y) = 1 - R(x,y)$

Composition of relations

→ Given R to be a relation on X, Y & S to be a relation on Y, Z then RoS is composition of relation on X, Z defined as:

$$R \circ S = \{(x,z) \mid (x,z) \in X \times Z,\ \exists\, y \in Y \text{ such that } (x,y) \in R \ \& \ (y,z) \in S\}$$

→ A common form of the composition relation is the max-min composition.

Max-min composition:-

~~Let R, S be defined on the sets {1,3,5} X {1,3,5}~~

Given the relation matrices of the relation R & S, the max-min composition is defined as

$$T = R \circ S$$

$$T(x,z) = \max_{y \in Y}(\min(R(x,y), S(y,z)))$$

ex: Let R, S be defined on the sets $\{1,3,5\} \times \{1,3,5\}$

$R: \{(x,y) \mid y = x+2\}$, $S: \{(x,y) \mid x < y\}$

$R = \{(1,3), (3,5)\}$, $S = \{(1,3), (1,5), (3,5)\}$

The relation matrices are

$$R: \begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad S: \begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Using max-min composition

$$R \circ S \ \begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$RoS(1,1) = max\{min(0,0), min(1,0), min(0,0)\} = max(0,0,0) = 0$

$RoS(1,3) = max\{0,0,0\} = 0$

$RoS(1,5) = max\{0,1,0\} = 1$

$RoS(3,1) = 0$, $RoS(3,3) = RoS(3,5) = RoS(5,1) = RoS(5,3) = RoS(5,5) = 0$

RoS from the relation matrix is $\{(1,5)\}$

$$SoR = \begin{array}{c} 1 \\ 3 \\ 5 \end{array} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Fuzzy Relation:

- Fuzzy relation is a fuzzy set defined on the cartesian product of crisp sets $X_1, X_2, \ldots X_n$ where the n-tuples $(x_1, x_2, \ldots, x_n)$ may have varying degrees of membership within the relation. The membership values indicate the strength of the relation b/w the tuples.

ex: Let R be the fuzzy relation b/w sets X_1 & X_2 where X_1 is the set of diseases & X_2 is the set of symptoms.

$X_1 = \{$ typhoid, viral fever, common cold $\}$

$X_2 = \{$ running nose, high temperature, shivering $\}$

The fuzzy relation R may be defined as:

	Running Nose	High temperature	shivering
Typhoid	0.1	0.9	0.8
viral fever	0.2	0.9	0.7
Common cold	0.9	0.4	0.6

Fuzzy Cartesian Product:

Let $\tilde{A}$ be a fuzzy set defined on the Universe X & $\tilde{B}$ be a fuzzy set defined on the universe Y, the cartesian product b/w the fuzzy sets $\tilde{A}$ & $\tilde{B}$ indicated as $\tilde{A} \times \tilde{B}$ & resulting in a fuzzy relation $\tilde{R}$ given by

$$\tilde{R} = \tilde{A} \times \tilde{B} \subset X \times Y$$

where $\tilde{R}$ has its membership function given by

$$\mu_{\tilde{R}}(x,y) = \mu_{\tilde{A} \times \tilde{B}}(x,y)$$
$$= \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y))$$

ex:

$\tilde{A} = \{(x_1, 0.2), (x_2, 0.7), (x_3, 0.4)\}$ $\tilde{B} = \{(y_1, 0.5), (y_2, 0.6)\}$ be two fuzzy sets defined on the universes of discourse $X = \{x_1, x_2, x_3\}$ & $Y = \{y_1, y_2\}$ respectively. Then the fuzzy relation $\tilde{R}$ resulting out of the fuzzy cartesian product $\tilde{A} \times \tilde{B}$ is given by

$$\tilde{R} = \tilde{A} \times \tilde{B} = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

Since

$\tilde{R}(x_1, y_1) = \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{B}}(y_1)) = \min(0.2, 0.5) = 0.2$

$R(x_1, y_2) = \min(0.2, 0.6) = 0.2$ $\tilde{R}(x_3, y_1) = \min(0.4, 0.5) = 0.4$

$R(x_2, y_1) = \min(0.7, 0.5) = 0.5$ $\tilde{R}(x_3, y_2) = \min(0.4, 0.6) = 0.4$

$R(x_2, y_2) = \min(0.7, 0.6) = 0.6$

Operations on Fuzzy Relations

Let $\tilde{R}$ & $\tilde{S}$ be fuzzy relations on $X \times Y$

Union $\mu_{\tilde{R} \cup \tilde{S}}(x,y) = \max(\mu_{\tilde{R}}(x,y); \mu_{\tilde{S}}(x,y))$

Intersectⁿ $\mu_{\tilde{R} \cap \tilde{S}}(x,y) = \min(\mu_{\tilde{R}}(x,y), \mu_{\tilde{S}}(x,y))$

Compliment $\mu_{\tilde{R}^c}(x,y) = 1 - \mu_{\tilde{R}}(x,y)$

Composition of relation

$$\mu_{\tilde{R} \circ \tilde{S}}(x,z) = \max(\min(\mu_{\tilde{R}}(x,y), \mu_{\tilde{S}}(y,z)))$$

ex: $X = \{x_1, x_2, x_3\}$ $Y = \{y_1, y_2\}$ $Z = \{z_1, z_2, z_3\}$

Let $\tilde{R}$ be a fuzzy relation Let $\tilde{S}$ be a fuzzy relation

$$\begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 0.5 & 0.1 \\ x_2 & 0.2 & 0.9 \\ x_3 & 0.8 & 0.6 \end{array} \qquad \begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline y_1 & 0.6 & 0.4 & 0.7 \\ y_2 & 0.5 & 0.8 & 0.9 \end{array}$$

Then $R \circ S$ by max-min composition yields

$$R \circ S = \begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline x_1 & 0.5 & 0.4 & 0.5 \\ x_2 & 0.5 & 0.8 & 0.9 \\ x_3 & 0.6 & 0.6 & 0.7 \end{array}$$

$\mu_{\tilde{R} \circ \tilde{S}}(x_1, z_1) = \max(\min(0.5, 0.6), \min(0.1, 0.5)) = \max(0.5, 0.1) = 0.5$

$\mu_{\tilde{R} \circ \tilde{S}}(x_1, z_2) = \max(\min(0.5, 0.4), \min(0.1, 0.8)) = \max(0.4, 0.1) = 0.4$

$\mu_{R \circ S}(x_1, z_3) = \max(0.5, 0.1) = 0.5$

$(x_2, z_1) = \max(0.2, 0.5) = 0.5$

$(x_2, z_2) = \max(0.2, 0.8) = 0.8$

$(x_2, z_3) = \max(0.2, 0.9) = 0.9$

$(x_3, z_1) = \max(0.6, 0.5) = 0.6$

$(x_3, z_2) = \max(0.4, 0.6) = 0.6$

$(x_3, z_3) = \max(0.7, 0.6) = 0.7$

ex 3 $P = \{P_1, P_2, P_3, P_4\}$ of four varieties of paddy plants, set $D = \{D_1, D_2, D_3, D_4\}$ of the various diseases affecting the plants & $S = \{S_1, S_2, S_3, S_4\}$ be the common symptoms of the diseases.

Let $\tilde{R}$ be a relation on $P \times D$ & $\tilde{S}$ be a relation on $D \times S$

$$\tilde{R} = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{bmatrix} 0.6 & 0.6 & 0.9 & 0.8 \\ 0.1 & 0.2 & 0.9 & 0.8 \\ 0.9 & 0.3 & 0.4 & 0.8 \\ 0.9 & 0.8 & 0.1 & 0.2 \end{bmatrix} \end{array}$$

$$\tilde{S} = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0.9 \\ 1 & 1 & 0.4 & 0.6 \\ 0 & 0 & 0.5 & 0.9 \\ 0.9 & 1 & 0.8 & 0.2 \end{bmatrix} \end{array}$$

obtain the association of the plants with the different symptoms of the diseases using max-min composition.

sol

$$R \circ S = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.7 & 0.9 \end{bmatrix} \end{array}$$