Note : This problem is taken from the book "Computational Fluid Dynamics the Basics with Applications" by JD Anderson (Page 356-373)

<u>Numerical Solutions of Quasi-One-Dimensional Nozzle Flows With Shock Capturing</u>

For the shock capturing method, experience has shown that the conservation form of the governing equations should be used. When the conservation form is used, the computed flow-field results are generally smooth and stable. However, when the nonconservation form is used for a shock-capturing solution, the computed flow-field results usually exhibit unsatisfactory spatial oscillations upstream and downstream of the shock wave, the shocks may appear in the wrong location, and the solution may even become unstable. Why is the use of the conservation form of the equations so important for the shock-capturing method ? The answer can be seen by considering the flow across a normal shock wave, as illustrated in Fig.1.

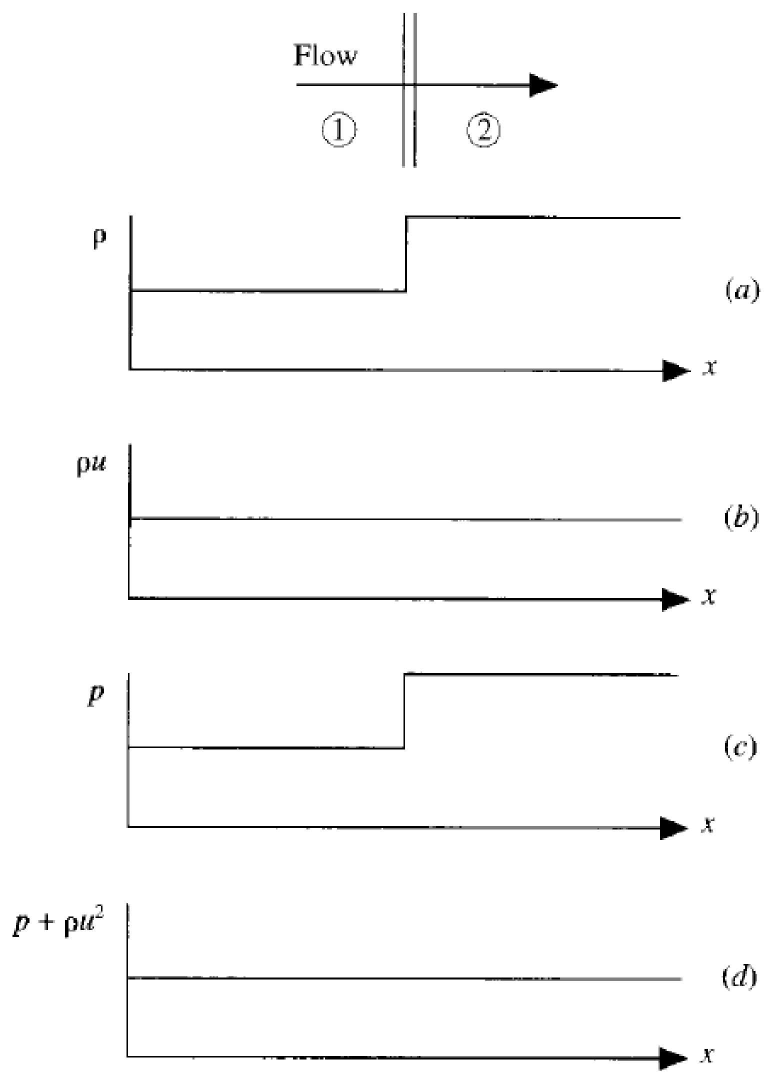


Fig.1. Variation of flow properties through a normal shock

Clearly, there is a discontinuous increase in ρ across the shock. If the nonconservation form of the governing equations were used to calculate this flow, where the primary dependent variables are the primitive variables such as ρ and p, then the equations would see a large discontinuity in the dependent variable ρ. On the other hand, recall the continuity equation for a normal shock wave

$$\rho_1 u_1 = \rho_2 u_2 \tag{1}$$

From Eq. (1) , the mass flux ρu is constant across the shock wave, as illustrated in Fig.1b. The conservation form of the governing equations uses the product ρu as a dependent variable, and hence the conservation form of the equations see no discontinuity in this dependent variable in the normal direction across the shock wave. In turn, the numerical accuracy and stability of the solution should be greatly enhanced. To reinforce this discussion, consider the momentum equation across a normal shock wave.

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \tag{2}$$

As shown in Fig.1c, the pressure itself is discontinuous across the shock; however, from eq. (2) the flux variable $p + \rho u^2$ is constant across the shock. This is illustrated in Fig.1d. The conservation form of the equations would see no discontinuity in this dependent variable in a normal direction across the shock. Although this example of the flow across a normal shock wave is somewhat simplistic, it serves to explain why the use of the conservation form of the governing equations is so important for calculations using the shock-capturing method. Because the conservation form uses flux variables as the dependent variables and because the changes in these flux variables are either zero or small across a shock wave, the numerical quality of a shock-capturing method will be enhanced by the use of conservation form in contrast to the nonconservation form, which uses the primitive variables as dependent variables.

Use of the Governing Equations in Conservation Form

We derived the following continuity equation when we dealt with numerical solution quasi-one-dimensional flow without shock capturing case.

$$\frac{\partial(\rho A)}{\partial t} + \frac{\partial(\rho A V)}{\partial x} = 0 \tag{3}$$

This is the continuity equation for quasi-one-dimensional flow. It is already in conservation form. Nondimensionalizing the variables according to the forms given, we have

$$\frac{\partial\left(\frac{\rho}{\rho_0}\frac{A}{A^*}\right)}{\partial\left(\frac{t}{L/a_0}\right)}\left(\frac{\rho_0 A^* a_0}{L}\right) + \frac{\partial\left(\frac{\rho}{\rho_0}\frac{A}{A^*}\frac{V}{a_0}\right)}{\partial(x/L)}\left(\frac{\rho_0 A^* a_0}{L}\right) = 0$$

$$\frac{\partial(\rho' A')}{\partial t'}+\frac{\partial(\rho' A' V')}{\partial x'}=0 \tag{4}$$

The primes in Eq. (4) denote the nondimensional variables.

The momentum equation that was derived in last session is repeated below:

$$\frac{\partial(\rho V A)}{\partial t}+\frac{\partial(\rho V^2 A)}{\partial x}=-A\frac{\partial p}{\partial x} \tag{5}$$

This is momentum equation for quasi-one-dimensional flow. It is already in conservation form. Let us combine two x derivatives in Eq. 5 as follows. Since,

$$\frac{\partial(pA)}{\partial x}=p\frac{\partial A}{\partial x}+A\frac{\partial p}{\partial x} \tag{6}$$

We can add Eq. (6) to Eq. (5), obtaining

$$\frac{\partial(\rho A V)}{\partial t}+\frac{\partial(\rho A V^2+pA)}{\partial x}=p\frac{\partial A}{\partial x} \tag{7}$$

Nondimensionalizing Eq. (7), we have

$$\frac{\partial\left(\frac{\rho}{\rho_0}\frac{A}{A^*}\frac{V}{a_0}\right)}{\partial\left(\frac{t}{L/a_0}\right)}\left(\frac{\rho_0 A^* a_0^2}{L}\right)+\frac{\partial\left[\frac{\rho}{\rho_0}\frac{A}{A^*}\frac{V^2}{a_0^2}\left(\rho_0 A^* a_0^2\right)+\frac{p}{p_0}\frac{A}{A^*}\left(p_0 A^*\right)\right]}{\partial\left(\frac{x}{L}\right)L}=\frac{p}{p_0}\frac{\partial\left(A/A^*\right)}{\partial\left(x/L\right)}\left(\frac{p_0 A^*}{L}\right)$$

or,

$$\frac{\partial(\rho' A' V')}{\partial t'}+\frac{\partial\left[\rho' A' V'^2+p' A'\left(p_0/\rho_0 a_0^2\right)\right]}{\partial x'}=p'\frac{\partial A'}{\partial x'}\left(\frac{p_0}{\rho_0 a_0^2}\right) \tag{8}$$

However, $\frac{p_0}{\rho_0 a_0^2}=\frac{\rho_0 R T_0}{\rho_0 a_0^2}=\frac{\rho_0 R T_0}{\rho_0 \gamma R T_0}=\frac{1}{\gamma}$

Thus, Eq. (8) becomes

$$\frac{\partial(\rho' A' V')}{\partial t'}+\frac{\partial\left[\rho' A' V'^2+(1/\gamma)p' A'\right]}{\partial x'}=\frac{1}{\gamma}p'\frac{\partial A'}{\partial x'} \tag{9}$$

The energy equation in conservation form is repeated below,

$$\frac{\partial\left[\rho\left(e+V^2/2\right)A\right]}{\partial t}+\frac{\partial\left[\rho\left(e+V^2/2\right)AV\right]}{\partial x}=-\frac{\partial(pAV)}{\partial x} \tag{10}$$

This is the energy equation for quasi-one-dimensional flow. It is already in conservative form. Combining the x derivatives in Eq. (10), we have

$$\frac{\partial\left[\rho\left(e+V^2/2\right)A\right]}{\partial t}+\frac{\partial\left[\rho\left(e+V^2/2\right)AV+pAV\right]}{\partial x}=0 \tag{11}$$

Let us define a nondimensional internal energy as follows :

$$e'=\frac{e}{e_0} \quad \text{where} \qquad e_0=c_vT_0=\frac{RT_0}{\gamma-1}$$

With this, the nondimensional form of Eq. (11) is obtained as follows:

$$\frac{\partial\left\{\frac{\rho}{\rho_0}\left[\frac{e}{e_0}(e_0)+\frac{V^2}{2a_0^2}\left(a_0^2\right)\right]\frac{A}{A^*}\right\}}{\partial\left(\frac{t}{L/a_0}\right)}\left(\frac{\rho_0A^*a_0}{L}\right)$$

$$+\frac{\partial\left\{\frac{\rho}{\rho_0}\left[\frac{e}{e_0}(e_0)+\frac{V^2}{2a_0^2}\left(a_0^2\right)\right]\frac{V}{a_0}\frac{A}{A^*}\left(\rho_0a_0A^*\right)+\left(\frac{p}{p_0}\frac{A}{A^*}\frac{V}{a_0}\right)\left(p_0A^*a_0\right)\right\}}{\partial\left(\frac{x}{L}\right)L}=0 \tag{12}$$

Since $e_0=RT_0/\left(\gamma-1\right)$, Eq. (12) becomes

$$\frac{\partial\left[\rho'\left(\frac{e'}{\gamma-1}+\frac{\gamma}{2}V'^2\right)A'\right]}{\partial t'}\left(\frac{\rho_0A^*a_0RT_0}{L}\right)$$

$$+\frac{\partial\left[\rho'\left(\frac{e'}{\gamma-1}+\frac{\gamma}{2}V'^2\right)V'A'\left(\frac{\rho_0a_0A^*RT_0}{L}\right)+\left(p'A'V'\right)\left(\frac{p_0A^*a_0}{L}\right)\right]}{\partial x'}=0 \tag{13}$$

Divide Eq. (13) by $\dfrac{\rho_0a_0A^*RT_0}{L}$

$$\frac{\partial\left[\rho'\left(\frac{e'}{\gamma-1}+\frac{\gamma}{2}V'^2\right)A'\right]}{\partial t'}+\frac{\partial\left[\rho'\left(\frac{e'}{\gamma-1}+\frac{\gamma}{2}V'^2\right)V'A'+p'A'V'\left(\frac{p_0}{\rho_0RT_0}\right)\right]}{\partial x'}=0 \tag{14}$$

However, in Eq. (14),

$$\frac{p_0}{\rho_0 R T_0} = \frac{\rho_0 R T_0}{\rho_0 R T_0} = 1$$

Thus, Eq. (14) becomes

$$\frac{\partial\left[\rho'\left(\frac{e'}{\gamma-1}+\frac{\gamma}{2}V'^2\right)A'\right]}{\partial t'} + \frac{\partial\left[\rho'\left(\frac{e'}{\gamma-1}+\frac{\gamma}{2}V'^2\right)V'A' + p'A'V'\right]}{\partial x'} = 0 \tag{15}$$

Equations (4), (9) and (15) are the nondimensional conservation form of the continuity, momentum and energy equations for quasi-one-dimensional flow, respectively. The equations for quasi-one-dimensional flow can be expressed in a generic form. Let us define the elements of the solutions vector U, the flux vector F, and the source term J as follows.

$$U_1 = \rho' A'$$

$$U_2 = \rho' A' V'$$

$$U_3 = \rho'\left(\frac{e'}{\gamma-1}+\frac{\gamma}{2}V'^2\right)A'$$

$$F_1 = \rho' A' V'$$

$$F_2 = \rho' A' V'^2 + \frac{1}{\gamma} p' A'$$

$$F_3 = \rho'\left(\frac{e'}{\gamma-1}+\frac{\gamma}{2}V'^2\right)V'A' + p'A'V'$$

$$J_2 = \frac{1}{\gamma} p' \frac{\partial A'}{\partial x'}$$

With these elements, Eqs. (3), (8) , and (14) can be written, respectively, as

$$\frac{\partial U_1}{\partial t'} = -\frac{\partial F_1}{\partial x'} \tag{16a}$$

$$\frac{\partial U_2}{\partial t'} = -\frac{\partial F_2}{\partial x'} + J_2 \tag{16b}$$

$$\frac{\partial U_3}{\partial t'} = -\frac{\partial F_3}{\partial x'} \tag{16c}$$

We are now finished with the governing equations for quasi-one-dimensional flow. Eqs (16a) to (16c) represent the continuity, momentum, and energy equations for quasi-one-dimensional flow, in conservation form. These are the equations we wish to numerically solve using MacCormack's technique.

In conservation form of the equations, the dependent variables (the variables for which we directly obtain numbers) are not the primitive variables. For example, Eqs. (16a) to (16c), our numerical solution will give us numbers directly for U_1 , U_2 and U_3 given above, we have

$$\rho' = \frac{U_1}{A'} \tag{17}$$

$$V' = \frac{U_2}{U_1} \tag{18}$$

$$T' = e' = (\gamma - 1)\left(\frac{U_3}{U_1} - \frac{\gamma}{2} V'^2\right) \tag{19}$$

$$p' = \rho' T' \tag{20}$$

Note in Eq. (19) that we have recognized the fact that $e' = T'$, or

$$e' \equiv \frac{e}{e_0} = \frac{c_v T}{c_v T_0} = \frac{T}{T_0} = T'$$

Therefore, after we obtain U_1 , U_2 and U_3 at each time step from the numerical solution of Eqs. (16a) to (16c), we can immediately calculate the corresponding primitive variables at each time step, ρ' , V' , T' and p' , from Eqs. (17) to (20).

The Setup

Return to Eqs. (16a) to (16c) for a moment; we note that the flux vector elements F_1 , F_2 and F_3 are couched in terms of the primitive variables. The relations for F_1 , F_2 and F_3 immediately preceding Eqs. (16a) to (16c). When the computer program is written with F_1 , F_2 and F_3 expressed directly in terms of ρ', V' , p' and e', instabilities develop during the course of the time-marching solution. For example, in the present example of quasi-one-dimensional flow, instabilities develop in the subsonic section which finally cause the program to blow up after about 300 time steps. This behavior is an example of a lack of "purity" in the formulation of the governing equations in conservation form, a lack which eventually causes numerical problems. If we were to write computer program to implement the equations as written in previous section, we would set up the numerical solution of Eqs. (16a) to (16c) for U_1 , U_2 and U_3 at each time step. We would then decode these elements of the solutions vector to obtain the primitive variables at each time step, as shown in Eqs. (17) to (20) . These primitive variables ρ', V' , p' and e' would, in turn, be used to construct F_1 , F_2 and F_3 for use in the solutions of Eqs. (16a) to (16c) for the next time step, and so forth. As stated above, when the primitive variables are used to construct F_1 , F_2 and F_3, numerical difficulties occasionally arise. This is somehow connected to the fact that the dependent variables which appear explicitly in Eqs. (16a) to (16c) are U_1 , U_2 and U_3 – not the primitive variables. For this reason , it is best to couch F_1 , F_2 and F_3 directly in terms of the dependent variables U_1 , U_2 and U_3 and avoid the use of the primitive variables in Eqs. (16a) to (16c). That is , in Eqs. (16a) to (16c), we will write

$$F_1 = F_1(U_1, U_2, U_3) \tag{21a}$$

$$F_2 = F_2(U_1, U_2, U_3) \tag{21b}$$

$$F_3 = F_3(U_1, U_2, U_3) \tag{21c}$$

$$J_2 = J(U_1, U_2, U_3) \tag{21d}$$

such that the governing equations are "purely" in terms of the elements of the solution vector, i.e., in terms of U_1 , U_2 and U_3 only. Let us proceed to obtain the specific forms indicated by Eqs. (21a) to (21d).

"PURE" FORM OF THE FLUX TERMS

Consider the flux terms F_1 , given in previous section by

$$F_1 = \rho' A' V' \tag{22}$$

Substituting Eqs. (17) and (18) for ρ' and V', respectively, into Eq. (22),we have

$$F_1 = U_2 \tag{23}$$

Consider the flux term F_2 , given in previous section by

$$F_2 = \rho' A' V'^2 + \frac{1}{\gamma} p' A' \tag{24}$$

From Eq. (20), the pressure in Eq. (24) can be replaced by the product $\rho' T'$. In turn, ρ', V' and T' can be expressed in terms of U_1 , U_2 and U_3 via Eqs. (17) to (19). Hence, Eq. (24) becomes

$$F_2 = \frac{U_2^2}{U_1} + \frac{1}{\gamma} U_1 (\gamma - 1) \left[\frac{U_3}{U_1} - \frac{\gamma}{2} \left(\frac{U_2}{U_1} \right)^2 \right]$$

or,

$$F_2 = \frac{U_2^2}{U_1} + \frac{\gamma - 1}{\gamma} \left[U_3 - \frac{\gamma}{2} \frac{U_2^2}{U_1} \right] \tag{25}$$

Consider the flux term F_3 , given in previous section by

$$F_3 = \rho' \left(\frac{e'}{\gamma - 1} + \frac{\gamma}{2} V'^2 \right) V' A' + p' A' V' \tag{26}$$

Substituting Eqs. (17) to (20) into Eq. (26), we have

$$F_3 = U_2 \left(\frac{U_3}{U_1} - \frac{\gamma}{2} V'^2 + \frac{\gamma}{2} V'^2 \right) + U_2 T'$$

$$= \frac{U_2 U_3}{U_1} + (\gamma - 1) U_2 \left[\frac{U_3}{U_1} - \frac{\gamma}{2} \left(\frac{U_2}{U_1} \right)^2 \right]$$

or ,

$$F_3 = \gamma \frac{U_2 U_3}{U_1} - \frac{\gamma(\gamma-1)}{2} \frac{U_2^3}{U_1^2} \tag{27}$$

Finally the source term J_2 was given in previous section as

$$J_2 = \frac{1}{\gamma} p' \frac{\partial A'}{\partial x'} \tag{28}$$

From Eq. (20) , this becomes

$$J_2 = \frac{1}{\gamma} \rho' T' \frac{\partial A'}{\partial x'} \tag{29}$$

Substituting Eqs. (17) and (19) into (29),we have

$$J_2 = \frac{1}{\gamma} \frac{U_1}{A'} (\gamma - 1) \left[\frac{U_3}{U_1} - \frac{\gamma}{2} \left(\frac{U_2}{U_1} \right)^2 \right] \frac{\partial A'}{\partial x'}$$

or,

$$J_2 = \frac{\gamma - 1}{\gamma} \left[U_3 - \frac{\gamma}{2} \frac{U_2^2}{U_1} \right] \frac{\partial(\ln A')}{\partial x'} \tag{30}$$

We now return to our governing equations in conservation form as given by Eqs. (16a) to (16c). With F_1 , F_2 ,F_3 and J_2 given by Eqs. (23) , (25) , (27) and (30), respectively, then Eqs. (16a) to (16c) are expressed in terms of U_1 , U_2 and U_3 only- the primitive variables are nowhere to be found. This is the "pure" form of the governing equations in conservation form; it is the form which we will use in the following sections. When a computer program is written to solve the equations in this pure form, the solution is stable and convergence to a steady state is achieved.

A CASE WITH SHOCK CAPTURING

In the present section, we will numerically solve a nozzle flow with p_e specified such that a normal shock wave will form within the nozzle.

The Setup

Consider the nomenclature shown in Fig. 2.

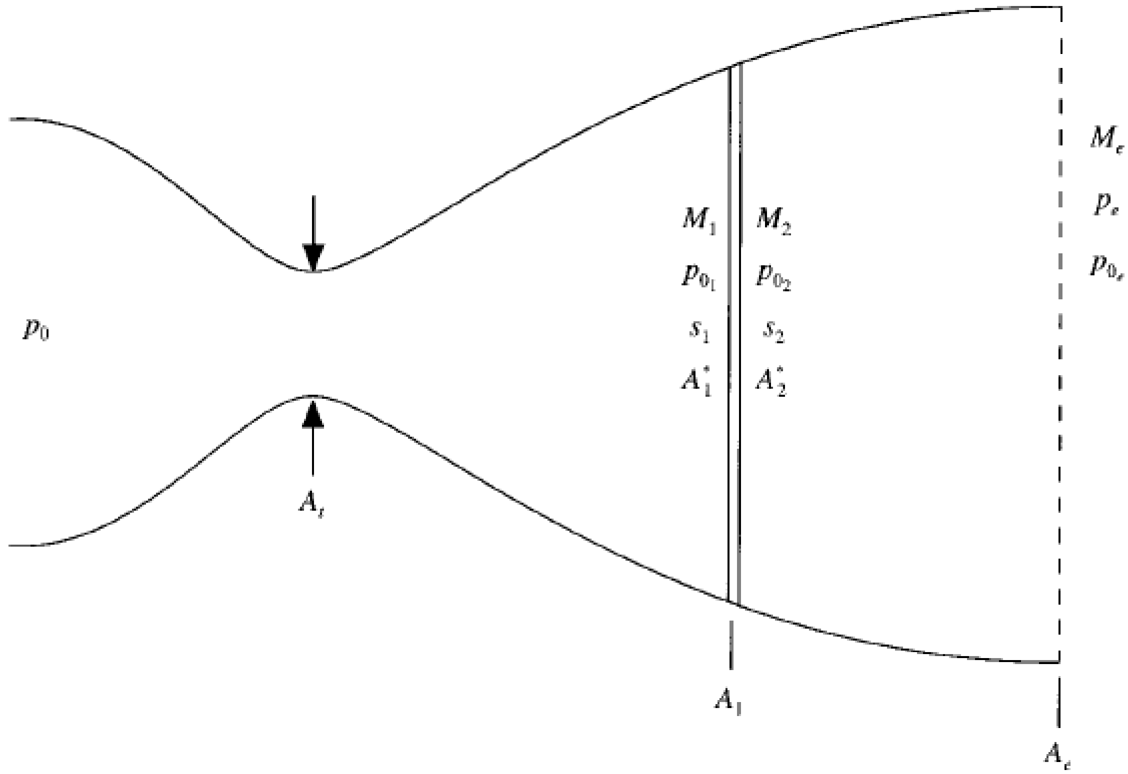


Fig.2. Nomenclature for the normal shock case

The normal shock wave is located at area A_1. Conditions immediately upstream of the shock are denoted with a subscript 1, and those immediately downstream of the shock are denoted with a subscript 2. The flow from the reservoir, where the pressure is p_0 , to station 1 is isentropic (with constant entropy s_1). Hence, the total pressure is constant in this flow; that is , $p_{0_1} = p_0$. The total pressure decreases across the shock (due to entropy increase across the shock). The flow from station 2 downstream of the shock to the nozzle exit is also isentropic (with constant entropy s_2 , where $s_2 > s_1$). Hence, the total pressure is constant in this portion of the flow, with $(p_0)_e = p_{0_2}$. Keep in mind that $p_{0_2} < p_{0_1}$. For this flow in front of the shock, $A_1^* = A_t$. However, due to entropy increase across the shock, the value of A^* in the subsonic flow downstream of the shock, denoted by A_2^* , takes on the role of a reference value. Indeed $A_2^* > A_1^*$.

In this section, we will numerically calculate the flow through a convergent-divergent nozzle under the condition where a normal shock wave exists in the divergent portion. The nozzle shape will be the same as used in previous problem where shock capturing method was not applied. We will use the governing equations in conservation form and will employ the philosophy of shock capturing. However, before jumping into the numerical solution, let us examine the exact analytical solutions.

EXACT ANALYTICAL RESULTS :

For the nozzle shape specified, the area of the exit is $A_e / A_t = 5.95$. Let us calculate the flow where p_e is specified as follows :

$$\frac{p_e}{p_{0_1}} = 0.6784 \qquad \text{(specified)} \tag{31}$$

The value of $p_e / p_{0_1} = 0.6784$ specified in the current section should be abought right to force a normal shock wave to stand somewhere inside the divergent portion of the nozzle. Let us first calculate the precise location, i.e., the precise area ratio inside the nozzle, where the normal shock wave will be located, compatible with the specified exit pressure given in Eq. (31). This calculation can be done in a direct fashion as follows .

The mass flow through the nozzle can be expressed as

$$\dot{m} = \frac{p_{0_1} A_1^*}{\sqrt{T_0}} \sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}} \tag{32}$$

For a given T_0,

$$\dot{m} \propto p_0 A^*$$

Since the mass flow is constant across the normal shock wave in Fig. 2, we have

$$p_{0_1} A_1^* = p_{0_2} A_2^* \tag{33}$$

Forming the ratio $p_e A_e / p_{0_e} A_2^*$, where $A_e^* = A_2^*$, and invoking Eq. (33), we have

$$\frac{p_e A_e}{p_{0_e} A_e^*} = \frac{p_e A_e}{p_{0_2} A_2^*} = \frac{p_e A_e}{p_{0_1} A_1^*} = \frac{p_e A_e}{p_{0_1} A_t} \tag{34a}$$

The right hand side of Eq. (34a) is known, because $\frac{p_e}{p_{0_1}}$ is specified as 0.6784 and $\frac{A_e}{A_t} = 5.95$. Thus, from Eq. (34a)

$$\frac{p_e A_e}{p_{0_e} A_e^*} = 0.6784(5.95) = 4.03648 \tag{34b}$$

From the isentropic relations, we have, respectively

$$\frac{A_e}{A_e^*} = \frac{1}{M_e}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_e^2\right)\right]^{(\gamma+1)/2(\gamma-1)} \tag{35}$$

and $$\frac{p_e}{p_{0_e}} = \left(1+\frac{\gamma-1}{2}M_e^2\right)^{-\gamma/(\gamma-1)} \tag{36}$$

Substituting Eqs. (35) and (36) in (34b), we have

$$\frac{1}{M_e}\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/2(\gamma-1)}\left[1+\frac{\gamma-1}{2}M_e^2\right]^{-1/2} = 4.03648 \tag{37}$$

Solving Eq. (37) for M_e , we have

$$M_e = 0.1431 \tag{38}$$

From Eq. (38), we have

$$\frac{p_e}{p_{0_e}} = \left[1+\frac{\gamma-1}{2}(0.1431)^2\right]^{-3.5} = 0.9858 \tag{39}$$

The total pressure ratio across the normal shock can be written as

$$\frac{p_{0_2}}{p_{0_1}} = \frac{p_{0_e}}{p_{0_1}} = \frac{p_{0_e}}{p_e}\frac{p_e}{p_{0_1}} \tag{40}$$

Substituting the numbers from Eqs. (31) and (39) into Eq. (40), we have

$$\frac{p_{0_2}}{p_{0_1}} = \frac{0.6784}{0.9858} = 0.6882 \tag{41}$$

The total pressure ratio across a normal shock is a function of M_1 in front of the shock, given by

$$\frac{p_{0_2}}{p_{0_1}} = \left[\frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}\right]^{\gamma/(\gamma-1)}\left[\frac{\gamma+1}{2\gamma M_1^2-(\gamma-1)}\right]^{1/(\gamma-1)} \tag{42}$$

Combining Eqs. (41) and (42) and solving for M_1 , we have

$$M_1 = 2.07 \tag{43}$$

Substituting Eq. (43) into the equation $\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{(\gamma+1)/(\gamma-1)}$, we get

$$\frac{A_1}{A_1^*} = \frac{A_1}{A_t} = 1.790 \tag{44}$$

The exact, analytical location of the normal shock wave is now known-it stands at a location in the nozzle where the area ratio 1.79. For our nozzle shape, this corresponds to a station of x/L = 2.1. All other properties across the shock wave now fall out from the result that $M_1 = 2.07$ .The static pressure ratio across the shock and the Mach number immediately behind the shock are obtained from

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right) = 1 + 1.167\left[\left(2.07\right)^2 - 1\right] = 4.83 \tag{45}$$

and

$$M_2 = \left\{\frac{1 + \left[\left(\gamma-1\right)/2\right]M_1^2}{\gamma M_1^2 - \left(\gamma-1\right)/2}\right\}^{1/2} = \left[\frac{1 + 0.2\left(2.07\right)^2}{1.4\left(2.07\right)^2 - 0.2}\right]^{1/2} = 0.566 \tag{46}$$

The exact, analytical solution obtained above will be compared with the numerical solution.

BOUNDARY CONDITIONS

The subsonic inflow boundary conditions are theoretically same as discussed in the Boundary Conditions subsection in previous problem; i.e., at the subsonic inflow boundary two properties are held fixed and one is allowed to float, and at the supersonic outflow boundary all properties are allowed to float. In the present formulation, as before , we hold ρ' and T' fixed at the inflow boundary, both equal to 1.0, and allow V' to float. By holding ρ' fixed, then U_1 at grid point i = 1 is fixed, independent of time, via $U_1 = \rho' A'$ .That is,

$$U_{1(i=1)} = \left(\rho' A'\right)_{i=1} = A'_{i=1} = \text{fixed value} \tag{47}$$

The floating value of V' at the inflow boundary is calculated at the end of each time step by linearly extrapolating U_2 from the known values at the internal grid points i = 2 and 3, that is,

$$U_{2(i=1)} = 2U_{2(i=2)} - U_{2(i=3)} \tag{48}$$

and then obtaining V' at i = 1 from Eq. (17) . Since V' floats at the inflow boundary, so does the value of U_3, which is given by

$$U_3 = \rho'\left(\frac{e'}{\gamma-1} + \frac{\gamma}{2}V'^2\right)A' \tag{49}$$

Since $\rho' A' = U_1$ and $e' = T'$, Eq. (49) is written as

$$U_3 = U_1\left(\frac{T'}{\gamma-1} + \frac{\gamma}{2}V'^2\right) \tag{50}$$

The value of U_3(i = 1) is found by inserting the value of V' at i = 1, calculated above, as well as the fixed value T' = 1, into Eq. (50). Note that the values of U_1 , U_2 and U_3 calculated at grid point i = 1 are used in turn to obtain the values of the flux terms F_1 , F_2 and F_3 at grid point i = 1. These values of the flux terms at the inflow boundary are needed to form the rearward differences that appears in Eqs. (16a) to (16c) during the corrector step of MacCormack's technique. The values of F_1 , F_2 and F_3 at the inflow boundary are calculated from from Eqs. (23), (25) and (27), respectively, using U_1 , U_2 and U_3 at grid point i = 1.

The outflow boundary condition for the present problem is also subsonic. A generic discussion of a subsonic outflow boundary as given in previous problem, where we emphasized that the exit pressure p_e must be specified, but all other properties are allowed to float. The same applies to the present calculation. Keep in mind that U_1 , U_2 and U_3 are the primary dependent variables in the governing equations. Hence, we obtain U_1 and U_2 at the downstream boundary by linear extrapolation from the adjacent two interior points.

$$\left(U_1\right)_N = 2\left(U_1\right)_{N-1} - \left(U_1\right)_{N-2} \tag{51a}$$

$$\left(U_2\right)_N = 2\left(U_2\right)_{N-1} - \left(U_2\right)_{N-2} \tag{51b}$$

Next, we decode $V_N^{'}$ from $\left(U_1\right)_N$ and $\left(U_2\right)_N$ using Eq. (18) .

$$V_N^{'} = \frac{\left(U_2\right)_N}{\left(U_1\right)_N} \tag{52}$$

The value of U_3 at grid point i = N is determined from the specified value of $p_N^{'} = 0.6784$ as follows. From the definition of U_3,

$$U_3 = \rho'\left(\frac{e'}{\gamma - 1} + \frac{\gamma}{2}V'^2\right)A' \tag{53}$$

However, $e' = T'$,and form the equation of state, $p' = \rho' T'$.Hence, Eq. (53) becomes

$$U_3 = \frac{p'A'}{\gamma - 1} + \frac{\gamma}{2}\rho' A' V'^2 \tag{54}$$

Since, $U_2 = \rho' A' V'$, Eq. (54) becomes

$$U_3 = \frac{p'A'}{\gamma - 1} + \frac{\gamma}{2}U_2 V' \tag{55}$$

Evaluating Eq. (55) at the downstream boundary, we have

$$\left(U_3\right)_N = \frac{p_N^{'} A'}{\gamma - 1} + \frac{\gamma}{2}\left(U_2\right)_N V_N^{'} \tag{56}$$

Since $p_N^{'}$ is specified as 0.6784 , Eq. (56) becomes

$$\left(U_3\right)_N = \frac{0.6784A'}{\gamma - 1} + \frac{\gamma}{2}\left(U_2\right)_N V_N^{'} \tag{57}$$

Equation (57) is the manner in which the specified exit pressure is folded into the numerical solution.

INITIAL CONDITIONS

For the present calculations, we choose the following initial conditions, which are qualitatively similar to the final solution.

$$\left.\begin{array}{l}\rho'=1\\ T'=1\end{array}\right\}\text{ for } 0\le x'\le 0.5 \qquad \begin{array}{r}(58a)\\(58b)\end{array}$$

$$\left.\begin{array}{l}\rho'=1.0-0.366(x'-0.5)\\ T'=1.0-0.167(x'-0.5)\end{array}\right\}\text{ for } 0.5\le x'\le 1.5 \qquad \begin{array}{r}(58c)\\(58d)\end{array}$$

$$\left.\begin{array}{l}\rho'=0.634-0.702(x'-1.5)\\ T'=0.833-0.4908(x'-1.5)\end{array}\right\}\text{ for } 1.5\le x'\le 2.1 \qquad \begin{array}{r}(58e)\\(58f)\end{array}$$

$$\left.\begin{array}{l}\rho'=0.5892+0.10228(x'-2.1)\\ T'=0.93968+0.0622(x'-2.1)\end{array}\right\}\text{ for } 2.1\le x'\le 3.0 \qquad \begin{array}{r}(58g)\\(58h)\end{array}$$

Pseudocode

```
% Initial Condition

for i = 1:k
    if x(i) >= 0 && x(i) < 0.5
        rho(i) = 1;                           % Eq. (58a)
        T(i) = 1;                             % Eq. (58b)
    elseif x(i) >= 0.5 && x(i) < 1.5
        rho(i) = 1-0.366*(x(i)-0.5);          % Eq. (58c)
        T(i) = 1-0.167*(x(i)-0.5);            % Eq. (58d)
    elseif x(i) >= 1.5 && x(i) < 2.1
        rho(i) = 0.634-0.702*(x(i)-1.5);      % Eq. (58e)
        T(i) = 0.833-0.4908*(x(i)-1.5);       % Eq. (58f)
    else
        rho(i) = 0.5892+0.10228*(x(i)-2.1);% Eq. (58g)
        T(i) = 0.93968+0.0622*(x(i)-2.1);     % Eq. (58h)
    end
end
```

As before, the initial condition for V’ is determined by assuming a constant mass flow; it is calculated from following equation

$$V'=\frac{U_2}{\rho'A'}=\frac{0.59}{\rho'A'} \qquad (59)$$

Pseudocode

```
V = 0.59./(rho.*A);  % Velocity, V/a0 [Eq.(59)]
```

The value 0.59 is chosen for U_2 because it is close to the exact analytical value of the steady-state mass flow (which for this case is 0.579). Therefore, the initial conditions for V' as a function of x' is obtained by substituting the ρ' variation given by Eqs. (58a), (58c) , (58e) and (58g) into Eq. (59). Finally, the initial conditions U_1 , U_2 , U_3 are obtained by substituting the above variations for ρ', T' and V' into the equations given below.

$$U_1 = \rho' A'$$

$$U_2 = \rho' A' V'$$

$$U_3 = \rho' \left(\frac{e'}{\gamma - 1} + \frac{\gamma}{2} V'^2 \right) A'$$

where $e' = T'.V'$ is calculated such that $U_2 = \rho' A' V' = 0.59$.

Pseudocode

```
% Initial condition of solution vectors
U1 = rho.*A;
U2 = rho.*A.*V;
U3 = rho.*A.*((T./gamma2)+(0.5*gamma.*V.^2));

% Initial condition of flux vectors
F1 = U2;                                          % eq (23)
F2 = (U2.^2./U1)+(1-gamma1).*(U3-0.5*gamma
      .*U2.^2./U1);                               % eq (25)
F3 = (gamma.*U2.*U3./U1)-
0.5*gamma*gamma2.*U2.^3./U1.^2;                   % eq (27)
```

TIME STEP CALCULATION

The governing equations for unsteady, quasi-one-dimensional flow in conservation form are hyperbolic partial differential equations, just as are the governing equations in nonconservation form. Therefore, for an explicit finite-difference solution, the stability criterion for the time step increment Δt is specified by the CFL criterion. In turn, for the calculations in the present section, the value of Δt is obtained precisely as described in previous problem where we didn't apply shock-capturing method and given by following equations.

$$\left(\Delta t\right)_i^t = C\frac{\Delta x}{\left(\sqrt{T}\right)_i^t + V_i^t}$$

$$\Delta t = \text{minimum}\left(\Delta t_1^t, \Delta t_2^t, ..., \Delta t_i^t, ..., \Delta t_N^t\right)$$

pseudocode

```
dta = (c*dx)./(V+T.^0.5);
dt = min(dta);
```

THE INTERMEDIATE TIME-MARCHING PROCEDURE: THE NEED FOR ARTIFICIAL VISCOSITY

Perhaps the most dramatic distinction between the present shock-capturing case and our previous calculations is the matter of artificial viscosity. Think back about our calculations so far; they have been carried out with no artificial viscosity explicitly added to the numerical calculations. When we practice the art of shock capturing, the smoothing and stabilization of the solution by the addition of some type of numerical dissipation is necessary.

To proceed with this solution, we will add artificial viscosity in the manner described above. Specifically following Eq.(), we form an expression

$$S_i^{t'} = \frac{C_x\left|\left(p'\right)_{i+1}^{t'} - 2\left(p'\right)_i^{t'} + \left(p'\right)_{i-1}^{t'}\right|}{\left(p'\right)_{i+1}^{t'} + 2\left(p'\right)_i^{t'} + \left(p'\right)_{i-1}^{t'}}\left(U_{i+1}^{t'} - 2U_i^{t'} + U_{i-1}^{t'}\right) \tag{60}$$

whereas beforehand we would calculate a predicted value (using MacCormack's technique) from

$$\overline{U}^{t_i'+\Delta t'} = \left(U\right)_i^{t'} + \left(\frac{\partial U}{\partial t'}\right)_i^{t'}\Delta t'$$

we now replace this with

$$\left(\overline{U}_1\right)_i^{t'+\Delta t'} = \left(U_1\right)_i^{t'} + \left(\frac{\partial U_1}{\partial t'}\right)_i^{t'}\Delta t' + \left(S_1\right)_i^{t'} \tag{61}$$

$$\left(\overline{U}_2\right)_i^{t'+\Delta t'} = \left(U_2\right)_i^{t'} + \left(\frac{\partial U_2}{\partial t'}\right)_i^{t'}\Delta t' + \left(S_2\right)_i^{t'} \tag{62}$$

$$\left(\overline{U}_3\right)_i^{t'+\Delta t'} = \left(U_3\right)_i^{t'} + \left(\frac{\partial U_3}{\partial t'}\right)_i^{t'}\Delta t' + \left(S_3\right)_i^{t'} \tag{63}$$

where U_1, U_2 and U_3 are our dependent variables in Eqs. (16a) to (16c) and S_1, S_2 and S_3 in Eqs. (61) to (63) are obtained from Eq. (60) by using respectively, U_1, U_2 and U_3 on the right-hand side. Similarly, on the corrector step, whereas beforehand we would calculate the corrected values from

$$U_i^{t'+\Delta t'} = U_1^{'} + \left(\frac{\partial U}{\partial t'}\right)_{av} \Delta t'$$

We now replace this with

$$(U_1)_i^{t'+\Delta t'} = (U_1)_i^{t'} + \left(\frac{\partial U_1}{\partial t}\right)_{av} \Delta t' + (\bar{S}_1)_i^t \tag{64}$$

$$(U_2)_i^{t'+\Delta t'} = (U_2)_i^{t'} + \left(\frac{\partial U_2}{\partial t}\right)_{av} \Delta t' + (\bar{S}_2)_i^t \tag{65}$$

$$(U_3)_i^{t'+\Delta t'} = (U_3)_i^{t'} + \left(\frac{\partial U_3}{\partial t}\right)_{av} \Delta t' + (\bar{S}_3)_i^t \tag{66}$$

Where $\bar{S}_1$, $\bar{S}_2$, and $\bar{S}_3$ are obtained from the following equation

$$\bar{S}_i^{t'+\Delta t'} = \frac{C_x \left|(\bar{p}')_{i+1}^{t'+\Delta t'} - 2(\bar{p}')_i^{t'+\Delta t'} + (\bar{p}')_{i-1}^{t'+\Delta t'}\right|}{(\bar{p}')_{i+1}^{t'} + 2(\bar{p}')_i^{t'} + (\bar{p}')_{i-1}^{t'}} \times \left[(\bar{U})_{i+1}^{t'+\Delta t'} - 2(\bar{U})_i^{t'+\Delta t'} + (\bar{U})_{i-1}^{t'+\Delta t'}\right] \tag{67}$$

The values of $\bar{S}_1$, $\bar{S}_2$ and $\bar{S}_3$ are obtained from Eq. (67) by using, respectively , the values of $\bar{U}_1$, $\bar{U}_2$ and $\bar{U}_3$ on the right-hand side.

The matlab code of the given problem is given below :

```
clc
clear all
close all
%%%%%-------------------------------------------------
%%%%%
%%%%%-------------------------------------------------
%%%%%

% This program deals with the quasi-one-dimensional
% flow through a convergent-divergent nozzle using
shock
% capturing method.
% The finite difference expression is set up using
% MacCormack's explicit technique for the numerical
solution
% of governing equations in conservation form.

%%%%%-------------------------------------------------
%%%%%
```

```
%%%%%--------------------------------------------------
%%%%%

% Note : All parameters are nondimensionalized.

dx = 0.05;                % grid size
c  = 0.5;                 % courant number
gamma = 1.4;              % Ratio of specific heats

% Discretization of nondimensional distance along the
nozzle
x = 0:dx:3.0;
k = length(x);
n = 25;

Cx = 0.2;     % artificial viscosity arbitrary parameter

% Parabolic area distrubition along the nozzle
A = 1+2.2.*(x-1.5).^2;    % Area,A/A*

% Grid distribution in the nozzle
for i = 1:k
    y = linspace(-A(i)/2,A(i)/2,n);
    for j = 1:n
        xx(i,j) = x(i);
        yy(i,j) = y(j);
    end
end

% Initial Condition

for i = 1:k
    if x(i) >= 0 && x(i) < 0.5
        rho(i) = 1;                        % Eq. (58a)
        T(i) = 1;                          % Eq. (58b)
    elseif x(i) >= 0.5 && x(i) < 1.5
        rho(i) = 1-0.366*(x(i)-0.5);       % Eq. (58c)
        T(i) = 1-0.167*(x(i)-0.5);         % Eq. (58d)
    elseif x(i) >= 1.5 && x(i) < 2.1
        rho(i) = 0.634-0.702*(x(i)-1.5);    % Eq. (58e)
        T(i) = 0.833-0.4908*(x(i)-1.5);     % Eq. (58f)
    else
```

```
        rho(i) = 0.5892+0.10228*(x(i)-2.1); % Eq. (58g)
        T(i) = 0.93968+0.0622*(x(i)-2.1);   % Eq. (58h)
    end
end

V = 0.59./(rho.*A);                 % Velocity,V/a0,[Eq.(59)]
P = rho.*T;                         % pressure,p/p0
mf_0 = rho.*A.*V;                   % mass flow rate @ t = 0

gamma1 = 1/gamma;
gamma2 = gamma-1;

% Initial condition of solution vectors
U1 = rho.*A;
U2 = rho.*A.*V;
U3 = rho.*A.*((T./gamma2)+(0.5*gamma.*V.^2));

% Initial condition of flux vectors
F1 = U2;                                        % Eq.(23)
F2 = (U2.^2./U1)+(1-gamma1).*                   % Eq.(25)
(U3-0.5*gamma.*U2.^2./U1);
F3 = (gamma.*U2.*U3./U1)-
0.5*gamma*gamma2.*U2.^3./U1.^2;                 % Eq.(27)

resmax = 10^-6;       % maximum error
res = 1;
t = 0;
nstep = 0;

while res > resmax
    dta = (c*dx)./(V+T.^0.5);
    dt = min(dta);

    nstep = nstep+1;

    % Predictor Step
     for i = 2:k-1
         J2_p = gamma1*rho(i)*T(i)*((A(i+1)-A(i))/dx);
         dU1dt_p(i) = -(F1(i+1)-F1(i))/dx;
         dU2dt_p(i) = -(F2(i+1)-F2(i))/dx + J2_p;
         dU3dt_p(i) = -(F3(i+1)-F3(i))/dx;
     end
```

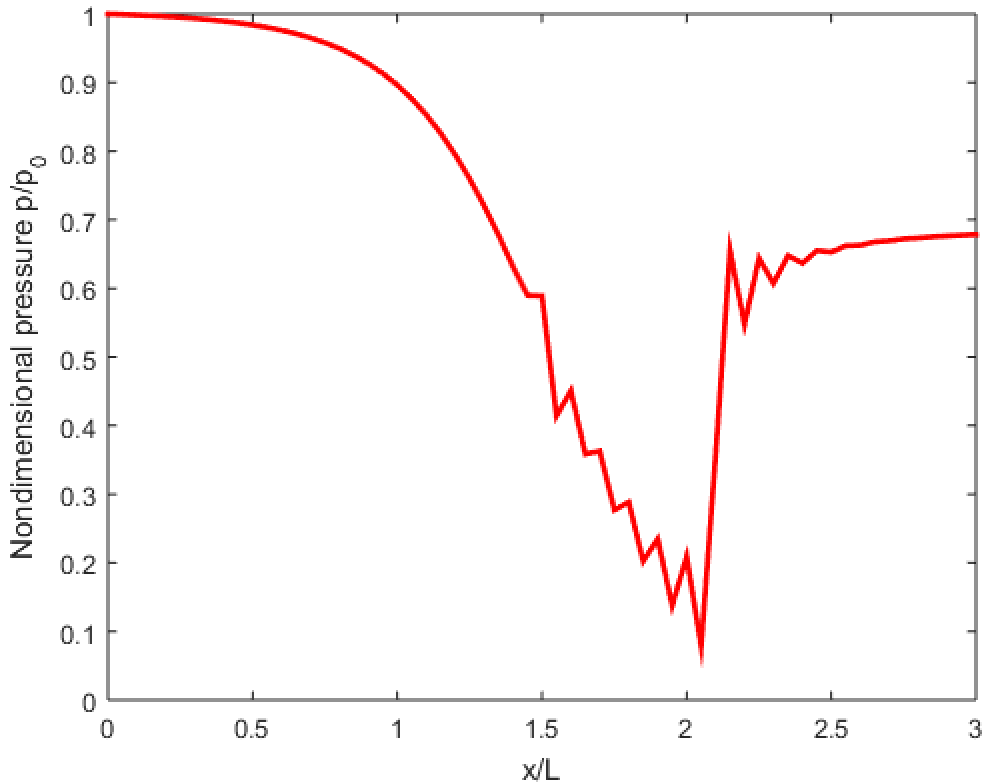
```
    for i = 2:k-1
        num = abs(P(i+1)-2*P(i)+P(i-1));
        den = P(i+1)+2*P(i)+P(i-1);
        S1_p(i) = (Cx*num*(U1(i+1)-2*U1(i)+U1(i-
1)))/den;
        S2_p(i) = (Cx*num*(U2(i+1)-2*U2(i)+U2(i-
1)))/den;
        S3_p(i) = (Cx*num*(U3(i+1)-2*U3(i)+U3(i-
1)))/den;
    end

    for i = 2:k-1
     U1_p(i) = U1(i)+(dU1dt_p(i)*dt)+S1_p(i);% Eq.(61)
     U2_p(i) = U2(i)+(dU2dt_p(i)*dt)+S2_p(i);% Eq.(62)
     U3_p(i) = U3(i)+(dU3dt_p(i)*dt)+S3_p(i);% Eq.(63)
    end

    U1_p(1) = U1(1);
    U2_p(1) = U2(1);
    U3_p(1) = U3(1);
    U1_p(k) = U1(k);
    U2_p(k) = U2(k);
    U3_p(k) = U3(k);


for i = 1:k
    rho_p(i) = U1_p(i)/A(i);                    % Eq.(17)
    V_p(i) = U2_p(i)/U1_p(i);                   % Eq.(18)
    T_p(i) = gamma2*((U3_p(i)/U1_p(i))-
0.5*gamma*V_p(i)^2);                            % Eq.(19)
    P_p(i) = rho_p(i)*T_p(i);                   % Eq.(20)
    F1_p(i) = U2_p(i);                          % Eq.(23)
    F2_p(i) = (U2_p(i)^2/U1_p(i))+(1-gamma1)*...
      (U3_p(i)-0.5*gamma*U2_p(i)^2/U1_p(i)); % Eq.(25)
    F3_p(i) = (gamma*U2_p(i)*U3_p(i)/U1_p(i))-...
 (0.5*gamma*gamma2*U2_p(i)^3/U1_p(i)^2);       % Eq.(27)
end


    % corrector Step
end
```

```
    for i = 2:k-1
        J2_c = gamma1*rho_p(i)*T_p(i)*((A(i)-A(i-
1))/dx);
        dU1dt_c(i) = -(F1_p(i)-F1_p(i-1))/dx;
        dU2dt_c(i) = -(F2_p(i)-F2_p(i-1))/dx + J2_c;
        dU3dt_c(i) = -(F3_p(i)-F3_p(i-1))/dx;
    end

    for i = 2:k-1
        num = abs(P_p(i+1)-2*P_p(i)+P_p(i-1));
        den = P_p(i+1)+2*P_p(i)+P_p(i-1);
        S1_c(i) = (Cx*num*(U1_p(i+1)-2*U1_p(i)+U1_p(i-
1)))/den;
        S2_c(i) = (Cx*num*(U2_p(i+1)-2*U2_p(i)+U2_p(i-
1)))/den;
        S3_c(i) = (Cx*num*(U3_p(i+1)-2*U3_p(i)+U3_p(i-
1)))/den;
    end

     % Average time derivatives

     for i = 2:k-1
         dU1dt_av(i) = 0.5*(dU1dt_p(i)+ dU1dt_c(i));
         dU2dt_av(i) = 0.5*(dU2dt_p(i)+ dU2dt_c(i));
         dU3dt_av(i) = 0.5*(dU3dt_p(i)+ dU3dt_c(i));
     end

     for i = 2:k-1
    U1(i) = U1(i)+ (dU1dt_av(i)*dt)+S1_c(i);   % Eq.(64)
    U2(i) = U2(i)+ (dU2dt_av(i)*dt)+S2_c(i);   % Eq.(65)
    U3(i) = U3(i)+ (dU3dt_av(i)*dt)+S3_c(i);   % Eq.(66)
     end

  % Boundary condition @ first node
     U1(1) = A(1);                          % Eq. (47)
     U2(1) = 2*U2(2)-U2(3);                 % Eq. (48)
     ve = U2(1)/U1(1);
   U3(1) = U1(1)*((1/gamma2)+0.5*gamma*ve^2); % Eq.(50)

  % Boundary condition @ last node
    U1(k) = 2*U1(k-1)-U1(k-2);               % Eq. (51a)
```

```
    U2(k) = 2*U2(k-1)-U2(k-2);                 % Eq. (51b)
    U3(k) = (0.6784*A(k)/gamma2)+...           % Eq. (55)
            (0.5*gamma*U2(k)^2/U1(k));

    for i = 1:k
        F1(i) = U2(i);
        F2(i) = (U2(i)^2/U1(i))+(1-gamma1)*(U3(i)-
0.5*gamma...
                *U2(i)^2/U1(i));
        F3(i) = (gamma*U2(i)*U3(i)/U1(i))-
0.5*gamma*gamma2...
                *U2(i)^3/U1(i)^2;
    end


    % corrected values of primitive variables
    for i = 1:k
    rho(i) = U1(i)/A(i);                          % Eq.(17)
    V(i) = U2(i)/U1(i);                           % Eq.(18)
    T(i) = gamma2*((U3(i)/U1(i))-0.5*gamma*V(i)^2);%
Eq.(19)
    M(i) = V(i)*T(i)^-0.5;
    P(i) = rho(i)*T(i);                            % Eq.(20)
    end

    % mass flow rate after t+dt
    mf = rho.*A.*V;

    res1 = abs(dU1dt_av(16));
    res2 = abs(dU2dt_av(16));

    if res1 > res2
        res = res1;
    else
        res = res2;
    end


for i = 1:k
    for j = 1:n
        rho1(i,j) = rho(i);
        T1(i,j) = T(i);
```

```
        P1(i,j) = P(i);
        M1(i,j) = M(i);
        V1(i,j) = V(i);
    end
end

figure(1)
if mod(nstep,5) == 0
surf(xx,yy,M1);view(2)
shading interp
colormap(jet(256))
colorbar
axis off
drawnow
end

if nstep == 2600,break,end

end
figure(2)
plot(x,P,'-r','Linewidth',2)

 figure(3)
 plot(x,M,'-b','Linewidth',2)

 figure(4)
 plot(x,mf,'-k','Linewidth',2)
```

The graph of pressure distribution, mach number distribution and mass flow through nozzle is plotted without artificial velocity (Cx = 0).



Nondimensional pressure p/p0
x/L
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.5
1.5
2
2.5
3

Fig. 3

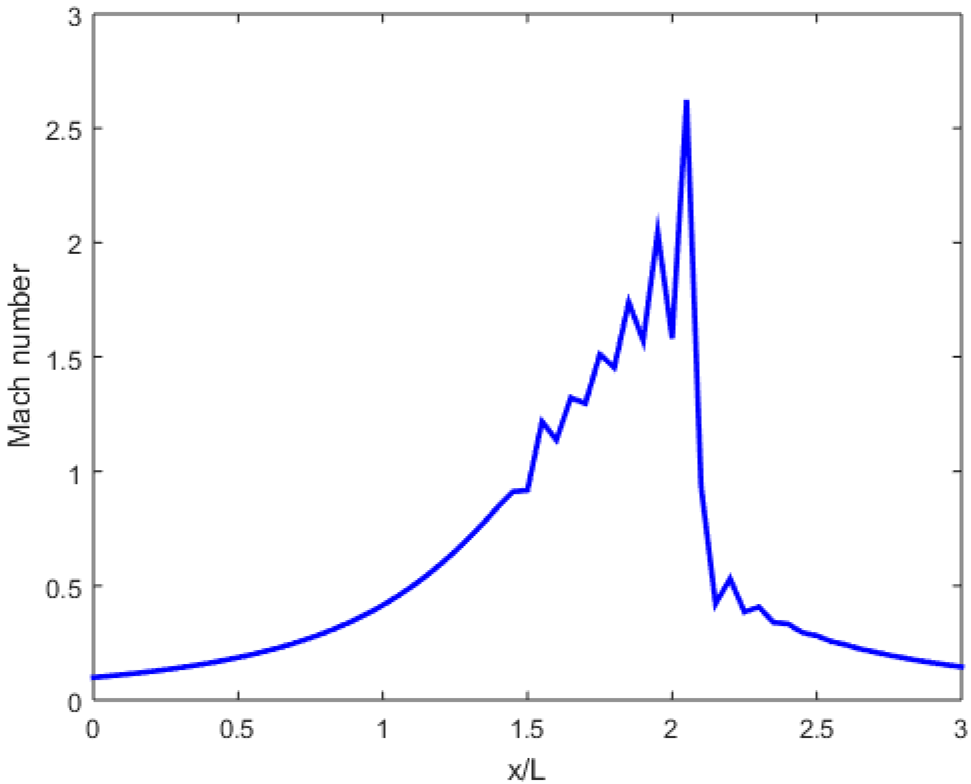


Fig. 4

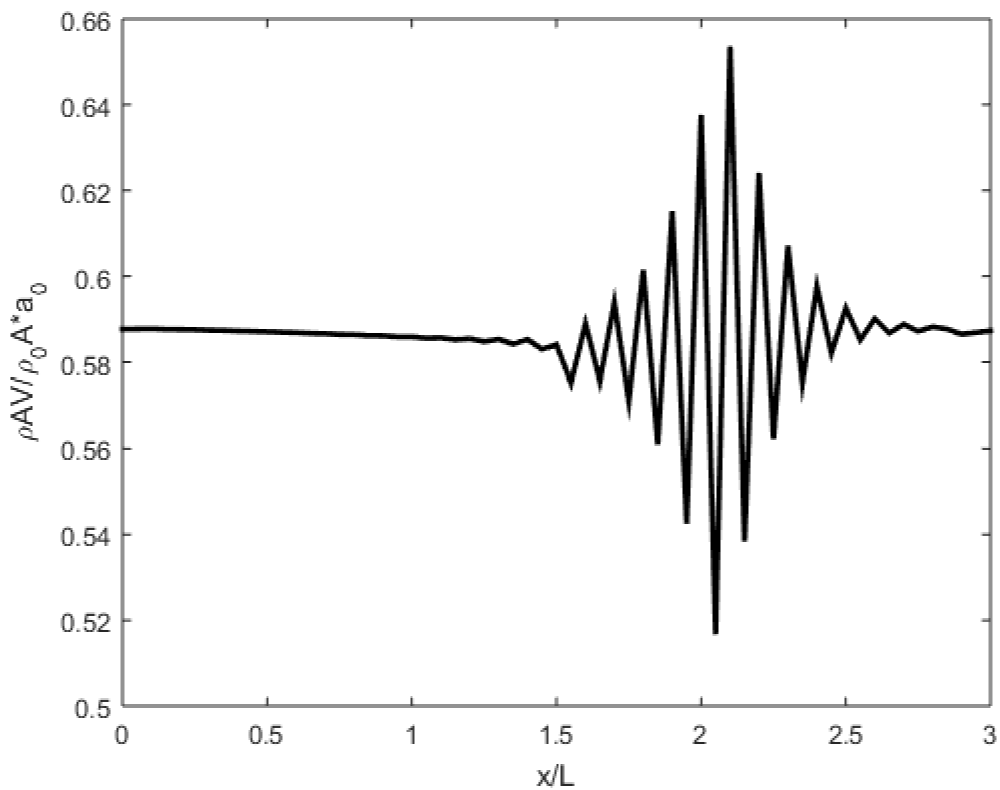
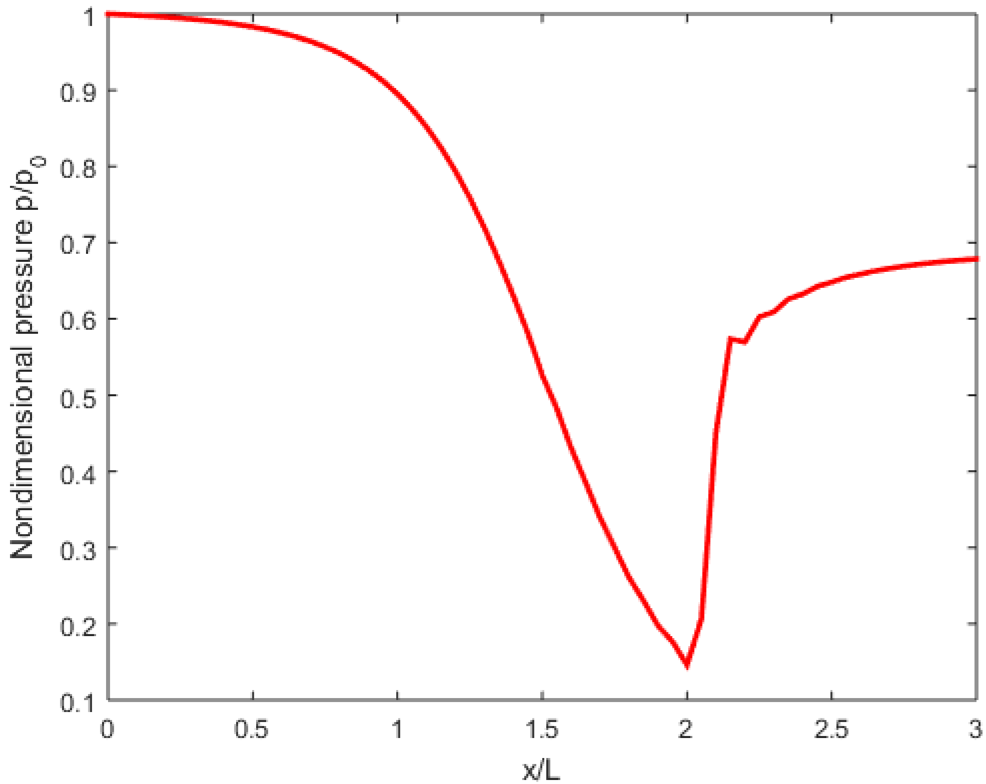


Fig. 5

The graph of pressure distribution, mach number distribution and mass flow through nozzle is plotted without artificial velocity (Cx = 0.2).



Nondimensional pressure p/p0
x/L
0
0.5
1
1.5
2
2.5
3
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Fig. 6

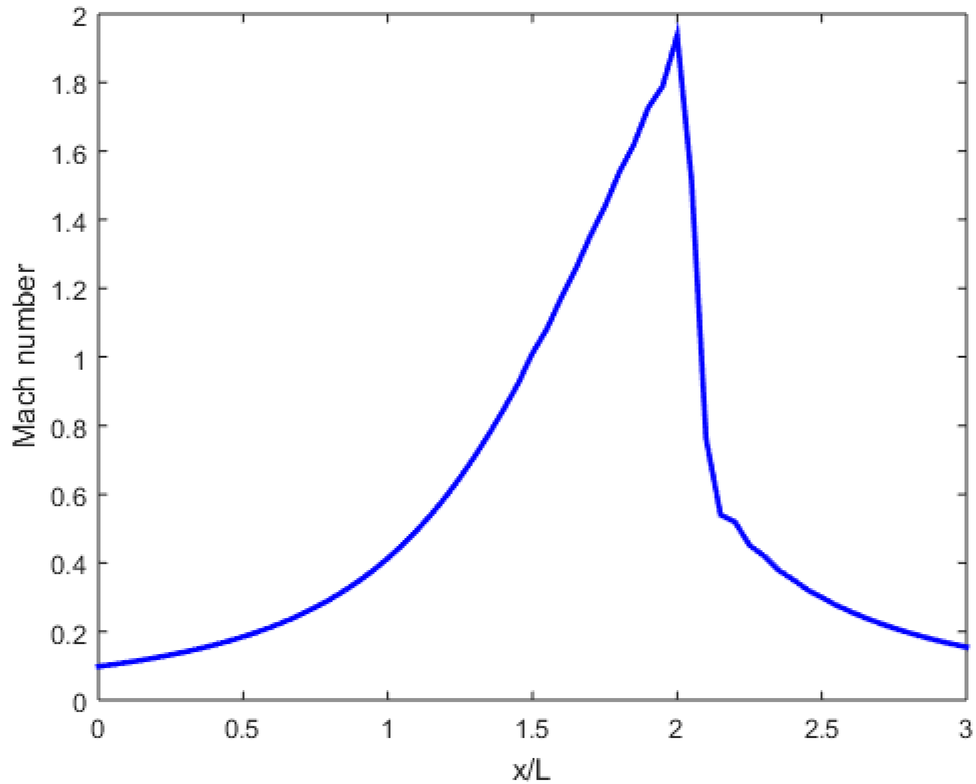

Mach number
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0
0.5
1
1.5
2
2.5
3
x/L

Fig. 7

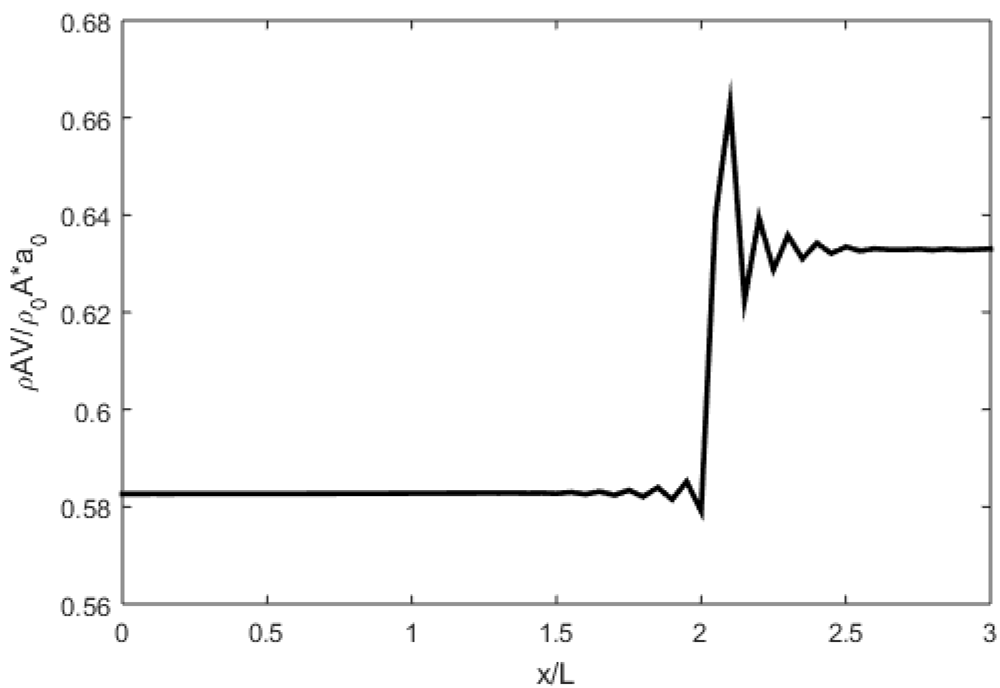

$\rho AV/\rho_0 A^*a_0$
0.56
0.58
0.6
0.62
0.64
0.66
0.68
0
0.5
1
1.5
2
2.5
3
x/L

Fig. 8

1. The addition of artificial viscosity has just about eliminated the oscillations that were encountered in the case with no artificial viscosity. The contrast between the numerical results with Cx = 0.2 and Cx = 0 is dramatic. That is what artificial viscosity does-smooth the results and decrease (if not virtually eliminate) the oscillations.
2. Close examination of Fig. 8 shows that the oscillations are not completely eliminated. There is a small oscillation in the pressure distribution just down-stream of the shock; however, it is not that bothersome. Results obtained with more artificial viscosity (Cx = 0.3) show that even this small oscillation virtually disappears. However, too much artificial viscosity can compromise other aspects of the solution,as noted below.
3. The numerical results in Fig. 8 show that artificial viscosity tends to smear the captured shock wave over more grid points. The more extreme changes across the shock that are predicted by the exact, analytical results are slightly diminished by the inclusion of artificial viscosity in the numerical results. This increased smearing of the shock wave due to increased artificial viscosity is one of the undesirable aspects of adding extra numerical dissipation to the solution.