Report: Assignment 3

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Ques1:
Here, we have to design an IMRT setup with the following constraints in mind:

- We want to achieve close to 10 units of radiation in the red region (denoting a tumour)
- Radiation as much below 4 units as possible in the remaining portion.
- You may assume the radiation intensity diminishes to half every 4 units
- Each beam is 1 unit wide

We assume that the beams are not placed at specific angles but are present corresponding to every edge block i.e.in this case of the given graph, we would have $7*2+6*2 = 26$ beams with each beam having 20 beam-lets. This is assumed to make the modelling of multiple beams problem relatively easy since the beams when placed equi-angled are not only difficult to incorporate in the model but also not very efficient to cure this tumour, which apparently is I-shaped.
We make the graph finer by introducing the mesh of beam-lets which makes the original 6x7 graph into 120x140 graph. We use the following notations for our model:
Due to such fineness in the graph, we refer to each of the block in 120x140 graph by their mid-point(i,j). We now introduce:

$$D_{ij} = w_{i,0} + w_{0,j} + w_{i,120} + w_{140,j}$$

D_{ij} represents the intensity of radiations received by block defined by row i and column j.
Each of the $w_{i,0}$ represents the intensity of beam-lets on the left axis of the original graph reaching block (i,j), similarly $w_{i,120}$ denotes the intensity of beam-lets on the right reaching block (i,j), just as $w_{0,j}$ represents the intensity of beam-lets from the bottom (as (0,0) is considered to be at the bottom left of the graph) reaching block (i,j) and $w_{140,j}$ represents the intensity of beam-lets from the top reaching block (i,j).
Now, as we have been given that the intensity of the beam-let reduces to half after they have traversed 4 units of distance on the original graph hence, though our decision variables would be the intensities sent out by each beam-lets, which we represent by $\bar{w}_{i,j}$, the value of the above defined $w_{i,j}$ would vary according to the region they lie in, which is depicted as follows:

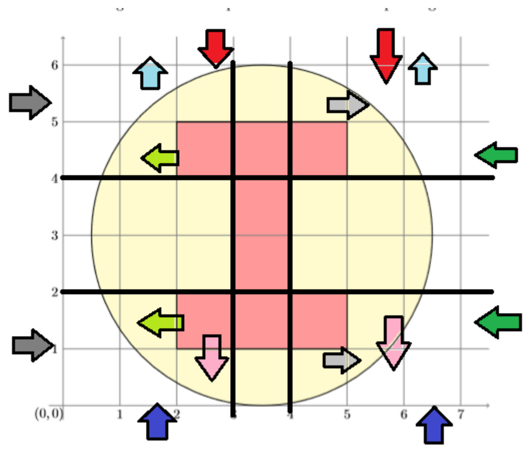

Figure 1: the decrease in intensity is shown by the bold lines on the graph and fading colour represents the halving of the intensities in those regions

We include conditions in our program so that the values of $w_{i,j}$ are as per the intensity received in that block by the respective $\bar{w}_{i,j}$ beamlet
We define a set of coordinates of the tumour cells and name it as T. for all such tumour cells, the respective $D_{i,j} \geq 10$ and similarly for every other cell, $D_{i,j} \leq 4$. We model this problem as below:

$$\underset{\bar{w}_{i,j}}{minimize} \sum_i \sum_j D_{i,j}$$

$$s.t. \;\; D_{i,j} \geq 10 \;\; \forall \;\; (i,j) \;\; \in \;\; T$$

$$D_{i,j} \leq 4 \;\; \forall \;\; (i,j) \;\; \in \;\; T^{'}$$

$$\bar{w}_{i,j} \geq 0$$

where $D_{i,j}$ are defined as mentioned earlier. We see that each of the $D_{i,j} \;\; and \;\; w_{i,j}$ can be expressed in terms of the intensity of the beamlets i.e. $\bar{w}_{i,j}$. Our decision variables hence, are only $\bar{w}_{i,j}$.
Code file of the same has been attached with the name **ques1.py**.

Ques2:
To Show: Any solution of the original LP is also a solution of the network flow LP.

Proof. **Original LP:**

$$\min_x \sum_{k=1}^{L} x_k$$

$$s.t. \quad \sum_{k=1}^{L} S_k x_k = M$$

$$S_k \ \ is \ \ a \ \ valid \ \ shape \ \ matrix \ \ and \ \ x_k \geq 0$$

where S_k represents the shape matrix and M is the intensity matrix.

Network Flow LP:

$$min \quad \sum_{(u,v)\in A} x_{uv}$$

$$s.t. \quad \sum_{(u,v)\in A} R'_{uv} x_{uv} = b'$$

$$x_{uv} \geq 0 \ \ \forall \ \ (u,v) \in A$$

From the paper by Ahuja and Hemacher, we use the fact as mentioned in the paper that, the following LP is equivalent to the Network flow LP.(discussed in the paper)
Equivalent LP to the network Flow:

$$min \sum_{k=1}^{K'} x_{ik}$$

$$s.t. \quad \sum_{k=1}^{K'} R_k x_{ik} = b_i$$

$$x_{ik} \geq 0 \ \ \forall \ \ k = 1, 2, \ldots K'$$

where x_{ik} represents the time for which row i of the Shape matrix k is exposed to the radiation and R_k represents the i the row of each k the Shape matrix. Similarly b_i represents the i th row of the intensity matrix. We extended these representations to the final network flow model where R'_{uv} is the representation of the R_k row matrix such that u is the minimum index with value 1 and v-1 is the highest index with value 1 and hence x_{uv} represents the time for which the area uv is exposed to the intensity.

Since the paper shows clear equivalence between the above LP and the network flow LP, we would use the Equivalent Network flow LP to show equivalence with the original LP.
$\sum_{k=1}^{L} S_k x_k = M$ can be broken down into m equations as below, m being the number of rows in the intensity matrix M

$$S_{i1}x_1 + S_{i2}x_2 + \cdots + S_{iL}x_L = M_i \quad \forall \;\; i \;\; \in \;\; |rows(M)|$$

Now, the solution set of the original LP would look like:

$$Sol_{LP} = \{(x_1, x_2, \ldots, x_L) : \sum_j S_{ij}x_j = M_i; x_j \geq 0 \;\; \forall j\}$$

and the solution set of the Network flow LP would look like:

$$Sol_{NetLP} = \{(x_1, x_2, \ldots, x_{K'}) : \sum_{k=1}^{K'} R_k x_k = b_i; x_k \geq 0 \;\; \forall \;\; j\}$$

Here K'=cardinality of set of all possible row vectors and L=cardinality of set of all possible shape matrices. From here, we can say that L > K'. We essentially show that any solution of the original LP can be used to generate a solution in the network flow LP. Consider $(x_1, x_2, \ldots, x_L)$ as a solution of the original LP. Then, we define:

$$A_{\bar{k}} = \{x_j : S_{ij} = R_{\bar{k}} \;\; \forall \;\; j\}$$

A solution $(x_1, x_2, \ldots, x_L)$ can be transformed to $(x_1, x_2, \ldots, x_{K'})$ by defining each $x_{\bar{k}}$ as:

$$x_{\bar{k}} = \sum_{j \in A_{\bar{k}}} x_j$$

Since this can be done for any solution obtained from original LP, hence we have proved our hypothesis that any solution of the original LP is also a solution of the network flow LP. $\square$

To check: Is the converse true?
Going by the argument stated above, any solution of the network flow LP say $(x_1, x_2, \ldots, x_{K'})$, we can translate this solution to the original LP by solving the following system of linear equations:

$$x_{\bar{k}} = \sum_{j \in A_{\bar{k}}} x_j \quad \forall \;\; k \in K'$$

This system of equations would have infinite solutions since the number of variable come from space $\mathbb{R}^L$ and the number of equations come from space $\mathbb{R}^{K'}$. Since we stated above that $K' < L$, we can say that this system of linear equations would always have a solution due to presence of free variables. Hence the converse is true as well.