ECE 2200: Signals and Information

Sinusoids

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Complex numbers

- Imaginary unit: Let j be the solution of $x^2 = -1$.

$$j = \sqrt{-1}, j^2 = -1, j^3 = -j, j^4 = 1, \cdots$$

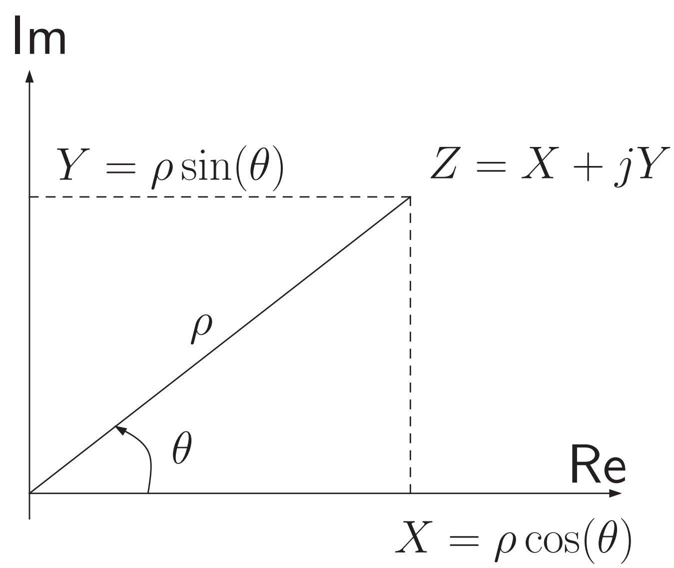
Note that j^n is a periodic sequence with period 4.

- Complex number:

$$Z = X + jY \quad \longleftrightarrow \quad \mathrm{Re}(Z) \triangleq X,\ \mathrm{Im}(Z) \triangleq Y$$

- Polar representation:

$$Z = \rho\cos(\theta) + j\rho\sin(\theta)$$



We call ρ the magnitude of Z and is denoted by $|Z|$.
We call θ the phase of Z and is denoted by $\angle Z$.

$$|Z| = \sqrt{X^2 + Y^2}, \quad \angle Z = \arctan\left(\frac{Y}{X}\right)$$

We write $Z = \rho\angle\theta$.

Euler's Identity

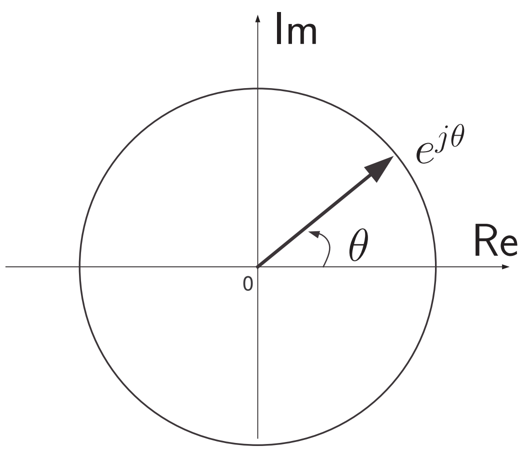
Euler's Identity:

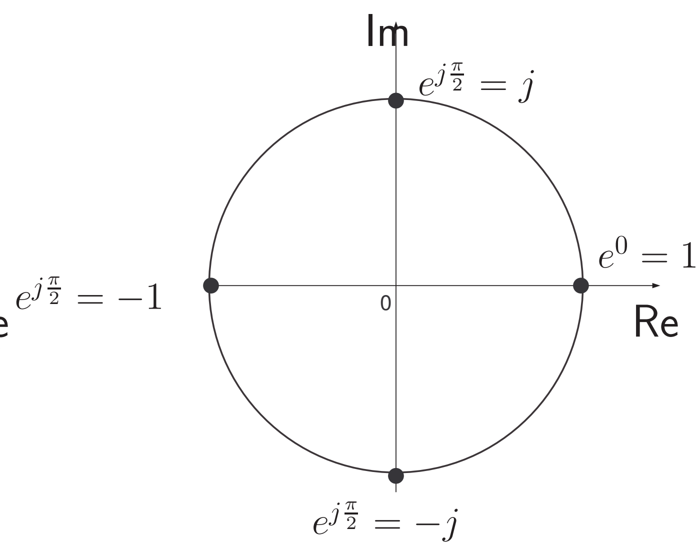
$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Proof: Recall the power series expansion:

$$\begin{aligned}
\cos\theta &= 1 - \frac{1}{2!}(\theta)^2 + \frac{1}{4!}\theta^4 - \cdots \\
\sin\theta &= \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots \\
e^{j\theta} &= 1 + j\theta + \frac{1}{2!}(j\theta)^2 + \frac{1}{3!}(j\theta)^3 + \cdots
\end{aligned}$$

The identity is verified by computing $\cos\theta + j\sin\theta$.





Inverse Euler's Formula

$$\begin{aligned}
\cos(\theta) &= \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \\
\sin(\theta) &= \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)
\end{aligned}$$

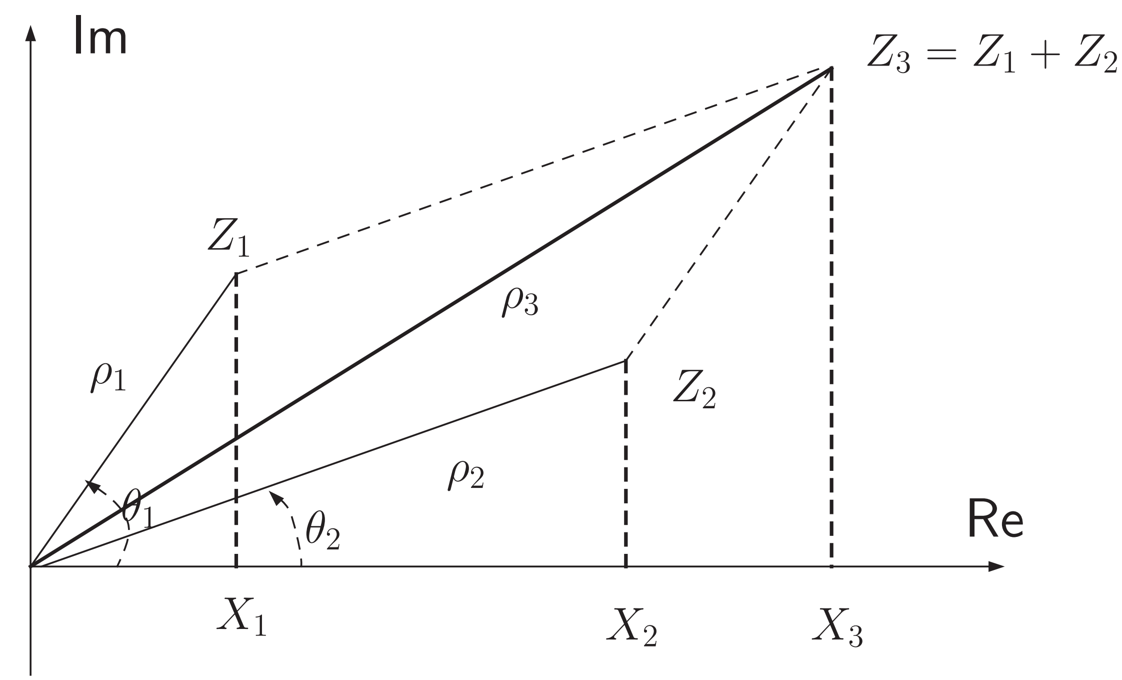
Complex Arithmetic: Additions

Given $Z_1 = X_1 + jY_1 = \rho_1 e^{j\theta_1}$ and $Z_2 = X_2 + jY_2 = \rho_2 e^{j\theta_2}$,

$$Z_3 \;=\; Z_1 + Z_2 = \underbrace{(X_1 + X_2)}_{X_3 = \text{Re}(Z_3)} + j\,\underbrace{(Y_1 + Y_2)}_{Y_3 = \text{Im}(Z_3)} = \rho_3 e^{j\theta_3}$$

where

$$\rho_3 = \sqrt{(X_1 + X_2)^2 + (Y_1 + Y_2)^2}, \quad \theta_3 = \arctan \frac{Y_1 + Y_2}{X_1 + X_2}.$$



Multiplication and Division

Given $Z_1 = X_1 + jY_1 = \rho_1 e^{j\theta_1}$ and $Z_2 = X_2 + jY_2 = \rho_2 e^{j\theta_2}$,

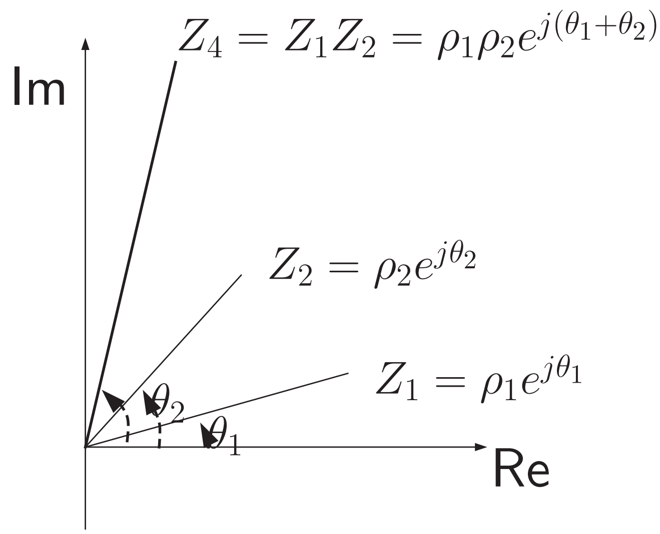
$$\begin{aligned} Z_4 &= Z_1 \times Z_2 = (X_1 + jY_1) \times (X_2 + jY_2) \\ &= \underbrace{(X_1X_2 - Y_1Y_2)}_{\text{Re}(Z_4)} + j\underbrace{(X_1Y_2 + Y_1X_2)}_{\text{Im}(Z_4)} \end{aligned}$$

It is simpler in the polar form:

$$\begin{aligned} Z_4 &= \rho_1 e^{j\theta_1} \times \rho_2 e^{j\theta_2} = \rho_1\rho_2 e^{j(\theta_1+\theta_2)} \\ &= \rho_1\rho_2 \cos(\theta_1+\theta_2) + j\rho_1\rho_2 \sin(\theta_1+\theta_2) \end{aligned}$$

Substituting $X_i = \rho_i \cos\theta_i$ and $Y_i = \rho_i \sin\theta_i$, we have the well known known trigonometry identities,

$$\begin{aligned} \cos(\theta_1+\theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \\ \sin(\theta_1+\theta_2) &= \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 \end{aligned}$$



The division is also easier in the polar form:

$$Z_6 = Z_1 \div Z_2 = \frac{\rho_1 e^{j\theta_1}}{\rho_2 e^{j\theta_2}} = \frac{\rho_1}{\rho_2} e^{j(\theta_1-\theta_2)}$$

In the rectangular form,

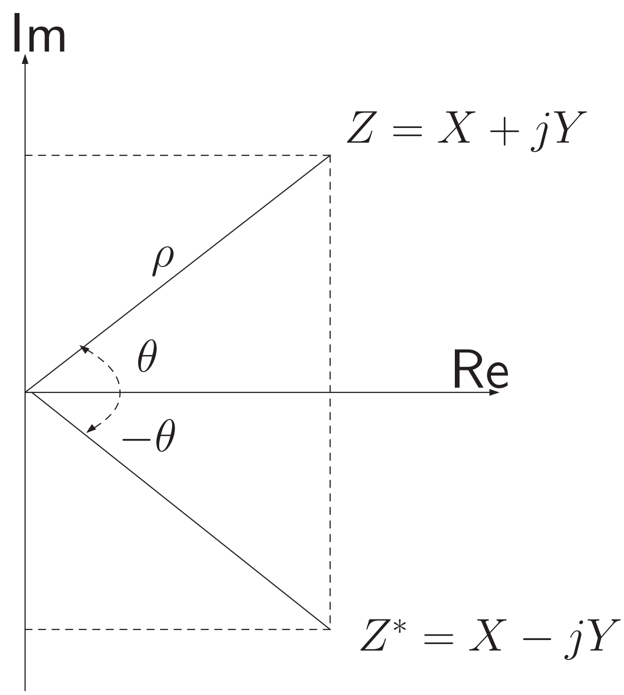
$$
\begin{aligned}
Z_6 &= \frac{X_1 + jY_1}{X_2 + jY_2} \\
&= \frac{(X_1 + jY_1)(X_2 - jY_2)}{(X_2 + jY_2)(X_2 - jY_2)} \\
&= \underbrace{\frac{X_1X_2 + Y_1Y_2}{X_2^2 + Y_2^2}}_{\text{Re}(Z_6)} + j\,\underbrace{\frac{Y_1X_2 - X_1Y_2}{X_2^2 + Y_2^2}}_{\text{Im}(Z_6)}
\end{aligned}
$$

If we substitute, for $i = 1, 2$, $X_i = \rho_i \cos\theta_i$ and $Y_i = \rho_i \sin\theta_i$, we get the trigonometry identifies for $\cos(\theta_1 - \theta_2)$ and $\sin(\theta_1 - \theta_2)$.

Complex Conjugate

Let $Z = X + jY = \rho e^{j\theta}$. The complex conjugate of Z, denoted by Z^*, is defined as

$$Z^* = X - jY = \rho e^{-j\theta}$$



Properties

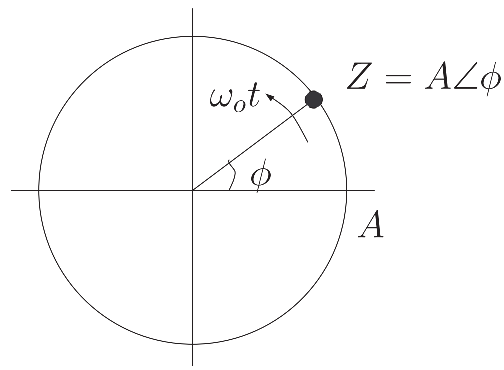
- $(Z^*)^* = Z$
- $(Z_1 + Z_2)^* = Z_1^* + Z_2^*$
- $Z + Z^* = 2\text{Re}(Z),\ Z - Z^* = 2j\text{Im}(Z)$
  $\text{Re}(Z) = \frac{1}{2}(Z + Z^*), \text{Im}(Z) = \frac{1}{2j}(Z - Z^*)$
- $(Z_1 Z_2)^* = Z_1^* Z_2^*$
- $ZZ^* = |Z|^2$

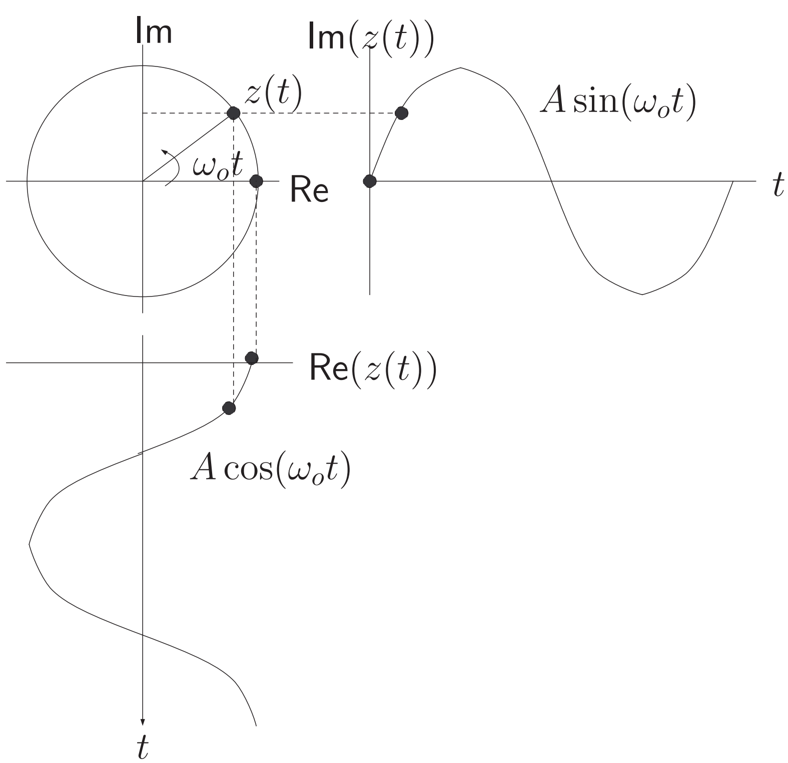
Sinusoids and Phasors

Phasor representation of complex sinusoids

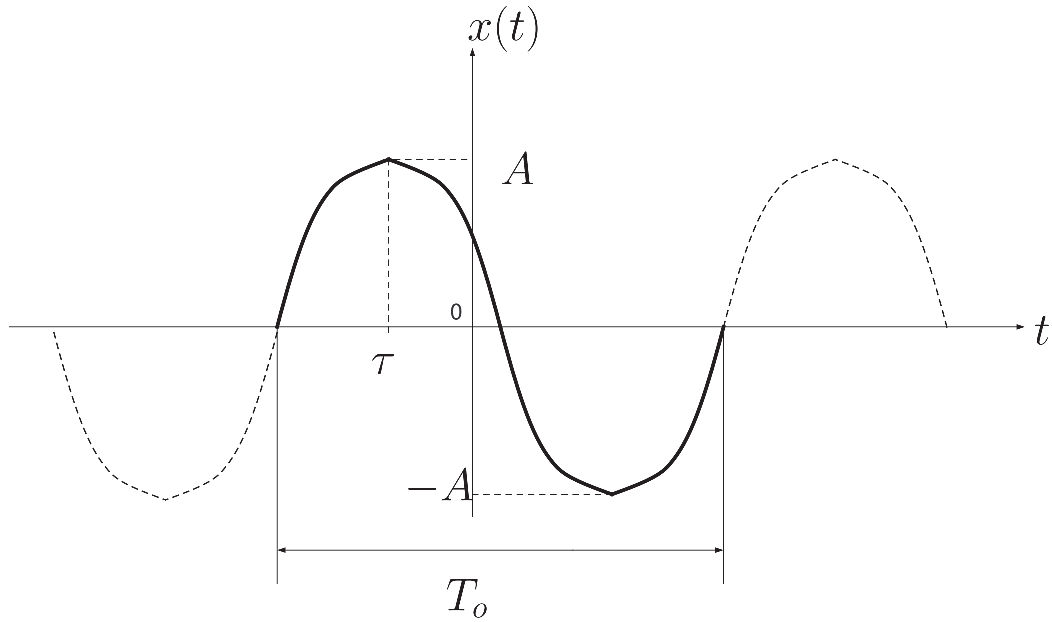
$$z(t) = Ae^{j(\omega_o t+\phi)} = \left(Ae^{j\phi}\right) e^{j\omega_o t}$$

The complex magnitude $Z = Ae^{j\phi}$ is the phasor.





Sinsusoidal Function



A continuous-time sinusoid has the form

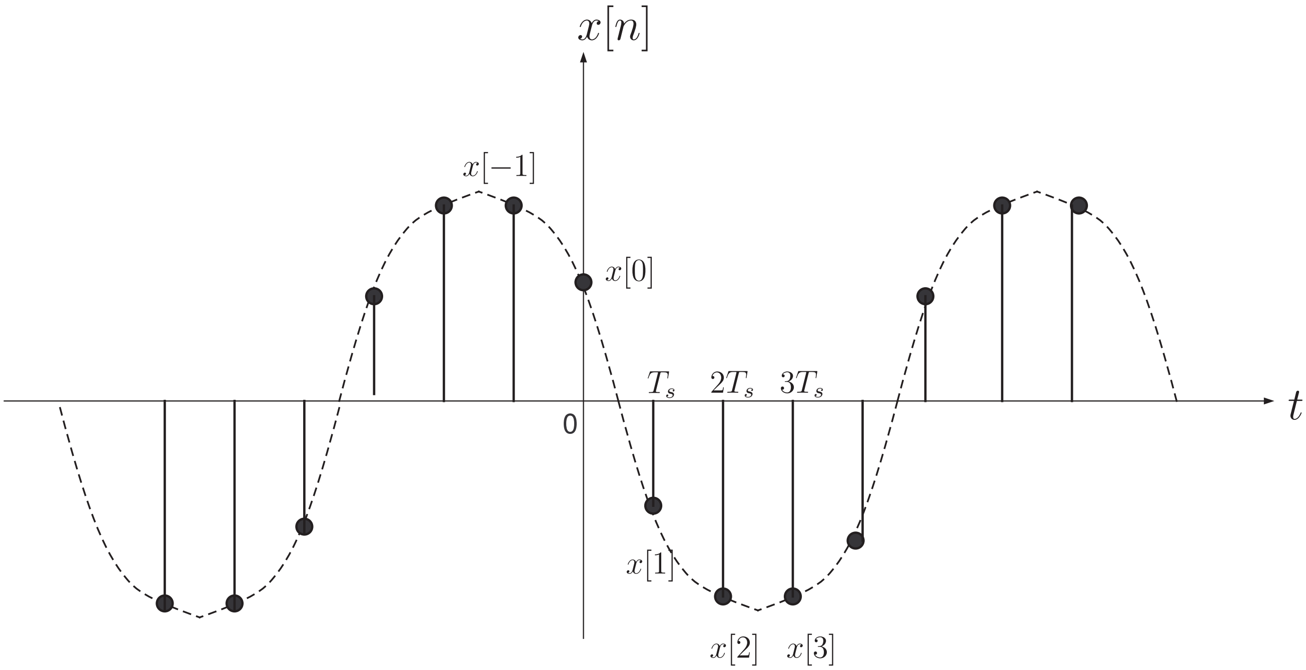
$$\begin{aligned} x(t) &= A\cos(2\pi f_o t + \phi) \\ &= A\cos(\omega_o t + \phi) \end{aligned}$$

where

- A: amplitude.
- T_o: period.
- f_o: frequency (cycles/sec, or Hertz).
  $\omega_0 = 2\pi f_o$: angular frequency (radiances/sec)
- τ: delay
  $\phi = -\omega_o \tau$: relative advance (radiance).
- Sine vs. cosine:

$$A\sin(\omega_o t + \phi) = A\cos(\omega_o t + \phi - \frac{\pi}{2})$$

Plot of Sinusoids



A discrete-time sinusoid has the form

$$\begin{aligned} x[n] &= A\cos(2\pi f_o n T_s + \phi) \\ &= A\cos(\hat{\omega}_o n + \phi) \end{aligned}$$

where

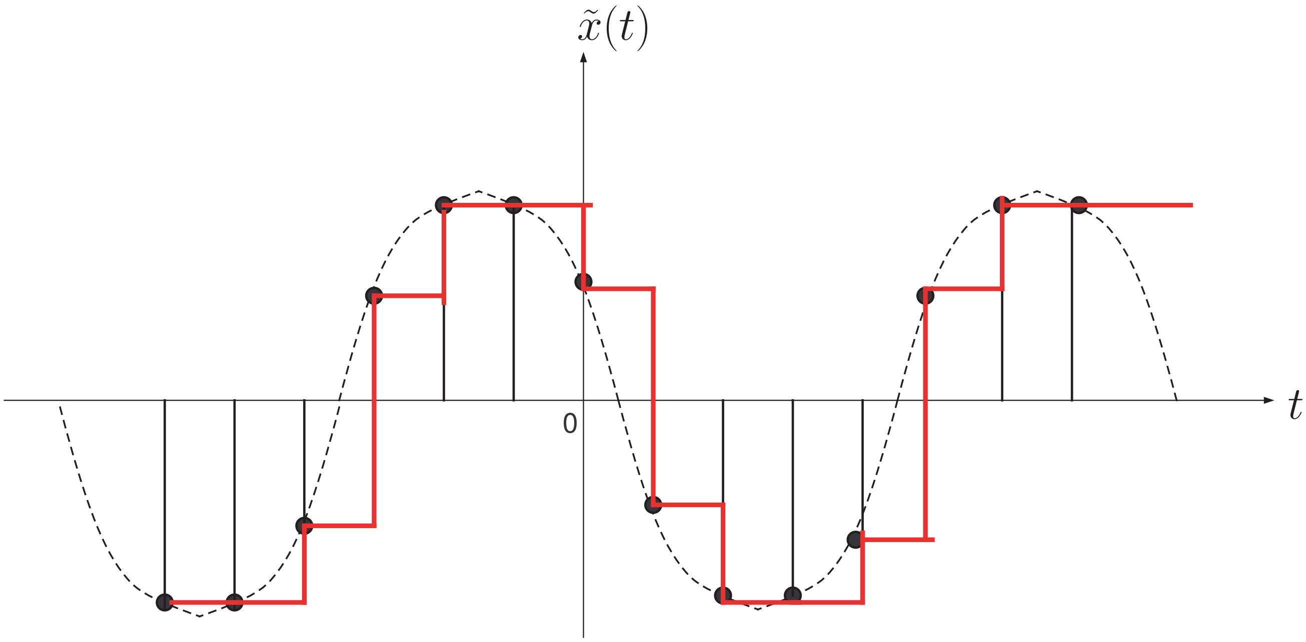
- sampling period: T_s.
- sampling frequency: $f_s = 1/T_s$.
- discrete-time frequency:

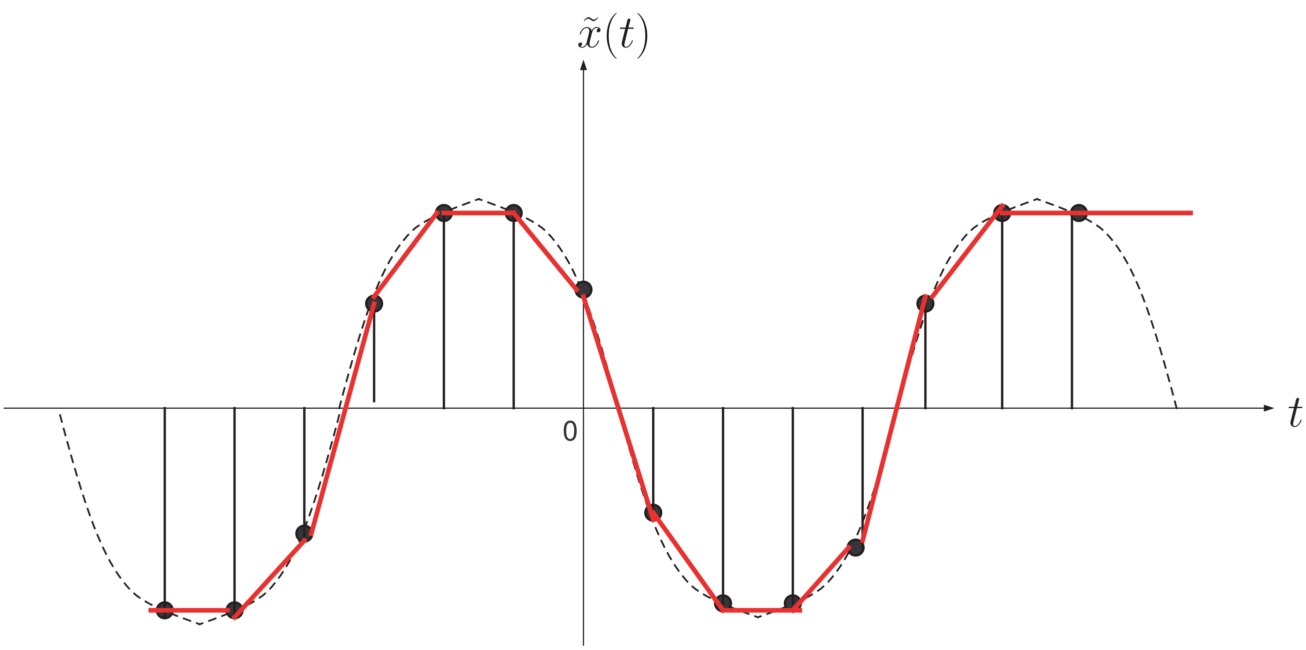
$$\begin{aligned} \hat{\omega}_o &\stackrel{\Delta}{=} 2\pi f_o T_s \\ &= 2\pi \frac{f_o}{f_s} = \frac{\omega_o}{f_s} \end{aligned}$$

Plot of Sinusoids

```
Fs=44100; D=1/Fs;                %sample frequency 44.1KHz

f0=1, A=1, Phi=0.3*pi            %Frequency, amplitude, phase

t=[-0.5:D:1];                    % Time horizon
y=A*cos(2*pi*f0*t+Phi);

tplot=t;                         % interval of plot
plot(tplot,y(1:length(tplot)))

sound(y,Fs)
```

Digital to Analog Conversion





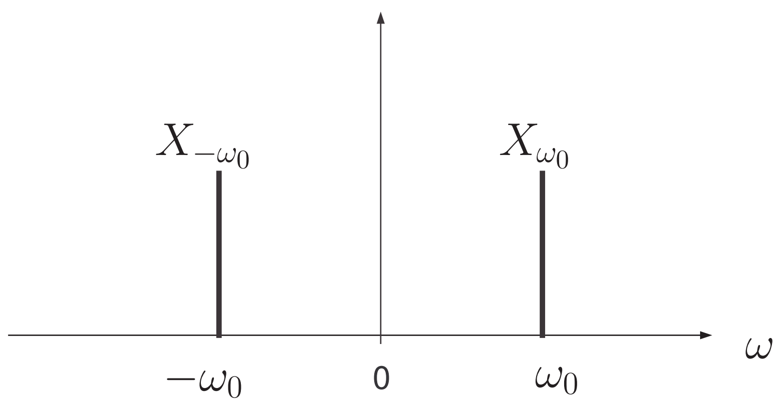
Spectral Lines

- Sinusoids as conjugating rotating phasors

$$A\cos(\omega_o t+\phi) \;=\; \frac{A}{2}e^{j\phi}e^{j\omega_o t}+\frac{A}{2}e^{-j\phi}e^{-j\omega_o t}$$

one with positive frequency ω_o, the other with negative frequency $-\omega_o$.

- Spectral lines: graphically, we can show the frequency components of $x(t)$ in a spectrum plot with two spectral lines and phasors $X_{\omega_0}=\frac{A}{2}e^{j\phi}$ and $X_{-\omega_0}\frac{A}{2}e^{-j\phi}$.



Sum of sinusoids with the same frequency

Sum of complex exponentials and sum of phasors:

$$\sum_{k=1}^{n} A_k e^{j(\omega_0 t+\phi_k)} = (\sum_{k=1}^{n} Z_k) e^{j\omega_0 t} = Z e^{j\omega t}$$

where $Z_k = A_k e^{j\phi_k}$ are phasors of individual complex exponentials, and Z is the phasor of the sum of the complex exponentials. Thus the phasor of the sum is the sum of the phasors.

Sum of sinusoids as sum of complex exponentials

$$\begin{aligned} \sum_{k=1}^{n} A_k \cos(\omega_0 t + \phi_k) &= \text{Re}\left(\sum_{k=1}^{n} Z_k e^{j\omega_0 t}\right) \\ &= \text{Re}\{(\sum_{k=1}^{n} Z_k) e^{j\omega_0 t}\} \end{aligned}$$

If we want to compute the amplitude A and the phase ϕ of

$$\sum_{k=1}^{n} A_k \cos(\omega_0 t + \phi_k) = A\cos(\omega_0 t + \phi)$$

we can apply the "phasor rule":

- Compute phasor representation:

$$A_k \cos(\omega_o t + \phi_k) \Rightarrow Z_k = A_k \angle \phi_k$$

- Phasor addition:

$$Z = \sum_k A_k e^{j\phi_k} = A\angle\phi$$

Examples

P-2.10: Given

$$x(t) = 5\cos(\omega t) + 5\cos(\omega t + 120^o) + 5\cos(\omega t - 120^o)$$

Find the phasor of $x(t)$.

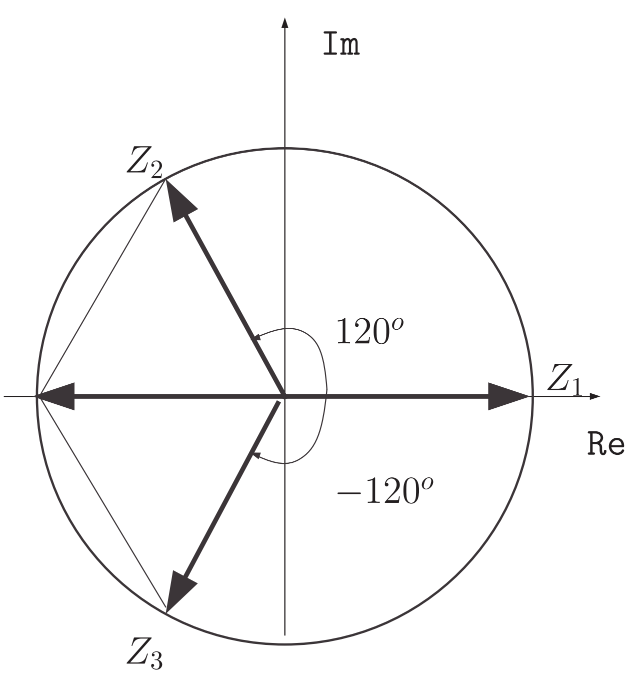
Solution: We write the phasor representation of each term:

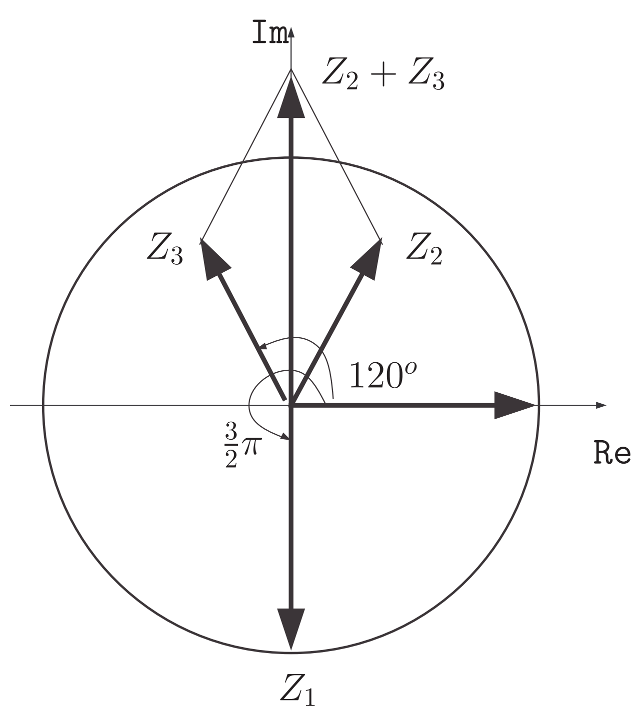
$$Z_1 = 5\angle 0, \quad Z_2 = 5\angle\frac{2\pi}{3}, \quad Z_3 = 5\angle\frac{-2\pi}{3}$$

By the phasor rule, see the figure below,

$$Z = Z_1 + Z_2 + Z_3 = 0$$

Thus we have $A = 0$, ϕ indeterminate, $x(t) = 0$





P-2.17: Define

$$x(t) = 5\cos(\omega t + \frac{3}{2}\pi) + 4\cos(\omega t + \frac{2}{3}\pi) + 4\cos(\omega t + \frac{1}{3}\pi)$$

Write $x(t) = A\cos(\omega t + \phi)$ and show the phasor.

Solution: We write the phasor representation of each term:

$$Z_1 = 5\angle\frac{3}{2}\pi, \quad Z_2 = 4\angle\frac{\pi}{3}, \quad Z_3 = 4\angle\frac{2\pi}{3}$$

We then have

$$Z = Z_1 + Z_2 + Z_3 = (4\sqrt{3} - 5)^{j\frac{\pi}{2}}$$

Thus we have

$$x(t) = (4\sqrt{3} - 5)\cos(\omega t + \frac{\pi}{2})$$

Piano key

- A piano has 88keys
- The keys are divided into octaves with 12 keys each
- One octave above doubles the frequency
- The A4 (key 49) has base frequency 440Hz.
- Neighboring keys has a constant frequency ratio, which is given by

$$r^{12} = 2 \quad \Longrightarrow \quad r = 2^{1/12} = 1.0595$$

- The frequency of middle-C is given by

$$f_{C_4} \times r^{(49-45)} = 440 \quad \Longrightarrow \quad f_{C_4} = 440 \times 2^{-9/12} = 261.6$$

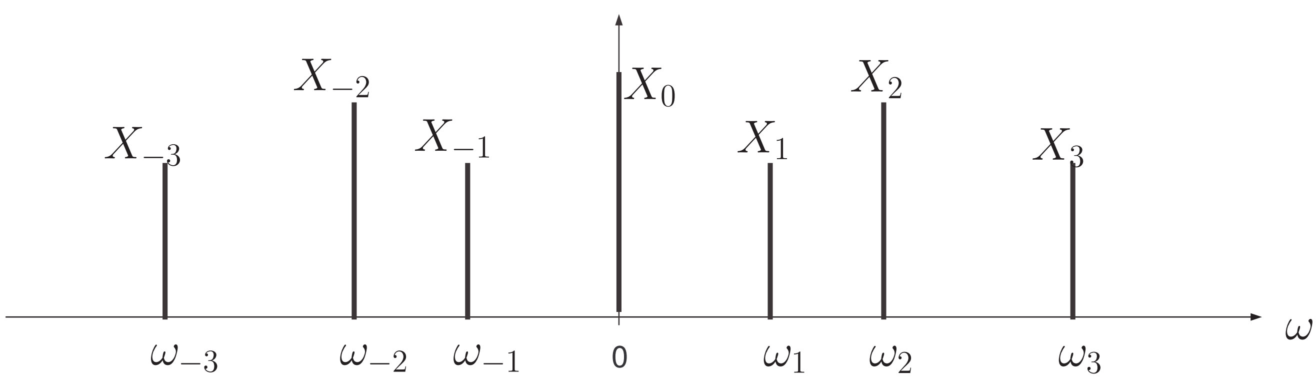
approximately 261.6Hz

Spectrum Representation of Sinusoids

Consider a linear combination of sinusoids

$$x(t) = A_0 + \sum_{k=1}^{n} A_k \cos(\omega_k + \phi).$$

We can plot the spectral lines of $x(t)$ by



In particular, we can write

$$x(t) = \sum_{k=-n}^{n} X_k e^{j\omega_k t}$$

where

- $k = 0$: the DC component: $\omega_0 = 0, X_0 = A_0$
- $k > 0$: the positive frequency components at ω_k, $X_k = \frac{A_k}{2} e^{j\phi_k}$.
- $k < 0$: the negative frequency components at $\omega_k = -\omega_{|k|}$, $X_k = \frac{A_k}{2} e^{-j\phi_k}$.

Remark: Note that not all signals can be written as a linear combination of complex sinusoids.