Probability Notes

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Contents

Preface

Large part of the notes in Part I are based off ECSE 509: "*Probability and Random Signals 2*" course taught at McGill University in Fall 2019 by Prof. I. Psaromiligkos [1].

Notations

Notation	Definition
S	Sample space
$\mathcal{P}$	Power set
$\mathcal{F}$	Events algebra
$P : \mathcal{P} \to [0, 1]$	Probability function
$f : \mathbb{R}^n \to \mathbb{R}$	Probability density function (PDF)
$F : \mathbb{R}^n \to \mathbb{R}$	Cumulative distribution function (CDF)
$p : S \subset \mathbb{R}^n \to \mathbb{R}$	Cumulative distribution function (CDF) Usually used for discrete random variables
$\underline{(\cdot)} : S \to \mathbb{R}^n$	A random variable (RV). S is referred to as the *domain*
$\mathcal{R}_x$	Set of possible points that random variable $\underline{x}$ can take
$\underline{x}$	Single random variable
$\underline{\mathbf{x}}$	Multivariate random variable
$\mathbb{E}[\cdot]$	Expectation operator
$\mathrm{Var}[\cdot]$	Variance
$\mathrm{Cov}[\cdot]$	Covariance
$\jmath$	$\sqrt{-1}$

State Estimation

$\underline{x}$	Estimatnd; the random variable for which its parameters are to be estimated
$\underline{\hat{x}}$	Estimator of the estimand $\underline{x}$
$\hat{x}$	Estimate of the random variable (estimand) $\underline{x}$

Hypothesis Testing

H_m	Hypothesis m
H_0	Null hypothesis
H_1	Alternate hypothesis
$P(H_1 \mid H_0) = \alpha$	Significance of test (valid only for binary testing)
$P(H_0 \mid H_0) = 1 - \alpha$	Confidence of test (valid only for binary testing)
π_m	Prior probability of hypothesis H_m
Λ	Likelihood ratio
η	Likelihood test threshold

Random Processes

$\underline{x}(t)$	Random process
$\eta_x(t)$	Mean function of the random process $\underline{x}(t)$
$R_x(t_1, t_2)$	Autocorrelation of the random process $\underline{x}(t)$
$C_x(t_1, t_2)$	Autocovariance of the random process $\underline{x}(t)$

Stochastic Convergence

$\underline{x}_n$	Random sequence
$\underline{x}_n \to \underline{\chi}$ (s)	Sequence $\underline{x}_n$ converges to $\underline{\chi}$ *surely* or *everywhere*
$\underline{x}_n \to \underline{\chi}$ (a.s.)	Sequence $\underline{x}_n$ converges to $\underline{\chi}$ *almost surely*
$\underline{x}_n \to \underline{\chi}$ (p)	Sequence $\underline{x}_n$ converges to $\underline{\chi}$ *in probability*
$\underline{x}_n \xrightarrow{p} \underline{\chi}$	Sequence $\underline{x}_n$ converges to $\underline{\chi}$ *in probability*
$\underline{x}_n \to \underline{\chi}$ (m.s.)	Sequence $\underline{x}_n$ converges to $\underline{\chi}$ *in the mean square sense*
$\underline{x}_n \to \underline{\chi}$ (d)	Sequence $\underline{x}_n$ converges to $\underline{\chi}$ *in distribution*

Part I

Fundamentals

Chapter 1

Single Variable Probability

1.1 Definitions and intuition

Probability is a measure of:

- Our certainty or belief that a statement is true. My belief could be different than yours;
- How frequently an event will occur.

Definition 1.1.1 (*Random experiment*)**.** A *random experiment* is an experiment (or a physical process) whose outcome is not certain but all of its possible outcomes are known and predictable in advance.

A random experiment is modeled as a probability space (check Definition 1.1.7).

Definition 1.1.2 (*Sample space S*)**.** The *sample space* S is the set of *all* possible *outcomes*.

Definition 1.1.3 (*Event*)**.** An *event* A is a *subset* of S. That is, $A \subseteq S$. It's NOT an element of S! It is rather an element of the power set $\mathcal{P}_S$ (Definition 1.1.4).

Thus, an event A is a *set* of outcomes.

Definition 1.1.4 (*Power set of S*)**.** The *power set* of the sample space S is the set containing all subsets of a set S. It's denoted by $\mathcal{P}_S$, or simply $\mathcal{P}$.

- $\mathcal{P}_S$ is the set of all events.
- $\varnothing \subseteq S$: Impossible event.
- $S \subseteq S$: Certain event.

Example 1.1.1 (Flipping a coin)**.** Flipping a coin *once* has the sample space

$$S = \{H, T\}, \tag{1.1}$$

H and T are all the possible *outcomes* (NOT events) of the sample space S. The power set is thus given by

$$\mathcal{P}_S = \{\{\varnothing\}, \{H\}, \{T\}, \{H, T\}\}. \tag{1.2}$$

Note that the last element of $\mathcal{P}_S$, $\{H, T\}$ is the sample space and it is thus a *certain* event. △

Example 1.1.2 (Number of occurences)**.** Relative frequency interpretation of probability (which is very intuitive). Say a random experiment gives an outcome a in an event A (*i.e.*, $a \in A \subseteq S$) from the sample space. Repeat the experiment N times. Then, define the number of occurrences of event A after repeating the experiment N times by

$$n(A, N) = \#\text{ of times that } A \text{ happens}. \tag{1.3}$$

Note that "# of times that A happens" implies that the outcome a_k of experiment k belongs to the event A. That is, $a_k \in A$. Then,

$$\underbrace{\lim_{N\to\infty} \frac{n(A, N)}{N}}_{\text{Probability of event } A.} = \text{constant}. \tag{1.4}$$

△

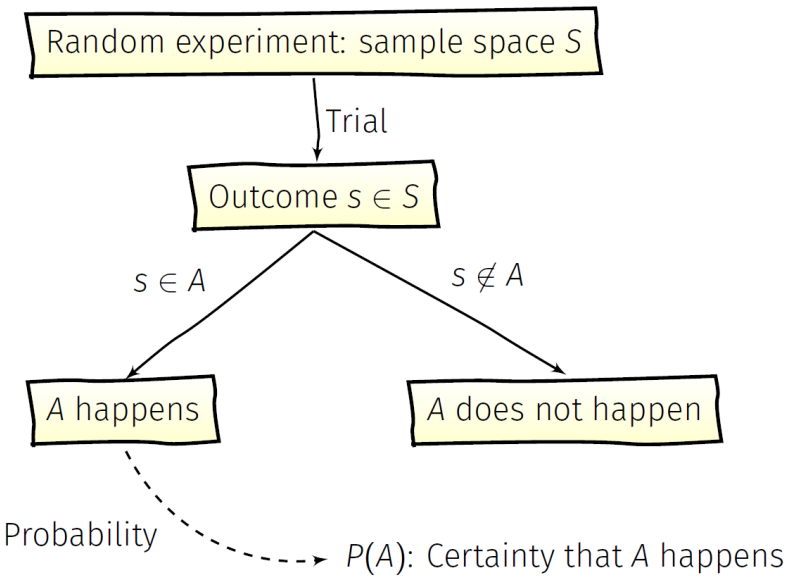


Figure 1.1: Intuition of probability. Figure obtained from [1].

Definition 1.1.5 (*Probability*)**.** *Probability* is a set function $P : \mathcal{P}_S \mapsto [0, 1]$ that assigns to an event

$A \in \mathcal{P}_S$ (recall $A \subseteq S$) a number $P(A)$ that satisfies the following three axioms:

1. $P(A) \geq 0$,
2. $P(S) = 1$, and
3. For any sequence of mutually exclusive events $A_1, A_2, \ldots$ (*i.e.*, for which $A_i \bigcap A_j = \emptyset$ for $i \neq j$), we have

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i). \tag{1.5}$$

The number $P(A)$ is called the probability of A.

Remark 1.1.1. It is NOT always possible to define a function $P : \mathcal{P} \mapsto [0, 1]$ that satisfies the three axioms. Workaround: focus on the events in a σ-algebra $\mathcal{F}$.

Here are some other remarks about the probability function.

- The probability function $P : \mathcal{P}_S \mapsto [0, 1]$ maps the power set $\mathcal{P}_S$ (or σ-algebra) to $[0, 1]$. That is, the argument of P should be an *element* of $\mathcal{P}_S$. And since $\mathcal{P}_S$ is a *set of sets* of S, then the argument is the event which is denoted by $\{a_1, a_2, \ldots, a_n\} = A \in \mathcal{P}_S$, where $a_1, \ldots, a_n \in S$. Thus the proper way to write "the probability of event A" is $P(\{A\})$ but the notation $P(A)$ will often be used for ease of reading and writing.
- Note that $P(A) = 0$ implies that event A is *improbable*; not necessarily impossible! It's still possible to get A. This might be a little unintuitive, but it can be thought of that the chance of A happening is so small that it seems that event A is highly unlikely to occur.
- Similarly, $P(A) = 1$ does **not** imply that A will certainly occur! It just implies that it is very probably.

Definition 1.1.6 (σ-*algebra*)**.** A set $\mathcal{F}$ of subsets of S, that is, $\mathcal{F} \subseteq \mathcal{P}$ is σ-*algebra* if and only if

1. $S \in \mathcal{F}$,
2. $E \in \mathcal{F} \implies E^{\mathsf{C}} \in \mathcal{F}$, and
3. If the sets $A_1, A_2, \ldots$ belong to $\mathcal{F}$, then so does $\bigcup_{i=1}^{\infty} A_i$.

If S is countable, then $\mathcal{F} = \mathcal{P}_S$ is used.

A random experiment is modeled as a probability space.

Definition 1.1.7 (*Probability space*)**.** A *probability space* is a triplet

$$(S, \mathcal{F}, P), \tag{1.6}$$

where

1. S: is the sample set,
2. $\mathcal{F}$ is the events algebra, and
3. $P : \mathcal{F} \mapsto [0, 1]$ is the probability function.

1.2 Basic theorems

Theorem 1.2.1 (Basic theorems)**.** Below is a list of basic theorems of probability.

1. For any event $A \in \mathcal{F}$,
$$P\left(A^{\mathsf{C}}\right) = 1 - P(A). \tag{1.7}$$
2. For any event $A \in \mathcal{F}$,
$$0 \leq P(A) \leq 1. \tag{1.8}$$
3. $P(\emptyset) = 0$.
4. For any events $A, B \in \mathcal{F}$,
$$P(A - B) = P(A) - P(A \cap B). \tag{1.9}$$
Special case: if $B \subseteq A$, then
$$P(A - B) = P(A) - P(B). \tag{1.10}$$
5. For any $A, B \in \mathcal{F}$,
$$P(A) = P(AB) + P\left(AB^{\mathsf{C}}\right). \tag{1.11}$$
6. For any $A, B \in \mathcal{F}$,
$$P(A \cap B) = P(A) + P(B) - P(A \cup B), \tag{1.12}$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \tag{1.13}$$

1.3 Conditional probabilities

Events can affect other events. Therefore, the question "what's the probability of event A happening?" could be quite different from "what's the probability of event A *given that event* B *happened*?"[1]. The notation

$$P\left(A|\ B\right) \tag{1.14}$$

reads "probability of event A happening given that event B happened".

Given

1. a random experiment $(S, \mathcal{F}, P\left(\right))$, and
2. $A,\ B \in \mathcal{F}$ with $P\left(B\right) \neq 0$.

Definition 1.3.1 (*Conditional probability*)**.** The conditional probability of A given B, $P\left(A|\ B\right)$, defined as

$$P\left(A|B\right) = \frac{P\left(A \bigcap B\right)}{P\left(B\right)}. \tag{1.15}$$

Theorem 1.3.1 (Conditional probability)**.** Let $B \in \mathcal{F}$ with $P\left(B\right) \neq 0$. The mapping

$$P\left(\cdot|B\right) : \mathcal{F} \to \mathbb{R}, \tag{1.16}$$

satisfies the axioms of probability

1. $P\left(A|B\right) \geq 0, \quad \forall A \in \mathcal{F}$
2. $P\left(S|B\right) = 1$
3. If $A_1, A_2, \ldots \in \mathcal{F}$ is a sequence of mutually exclusive events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i|\ B\right) = \sum_{i=1}^{\infty} P\left(A_i|B\right). \tag{1.17}$$

Therefore,

- For a given event B with $P\left(B\right) \neq 0$, the mapping

$$P\left(\cdot|B\right) : \mathcal{F} \to \mathbb{R}, \tag{1.18}$$

  defines a valid probability function.

- All the basic theorems of probability listed in Theorem 1.2.1apply to $P\left(\cdot|B\right)$.

[1] In some cases, the two questions are the same. This brings the notion of independence.

Here are some properties of conditional probability

1. For $A, B \in \mathcal{F}$ with $P(A), P(B) \neq 0$:

$$P(AB) = P(A|B)\,P(B) \tag{1.19}$$
$$= P(B|A)\,P(A). \tag{1.20}$$

2. **Total probability**. Let $B_1, B_2, \ldots, B_n$ be a partition of S, with $P(B_i) \neq 0, \forall i$:

$$P(A) = P(A|B_1)\,P(B_1) + P(A|B_2)\,P(B_2) + \ldots + P(A|B_n)\,P(B_n) \tag{1.21}$$
$$= \sum_{i=1}^{n} P(A|B_i)\,P(B_i). \tag{1.22}$$

3. **Bayes rule**. Let $B_1, B_2, \ldots, B_n$ be a partition of S, with $P(B_i) \neq 0, \forall i$:

$$P(B_i|A) = \frac{P(A|B_i)\,P(B_i)}{\sum_{j=1}^{n} P(A|B_j)\,P(B_j)}. \tag{1.23}$$

Special case for $n = 1$:

$$P(B|A) = \frac{P(A|B)\,P(B)}{P(A)}. \tag{1.24}$$

Definition 1.3.2 (*Indpendence*)**.** The events $A, B \in \mathcal{F}$ are called *independent* if

$$\underbrace{P(A|B) = P(A)}_{\text{requires } P(B)\neq 0} \text{ or } \underbrace{P(B|A) = P(B)}_{\text{requires } P(A)\neq 0} \text{ or } P(AB) = P(A)\,P(B) \tag{1.25}$$

Definition 1.3.3 (*Independence (multiple events)*)**.** The events $A, B, C \in \mathcal{F}$ are called *independent* if

- They are independent in pairs, and
- $P(ABC) = P(A)\,P(B)\,P(C)$.

Definition 1.3.4 (*Conditional independence*)**.** Events $A, B \in \mathcal{F}$ are called *conditionally independent given $C \in \mathcal{F}$, with $P(C) \neq 0$*, if

$$P(AB|C) = P(A|C)\,P(B|C). \tag{1.26}$$

Remark: Conditional independence is quite different from independence. They do not imply each other. Here are some examples.

Example 1.3.1 (Independent but conditionally dependent)**.** Two fair coins are flipped. Define the fol-

lowing events

- A - Your first coin flip is heads,
- B - Your second coin flip is heads, and
- C - Your first two flips were the same.

Then A and B are independent. However, A and B are conditionally dependent given C, since if you know C then your first coin flip will inform the other one.

Example 1.3.2 (Dependent but conditionally independent)**.** Consider two brothers John and Joseph, both having a genetic disease. These two events are dependent as they are brothers. However, given the condition that Joseph is an adopted son of the family makes the events independent.

1.4 Random variables and distributions

In many random experiments, it is convenient to assign numerical labels to outcomes.

Example 1.4.1. Experiment: pick a car out of a black Escort (Eb), a red Escort (Er), and a red Mazda (Mr). The sample space is then

$$S = \{Eb, Er, Mr\}, \tag{1.27}$$

and the events algebra is $\mathcal{F} = \mathcal{P}$.

Say we are interested in the make of the car and not the color. Then we can assign labels as can be seen in Figure 1.2.

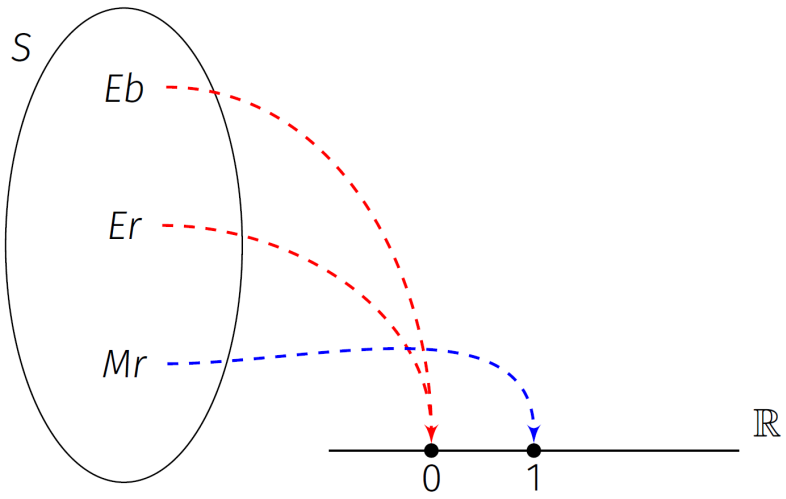


Figure 1.2: Assigning numerical labels to outcomes; each color is an outcome. From [1]

Definition 1.4.1 (*Random variable (RV)*)**.** A *random variable (RV)* (denoted by an underline) $\underline{x} : S \to \mathbb{R}$ is a mapping from S to $\mathbb{R}$ with the following properties.

1. $A(x) = \{s \in S : \underline{x}(s) \leq x\} \in \mathcal{F}, \forall x \in \mathbb{R}$
2. $P(\{s \in S : \underline{x}(s) = \infty\}) = 0$ and $P(\{s \in S : \underline{x}(s) = -\infty\}) = 0$

S is referred to as the *domain* of the RV $\underline{x}(\cdot)$ (since a RV is a mapping, then it has a domain and a range). [The domain can be thought of as the sample space[a].]

[a] Not quite sure of this statement.

Definition 1.4.2 (*Cumulative distribution function (CDF)*)**.** The function $F : \mathbb{R} \to [0, 1]$ defined by

$$F(x) = P(s \in S : \underline{x}(s) \leq x) \tag{1.28}$$
$$= P(\underline{x} \leq x), \quad \forall x \in \mathbb{R} \tag{1.29}$$

is called the *cumulative distribution function (CDF)* of $\underline{x}$. Note that (1.29) is a shorthand for (1.28).

Note: sometimes the CDF will be denoted with a subscript of the RV (*e.g.*, $F_x(x)$). This is for clarity as to make sure that the CDF is over the RV $\underline{x}$. This come more in handy when there are multiple variables in hands (such as in integration).

1.4.1 Properties of random variables

Here are some properties of random variables.

1. $\lim_{x\to\infty} F_x(x) = 1$ and $\lim_{x\to-\infty} F_x(x) = 0$
2. $F_x(x)$ is non-decreasing
3. $F_x(x)$ is continuous from the right. That is

$$\lim_{x\to x_0^+} F_x(x) = F_x(x_0) \tag{1.30}$$

4. $P(\{x_1 < \underline{x} \leq x_2\}) = F_x(x_2) - F_x(x_1)$
5. $P(\{\underline{x} = x_1\}) = F_x(x_1) - F_x(x_1^-)$, where $F_x(x_1^-) := \lim_{x\to x_1^-} F_x(x)$
6. $P(\{x_1 \leq \underline{x} \leq x_2\}) = F_x(x_2) - F_x(x_1^-)$

Definition 1.4.3 (*Types of RVs*)**.** A RV $\underline{x}$ is called

- *Discrete* if $F_x(x)$ is constant except of countable number of discontinuities (piecewise constant). Figure 1.3 shows an example of a CDF of a discrete RV.
- *Continuous* if $F_x(x)$ is
  1. continuous, and
  2. differentiable (with the exception of a countable number of point).
- *Mixed* if it is neither continuous nor discrete.

Definition 1.4.4 (*Probability mass function (PMF)*)**.** Let $\underline{x}$ be a discrete RV. Then,

- $P(\underline{x} = x) = 0$ when $x \neq x_i, i = 1, 2, \ldots,$
- $\mathcal{R}_x = \{x_i : i = 1, , 2, \ldots\}$ is the set of possible points.

The *probability mass function (PMF)* of a RV $\underline{x}$, $p(x)$ or $p_x(x)$, $x \in \mathbb{R}$, is defined as

$$p(x) = P(\underline{x} = x) \qquad = \begin{cases} 0 & \text{if } x \notin \mathcal{R}_x, \\ P(\underline{x} = x_i) & \text{if } x \in \mathcal{R}_x, \text{ i.e., } x = x_i, i = 1, 2, \ldots. \end{cases} \tag{1.31}$$

Also,

- $\sum_{i=1}^{\infty} p(x_i) = 1$ and $p(x) \geq 0, \forall x$ (defining properties),
- $P(\underline{x} \in \mathcal{A}) = \sum_{x_i \in \mathcal{A}} p(x_i)$.

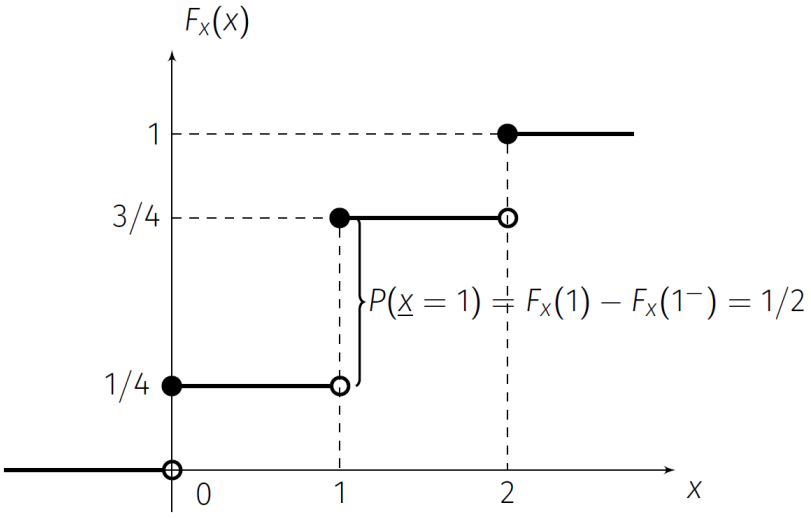


Figure 1.3: Example of a CDF of a discrete RV. From [1].

Definition 1.4.5 (*Porbability density function (PDF)*)**.** The *probability density function (PDF)*, $f(x)$ or $f_x(x)$, of a RV $\underline{x}$ is defined as

$$f_x(x) = \frac{\mathrm{d}F_x(x)}{\mathrm{d}x}. \tag{1.32}$$

1.4.2 Properties of PDFs

If $\underline{x}$ is continuous with PDF $f(x)$

1. $f_x(\underline{x}) \geq 0, \forall x$ (defining property)
2. $\int_{-\infty}^{x} f_x(\lambda)\mathrm{d}\lambda = F_x(x)$
3. $\int_{-\infty}^{\infty} f_x(x)\mathrm{d}\lambda = 1$ (defining property)

4.

$$P\left(\{a \leq \underline{x} \leq b\}\right) = \int_a^b f_x(\lambda)\mathrm{d}\lambda \tag{1.33}$$

$$P\left(\{a < \underline{x} \leq b\}\right) = P\left(\{a \leq \underline{x} < b\}\right) = P\left(\{a < \underline{x} < b\}\right). \tag{1.34}$$

Interpretation of PDF: For a continuous $\underline{x}$: $P\left(\underline{x} = x\right) = 0$ for all $x \in \mathbb{R}$ which implies that the PDF $f\left(x\right)$ is *not* a probability.

For small $\epsilon > 0$:

$$P\left(|\underline{x} - x| < \epsilon\right) = P\left(x - \epsilon < \underline{x} < x + \epsilon\right) \tag{1.35}$$

$$= \int_{x-\epsilon}^{x+\epsilon} f\left(t\right)\mathrm{d}t \tag{1.36}$$

$$\approx 2\epsilon f\left(x\right) \tag{1.37}$$

assuming $f\left(x\right)$ is continuous at x. Equivalently,

$$f\left(x\right) \approx \frac{1}{2\epsilon} P\left(x - \epsilon < \underline{x} < x + \epsilon\right). \tag{1.38}$$

$f\left(x\right)$ is proportional to the probability that $\underline{x}$ lies in a small neighbourhood of x.

1.4.3 Conditional distributions

Definition 1.4.6 (*Conditional distributions*)**.** Let $M \in \mathcal{F}$ with $P\left(M\right) \neq 0$. Then

1. *Conditional CDF* of $\underline{x}$ given event M is

$$F_x\left(x|M\right) = P\left(\underline{x} \leq x|M\right) \tag{1.39}$$

$$= \frac{P\left(\underline{x} \leq x, M\right)}{P\left(M\right)}. \tag{1.40}$$

2. *Conditional PDF* of $\underline{x}$ given event M is

$$f_x\left(x|M\right) = \frac{\mathrm{d}F_x\left(x|M\right)}{\mathrm{d}x}. \tag{1.41}$$

3. *Conditional PMF* of $\underline{x}$ ($\underline{x}$ discrete given M) is

$$p\left(x|M\right) = \begin{cases} 0, & x \notin \{x_i : i = 1, 2, \ldots\} \\ P\left(\underline{x} = x_i|M\right), & x \in \{x_i : i = 1, 2, \ldots\}. \end{cases} \tag{1.42}$$

Note that conditional CDFs, PDFs, and PMFs behave like unconditional ones.

Properties of conditional distributions

Let $B_1, B_2, \ldots, B_n$ be a partition of S, $P(B_i) \neq 0$. Let $A \in \mathcal{F}$, $P(A) \neq 0$.

- **Total probability**:

$$F_x(x) = \sum_{i=1}^{n} F_x(x|B_i) P(B_i), \tag{1.43}$$

$$f_x(x) = \sum_{i=1}^{n} f_x(x|B_i) P(B_i), \tag{1.44}$$

$$p_x(x) = \sum_{i=1}^{n} p_x(x|B_i) P(B_i). \tag{1.45}$$

- **Bayes rule**:

$$P(A|\underline{x} \leq x) = \frac{P(\underline{x} \leq x, A)}{P(\underline{x} \leq x)} \tag{1.46}$$

$$= \frac{P(\underline{x} \leq x|A) P(A)}{P(\underline{x} \leq x)} \tag{1.47}$$

$$= \frac{F_x(\underline{x} \leq x|A) P(A)}{F_x(\underline{x} \leq x)}. \tag{1.48}$$

When $\underline{x}$ is discrete with $\mathcal{R}_x = \{x_1, x_2, \ldots\}$, $p(x_i) \neq 0$:

- **Conditional probability** using PMF:

$$P(A|\underline{x} = x_i) = \frac{P(A, \underline{x} = x_i)}{P(\underline{x} = x_i)} \tag{1.49}$$

$$= \frac{P(A, \underline{x} = x_i)}{p_x(\underline{x} = x_i)}. \tag{1.50}$$

- **Total probability**:

$$P(A) = \sum_{i=1}^{\infty} P(A|\underline{x} = x_i) p(x_i). \tag{1.51}$$

- **Bayes rule**:

$$p(x|A) = \frac{P(A|\underline{x} = x) p(x)}{P(A)} \tag{1.52}$$

$$= \frac{P(A|\underline{x} = x) p(x)}{\sum_{i=1}^{\infty} P(A|\underline{x} = x_i) P(x_i)}. \tag{1.53}$$

When $\underline{x}$ is continuous, it is not possible to divide by $P(\underline{x} = x) = 0$. Therefore, another definition for conditional PDF is needed.

Definition 1.4.7 (*Conditional PDF*)**.** The *conditional PDF* of $A \in \mathcal{F}$ given $\underline{x} = x$ (assuming $f_x(x) \neq 0$) is

$$P\left(A|\underline{x} = x\right) = \frac{f_x\left(x|A\right) P\left(A\right)}{f_x\left(x\right)}. \tag{1.54}$$

The properties of such conditional PDF are

- **Total probability**: $P\left(A\right) = \int_{-\infty}^{\infty} P\left(A|\underline{x} = x\right) f_x\left(x\right) \mathrm{d}x$*/*
- **Bayes rule**:

$$f_x\left(x|A\right) = \frac{P\left(A|\underline{x} = x\right) f_x\left(x\right)}{P\left(A\right)} \tag{1.55}$$

$$= \frac{P\left(A|\underline{x} = x\right) f_x\left(x\right)}{\int_{-\infty}^{\infty} P\left(A|\underline{x} = x\right) f_x\left(x\right) \mathrm{d}x}. \tag{1.56}$$

1.5 Transformed random variables

Say there exists a random variable $\underline{x} : S \to \mathbb{R}$ that is applied on the probability space $(S, \mathcal{F}, P)$. Further, say there exists a [deterministic] function $g : \mathbb{R} \to \mathbb{R}$. Then, there is a new RV $\underline{y}$ defined as

$$\underline{y}(s) = g(\underline{x}(s)) \quad \forall s \in S. \tag{1.57}$$

Furthermore, say the distribution (*i.e.*, CDF and PDF (or PMF)) of $\underline{x}$ is known. THen what is the distribution of $\underline{y} = g(\underline{x})$?

The foundation is the following

$$P\left(\underline{y} \in A\right) = P\left(\left\{s \in S : \underline{y}(s) \in A\right\}\right) \tag{1.58}$$

$$= P\left(\left\{s \in S : g(\underline{x}(s)) \in A\right\}\right). \tag{1.59}$$

There are different methods to find the distribution of a transformed random variable (*i.e.*, $\underline{y} = g(\underline{x})$) depending on the types of variables.

1. **Method of distributions** works when for *any* types of random variables (continuous or discrete). However, it is the most exhaustive method.
2. **Method of transformations** works for *continuous* random variables; *i.e.*, both $\underline{x}$ and $\underline{y}$ are continuous.
3. **Discrete transformation** for *discrete* transformed variable $\underline{y}$ and *continuous* or *discrete* domain random variable $\underline{x}$.

1.5.1 Method of distribution

The method will be motivated by an example.

Example 1.5.1. Consider the transformed random variable

$$\underline{y} = g(\underline{x}) \tag{1.60}$$
$$= \underline{x}^2. \tag{1.61}$$

Evaluate the CDF as follows

$$F_y(y) = P\left(\underline{y} \leq y\right) = P\left(g(\underline{x}) \leq y\right) = P\left(\underline{x}^2 \leq y\right) \tag{1.62}$$
$$= P\left(\left\{s \in S : \underline{x}(s)^2 \leq y\right\}\right). \tag{1.63}$$

There are three cases

1. $y < 0$: $P\left(\underline{x}^2 \leq y\right) = 0 \implies F_y(y) = 0$.
2. $y = 0$: $P\left(\underline{x}^2 \leq 0\right) = P\left(\underline{x}^2 = 0\right) = P\left(\underline{x} = 0\right) \implies F_y(0) = P\left(\underline{x} = 0\right)$.
3. $y > 0$: $P\left(\underline{x}^2 \leq y\right) = P\left(|\underline{x}| \leq \sqrt{y}\right) = P\left(-\sqrt{y} \leq \underline{x} \leq \sqrt{y}\right) \implies$

$$F_y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} f_x(x)\,\mathrm{d}x. \tag{1.64}$$

1.5.2 Method of transformation

The process is outlined by the following steps.

For any y (treat it as a deterministic variable):

1. Solve $g(x) = y$ for x. List *all* possible solutions $x_1, x_2, \ldots, x_n$.
2. Calculate $g'(x) = \frac{\mathrm{d}g(x)}{\mathrm{d}x}$.
3. Evaluate

$$f_y(y) = \begin{cases} 0, & \text{no solutions to } g(x) = y, \\ \sum_{i=1}^{n} \frac{f_x(x_i)}{|g'(x_i)|}, & \text{otherwise.} \end{cases} \tag{1.65}$$

The method is derived in [2, Sec. 2.6.2].

Example 1.5.2 (Method of transformation)**.** Consider

$$\underline{y} = g(\underline{x}) \tag{1.66}$$
$$= \underline{x}^2. \tag{1.67}$$

Carry the steps:

1. Solve for x in $y = g(x)$ for all possible cases of y:

   1.1. $y < 0$: No solutions.

1.2. $y = 0$: One solution $x_1 = 0$.

1.3. $y > 0$: Two solutions $x_1 = -\sqrt{y}$ and $x_2 = \sqrt{y}$.

2. Calculate the derivative of the function

$$g'(x) = 2x, \quad \forall x. \tag{1.68}$$

3. Calculate $f_y(y)$:

3.1. $y < 0$: $f_y(y) = 0$.

3.2. $y = 0$: $f_y(0) = 0$.

3.3. $y > 0$:

$$f_y(y) = \frac{f_x\left(-\sqrt{y}\right)}{\left|-2\sqrt{y}\right|} + \frac{f_x\left(\sqrt{y}\right)}{\left|2\sqrt{y}\right|} \tag{1.69}$$

$$= \frac{1}{2\sqrt{y}}\left[f_x\left(-\sqrt{y}\right) + f_x\left(\sqrt{y}\right)\right]. \tag{1.70}$$

△

1.5.3 Discrete transformation

The method will be motivated by two examples:

1. continuous to discrete transformation, and
2. discrete to discrete transformation.

Example 1.5.3 (Continuous to discrete transformation)**.** Let $\underline{x}$ be continuous with $f(x) = 0$, $x < 0$ and

$$\underline{y} = g(\underline{x}) \tag{1.71}$$

$$= \lfloor \underline{x} \rfloor \quad \text{(floor of } \underline{x}\text{)}. \tag{1.72}$$

Then the transformed variable $\underline{y}$ is discrete. *i.e.*, $\mathcal{R}_y = \{0, 1, 2, \ldots\}$.

The PMF of $\underline{y}$ is then given by

$$p_y(y) = P\left(\underline{y} = y\right) \tag{1.73}$$

$$= P\left(\lfloor \underline{x} \rfloor = y\right) \tag{1.74}$$

$$= P\left(y \leq \underline{x} < y + 1\right) \tag{1.75}$$

$$= \int_y^{y+1} f_x(x)\mathrm{d}x, \quad y = 0, 1, 2, \ldots. \tag{1.76}$$

△

Example 1.5.4 (Discrete to discrete transformation)**.** Let $\underline{x}$ be discrete with $\mathcal{R}_x = \mathbb{Z}$, and

$$\underline{y} = g(\underline{x}) \tag{1.77}$$
$$= |\underline{x}| \,. \tag{1.78}$$

Then $\underline{y}$ is also discrete random variable with $\mathcal{R}_y = \{0, 1, 2, \ldots\}$.

The PMF of $\underline{y}$ can then be given by

$$p_y(y) = P\left(\underline{y} = y\right) \tag{1.79}$$
$$= P\left(|\underline{x}| = y\right) \tag{1.80}$$
$$= P\left(\underline{x} = \pm y\right) \tag{1.81}$$
$$= \begin{cases} p_x(0)\,, & y = 0, \\ p_x(-y) + p_x(y)\,, & y = 1, 2, \ldots. \end{cases} \tag{1.82}$$

△

1.6 Statistics of random variables

Sometimes, information about the random variables such as the mean is sought after. Information about a random variables are referred to as the *statistics* of it.

Definition 1.6.1 (*Expectation operator*)**.** The expectation operator $\mathbb{E}[\cdot]$ is a functional operator that operates on random variables. Specifically:

- For continuous random variables

$$\mathbb{E}[g(\underline{x})] = \int_{-\infty}^{\infty} g(x) f(x)\, \mathrm{d}x. \tag{1.83}$$

- For discrete random variables

$$\mathbb{E}[g(\underline{x})] = \sum_{-\infty}^{\infty} g(x) p(x)\,. \tag{1.84}$$

Definition 1.6.2 (*Mean of a random variable*)**.** The mean of a random variable $\underline{x}$, μ_x, is defined as

$$\mu_x = \mathbb{E}[\underline{x}] \tag{1.85}$$

which

- for continuous random variable

$$\mu_x = \mathbb{E}[\underline{x}] \tag{1.86}$$

$$= \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x, \tag{1.87}$$

- while for discrete random variable

$$\mu_x = \mathbb{E}[\underline{x}] \tag{1.88}$$

$$= \sum_{-\infty}^{\infty} x p(x). \tag{1.89}$$

and the conditional mean of $\underline{x}$ given event M is

$$\mathbb{E}[\underline{x}|M] = \int_{-\infty}^{\infty} x f(x|M)\,\mathrm{d}x. \tag{1.90}$$

Properties of the mean

- Mean of $\underline{y} = g(\underline{x})$ is

$$\mathbb{E}\left[\underline{y}\right] = \mathbb{E}[g(\underline{x})] \tag{1.91}$$

$$= \int_{-\infty}^{\infty} g(x) f_x(x)\mathrm{d}x. \tag{1.92}$$

- Linear property:

$$\mathbb{E}\left[\sum_{i=1}^{n} \alpha_i g_i(\underline{x})\right] = \sum_{i=1}^{n} \alpha_i \mathbb{E}[g_i(\underline{x})]. \tag{1.93}$$

Note that in general,

$$\mathbb{E}[g(\underline{x})] \neq g(\mathbb{E}[\underline{x}]). \tag{1.94}$$

Definition 1.6.3 (*Variance and standard deviation*)**.** Let $\underline{x}$ be a random variable. Then the *variance* of $\underline{x}$, $\mathrm{Var}[\underline{x}]$, is defined as

$$\mathrm{Var}[\underline{x}] = \sigma_x^2 \tag{1.95}$$

$$= \mathbb{E}\left[(\underline{x} - \mathbb{E}[\underline{x}])^2\right] \tag{1.96}$$

$$= \mathbb{E}\left[\underline{x}^2\right] - \mathbb{E}[\underline{x}]^2. \tag{1.97}$$

The *standard deviation* of $\underline{x}$ is $\sigma_x = \sqrt{\mathrm{Var}[\underline{x}]}$.

A useful property of the variance is

$$\mathrm{Var}[a\underline{x} + b] = a^2 \mathrm{Var}[\underline{x}]. \tag{1.98}$$

Theorem 1.6.1 (Markov's inequality)**.** Let $\underline{x}$ be a non-negative random variable. That is, $P(\underline{x} < 0) = 0$. For any $\epsilon > 0$:

$$P(\underline{x} \geq \epsilon) \leq \frac{\mathbb{E}[\underline{x}]}{\epsilon}. \tag{1.99}$$

Theorem 1.6.2 (Chebyshev's inequality)**.** Let $\underline{x}$ be a random variable with mean μ and variance σ^2. Then, for any $\epsilon > 0$:

$$P(|\underline{x} - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon}. \tag{1.100}$$

Theorem 1.6.3 (Jensen's inequality)**.** Let $g(\cdot)$ be a convex function. Then,

$$\mathbb{E}[g(\underline{x})] \geq g(\mathbb{E}[\underline{x}]). \tag{1.101}$$

1.6.1 Moments

Definition 1.6.4 (*Moments*)**.** The different moments of a random variable are defined as follows.

- *n-th moment* of $\underline{x}$: $m_n = \mathbb{E}[\underline{x}^n]$.
- *n-th central moment* of $\underline{x}$: $\mu_n = \mathbb{E}[(\underline{x} - \mathbb{E}[\underline{x}])^n]$.
- *n-th absolute moment* of $\underline{x}$: $\mathbb{E}[|\underline{x}|^n]$.
- *n-th absolute central moment* of $\underline{x}$: $\mu_n = \mathbb{E}[|\underline{x} - \mathbb{E}[\underline{x}]|^n]$.

1.6.2 Characteristic functions

Definition 1.6.5 (*Characteristic function*)**.** The *characteristic function* of a random variable $\underline{x}$ is the Fourier transform of its PDF. Specifically,

$$\Phi_x(\omega) = \int_{-\infty}^{\infty} e^{\jmath\omega x} f_x(x) \mathrm{d}x, \tag{1.102}$$

where the inverse expression is given by

$$f_x(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\jmath\omega x} \Phi_x(\omega) \mathrm{d}\omega. \tag{1.103}$$

Definition 1.6.6 (*Moment generating function*)**.** The *moment generating function* of $\underline{x}$ is the Laplace transform of its PDF. Specifically,

$$\bar{\Phi}_x(s) = \int_{-\infty}^{\infty} e^{Sx} f_x(x) \mathrm{d}x. \tag{1.104}$$

Theorem 1.6.4 (Moment theorem)**.**

$$\mathbb{E}\left[\underline{x}^n\right] = \left.\frac{\mathrm{d}^n}{\mathrm{d}s^n}\bar{\Phi}_x\right|_{s=0} = \left.\frac{1}{\jmath^n}\frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\Phi_x(\omega)\right|_{\omega=0}. \tag{1.105}$$

Chapter 2

Multivariate Probability

2.1 Multivariate random variables and distributions

In this chapter, we are dealing with multiple random variables.

Definition 2.1.1 (*Multiple random variable*)**.** Let $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$ be n random variables defined on the *same* probability space $(S, \mathcal{F}, P)$. Then, the mapping

$$s \in S \to [\underline{x}_1(s), \underline{x}_2(s), \ldots, \underline{x}_n(s)]^{\mathsf{T}} \in \mathbb{R}^n \tag{2.1}$$

defines an n-dimensional random variable (random vector or column matrix). Random vectors will be denoted by boldface underlined alphabet. Specifically,

$$\underline{\mathbf{x}} = \begin{bmatrix} \underline{x}_1 & \underline{x}_2 & \ldots & \underline{x}_n \end{bmatrix}^{\mathsf{T}}, \tag{2.2}$$

where $\underline{x}_i, i = 1, \ldots, n$ are random variables.

2.1.1 Joint cumulative distribution function (JCDF)

Definition 2.1.2 (*Joint cumalitive distribution function (JCDF)*)**.** Let $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$ be n random variables defined on the same probability space $(S, \mathcal{F}, P)$. Then, the *joint cumulative distribution function (JCDF)* of these random variables are

$$\begin{aligned} F_{x_1,x_2,\ldots,x_n}(x_1, x_2, \ldots, x_n) &= P(\underline{x}_1 \leq x_1, \underline{x}_2 \leq x_2, \ldots, \underline{x}_n \leq x_n) && (2.3)\\ &= F_{\mathbf{x}}(\mathbf{x}) && (2.4)\\ &= F(\mathbf{x}) && (2.5)\\ &= P(\underline{\mathbf{x}} \leq \mathbf{x}). && (2.6) \end{aligned}$$

Properties of JCDFs

1. $F(\mathbf{x})$ is non-decreasing in each of its arguments.

2. $F(\mathbf{x})$ is right-continuous in each of its arguments.
3. For a fixed i, $\lim_{x_i \to -\infty} F(\mathbf{x}) = 0$.
4. When $\underline{x}_i \to \infty$ for *all* i, then $F(\mathbf{x}) \to 1$.

Definition 2.1.3 (*Marginal CDF*)**.** Let $\underline{\mathbf{x}} = \begin{bmatrix} \underline{\mathbf{x}}_1^\mathsf{T} & \underline{\mathbf{x}}_2^\mathsf{T} \end{bmatrix}^\mathsf{T} \in \mathbb{R}^n$ with joint CDF $F(\mathbf{x})$. Say we want to know the joint CDF of a subset of these random variables $\underline{\mathbf{x}}_1$. That is, we want to *marginalize out* the random variables $\underline{\mathbf{x}}_2$. The joint CDF of $\underline{\mathbf{x}}_1$ is computed by taking the limit

$$F(\mathbf{x}_1) = \lim_{\mathbf{x}_2 \to \infty} F(\mathbf{x}). \tag{2.7}$$

Here are the definitions:

- The process is called *marginalization*.
- The retained random variables $\underline{\mathbf{x}}_1$ are called *marginal random variables*.
- The joint CDF of the marginal random variables is called the *marginal CDF*.
- The discarded random variables $\underline{\mathbf{x}}_2$ is said to be *marginalized out*.

Example 2.1.1 (Marginalizing a CDF)**.** Given the joint CDF $F(x_1, x_2, x_3)$, then the joint CDF $F(x_1, x_3)$ is given by

$$F(x_1, x_3) = \lim_{x_2 \to \infty} F(x_1, x_2, x_3). \tag{2.8}$$

△

2.1.2 Joint probability mass function (JPMF)

Definition 2.1.4 (*Joint probability mass function (JPMF)*)**.** This is the analog to joint CDF for discrete random variables. The *joint probability mass function (JPMF)* of the discrete random variables $\underline{x}_1, \ldots, \underline{x}_n$ is

$$\begin{aligned} p_{x_1,\ldots,x_n}(x_1, \ldots, x_n) &= P(\underline{x}_1 = x_1, \ldots, \underline{x}_n = x_n) && (2.9)\\ &= p_{\mathbf{x}}(\mathbf{x}) && (2.10)\\ &= P(\underline{\mathbf{x}} = \mathbf{x}). && (2.11) \end{aligned}$$

Note that the subscripts may be omitted. That is, $p_{\mathbf{x}}(\mathbf{x}) = p(\mathbf{x})$.

Properties of JPMFs

1. **Total probability**: $0 \leq p(\mathbf{x}) \leq 1$.

2. **Normalization Property**:

$$\sum_{\mathbf{x}\in\mathcal{R}_{\mathbf{x}}} p(\mathbf{x}) = \sum_{x_1\in\mathcal{R}_{x_1}} \cdots \sum_{x_n\in\mathcal{R}_{x_n}} p(x_1,\ldots,x_n) \tag{2.12}$$

$$= 1. \tag{2.13}$$

3. Probability that $\underline{\mathbf{x}}$ is in a region $\mathcal{D}$ (*i.e.*, $\underline{\mathbf{x}} \in \mathcal{D}$) is given by

$$P(\underline{\mathbf{x}} \in \mathcal{D}) = \sum_{\underline{\mathbf{x}}\in\mathcal{D}} p(\mathbf{x}). \tag{2.14}$$

4. **Marginalization property**: Let $\underline{\mathbf{x}} = \begin{bmatrix}\underline{\mathbf{x}}_1^\mathsf{T} & \underline{\mathbf{x}}_2^\mathsf{T}\end{bmatrix}^\mathsf{T}$ be a discrete random variable with joint PMF $p(\mathbf{x})$. Then the joint PMF of $\underline{\mathbf{x}}_1$ is given by marginalization. Specifically,

$$p(\mathbf{x}_1) = \sum_{\mathbf{x}_2\in\mathcal{R}_{\mathbf{x}_2}} p(\mathbf{x}). \tag{2.15}$$

2.1.3 Joint probability density function (JPDF)

Definition 2.1.5 (*Joint probability density function (JPDF)*)**.** The *joint PDF* of random variables $\underline{\mathbf{x}} = \begin{bmatrix}\underline{x}_1 & \cdots & \underline{x}_n\end{bmatrix}^\mathsf{T}$, $f(\mathbf{x})$, is given by

$$f(\mathbf{x}) = f(x_1,\ldots,x_n) \tag{2.16}$$

$$= \frac{\partial^n F(x_1,\ldots,x_n)}{\partial x_1\cdots\partial x_n}. \tag{2.17}$$

Often, subscripts will be dropped (*i.e.*, $f_{\mathbf{x}}(\mathbf{x}) = f(\mathbf{x})$).

Properties of JPDFs

1. $f(\mathbf{x}) \geq 0$.

2. **Normalization**:

$$\int_{\mathbb{R}^n} f(\mathbf{x}) = \int_{\mathbb{R}^n} f(x_1,\ldots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n \tag{2.18}$$

$$= 1. \tag{2.19}$$

3. Probability that $\underline{\mathbf{x}}$ is in a region $\mathcal{D}$:

$$P(\underline{\mathbf{x}} \in \mathcal{D}) = \int_{\mathcal{D}} f(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{2.20}$$

4. **Marginalization property**: Let $\underline{\mathbf{x}} = \begin{bmatrix}\underline{\mathbf{x}}_1^\mathsf{T} & \underline{\mathbf{x}}_n^\mathsf{T}\end{bmatrix}^\mathsf{T}$ be a continuous random variable with joint PDF

$f(\mathbf{x})$. Then the joint PDF of $\underline{\mathbf{x}}_1$ is

$$f(\mathbf{x}_1) = \int_{-\infty}^{\infty} f(\mathbf{x})\, \mathrm{d}\mathbf{x}. \tag{2.21}$$

2.1.4 Independence

Definition 2.1.6 (*Mutual independence*)**.** The random variables $\underline{x}_1, \ldots, \underline{x}_n$ are called *mutually independent* if the events $\{\underline{x}_1 \leq x_1\}, \ldots, \{\underline{x}_n \leq x_n\}$ are independent for any $x_1, \ldots, x_n \in \mathbb{R}^n$.

It follows that

$$F(x_1, \ldots, x_n) = F(x_1) \cdots F(x_n), \tag{2.22}$$
$$f(x_1, \ldots, x_n) = f(x_1) \cdots f(x_n), \tag{2.23}$$
$$p(x_1, \ldots, x_n) = p(x_1) \cdots p(x_n). \tag{2.24}$$

Remark 2.1.1. Here some remarks about mutual independence.

- Any subset of the mutually independent random variables $\{\underline{x}_1, \ldots, \underline{x}_n\}$ is a set of mutually independent random variables.
- If $\{\underline{x}_1, \ldots, \underline{x}_n\}$ are independent *in pairs*, they are not necessarily independent.
- The random variables $\underline{y}_1 = g_1(\underline{x}_1), \ldots, \underline{y}_n = g_n(\underline{x}_n)$ are independent if $\{\underline{x}_1, \ldots, \underline{x}_n\}$ are independent. Note that the functions $g_1, \ldots, g_n$ need not be the same.
- Care must be taken when talking about independence. For instance, "independence" of a group of random variables may signify pairwise independence which does not imply mutual independence.

Definition 2.1.7 (*Group independence*)**.** A group of random variables $\mathcal{G}_{\mathbf{x}} = \{\underline{x}_1, \ldots, \underline{x}_n\}$ is *independent* of a group of random variables $\mathcal{G}_{\mathbf{y}} = \left\{\underline{y}_1, \ldots, \underline{y}_m\right\}$ if

$$f(\mathbf{x}, \mathbf{y}) = f(\mathbf{x})\, f(\mathbf{y}). \tag{2.25}$$

Definition 2.1.8 (*Independent and identically distributed (i.i.d.) random variables*)**.** The random variables $\underline{x}_1, \ldots, \underline{x}_n$ are called *independent and identically distributed (i.i.d.)* if

1. they are independent, and
2. $F_{x_1}(x) = F_{x_2}(x) = \cdots = F_{x_n}(x)$, or equivalently

$$f_{x_1}(x) = \cdots = f_{x_n}(x), \quad \text{or} \tag{2.26}$$
$$p_{x_1}(x) = \cdots = p_{x_n}(x). \tag{2.27}$$

2.1.4.1 Complex random variables

The statistics (PDF, CDF, etc.) of the n complex random variables

$$\begin{aligned} \underline{z}_1 &= \underline{x}_1 + \jmath \underline{y}_1 \\ &\vdots \\ \underline{z}_n &= \underline{x}_n + \jmath \underline{y}_n \end{aligned} \tag{2.28}$$

are determined by the joint PDF $f(x_1, \ldots, x_n, y_1, \ldots, y_n)$ of the $2n$ real random variables $\mathbf{x}, \mathbf{y}$.

Definition 2.1.9 (*Mutually independent complex random variables*)**.** The complex random variables $\underline{z}_1, \ldots, \underline{z}_n$ as defined in (2.28) are *mutually independent* if

$$f(\mathbf{x}, \mathbf{y}) = f(x_1, y_1) \cdots f(x_n, x_n). \tag{2.29}$$

2.2 Transformed random variables

Consider n random variables $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$ with joint PDF $f(x_1, \ldots, x_n)$. Further, given a function $g : \mathbb{R}^n \to \mathbb{R}^k$

$$g(\mathbf{x}) = \begin{bmatrix} g_1(\mathbf{x}) \\ \vdots \\ g_k(\mathbf{x}) \end{bmatrix}. \tag{2.30}$$

Then there are k new random variables $\underline{\mathbf{y}} = \begin{bmatrix} \underline{y}_1 & \cdots & \underline{y}_k \end{bmatrix}^{\mathsf{T}}$. Specifically,

$$\underline{\mathbf{y}} = g(\underline{\mathbf{x}}). \tag{2.31}$$

Basic concepts

If $\underline{\mathbf{x}}$ is continuous, then

$$\begin{aligned} F(\mathbf{y}) &= F(y_1, \ldots, y_k) && (2.32) \\ &= P\left(\underline{y}_1 \leq y_1, \ldots, \underline{y}_k \leq y_k\right) && (2.33) \\ &= P\left(g_1(\underline{\mathbf{x}}) \leq y_1, \ldots, g_k(\underline{\mathbf{x}}) \leq y_k\right) && (2.34) \\ &= \int_{\mathcal{D}} f(\mathbf{x})\, \mathrm{d}\mathbf{x}, && (2.35) \end{aligned}$$

where

$$\begin{aligned} \mathcal{D} &= \{\mathbf{x} \in \mathbb{R}^n \text{ s.t. } g_1(\mathbf{x}) \leq y_1, \ldots, g_k(\mathbf{x}) \leq y_k\} && (2.36) \\ &= \{x_1, \ldots, x_n \text{ s.t. } g_1(\cdot) \leq y_1, \ldots, g_k(\cdot) \leq y_k\}. && (2.37) \end{aligned}$$

That is, $\mathcal{D}$ depends on $\mathbf{y}$.

If $\underline{\mathbf{x}}$ is discrete, then

$$p(\mathbf{y}) = P(y_1, \ldots, y_k) \tag{2.38}$$
$$= \sum_{\mathbf{x} \in \mathcal{D}} P(x_1, \ldots, x_n) \tag{2.39}$$

where

$$\mathcal{D} = \{\mathbf{x} \in \mathbb{R}^n \text{ s.t. } g_1(\mathbf{x}) = y_1, \ldots, g_k(\mathbf{x}) = y_k\} \tag{2.40}$$
$$= \{x_1, \ldots, x_n \text{ s.t. } g_1(\cdot) = y_1, \ldots, g_k(\cdot) = y_k\}. \tag{2.41}$$

Again, $\mathcal{D}$ depends on $\mathbf{y}$.

Transformed variables distribution

Given the distribution of $\underline{\mathbf{x}}$ (CDF, PDF, or PMF) and the function $g : \mathbb{R}^k \to \mathbb{R}^n$, then what is the distribution of $\underline{\mathbf{y}} = g(\underline{\mathbf{x}}) \in \mathbb{R}^k$?

Assumption: $k \leq n$. That is, the mapping g is surjective.

Just as is the case for a single transformed random variable (Section 1.5), there are different methods to find the distribution of a transformed random variable (*i.e.*, $\underline{y} = g(\underline{x})$) depending on the types of variables.

1. **Method of distributions** works for *any* type of random variables (continuous or discrete). However, it is the most exhaustive method.
2. **Method of transformations** works for *continuous* random variables. Specifically, both $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$ are continuous.

2.2.1 Method of distribution

Given the function $g : \mathbb{R}^n \to \mathbb{R}^k$, where $k \leq n$, then carry the following.

1. For all $\mathbf{y} \in \mathbb{R}^k$, find $\mathbf{x} \in D_y \subseteq \mathbb{R}^n$ such that

$$g(\mathbf{x}) \leq \mathbf{y}. \tag{2.42}$$

   Note that D_y is a function of $\mathbf{y}$.

2. The CDF of $\underline{\mathbf{y}}$ is then

$$F(\mathbf{y}) = P\left(\underline{\mathbf{y}} \leq \mathbf{y}\right) \tag{2.43}$$
$$= P(\underline{\mathbf{x}} \in D_y) \tag{2.44}$$
$$= \int_{D_y} f(\mathbf{x})\, \mathrm{d}\mathbf{x}. \tag{2.45}$$

3. The PDF of $\underline{\mathbf{y}}$ is then

$$f(\mathbf{y}) = \frac{\mathrm{d}F(\mathbf{y})}{\mathrm{d}\mathbf{y}}. \tag{2.46}$$

Example 2.2.1. Consider the transformed random variable

$$\underline{y} = g(\underline{x}_1, \underline{x}_2) \tag{2.47}$$
$$= \underline{x}_1 + \underline{x}_2. \tag{2.48}$$

To find the PDF of $\underline{y}$, the steps outlined above will be follows.

1. Find D_y:

$$D_y = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq y - x_1\right\}. \tag{2.49}$$

2. CDF of $\underline{y}$:

$$F(y) = \int\int_{D_y} f(x_1, x_2)\,\mathrm{d}x_1\mathrm{d}x_2 \tag{2.50}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{y-x_1} f(x_1, x_2)\,\mathrm{d}x_2\mathrm{d}x_1. \tag{2.51}$$

3. PDF of $\underline{y}$ is

$$f(y) = \frac{\mathrm{d}F(y)}{\mathrm{d}y} \tag{2.52}$$
$$= \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}y}\int_{-\infty}^{y-x_1} f(x_1, x_2)\,\mathrm{d}x_2\mathrm{d}x_1 \tag{2.53}$$
$$= \int_{-\infty}^{\infty} f(x_1, y - x_1)\,\mathrm{d}x_1. \tag{2.54}$$

△

Theorem 2.2.1 (Transformed variable PDF of linear combination of random variables)**.** Let $\underline{x}_1$ and $\underline{x}_2$ be jointly distributed random variables with joint PDF $f(x_1, x_2)$. The PDF of $\underline{y} = \underline{x}_1 + \underline{x}_2$ is given by

$$f(y) = \int_{-\infty}^{\infty} f(\lambda, y - \lambda)\,\mathrm{d}\lambda. \tag{2.55}$$

Theorem 2.2.2 (Transformed variable PDF of linear combination of independent random variables)**.** Let $\underline{x}_1$ and $\underline{x}_2$ be *independent* random variables with marginal PDFs $f_1(x_1)$ and $f_2(x_2)$, respectively. Then,

the PDF of $\underline{y} = \underline{x}_1 + \underline{x}_2$ is given by

$$f(y) = \int_{-\infty}^{\infty} f_1(\lambda) f_2(y - \lambda) \mathrm{d}\lambda \tag{2.56}$$

$$= f_1(y) * f_2(y), \tag{2.57}$$

where $*$ denotes the convolution operator.

2.2.2 Method of transformation

Note that the method is derived in [2, Sec. 2.6.2].

2.2.2.1 Approach 1

Given the transformed random variable

$$\underline{\mathbf{y}} = g(\underline{\mathbf{x}}), \tag{2.58}$$

where $\underline{\mathbf{y}} \in \mathbb{R}^k$ (i.e., $g : \mathbb{R}^n \to \mathbb{R}^k$) and given the PDF of the random variable $\underline{\mathbf{x}}$. For now, **assume** $k = n$. The distribution of $\underline{\mathbf{y}}$ is then found as follows.

1. For all $\mathbf{y}$, solve the system of equations

$$\mathbf{y} = g(\mathbf{x}) \tag{2.59}$$

for $\mathbf{x}$. For a given $\mathbf{y}$, the system has either

1.1. m solutions

$$\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}, \tag{2.60}$$

1.2. or no solutions.

Note that the system will not have infinite solutions since it is assumed that $k = n$.

2. Evaluate the Jacobian of g

$$J(\mathbf{x}) = \frac{\mathrm{d}g(\mathbf{x})}{\mathrm{d}\mathbf{x}}. \tag{2.61}$$

3. Then, the PDF of $\underline{\mathbf{y}}$ is

$$f(\mathbf{y}) = \sum_{i=1}^{m} \frac{f\left(\mathbf{x}^{(i)}\right)}{\left|\det\left(J(\mathbf{x}^{(i)})\right)\right|}, \tag{2.62}$$

where $\det(\cdot)$ is the determinant of a square matrix and $|\cdot|$ is the absolute value. **If for y the system**

has no solution, then

$$f(\mathbf{y}) = \mathbf{0}. \tag{2.63}$$

2.2.2.2 Approach 2

Given the transformed random variable

$$\underline{\mathbf{y}} = g(\underline{\mathbf{x}}), \tag{2.64}$$

where $\underline{\mathbf{y}} \in \mathbb{R}^k$ (i.e., $g : \mathbb{R}^n \to \mathbb{R}^k$) and given the PDF of the random variable $\mathbf{x}$. For now, **assume** $k = n$. The distribution of $\underline{\mathbf{y}}$ is then found as follows.

1. Identical to in Step 1 of approach 1 (Section 2.2.2.1): for all $\mathbf{y}$, solve the system of equations

   $$\mathbf{y} = g(\mathbf{x}) \tag{2.65}$$

   for $\mathbf{x}$. For a given $\mathbf{y}$, the system has either

   1.1. m solutions

   $$\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}, \tag{2.66}$$

   1.2. or no solutions.

   Note that the system will not have infinite solutions since it is assumed that $k = n$.

2. Evaluate the Jacobian

   $$\tilde{J}_i(\mathbf{y}) = \frac{\mathrm{d}\mathbf{x}^{(i)}}{\mathrm{d}\mathbf{y}} \tag{2.67}$$

   $$= \begin{bmatrix} \frac{\partial x_1^{(i)}}{\partial y_1} & \cdots & \frac{\partial x_1^{(i)}}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial x_n^{(i)}}{\partial y_1} & \cdots & \frac{\partial x_n^{(i)}}{\partial y_n} \end{bmatrix} \tag{2.68}$$

   for each solution $i = 1, \ldots, m$.

3. Then, the PDF of $\underline{\mathbf{y}}$ is

   $$f(y) = \sum_{i=1}^{m} \left| \det\left(\tilde{J}_i(\mathbf{x}^{(i)}) \right) \right| f\left(\mathbf{x}^{(i)} \right), \tag{2.69}$$

   where $\det(\cdot)$ is the determinant of a square matrix and $|\cdot|$ is the absolute value. **If for y the system has no solution**, then

   $$f(\mathbf{y}) = \mathbf{0}. \tag{2.70}$$

Example 2.2.2. Consider the transformed random variable $\underline{\mathbf{y}}$ given by

$$\underline{\mathbf{y}} = g(\underline{\mathbf{x}}) \tag{2.71}$$

$$= \begin{bmatrix} \underline{x}_1^2 \\ \underline{x}_1 + \underline{x}_2 \end{bmatrix}. \tag{2.72}$$

1. For all $\mathbf{y}$ solve

$$\mathbf{y} = g(\mathbf{x}) \tag{2.73}$$

$$= \begin{bmatrix} \underline{x}_1^2 \\ \underline{x}_1 + \underline{x}_2 \end{bmatrix} \tag{2.74}$$

for $\mathbf{x}$.

Case I $(y_1 < 0)$: No solutions.

Case II $(y_1 = 0)$: One solution

$$\mathbf{x} = \begin{bmatrix} 0 \\ y_2 \end{bmatrix}. \tag{2.75}$$

Case III $(y_1 > 0)$: two solutions:

$$\mathbf{x}^{(1)} = \begin{bmatrix} \sqrt{y_1} \\ y_2 - \sqrt{y_1} \end{bmatrix} \tag{2.76}$$

$$\mathbf{x}^{(2)} = \begin{bmatrix} -\sqrt{y_1} \\ y_2 + \sqrt{y_1} \end{bmatrix}. \tag{2.77}$$

2. For the ith solution (using approach 1):

$$\det\left(J_i(\mathbf{x})\right) = \det\left(\frac{\mathrm{d}g(\mathbf{x})}{\mathrm{d}\mathbf{x}}\right) \tag{2.78}$$

$$= \det\left(\begin{bmatrix} \frac{\partial x_1^{(i)}}{\partial y_1} & \frac{\partial x_1^{(i)}}{\partial y_2} \\ \frac{\partial x_2^{(i)}}{\partial y_1} & \frac{\partial x_2^{(i)}}{\partial y_2} \end{bmatrix}\right) \tag{2.79}$$

$$= \frac{\partial x_1^{(i)}}{\partial y_1}\frac{\partial x_2^{(i)}}{\partial y_2} - \frac{\partial x_2^{(i)}}{\partial y_1}\frac{\partial x_1^{(i)}}{\partial y_2}. \tag{2.80}$$

Case I $(y_1 < 0)$: no solutions.

Case II $(y_1 = 0)$: one solution:

$$\det\left(J(\mathbf{x})\right) = 0. \tag{2.81}$$

Case III ($y_1 > 0$): two solutions:

$$J_1(\mathbf{x}) = \frac{1}{2\sqrt{y_1}} \tag{2.82}$$

$$J_2(\mathbf{x}) = -\frac{1}{2\sqrt{y_1}}. \tag{2.83}$$

3. The JPDF of $\underline{\mathbf{y}}$ is then

$$f_{\mathbf{y}}(\mathbf{y}) = \sum_i f_{\mathbf{x}}(\mathbf{x}) \left|\det\left(J_i\right)\right| \tag{2.84}$$

or $f_{\mathbf{y}}(\mathbf{y}) = \mathbf{0}$ if there are no solutions.

$$f_{\mathbf{y}}(\mathbf{y}) = f_{\mathbf{x}^{(1)}}(\mathbf{x}) \left|\det J_1\right| + f_{\mathbf{x}^{(2)}}(\mathbf{x}) \left|\det J_2\right| \tag{2.85}$$

$$= f_{\mathbf{x}}(\sqrt{y_1}, y_2 - \sqrt{y_1}) \left|\frac{1}{2\sqrt{y_1}}\right| + f_{\mathbf{x}}(-\sqrt{y_1}, y_2 + \sqrt{y_1}) \left|-\frac{1}{2\sqrt{y_1}}\right|. \tag{2.86}$$

△

So far, in this method, we assumed that $k = n$. But if $k < n$, then follow the following steps.

1. Introduce $n - k$ "dummy" variables[a]:

$$\underline{y}_{k+1} = g_{k+1}(\underline{\mathbf{x}}) \tag{2.87}$$

$$\vdots \tag{2.88}$$

$$\underline{y}_n = g_n(\underline{\mathbf{x}}). \tag{2.89}$$

2. Evaluate $f\left(y_1, \ldots, y_k, \ldots, y_n\right)$ using the methods outlined before (now we have $k = n$).
3. Marginalize out $\underline{y}_{k+1}, \ldots, \underline{y}_n$ to obtain $f\left(\underline{y}_1, \ldots, \underline{y}_k\right)$.

[a] I believe it's better if the introduced mapping is bijective.

2.2.3 Linear transformation

If $\mathbf{g}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ (*i.e.*, $\mathbf{g}$ is linear), then the mean and the covariance are given by

$$\mathbb{E}\left[\underline{\mathbf{y}}\right] = \mathbf{A}\bar{\mathbf{x}}, \tag{2.90}$$

$$\mathrm{Cov}\left[\underline{\mathbf{y}}\right] = \mathbb{E}\left[\left(\underline{\mathbf{y}} - \bar{\mathbf{y}}\right)\left(\underline{\mathbf{y}} - \bar{\mathbf{y}}\right)^{\mathsf{T}}\right] \tag{2.91}$$

$$= \mathbb{E}\left[\mathbf{A}\left(\underline{\mathbf{x}} - \bar{\mathbf{x}}\right)\left(\mathbf{A}(\underline{\mathbf{x}} - \bar{\mathbf{x}})\right)^{\mathsf{T}}\right] \tag{2.92}$$

$$= \mathbf{A}\mathbb{E}\left[\left(\underline{\mathbf{x}} - \bar{\mathbf{x}}\right)\left(\underline{\mathbf{x}} - \bar{\mathbf{x}}\right)^{\mathsf{T}}\right]\mathbf{A}^{\mathsf{T}} \tag{2.93}$$

$$= \mathbf{A}\boldsymbol{\Sigma}_x\mathbf{A}^{\mathsf{T}}. \tag{2.94}$$

Transform	Acronym	Benefits	Drawbacks
First order Taylor expansion	TT1	Simple and fast to compute.	Inaccurate for highly nonlinear functions.
Second order Taylor expansion	TT2	Better accuracy than TT1.	Requires computation of Hessian of the function.
Monte-Carlo simulation	MCT	Very accurate.	Requires many samples, thus computationally inefficient.
Unscented transform	UCT	Accuracy close to TT2	Doesn't require second order information (Hessian).

Table 2.1: Summary of transforms that approximate the covariances of propagated random variables.

Note that the relation is an exact relation, not an approximation.

2.2.4 Nonlinear transformation (covariance propagation)

For the nonlinear case, getting the exact transformed covariance is tedious and cumbersome. Therefore, tools that approximate the covariance are used that trade off between accuracy and computational efficiency. Most of the information from this section is obtained from [3]. Table 2.1 summarizes the results.

2.2.4.1 First order Taylor expansion

For a nonlinear transformation, a simple approximation is linearization. Thus,

$$\underline{\mathbf{y}} \approx \mathbf{g}(\bar{\mathbf{x}}) + \mathbf{g}'(\bar{\mathbf{x}})\delta\underline{\mathbf{x}}. \tag{2.95}$$

The random variable becomes $\delta\underline{\mathbf{x}} \sim \mathcal{N}(\underline{\mathbf{x}} - \bar{\mathbf{x}}, \mathbf{\Sigma}_x)$. The covariance of $\underline{\mathbf{y}}$ is then given by

$$\begin{aligned}
\mathbb{E}\left[\underline{\mathbf{y}}\right] &= \mathbf{g}(\bar{\mathbf{x}}), && (2.96)\\
\mathrm{Cov}\left[\underline{\mathbf{y}}\right] &= \mathbb{E}\left[\left(\underline{\mathbf{y}} - \bar{\mathbf{y}}\right)\left(\underline{\mathbf{y}} - \bar{\mathbf{y}}\right)^{\mathsf{T}}\right] && (2.97)\\
&= \mathbb{E}\left[\left(\underline{\mathbf{y}} - \mathbf{g}(\bar{\mathbf{x}})\right)\left(\underline{\mathbf{y}} - \mathbf{g}(\bar{\mathbf{x}})\right)^{\mathsf{T}}\right] && (2.98)\\
&= \mathbb{E}\left[\mathbf{g}'(\bar{\mathbf{x}})\delta\underline{\mathbf{x}}\delta\underline{\mathbf{x}}^{\mathsf{T}}\mathbf{g}'(\bar{\mathbf{x}})^{\mathsf{T}}\right] && (2.99)\\
&= \mathbf{g}'(\bar{\mathbf{x}})\mathbf{\Sigma}_x\mathbf{g}'(\bar{\mathbf{x}})^{\mathsf{T}}. && (2.100)
\end{aligned}$$

2.2.4.2 Monte-Carlo simulation

Monte-Carlo simulation consists of simply sampling a *large* number (say in the order of $N = 10^4$) of the domain random variable $\underline{\mathbf{x}}$ and passing it through the nonlinear function $\mathbf{g}$ to get a large number of samples of $\underline{\mathbf{y}}$, $\mathbf{y}^{(i)}, i = 1, \ldots, N$. The mean and covariance are then given by

$$\bar{\mathbf{y}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{y}^{(i)}, \tag{2.101}$$

$$\mathrm{Cov}\left[\bar{\mathbf{y}}\right] \approx \frac{1}{N-1}\sum_{i=1}^{N}\left(\mathbf{y}^{(i)} - \bar{\mathbf{y}}\right)\left(\mathbf{y}^{(i)} - \bar{\mathbf{y}}\right)^{\mathsf{T}}, \tag{2.102}$$

where the $\frac{1}{N-1}$ term is to ensure that the estimator is *unbiased*.

2.2.4.3 Unscented transform

Let n_x be the length of the domain random variable (*i.e.*, $\underline{\mathbf{x}} \in \mathbb{R}^{n_x}$). Further, let $\mathbf{u}_i \in \mathbb{R}^{n_x}$ be the left singular vector of the singular value σ_i of the domain covariance matrix $\mathbf{\Sigma}_x$. That is,

$$\mathbf{\Sigma}_x = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^{\mathsf{T}} \tag{2.103}$$

$$= \sum_{i=1}^{n_x} \sigma_i^2 \mathbf{u}_i \mathbf{u}_i^{\mathsf{T}}, \tag{2.104}$$

where $\mathbf{\Sigma}$ is a diagonal matrix of the singular values.

Let $\mathbf{x}^{(i)}$ be the i-th realization of $\underline{\mathbf{x}}$. For this version of the unscented transform (UT), the number of realizations will be $2n_x + 1$. Specifically, set

$$\mathbf{x}^{(0)} = \bar{\mathbf{x}}, \tag{2.105}$$

$$\mathbf{x}^{(\pm i)} = \bar{\mathbf{x}} \pm \sqrt{n_x + \lambda}\sigma_i \mathbf{u}_i, \qquad i = 1, \ldots, n_x, \tag{2.106}$$

where λ is a user-defined parameter to be discussed and $\bar{\mathbf{x}}$ is the mean of $\underline{\mathbf{x}}$ (in estimation problems, this is usually the predicted value). Further, let

$$\omega^{(0)} = \frac{\lambda}{n_x + \lambda}, \tag{2.107}$$

$$\omega^{(\pm i)} = \frac{1}{2(n_x + \lambda)}, \qquad i = 1, \ldots, n_x. \tag{2.108}$$

Next, set

$$\mathbf{y}^{(\pm i)} = \mathbf{g}(\mathbf{x}^{(\pm i)}), i = 0, \ldots, n_x. \tag{2.109}$$

Then, the stats of the propagated random variable $\underline{\mathbf{y}}$ are

$$\bar{\mathbf{y}} = \sum_{i=-n_x}^{n_x} \omega^{(i)} \mathbf{y}^{(i)}, \tag{2.110}$$

$$\mathbf{\Sigma}_y = \left(1 - \alpha^2 + \beta\right) \left(\mathbf{y}^{(0)} - \bar{\mathbf{y}}\right) \left(\mathbf{y}^{(0)} - \bar{\mathbf{y}}\right)^{\mathsf{T}} \tag{2.111}$$

$$+ \sum_{i=-n_x}^{n_x} \omega^{(i)} \left(\mathbf{y}^{(i)} - \bar{\mathbf{y}}\right) \left(\mathbf{y}^{(i)} - \bar{\mathbf{y}}\right)^{\mathsf{T}}, \tag{2.112}$$

where α and β are user-defined parameters to be discussed. Further, $\omega^{(0)} + (1 - \alpha^2 + \beta)$ is often denoted $\omega_c^{(0)}$. The UT discussed above is the 'mod' version of the UT. There is another version denoted 'std' obtained by removing the first term in (2.111). I have no idea about the differences between the 'std' and 'mod' versions.

User-defined parameters

The authors of [3] discuss the values to set the user-defined parameters.

- α controls the spread of the sigma points and is suggested to be approximately 10^{-3}.

- β compensates for the distribution (not really sure what that means), and should be chosen as $\beta = 2$ when $\underline{\mathbf{x}}$ is Gaussian.
- λ is defined by $\lambda = \alpha^2 (n_x + \kappa) - n_x$, where κ is usually zero.
- $\omega^{(0)} = 1 - \frac{n_x}{3}$ for the 'std' version of the UT when $\underline{\mathbf{x}}$ is Gaussian.

2.2.4.4 Spherical cubature method

Given a function a random variable $\underline{\mathbf{x}} \in \mathbb{R}^{n_x}$ and a nonlinear function $\mathbf{g} : \mathbb{R}^{n_x} \to \mathbb{R}^{n_y}$, then the spherical cubature method[1] approximates the mean and the covariance of the transformed random variable $\underline{\mathbf{y}} = \mathbf{g}(\underline{\mathbf{x}})$. The method is summarized in Algorithm 1. More details are available in [4, Algorithm 6.8]

Algorithm 1 Nonlinear transformation: The spherical cubature

1: **Input**: The distribution on the random variable $\underline{\mathbf{x}} \sim \mathcal{N}(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x)$ where $\mathbf{x} \in \mathbb{R}^{n_x}$, and the nonlinear function $\mathbf{g} : \mathbb{R}^{n_x} \to \mathbb{R}^{n_y}$.

2: Generate the unit sigma points using

$$\boldsymbol{\xi} = \sqrt{n_x} \begin{bmatrix} \mathbf{1} & -\mathbf{1} \end{bmatrix} \in \mathbb{R}^{n_x \times 2 * n_x}. \tag{2.113}$$

3: Generate the sigma points

$$\mathcal{X}^{(i)} = \boldsymbol{\mu}_x + \sqrt{\boldsymbol{\Sigma}_x}\boldsymbol{\xi}^{(i)}, \quad , i = 1, \ldots, 2n_x, \tag{2.114}$$

where $\boldsymbol{\xi}^{(i)} \in \mathbb{R}^{n_x}$ is the i-th column of $\boldsymbol{\xi}$, and $\boldsymbol{\Sigma}_x = \sqrt{\boldsymbol{\Sigma}_x}\sqrt{\boldsymbol{\Sigma}_x}^{\mathsf{T}}$ is obtained using Cholesky decomposition.

4: Pass the sigma-points through the nonlinearity

$$\mathcal{Y}^{(i)} = \mathbf{g}(\mathcal{X}^{(i)}), \quad i = 1, \ldots, 2n_x. \tag{2.115}$$

5: Compute

$$\boldsymbol{\mu}_y = \frac{1}{2n_x} \sum_{i=1}^{2n_x} \mathcal{Y}^{(i)}, \tag{2.116}$$

$$\boldsymbol{\Sigma}_y = \frac{1}{2n_x} \sum_{i=1}^{2n_x} \left(\mathcal{Y}^{(i)} - \boldsymbol{\mu}_y\right) \left(\mathcal{Y}^{(i)} - \boldsymbol{\mu}_y\right)^{\mathsf{T}}. \tag{2.117}$$

2.3 Statistics of random variables

2.3.1 Order statistics

Definition 2.3.1 (*Order statistics*)**.** Let $\underline{x}_1, \ldots, \underline{x}_n$ be i.i.d. random variables with marginal CDF and

[1] It's a type of a sigma-point transformation.

PDF $F(\cdot)$ and $f(\cdot)$, respectively. Consider the transformation

$$\underline{x}_{(1)} = \text{ smallest value of } (\underline{x}_1, \ldots, \underline{x}_n) \tag{2.118}$$
$$\underline{x}_{(2)} = \text{ 2nd smallest value of } (\underline{x}_1, \ldots, \underline{x}_n) \tag{2.119}$$
$$\vdots \tag{2.120}$$
$$\underline{x}_{(n)} = n\text{th smallest (\textit{i.e.}, largest) value of} (\underline{x}_1, \ldots, \underline{x}_n), \tag{2.121}$$

then $\underline{x}_{(1)}, \ldots, \underline{x}_{(n)}$ are the *order statistics* of $\underline{x}_1, \ldots, \underline{x}_n$. Furthermore, $\underline{x}_{(k)}$ is the *kth order statistic* of $\underline{x}_1, \ldots, \underline{x}_n$.

Theorem 2.3.1 (CDF of order statistics)**.** The CDF of the kth order statistic of $\underline{x}_1, \ldots, \underline{x}_n$ is

$$F_{(k)}(x) = \sum_{i=k}^{n} \binom{n}{k} F(x)^i (1 - F(x))^{n-i}. \tag{2.122}$$

2.3.2 Mean

Theorem 2.3.2 (Mean of a function)**.** The *mean* of a function $g : \mathbb{R}^n \to \mathbb{R}$ of the n real random variables $\underline{x}_1, \ldots, \underline{x}_n$ is given by

$$\mathbb{E}[g(\mathbf{x})] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(\mathbf{x}) f(\mathbf{x}) \, \mathrm{d}\mathbf{x}, \tag{2.123}$$

where $\mathbb{E}[\cdot]$ is the multidimensional expectation operator (Definition 1.6.1).

Theorem 2.3.3 (Mean of a function of complex random variables)**.** Mean of a function $g : \mathbb{R}^n \to \mathbb{R}$ of the n complex random variables $\underline{z}_i = \underline{x}_i + \jmath \underline{y}_i, i = 1, \ldots, n$ is given by

$$\mathbb{E}[g(\underline{z}_1, \ldots, \underline{z}_n)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1 + \jmath y_1, \ldots, x_n + \jmath y_n) f(x_1, \ldots, x_n, y_1, \ldots, y_n) \, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}, \tag{2.124}$$

where $\mathbb{E}[\cdot]$ is the multidimensional expectation operator (Definition 1.6.1).

Definition 2.3.2 (*Mean of a random vector*)**.** The *mean* of a random vector $\underline{\mathbf{x}} = \begin{bmatrix} \underline{x}_1 & \cdots & \underline{x}_n \end{bmatrix}^{\mathsf{T}}$ is defined as

$$\mathbb{E}[\underline{\mathbf{x}}] = \begin{bmatrix} \mathbb{E}[\underline{x}_1] & \cdots & \mathbb{E}[\underline{x}_n] \end{bmatrix}^{\mathsf{T}}. \tag{2.125}$$

Note that the mean of a random vector is also a vector of the same length. That is, $\mathbb{E}[\underline{\mathbf{x}}] \in \mathbb{R}^n$.

Definition 2.3.3 (*Mean of a random matrix*)**.** Let

$$\underline{\mathbf{X}} = \begin{bmatrix} \underline{\mathbf{x}}_1^{\mathsf{T}} \\ \vdots \\ \underline{\mathbf{x}}_m^{\mathsf{T}} \end{bmatrix}_{m \times n}, \tag{2.126}$$

be a random matrix, where $\underline{\mathbf{x}}_i \in \mathbb{R}^n, i = 1, \ldots, m$. Then, the mean of $\underline{\mathbf{X}}$ is given by

$$\mathbb{E}\left[\underline{\mathbf{X}}\right] = \begin{bmatrix} \mathbb{E}\left[\underline{\mathbf{x}}_1\right]^{\mathsf{T}} \\ \vdots \\ \mathbb{E}\left[\underline{\mathbf{x}}_m\right]^{\mathsf{T}} \end{bmatrix}. \tag{2.127}$$

Properties of the expectation operator

The expectation operator is linear. That is

$$\mathbb{E}\left[\alpha_1 g_1(\underline{\mathbf{x}}) + \ldots + \alpha_m g_m(\underline{\mathbf{x}})\right] = \alpha_1 \mathbb{E}\left[g_1(\underline{\mathbf{x}})\right] + \ldots + \alpha_m \mathbb{E}\left[g_m(\underline{\mathbf{x}})\right] \tag{2.128}$$

for any $\underline{\mathbf{x}}$ (real or complex) and any *deterministic* $\alpha_1, \ldots, \alpha_m$ (real or complex). As a result,

$$\mathbb{E}\left[\mathbf{A}\underline{\mathbf{x}}\right] = \mathbf{A}\mathbb{E}\left[\underline{\mathbf{x}}\right], \tag{2.129}$$

for any deterministic matrix $\mathbf{A}$.

2.3.3 Covariance and correlation

Definition 2.3.4 (*Correlation matrix*)**.** The *correlation matrix* of the (complex) random variable $\underline{\mathbf{x}} = \begin{bmatrix} \underline{x}_1 & \cdots & \underline{x}_n \end{bmatrix}^{\mathsf{T}}$ is defined as

$$\mathbf{R} = \mathbb{E}\left[\underline{\mathbf{x}}\underline{\mathbf{x}}^*\right], \tag{2.130}$$

where $(\cdot)^*$ is the Hermitian[a] of a vector. The (i, j)th element of $\mathbf{R}$ is the correlation of $\underline{x}_i$ and $\underline{x}_j$.

[a] Hermitian: $\mathbf{x}^* = (\mathbf{x}^*)^{\mathsf{T}} = \left(\mathbf{x}^{\mathsf{T}}\right)^*$.

Definition 2.3.5 (*Covariance matrix*)**.** The *covariance matrix* of the random vector $\underline{\mathbf{x}} = \begin{bmatrix} \underline{x}_1 & \cdots & \underline{x}_n \end{bmatrix}^{\mathsf{T}}$ is

$$\mathbf{C} = \mathbb{E}\left[\left(\underline{\mathbf{x}} - \mathbb{E}\left[\underline{\mathbf{x}}\right]\right)\left(\underline{\mathbf{x}} - \mathbb{E}\left[\underline{\mathbf{x}}\right]\right)^*\right]. \tag{2.131}$$

The (i, j)th element of $\mathbf{C}$ is the covariance of $\underline{x}_i$ and $\underline{x}_j$. That is,

$$C_{ij} = \mathbb{E}\left[\left(\underline{x}_i - \mathbb{E}\left[\underline{x}_i\right]\right)\left(\underline{x}_j - \mathbb{E}\left[\underline{x}_j\right]\right)^*\right]. \tag{2.132}$$

The covariance matrix $\mathbf{C}$ of $\underline{\mathbf{x}}$ is equal to the correlation matrix of the "centered" random vector $\underline{\mathbf{x}} - \mathbb{E}[\underline{\mathbf{x}}]$.

Definition 2.3.6 (*Mutually orthogonal random variables*)**.** The random variables $\underline{x}_1, \ldots, \underline{x}_n$ are called *mutually orthogonal* if the correlation matrix $\mathbf{R}$ is diagonal. That is, $R_{ij} = 0, i \neq j$.

Definition 2.3.7 (*Mutually uncorrelated random variables*)**.** The random variables $\underline{x}_1, \ldots, \underline{x}_n$ are called *mutually uncorrelated* if the covariance matrix $\mathbf{C}$ is diagonal. That is, $C_{ij} = 0, i \neq j$.

Notes: If the random variables $\underline{x}_1, \ldots, \underline{x}_n$ are independent then they are uncorrelated. The converse is not true in general.

Definition 2.3.8 (*Uncorrelated random vectors*)**.** Two random vectors $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$ are called *uncorrelated* if

$$\mathbb{E}\left[\left(\underline{\mathbf{x}} - \mathbb{E}[\underline{\mathbf{x}}]\right)\left(\underline{\mathbf{y}} - \mathbb{E}\left[\underline{\mathbf{y}}\right]\right)^*\right] = \mathbf{0}. \tag{2.133}$$

Or equivalently

$$\mathbb{E}\left[\underline{\mathbf{x}}\underline{\mathbf{y}}^*\right] = \mathbb{E}[\underline{\mathbf{x}}]\,\mathbb{E}\left[\underline{\mathbf{y}}\right]^*. \tag{2.134}$$

Definition 2.3.9 (*Orthogonal random vectors*)**.** Two random vectors $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$ are called orthogonal if

$$\mathbb{E}\left[\underline{\mathbf{x}}\underline{\mathbf{y}}^*\right] = \mathbf{0}. \tag{2.135}$$

Properties

1. Relationship between the correlation and covariance matrix

$$\begin{aligned}\mathbf{C} &= \mathbf{R} - \mathbb{E}[\underline{\mathbf{x}}]\,\mathbb{E}[\underline{\mathbf{x}}]^* && (2.136)\\ &= \mathbb{E}[\underline{\mathbf{x}}\underline{\mathbf{x}}^*] - \mathbb{E}[\underline{\mathbf{x}}]\,\mathbb{E}[\underline{\mathbf{x}}]^*. && (2.137)\end{aligned}$$

2. The correlation and covariance matrix are positive semidefinite matrices.

   Proof. For any vector $\mathbf{a}$,

$$\begin{aligned}\mathbf{a}^*\mathbf{R}\mathbf{a} &= \mathbf{a}^*\mathbb{E}[\underline{\mathbf{x}}\underline{\mathbf{x}}^*]\,\mathbf{a} && (2.138)\\ &= \mathbb{E}[(\mathbf{a}^*\underline{\mathbf{x}})(\mathbf{a}^*\underline{\mathbf{x}})^*] && (2.139)\\ &= \mathbb{E}\left[|\mathbf{a}^*\underline{\mathbf{x}}|^2\right] && (2.140)\\ &\geq 0. && (2.141)\end{aligned}$$

   □

3. The eigenvalues of the correlation and covariance matrix are real and non-negative.

Theorem 2.3.4. If $\mathbf{R}$ is the correlation matrix of the n-dimensional random vector $\underline{\mathbf{x}}$. Then,

$$\mathbb{E}\left[\underline{\mathbf{x}}^*\mathbf{R}^{-1}\underline{\mathbf{x}}\right] = n. \tag{2.142}$$

Proof. Since $\underline{\mathbf{x}}^*\mathbf{R}^{-1}\underline{\mathbf{x}}$ is a scalar, we have

$$\begin{aligned}
\mathbb{E}\left[\underline{\mathbf{x}}^*\mathbf{R}^{-1}\underline{\mathbf{x}}\right] &= \mathbb{E}\left[\mathrm{tr}\left(\underline{\mathbf{x}}^*\mathbf{R}^{-1}\underline{\mathbf{x}}\right)\right] && (2.143)\\
&= \mathbb{E}\left[\mathrm{tr}\left(\underline{\mathbf{x}}\underline{\mathbf{x}}^*\mathbf{R}^{-1}\right)\right] && (2.144)\\
&= \mathrm{tr}\,\mathbb{E}\left[\underline{\mathbf{x}}\underline{\mathbf{x}}^*\mathbf{R}^{-1}\right] && (2.145)\\
&= \mathrm{tr}\left(\mathbb{E}\left[\underline{\mathbf{x}}\underline{\mathbf{x}}^*\right]\mathbf{R}^{-1}\right) && (2.146)\\
&= \mathrm{tr}\left(\mathbf{R}\mathbf{R}^{-1}\right) && (2.147)\\
&= \mathrm{tr}\,\mathbf{1} && (2.148)\\
&= n. && (2.149)
\end{aligned}$$

□

2.4 Moments and characteristic functions

Definition 2.4.1 (*Moments of random vectors*)**.** Consider the random variables $\underline{x}_1, \ldots, \underline{x}_N$ and $n_i = 0, 1, \ldots$.

- The $(n_1, n_2, \ldots, n_N)$th *moment* of $\underline{x}_1, \ldots, \underline{x}_N$ is

$$\mathbb{E}\left[\underline{x}_1^{n_1} \cdots \underline{x}_N^{n_N}\right]. \tag{2.150}$$

  Example ($N = 3$): (2,3,1)th moment is $\mathbb{E}\left[\underline{x}_1^2 \cdot \underline{x}_2^3 \cdot \underline{x}_3^1\right]$.

- The $(n_1, \ldots, n_N)$th *central moment* of $\underline{x}_1, \ldots, \underline{x}_N$ is

$$\mathbb{E}\left[(\underline{x}_1 - \mathbb{E}\left[\underline{x}_1\right])^{n_1} \cdots (\underline{x}_N - \mathbb{E}\left[\underline{x}_N\right])^{n_N}\right]. \tag{2.151}$$

- The $(n_1, \ldots, n_N)$th *absolute moment* of $\underline{x}_1, \ldots, \underline{x}_N$ is

$$\mathbb{E}\left[|\underline{x}_1|^{n_1} \cdots |\underline{x}_N|^{n_N}\right]. \tag{2.152}$$

Definition 2.4.2 (*Characteristic function*)**.** The *characteristic function* of a random vector $\underline{\mathbf{x}}$ is defined as

$$\begin{aligned}
\Phi_{\mathbf{x}}(\Omega) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{\jmath(\omega_1 x_1 + \ldots + \omega_N x_N)} f\left(x_1, \ldots, x_N\right) \mathrm{d}x_1 \cdots \mathrm{d}x_N && (2.153)\\
&= \mathbb{E}\left[e^{\jmath\Omega^{\mathsf{T}}\underline{\mathbf{x}}}\right] && (2.154)
\end{aligned}$$

where $\Omega = \begin{bmatrix} \omega_1 & \cdots & \omega_N \end{bmatrix}^\mathsf{T} \in \mathbb{R}^N$ and $\underline{\mathbf{x}} = \begin{bmatrix} \underline{x}_1 & \cdots & \underline{x}_N \end{bmatrix}^\mathsf{T}$. The inverse relation is given by

$$f(x_1, \ldots, x_N) = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_{\mathbf{x}}(\Omega) e^{-\jmath(\omega_1 x_1 + \ldots + \omega_N x_N)} \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_N. \tag{2.155}$$

Note that it looks similar to the Fourier transform of the PDF but it's not exactly the same because of the sign in the exponential.

Definition 2.4.3 (*Moment generating function*)**.** The *moment generating function* of a random vector $mbfrvx$ is defined as

$$\begin{aligned} \mathrm{Var}\left[\Phi\right]_{\mathbf{x}}(\mathbf{s}) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{s_1 x_1 + \ldots + s_N x_N} f(x_1, \ldots, x_N) \, \mathrm{d}x_1 \cdots \mathrm{d}x_N & (2.156) \\ &= \mathbb{E}\left[e^{\mathbf{s}^\mathsf{T} \underline{\mathbf{x}}}\right] & (2.157) \end{aligned}$$

where

$$\mathbf{s} = \begin{bmatrix} s_1 & \cdots & s_N \end{bmatrix}^\mathsf{T}. \tag{2.158}$$

Theorem 2.4.1 (Characteristic functions)**.** The $(n_1, \ldots, n_N)$th *moment* of $\underline{\mathbf{x}}$ is given by

$$\begin{aligned} m_{n_1, \ldots, n_N} &= \mathbb{E}\left[\underline{x}_1^{n_1 \cdots} \underline{x}_N^{n_N}\right] & (2.159) \\ &= \frac{\partial^{n_1 + \cdots + n_N}}{\partial s_1^{n_1} \cdots \partial s_N^{n_N}} \bar{\Phi}_{\mathbf{x}}(\mathbf{s}) \bigg|_{s_1 = \cdots = s_N = 0} & (2.160) \\ &= \frac{1}{\jmath^{n_1 + \cdots + n_N}} \frac{\partial^{n_1 + \cdots + n_N}}{\partial \omega_1^{n_1} \cdots \partial \omega_N^{n_N}} \bar{\Phi}_{\mathbf{x}}(\Omega) \bigg|_{\omega_1 = \cdots = \omega_N = 0}. & (2.161) \end{aligned}$$

Theorem 2.4.2 (PDF of independent random variables)**.** Let $\underline{x}_1, \ldots, \underline{x}_n$ be independent random variables with marginal PDFs $f_i(x), i = 1, \ldots, n$, respectively. The PDF of $\underline{y} = \underline{x}_1 + \cdots + \underline{x}_n$ is

$$f_y(y) = f_1(y) * \cdots * f_n(y), \tag{2.162}$$

where $*$ denotes the convolution.

2.5 Conditional distributions

Similar to the single random variable definition of conditional distributions (Definition 1.4.6), the following are extensions for such definitions to the multivariate case.

Definition 2.5.1 (*CDF of continuous RV conditioned on discrete RV*)**.** Let $\underline{x}$ and $\underline{y}$ be two random vari-

ables with $\underline{x}$ *discrete*. The *conditional CDF of* $\underline{y}$ *given* $\underline{x} = x$ is defined as

$$F\left(y|\ \underline{x} = x\right) = P\left(\underline{y} \leq y \middle|\ \underline{x} = x\right). \tag{2.163}$$

Remark 2.5.1. Some remarks about CDFs.

- Conditions: $p\left(x\right) = P\left(\underline{x} = x\right) \neq 0$.
- Notation: $F\left(y|\ \underline{x} = x\right) \rightarrow F_{y|x}(y|x)$, or $F\left(y|x\right)$.

Definition 2.5.2 (*PMF of discrete RV conditioned a discrete RV*)**.** Let $\underline{x}$ and $\underline{y}$ be two joint discrete random variables. The *conditional PMF of* $\underline{y}$ *given* $\underline{x} = x$ is defined as

$$p\left(y|\ \underline{x} = x\right) = P\left(\underline{y} = y \middle|\ \underline{x} = x\right) \tag{2.164}$$

$$= \frac{p_{xy}(x, y)}{p_x(x)}. \tag{2.165}$$

Notation: $p\left(y|\ \underline{x} = x\right) \rightarrow p_{y|x}(y|x)$, or $p\left(y|x\right)$.

Definition 2.5.3 (*PDF of continuous RV conditioned on continuous RV*)**.** Let $\underline{x}$ and $\underline{y}$ be two jointly continuous random variables. The *conditional PDF of* $\underline{y}$ *given* $\underline{x} = x$ is defined as

$$f\left(y|\ \underline{x} = x\right) = \frac{f_{xy}(x, y)}{f_x(x)}. \tag{2.166}$$

Remark 2.5.2. Some remarks about PDFs.

- Note: if $\underline{x}$ is continuous, then $P\left(\underline{x} = x\right) = 0$. Therefore, it is not possible to use a PDF definition similar to that of a PMF.
- Notation: $f\left(y|\ \underline{x} = x\right) \rightarrow f_{y|x}(y|x)$, or $f\left(y|x\right)$.

Definition 2.5.4 (*CDF of continous RV conditioned on continuous RV*)**.** Let $\underline{x}$, $\underline{y}$ be two jointly continuous random variables. The *conditional CDF of* $\underline{y}$ *given* $\underline{x} = x$ is defined as

$$F\left(y|\ \underline{x} = x\right) = \int_{-\infty}^{y} f_{y|x}(\lambda|x)\mathrm{d}\lambda. \tag{2.167}$$

- Notation: $F\left(y|\underline{x} = x\right) \rightarrow F_{y|x}(y|x)$, or $F\left(y|x\right)$.

Remark 2.5.3. Some remarks about CDFs.

1. Note that the conditional CDF is derived from the conditional PDF, rather than the other way around (deriving PDF from CDF as was done in the unconditional case).

2. For a given x, $f(y|x)$ is a valid PDF. That is,
   2.1. $f(y|x) \geq 0$
   2.2. $\int_{-\infty}^{\infty} f(y|x)\,\mathrm{d}y = 1$.
3. $f(x|y) = \frac{f(x,y)}{f(y)} = \frac{f(y|x)f(x)}{f(y)}$.
4. If $\underline{x}$ and $\underline{y}$ are independent, then $f(x,y) = f(x)\,f(y)$. Therefore,

$$f(y|x) = f(y), \tag{2.168}$$
$$f(x|y) = f(x). \tag{2.169}$$

Definition 2.5.5 (*Multivaraite conditional PDF*)**.** The *multivariate conditional PDF* of $\underline{x}_n, \ldots, \underline{x}_{k+1}$ given $\underline{k} = x_k, \ldots, \underline{x}_1 = x_1$ is

$$f(x_N, \ldots, x_{k+1}|\, x_k, \ldots, x_1) = \frac{f(x_n, \ldots, x_{k+1}, x_k, \ldots, x_1)}{f(x_k, \ldots, x_1)} \tag{2.170}$$

Definition 2.5.6 (*Multivariate conditional CDF*)**.** The *multivariate conditional CDF* of $\underline{x}_n, \ldots, \underline{x}_{k+1}$ given $\underline{k} = x_k, \ldots, \underline{x}_1 = x_1$ is

$$F(x_N, \ldots, x_{k+1}|\, x_k, \ldots, x_1) = \tag{2.171}$$
$$\int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_{k+1}} f_{x_n,\ldots,x_{k+1}|x_k,\ldots,x_1}(\lambda_n, \ldots, \lambda_{k+1}|\, x_k, \ldots, x_1)\mathrm{d}\lambda_n \cdots \mathrm{d}\lambda_{k+1}. \tag{2.172}$$

2.5.1 Properties of conditional probability

1. **Total probability**:

$$f(y) = \int_{-\infty}^{\infty} f(y|x)\,f(x)\,\mathrm{d}x. \tag{2.173}$$

2. **Bayes Theorem**:

$$f(x|y) = \frac{f(y|x)\,f(x)}{\int_{-\infty}^{\infty} f(y|x)\,f(x)\,\mathrm{d}x}. \tag{2.174}$$

3. **Chain rule**:

$$f(x_1, \ldots, x_n) = f(x_n|x_{n-1}, \ldots, x_1) \cdots f(x_3|x_2, x_1)\,f(x_2|x_1)\,f(x_1). \tag{2.175}$$

4. **Marginalization**: Removing any number of variables on the left of the conditional line can be done

by integrating the PDF with respect to them. For example,

$$f\left(\mathbf{x}_1 \middle|\, \mathbf{x}_3\right) = \int_{\infty}^{\infty} f\left(\mathbf{x}_1, \mathbf{x}_2 \middle|\, \mathbf{x}_3\right) \mathrm{d}\mathbf{x}_2. \tag{2.176}$$

5. **Chapman-Kolmogoroff equation**: Removing any number of variables to the right of the conditional line can be done by multiplying ty their conditional density given the remaining variables on the right, and then integrating the product. That is,

$$f\left(\mathbf{x}_1 \middle|\, \mathbf{x}_3\right) = \int_{-\infty}^{\infty} f\left(\mathbf{x}_1 \middle|\, \mathbf{x}_2, \mathbf{x}_3\right) f\left(\mathbf{x}_2 \middle|\, \mathbf{x}_3\right) \mathrm{d}\mathbf{x}_2. \tag{2.177}$$

2.5.2 Conditional covariance

These equations are obtained from [5, Eq. (2.53b)].

Given a the random variable

$$\underline{\mathbf{x}} = \begin{bmatrix} \underline{\mathbf{x}}_1 \\ \underline{\mathbf{x}}_2 \end{bmatrix} \sim \mathcal{N}\left(\bar{\mathbf{x}}, \boldsymbol{\Sigma}\right), \tag{2.178}$$

where

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{\mathbf{x}}_1 \\ \bar{\mathbf{x}}_2 \end{bmatrix}, \tag{2.179}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{12}^{\mathsf{T}} & \boldsymbol{\Sigma}_{22} \end{bmatrix}. \tag{2.180}$$

Then, what is the distribution of $\underline{\mathbf{x}}_1$ given $\underline{\mathbf{x}}_2 = \hat{\mathbf{x}}_2$?

The Schur's compliment can be used to compute this value. Specifically,

$$\mathbb{E}\left[\underline{\mathbf{x}}_1 | \underline{\mathbf{x}}_2 = \hat{\mathbf{x}}_2\right] = \bar{\mathbf{x}}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\left(\hat{\mathbf{x}}_2 - \bar{\mathbf{x}}_2\right), \tag{2.181}$$

$$\mathrm{Cov}\left[\underline{\mathbf{x}}_1 | \underline{\mathbf{x}}_2 = \hat{\mathbf{x}}_2\right] = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{12}^{\mathsf{T}}. \tag{2.182}$$

2.6 Conditional expectation and variance

Definition 2.6.1 (*Conditional mean*)**.** Given n random variables $\underline{x}_1, \ldots, \underline{x}_n$, the *conditional mean* of the random variable $h(\underline{x}_1, \ldots, \underline{x}_k)$ conditioned on $\underline{x}_{k+1}, \ldots, \underline{x}_n$ is

$$\mathbb{E}\left[h(\underline{x}_1, \ldots, \underline{x}_k) \middle|\, x_{k+1}, \ldots, x_n\right] = \tag{2.183}$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(x_1, \ldots, x_k) f\left(x_1, \ldots, x_k \middle|\, x_{k+1}, \ldots, x_n\right) \mathrm{d}x_1 \cdots \mathrm{d}x_k. \tag{2.184}$$

Remark 2.6.1. Some remarks about conditional means.

- Special case:

$$\mathbb{E}\left[\underline{x}_1 \middle| x_2, \ldots, x_n\right] = \int_{-\infty}^{\infty} x_1 f\left(x_1 \middle| x_2, \ldots, x_n\right) \mathrm{d}x_1. \tag{2.185}$$

- If $\underline{x}_1, \ldots, \underline{x}_n$ are independent of $\underline{x}_{k+1}, \ldots, \underline{x}_n$, then

$$\mathbb{E}\left[h(\underline{x}_1, \ldots, \underline{x}_k) \middle| x_{k+1}, \ldots, x_n\right] = \mathbb{E}\left[h(\underline{x}_1, \ldots, \underline{x}_k)\right]. \tag{2.186}$$

Theorem 2.6.1 (Nested expectation)**.** Let $\underline{x}$, $\underline{y}$ be two jointly distributed RVs. Further, let

$$g(y) = \mathbb{E}\left[\underline{x} \middle| \underline{y} = y\right] \tag{2.187}$$

$$= \int_{-\infty}^{\infty} x f\left(x|y\right) \mathrm{d}x. \tag{2.188}$$

Then

$$\mathbb{E}\left[g(\underline{y})\right] = \mathbb{E}\left[\underline{x}\right]. \tag{2.189}$$

In other words,

$$\mathbb{E}_y\left[\mathbb{E}_{x|y}\left[\underline{x} \middle| \underline{y}\right]\right] \to \mathbb{E}\left[\mathbb{E}\left[\underline{x} \middle| \underline{y}\right]\right] \tag{2.190}$$

$$= \mathbb{E}\left[\underline{x}\right]. \tag{2.191}$$

Proof.

$$\mathbb{E}\left[g(\underline{y})\right] = \int_{-\infty}^{\infty} g(y) f\left(y\right) \mathrm{d}y \tag{2.192}$$

$$= \int_{-\infty}^{\infty} \mathbb{E}\left[\underline{x} \middle| y\right] f\left(y\right) \mathrm{d}y \tag{2.193}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f\left(x|y\right) f\left(y\right) \mathrm{d}x \mathrm{d}y \tag{2.194}$$

$$= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f\left(x, y\right) \mathrm{d}y \mathrm{d}x \tag{2.195}$$

$$= \int_{-\infty}^{\infty} x f\left(x\right) \mathrm{d}x \tag{2.196}$$

$$= \mathbb{E}\left[\underline{x}\right]. \tag{2.197}$$

□

Theorem 2.6.2 (Multivariate nested expectation)**.** Let $\underline{x}_1, \ldots, \underline{x}_n$ be n random variables. Also, let

$$g(x_2, \ldots, x_n) = \mathbb{E}\left[\underline{x}_1 \middle| x_2, \ldots, x_n\right]. \tag{2.198}$$

Then,

$$\mathbb{E}\left[g(\underline{x}_2, \ldots, \underline{x}_n)\right] = \mathbb{E}\left[\underline{x}_1\right]. \tag{2.199}$$

Notation:

$$\mathbb{E}_{x_2,\ldots,x_n}\left[\mathbb{E}_{x_1|x_2,\ldots,x_n}\left[\underline{x}_1 \middle| \underline{x}_2, \ldots, \underline{x}_n\right]\right] \to \mathbb{E}\left[\mathbb{E}\left[\underline{x}_1 \middle| \underline{x}_2, \ldots, \underline{x}_n\right]\right]. \tag{2.200}$$

Theorem 2.6.3 (Multivariate nested expectation: general case)**.** Let $\underline{x}_1, \ldots, \underline{x}_n$ be n random variables and $h : \mathbb{R}^k \to \mathbb{R}$. Also, let

$$g(x_{k+1}, \ldots, x_n) = \mathbb{E}\left[h(\underline{x}_1, \ldots, \underline{x}_k) \middle| x_{k+1}, \ldots, x_n\right]. \tag{2.201}$$

Then,

$$\begin{aligned}\mathbb{E}\left[g(\underline{x}_{k+1}, \ldots, \underline{x}_n)\right] &= \mathbb{E}\left[\mathbb{E}\left[h(\underline{x}_1, \ldots, \underline{x}_k)\middle| \underline{x}_{k+1}, \ldots, \underline{x}_n\right]\right] &(2.202)\\ &= \mathbb{E}\left[h(\underline{x}_1, \ldots, \underline{x}_k)\right]. &(2.203)\end{aligned}$$

Definition 2.6.2 (*Conditional variance*)**.** Let $\underline{x}$, $\underline{y}$ be two random variables. Then the *conditional variance of* $\underline{x}$ *given* $\underline{y} = y$ is

$$\mathrm{Var}\left[\underline{x} \middle| \underline{y} = y\right] = \mathbb{E}\left[\left(\underline{x} - \mathbb{E}\left[\underline{x} \middle| \underline{y} = y\right]\right)^2 \middle| \underline{y} = y\right]. \tag{2.204}$$

2.6.1 Properties of conditional variance

1. $\mathrm{Var}\left[\underline{x} \middle| \underline{y} = y\right] = \mathbb{E}\left[\underline{x}^2 \middle| \underline{y} = y\right] - \mathbb{E}\left[\underline{x} \middle| \underline{y} = y\right]^2$.
2. $\mathrm{Var}\left[\underline{x} \middle| \underline{y} = y\right]$ is a function of y.
3. $g(\underline{y}) = \mathrm{Var}\left[\underline{x} \middle| \underline{y}\right]$ is a random variable.

Theorem 2.6.4 (Nested conditional variance)**.** Let $\underline{x}$, $\underline{y}$ be two random variables. Then

$$\mathrm{Var}\left[\underline{x}\right] = \mathbb{E}\left[\mathrm{Var}\left[\underline{x} \middle| \underline{y}\right]\right] + \mathrm{Var}\left[\mathbb{E}\left[\underline{x} \middle| \underline{y}\right]\right]. \tag{2.205}$$

Proof.

$$\begin{aligned}\mathbb{E}\left[\mathrm{Var}\left[\underline{x} \middle| \underline{y}\right]\right] &= \mathbb{E}\left[\mathbb{E}\left[\underline{x}^2 \middle| \underline{y}\right] - \mathbb{E}\left[\underline{x} \middle| \underline{y}\right]^2\right] &(2.206)\\ &= \mathbb{E}\left[\mathbb{E}\left[\underline{x}^2 \middle| \underline{y}\right]\right] - \mathbb{E}\left[\mathbb{E}\left[\underline{x} \middle| \underline{y}\right]^2\right] &(2.207)\end{aligned}$$

$$= \mathbb{E}\left[\underline{x}^2\right] - \mathbb{E}\left[\mathbb{E}\left[\underline{x}\middle|\, \underline{y}\right]^2\right]. \tag{2.208}$$

Further,

$$\operatorname{Var}\left[\mathbb{E}\left[\underline{x}\middle|\, \underline{y}\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\underline{x}\middle|\, \underline{y}\right]^2\right] - \mathbb{E}\left[\mathbb{E}\left[\underline{x}\middle|\, \underline{y}\right]^2\right] \tag{2.209}$$

$$= \mathbb{E}\left[\mathbb{E}\left[\underline{x}\middle|\, \underline{y}\right]^2\right] - \mathbb{E}\left[\underline{x}\right]^2. \tag{2.210}$$

Adding these two results proves the theorem. □

2.7 Passing measurements through a function

Theorem 2.7.1 (Passing measurements through a function)**.** Let $\underline{\mathbf{y}} \in \mathbb{R}^m$ be a (vector) random variable that depends on $\underline{\mathbf{x}} \in \mathbb{R}^n$ through

$$\underline{\mathbf{y}} = \mathbf{g}\left(\underline{\mathbf{x}}\right). \tag{2.211}$$

Furthermore, let $\underline{\mathbf{z}} \in \mathbb{R}^p$ be a transformed random variable given by

$$\underline{\mathbf{z}} = \mathbf{f}\left(\underline{\mathbf{y}}\right). \tag{2.212}$$

Then

$$f\left(\underline{\mathbf{x}}\middle|\, \mathbf{y}, \mathbf{z}\right) = f\left(\underline{\mathbf{x}}\middle|\, \mathbf{y}\right). \tag{2.213}$$

Furthermore, if **g** is invertible and **f** is invertible, then

$$f\left(\underline{\mathbf{x}}\middle|\, \mathbf{y}, \mathbf{z}\right) = f\left(\underline{\mathbf{x}}\middle|\, \mathbf{z}\right). \tag{2.214}$$

Note that if **g** is invertible but **f** is non-invertible, then (2.214) does not hold in general!

Proof. 1. The first part.

$$f\left(\mathbf{x}\middle|\, \mathbf{y}, \mathbf{z}\right) = \frac{f\left(\mathbf{y}, \mathbf{z}\middle|\, \mathbf{x}\right) f\left(\mathbf{x}\right)}{f\left(\mathbf{y}, \mathbf{z}\right)} \tag{2.215}$$

$$= \frac{f\left(\mathbf{z}\middle|\, \mathbf{y}, \mathbf{x}\right) f\left(\mathbf{y}\middle|\, \mathbf{x}\right) f\left(\mathbf{x}\right)}{f\left(\mathbf{z}\middle|\, \mathbf{y}\right) f\left(\mathbf{y}\right)} \tag{2.216}$$

$$= \frac{\cancel{f\left(\mathbf{z}\middle|\, \mathbf{y}\right)} f\left(\mathbf{y}\middle|\, \mathbf{x}\right) f\left(\mathbf{x}\right)}{\cancel{f\left(\mathbf{z}\middle|\, \mathbf{y}\right)} f\left(\mathbf{y}\right)} \tag{2.217}$$

$$= \frac{f\left(\mathbf{y}\middle|\, \mathbf{x}\right) f\left(\mathbf{x}\right)}{f\left(\mathbf{y}\right)} \tag{2.218}$$

$$= f\left(\mathbf{x}\middle|\, \mathbf{y}\right). \tag{2.219}$$

2. The second part. Let's expand $f(\mathbf{x}|\ \mathbf{y}, \mathbf{z})$.

$$f(\mathbf{x}|\ \mathbf{y}, \mathbf{z}) = \frac{f(\mathbf{y}, \mathbf{z}|\ \mathbf{x})\, f(\mathbf{x})}{f(\mathbf{y}, \mathbf{z})} \tag{2.220}$$

$$= \frac{f(\mathbf{y}|\ \mathbf{z}, \mathbf{x})\, f(\mathbf{z}|\ \mathbf{x})\, f(\mathbf{x})}{f(\mathbf{y}|\ \mathbf{z})\, f(\mathbf{z})} \tag{2.221}$$

$$= \frac{f(\mathbf{y}|\ \mathbf{z}, \mathbf{x})}{f(\mathbf{y}|\mathbf{z})} f(\mathbf{x}|\ \mathbf{z}). \tag{2.222}$$

Thus, (2.214) satisfied if and only if

$$f(\mathbf{y}|\ \mathbf{z}, \mathbf{x}) = f(\mathbf{y}|\mathbf{z}). \tag{2.223}$$

That is, if $\mathbf{y}$ is independent of $\mathbf{x}$ *given* $\mathbf{z}$. More rigorously, if there exists $\mathbf{y}_1 \neq \mathbf{y}_2$ such that

$$\begin{matrix} \mathbf{g}(\mathbf{x}_1) = \mathbf{y}_1 \\ \mathbf{g}(\mathbf{x}_2) = \mathbf{y}_2 \end{matrix} \implies \mathbf{x}_1 \neq \mathbf{x}_2, \tag{2.224}$$

and

$$\mathbf{z} = \mathbf{f}(\mathbf{y}_1) = \mathbf{f}(\mathbf{y}_2). \tag{2.225}$$

Then knowing $\mathbf{z}$ without knowing $\mathbf{x}$ will result in a non-singleton sample space S for the random variable $\underline{\mathbf{y}}$. That is, a set with more than one element which implies that $\underline{\mathbf{y}}$ cannot be determined uniquely. In such case, the sample space is the set of $\mathbf{y}$ that satisfy (2.225).

On the other hand, if in addition to knowing $\mathbf{z}$, $\mathbf{x}$ is also known, then the sample space of $\mathbf{y}$ is a singleton. That is, $\underline{\mathbf{y}}$ can be determined uniquely.

The condition (2.225) occurs only if the function $\mathbf{f}$ is non-invertible.

□

Chapter 3

Distributions

3.1 Bernoulli distribution

From [1].

Definition 3.1.1 (*Bernoulli drandom variable*)**.** A random variable $\underline{x}$ is called Bernoulli with parameter p if there exists an event A with probability $p = P(A)$ such that

$$x(s) = \begin{cases} 1, & s \in A \\ 0, & s \notin A, \end{cases} \tag{3.1}$$

where $s \in S$.

Remark 3.1.1. Some remarks on Bernoulli random variables.

- $\underline{x}$ is a *discrete* random variable with range $\mathcal{R}_x = \{0, 1\}$.
- PMF of $\underline{x}$

$$p(0) = P(\underline{x} = 0) = P\left(A^{\mathsf{C}}\right) = 1 - p = q \tag{3.2}$$

$$p(1) = P(\underline{x} = 1) = P(A) = p \tag{3.3}$$

$$p(x) = 0 \qquad \text{if } x \notin \{0, 1\}. \tag{3.4}$$

3.2 Geometric random variables

From [1].

Definition 3.2.1 (*Geometric random variable*)**.** A geometric random variable is a discrete random variable with a countable infinite set of possible values. The set of possible values for the geometric random

variable $\underline{x}$ is

$$\mathcal{R}_x = \{1, 2, \ldots\} = \mathbb{N}. \tag{3.5}$$

Theorem 3.2.1 (PMF of geometric random variables)**.** Let $\underline{x}$ be a geometric random variable with parameter p. The PMF of $\underline{x}$ is then

$$p(x) = \begin{cases} (1-p)^{x-1} p, & x = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases} \tag{3.6}$$

3.3 Binomial random variable

Definition 3.3.1 (*Binomial random variable*)**.** Consider a sequence of n identical and independent Bernoulli trials with probability of success p. The random variable $\underline{x}$ defined by

$$\underline{x} = \text{number of successes in the } n \text{ trials} \tag{3.7}$$

is called binomial with parameters n and p, or simply $B(n, p)$.

Remark 3.3.1. Some remarks on Binomial random variables.

- Set of possible values of $\underline{x}$:

$$\mathcal{R}_x = \{0, 1, \ldots, n\}. \tag{3.8}$$

- Notation:

$$\underline{x} \sim B(n, p). \tag{3.9}$$

- Basic examples
  - Number of heads in a sequence of 10 independent flips of a coin.
  - Number of defective IC chips in a production sample of size n.

Theorem 3.3.1 (PMF of a Binomial random variable)**.** Let $\underline{x} \sim B(n, p)$. The PMF of $\underline{x}$ is

$$p(x) = \begin{cases} \binom{n}{k} p^x (1-p)^{n-x}, & x = 0, 1, \ldots, n \\ 0, & \text{otherwise.} \end{cases} \tag{3.10}$$

3.4 Poisson distribution

A discrete random variable $\underline{x}$ is called Poisson with parameter $\lambda > 0$ if its PMF has the form

$$p(x) = \begin{cases} \frac{\lambda^x}{x!} e^{-\lambda}, & x \in \mathbb{N}_0 \\ 0, & \text{otherwise,} \end{cases} \tag{3.11}$$

where $\mathbb{N}_0 = \{0, 1, \ldots\}$.

3.5 Uniform random variables

Definition 3.5.1 (*Uniform random variable*)**.** A continuous random variable $\underline{x}$ is called uniform over the interval (a, b), written as $\underline{x} \sim U(a, b)$, where $a < b$ are real numbers, if its PDF is

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b, \\ 0, & \text{otherwise.} \end{cases} \tag{3.12}$$

Theorem 3.5.1 (Unoform random variable CDF)**.** The CDF is given by

$$F(x) = \begin{cases} 0, & x \leq a, \\ \frac{x-a}{b-a}, & a \leq x \leq b, \\ 1 & x \geq b. \end{cases} \tag{3.13}$$

3.6 Exponential random variable

Definition 3.6.1 (*Exponential random variable*)**.** A continuous random variable $\underline{x}$ is called exponential with parameter $\lambda > 0$ if its PDF is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases} \tag{3.14}$$

Theorem 3.6.1 (Exponential random variable CDF)**.** The CDF is given by

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x > 0, \\ 0, x \leq 0. \end{cases} \tag{3.15}$$

3.7 Gaussian random variables

Definition 3.7.1 (*Gaussian (normal) distribution*)**.** A random vector $\underline{\mathbf{x}}$ is called a *Gaussian (or normal) random vector* with mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{C}$ if it follows a *Gaussian (or normal) distribution*

given by

$$f(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det \mathbf{C}}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\mathsf{T}\mathbf{C}^{-1}(\mathbf{x}-\boldsymbol{\mu})} \tag{3.16}$$

$$= \mathcal{N}(\boldsymbol{\mu}, \mathbf{C}). \tag{3.17}$$

The n elements of $\mathbf{x}$, $\underline{x}_1, \ldots, \underline{x}_n$ are called *jointly normal random variables*.

Definition 3.7.2 (*Standard Normal Distribution*)**.** A *standard normal distribution* is a normal distribution with a special case of $\boldsymbol{\mu} = \mathbf{0}$ and $\mathbf{C} = \mathbf{1}$. That is,

$$f(\mathbf{x}) = \mathcal{N}(\mathbf{0}, \mathbf{1}). \tag{3.18}$$

3.7.1 Properties of Gaussian random variables

Theorem 3.7.1 (Linear combination of normal RVs)**.** A random vector $\underline{\mathbf{x}}$ is called a Gaussian (or normal) random vector if and only if the random variable $\mathbf{a}^\mathsf{T}\underline{\mathbf{x}}$ is a Gaussian random variable for any vector $\mathbf{a}$.

Remark 3.7.1 (Properties of Gaussian random variables)**.** Here are some properties of Gaussian random variables.

1. A Gaussian random vector consists of marginally Gaussian random variables.
2. However, marginally Gaussian random variables are *not* necessarily jointly Gaussian.[a]

[a] That is, if you take 3 Gaussian random variables, then they are not necessarily jointly Gaussian. Unless they are independent.

Definition 3.7.3 (*Complex Gaussian radnom vector*)**.** A complex random vector

$$\underline{\mathbf{z}} = \begin{bmatrix} \underline{x}_1 + \jmath \underline{y}_1 \\ \vdots \\ \underline{x}_n + \jmath \underline{y}_n \end{bmatrix} \tag{3.19}$$

is a *complex Gaussian random vector* if and only if the real random variables $\underline{x}_1, \underline{y}_1, \ldots, \underline{x}_n, \underline{y}_n$ are jointly normal.

Theorem 3.7.2 (Characteristic function of normal random variables)**.** If the random variables $\underline{x}_1, \ldots, \underline{x}_n$ are jointly Normal with mean 0 and covariance matrix $\mathbf{C}$. Then, their characteristic function is

$$\Phi_{\mathbf{x}}(\Omega) = e^{-\frac{1}{2}\Omega^\mathsf{T}\mathbf{C}\Omega}. \tag{3.20}$$

3.8 Chi-squared distribution

Definition 3.8.1 (*Chi-squared distribution*)**.** If $\underline{x}_1, \ldots, \underline{x}_n$ are ***independent*** *standard normal variables* (check Definition 3.7.2), then the sum of their squares

$$\underline{y} = \sum_{k=1}^{n} \underline{x}_k^2 \in \mathbb{R} \tag{3.21}$$

is distributed according to the Chi-square distribution with n degrees of freedom, denoted by χ_n^2. That is,

$$\underline{y} \sim \chi_n^2. \tag{3.22}$$

Remark 3.8.1. If we have a random vector $\underline{\mathbf{x}} \in \mathbb{R}^n$ with mean $\boldsymbol{\mu}_x = \mathbf{0}$ and a unit covariance $\mathbf{C} = \mathbf{1}$, then the sum of squares

$$\underline{y} = \underline{\mathbf{x}}^{\mathsf{T}} \underline{\mathbf{x}} \tag{3.23}$$

$$\tag{3.24}$$

follows a Chi-squared distribution with n degrees of freedom. That is

$$\underline{y} \sim \chi_n^2. \tag{3.25}$$

Remark 3.8.2. MATLAB has a built-in inverse CDF function called `chi2inv(p,nu)` that evaluates the inverse CDF of a Chi-squared distribution at the probability value $p \in [0, 1]$ and the degrees of freedom ν (which we denote as n in Definition 3.8.1).

3.8.1 Relation to the Mahalanobis distance

In estimation problems, the Mahalanobis distance is often encountered. Especially when testing for consistency of the estimates.

Definition 3.8.2 (*Mahalanobis distance*)**.** The *Mahalanobis distance* $D_M \in \mathbb{R}$ is defined as

$$D_M \triangleq \sqrt{\mathbf{r}^{\mathsf{T}} \boldsymbol{\Sigma}^{-1} \mathbf{r}}, \tag{3.26}$$

where $\mathbf{r} \in \mathbb{R}^n$ is a sample of

$$\underline{\mathbf{r}} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}). \tag{3.27}$$

It is a *Euclidean distance* normalized by *uncertainty*.

Example 3.8.1. Consider a random variable $\underline{\mathbf{x}} \in \mathbb{R}^n$ distributed according to $\underline{\mathbf{x}} \sim \mathcal{N}(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x)$. Say a realization $\mathbf{x} \in \mathbb{R}^n$ is sampled from the distribution. Then the Mahalanobis distance

$$D_M = \sqrt{(\mathbf{x} - \boldsymbol{\mu}_x)^{\mathsf{T}} \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu})} \tag{3.28}$$

is a measure of how far off the realization $\mathbf{x}$ is from the mean $\boldsymbol{\mu}_x$. △

Theorem 3.8.1. The squared Mahalanobis distance follows a Chi-squared distribution. That is,

$$\underline{D_M}^2 \sim \chi^2_n. \tag{3.29}$$

Proof. Given a random variable $\underline{\mathbf{r}} \in \mathbb{R}^n$ that follows the distribution $\underline{\mathbf{r}} \sim (\mathbf{0}, \boldsymbol{\Sigma})$, then the Mahalanobis distance squared is a random variable. Specifically,

$$\underline{D}_M^2 = \underline{\mathbf{r}}^\mathsf{T} \boldsymbol{\Sigma}^{-1} \underline{\mathbf{r}}. \tag{3.30}$$

Since the covariance matrix $\boldsymbol{\Sigma}$ is symmetric positive definite, then its eigenvectors are orthonormal and the eigenvalues are positive. Therefore, (3.30) can be written as

$$\begin{aligned}
\underline{D}_M^2 &= \underline{\mathbf{r}}^\mathsf{T} \boldsymbol{\Sigma}^{-1} \underline{\mathbf{r}} && (3.31)\\
&= \underline{\mathbf{r}}^\mathsf{T} \left(\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^\mathsf{T} \right)^{-1} \underline{\mathbf{r}} && (3.32)\\
&= \underline{\mathbf{r}}^\mathsf{T} \mathbf{U} \boldsymbol{\Lambda}^{-1} \mathbf{U}^\mathsf{T} \underline{\mathbf{r}} && (3.33)\\
&= \underline{\mathbf{r}}^\mathsf{T} \mathbf{U} \boldsymbol{\Lambda}^{-\frac{1}{2}} \boldsymbol{\Lambda}^{-\frac{1}{2}} \mathbf{U}^\mathsf{T} \underline{\mathbf{r}} && (3.34)\\
&= \left(\boldsymbol{\Lambda}^{-\frac{1}{2}} \mathbf{U}^\mathsf{T} \underline{\mathbf{r}} \right)^\mathsf{T} \left(\boldsymbol{\Lambda}^{-\frac{1}{2}} \mathbf{U}^\mathsf{T} \underline{\mathbf{r}} \right) && (3.35)\\
&= \underline{\mathbf{y}}^\mathsf{T} \underline{\mathbf{y}}, && (3.36)
\end{aligned}$$

where

$$\underline{\mathbf{y}} = \boldsymbol{\Lambda}^{-\frac{1}{2}} \mathbf{U}^\mathsf{T} \underline{\mathbf{r}} \in \mathbb{R}^n. \tag{3.37}$$

The next step is to check whether $\underline{\mathbf{y}}$ is a standard normal variable. First, let's check the mean.

$$\begin{aligned}
\boldsymbol{\mu}_y &= \mathbb{E}\left[\underline{\mathbf{y}}\right] && (3.38)\\
&= \boldsymbol{\Lambda}^{-\frac{1}{2}} \mathbf{U}^\mathsf{T} \underbrace{\mathbb{E}\left[\underline{\mathbf{r}}\right]}_{\mathbf{0}} && (3.39)\\
&= \mathbf{0}. && (3.40)
\end{aligned}$$

The second condition to check is whether the covariance of $\underline{\mathbf{y}}$ is the identity matrix or not.

$$\begin{aligned}
\mathrm{Cov}\left[\underline{\mathbf{y}}\right] &= \mathbb{E}\left[\underline{\mathbf{y}}\underline{\mathbf{y}}^\mathsf{T}\right] && (3.41)\\
&= \boldsymbol{\Lambda}^{-\frac{1}{2}} \mathbf{U}^\mathsf{T} \underbrace{\mathbb{E}\left[\underline{\mathbf{r}}\underline{\mathbf{r}}^\mathsf{T}\right]}_{\boldsymbol{\Sigma}} \mathbf{U} \boldsymbol{\Lambda}^{-\frac{1}{2}} && (3.42)\\
&= \boldsymbol{\Lambda}^{-\frac{1}{2}} \mathbf{U}^\mathsf{T} \underbrace{\boldsymbol{\Sigma}}_{\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\mathsf{T}} \mathbf{U} \boldsymbol{\Lambda}^{-\frac{1}{2}} && (3.43)
\end{aligned}$$

$$= \boldsymbol{\Lambda}^{-\frac{1}{2}} \cancel{\mathbf{U}^\top \mathbf{U}}^{\mathbf{1}} \boldsymbol{\Lambda} \cancel{\mathbf{U}^\top \mathbf{U}}^{\mathbf{1}} \boldsymbol{\Lambda}^{-\frac{1}{2}} \tag{3.44}$$

$$= \boldsymbol{\Lambda}^{-\frac{1}{2}} \boldsymbol{\Lambda} \boldsymbol{\Lambda}^{-\frac{1}{2}} \tag{3.45}$$

$$= \mathbf{1}. \tag{3.46}$$

Thus, $\underline{\mathbf{y}}$ is a standard normal random variable. Therefore, according to Definition 3.8.1, $\underline{D}_M^2$ follows a Chi-squared distribution with n degrees of freedom. □

Chapter 4

Graphs in probability

4.1 Graph definitions and terminology

Definition 4.1.1 (*Directed graphs*)**.** A directed graph consists of a set of vertices and a set of directed edges that connect pairs of vertices.

In probability, the vertices are random variables. Figure 4.1 shows an example of a directed graph.

Definition 4.1.2 (*Directed acyclic graph*)**.** A directed acyclic graph (DAG) is a directed graph that contains no cycles. That is, no directed paths from a vertex back to itself.

In case of probability, a DAG is called a *Bayesian network* (BN).

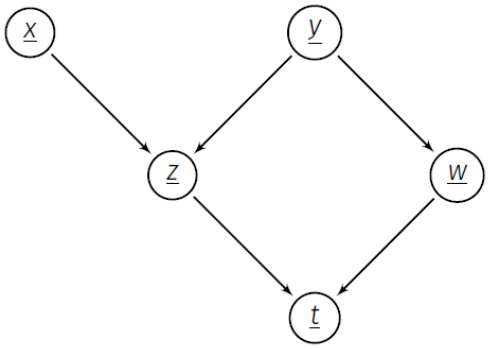


Figure 4.1: Illustration of a simple directed graph in probability.

The terms are illustrated in the figures below.

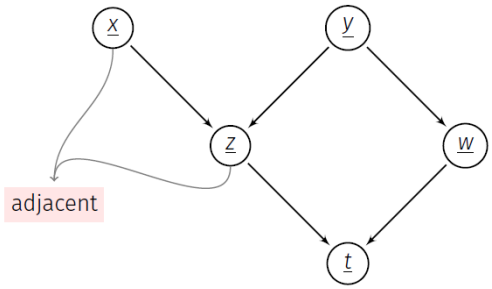


Figure 4.2: Directed graph terminology: adjacent vertices.

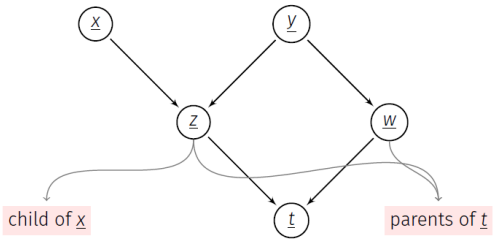


Figure 4.3: Directed graph terminology: child and parent vertices.

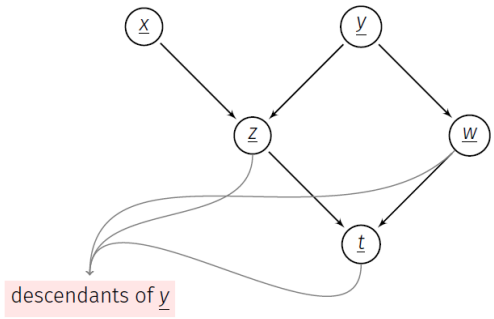


Figure 4.4: Directed graph terminology: descendants vertices.

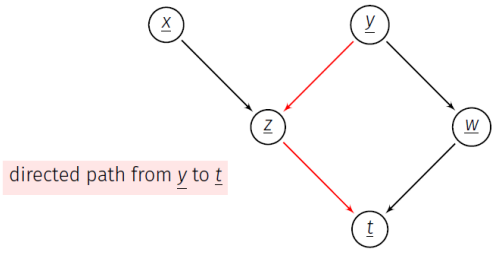


Figure 4.5: Directed graph terminology: directed path.

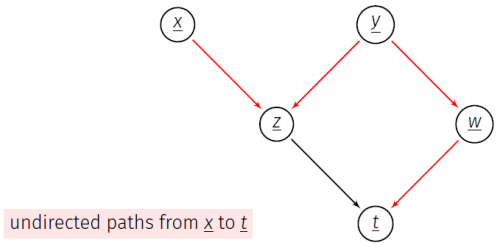


Figure 4.6: Directed graph terminology: undirected path.

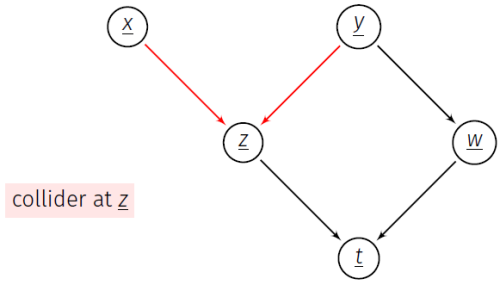


Figure 4.7: Directed graph terminology: collider.

4.2 Graphs in Probability

Definition 4.2.1 (*Bayesian networks*)**.** A Bayesian network is a directed acyclic graph (DAG) (check Def. 4.1.2) that has random variables as vertices.

Theorem 4.2.1 (Representing JPDFs using DAGs)**.** Representing JPDFs using DAGs Consider n random variables $\underline{x}_1, \ldots, \underline{x}_n$ with joint pdf (JPDF) $f(x_1, \ldots, x_n)$. Let G be a DAG with n vertices representing the random variables. Then, G represents the JPDF f if

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i | \text{ parents of } x_i). \tag{4.1}$$

Theorem 4.2.2 (Markov condition in DAG)**.** The DAG G represents the JPDF f if and only if the Markov condition holds, which states the following:

$$\forall \underline{x}_i : \quad \underline{x}_i \perp \bar{\mathcal{X}}_i | \text{parents of } \underline{x}_i, \tag{4.2}$$

where $\bar{\mathcal{X}}_i$ are all other random variables but the parents and descendants of $\underline{x}_i$.

Definition 4.2.2 (*D-donnection and d-separation*)**.** Consider a set of n random variables $\underline{x}_1, \ldots, \underline{x}_n$ whose JPDF is represented by G. Furthermore, let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be mutually exclusive sets with $\mathcal{A}, \mathcal{C} \neq \emptyset$.

The sets $\mathcal{A}$ and $\mathcal{C}$ are called *d-connected* given $\mathcal{B}$ if there exists an *undirected* path between some random

variable in $\mathcal{A}$ and some random variable in $\mathcal{C}$ such that

1. every collider along the path is in $\mathcal{B}$, or has a descendent in $\mathcal{B}$;
2. no non-collider along the path belongs in $\mathcal{B}$.

If $\mathcal{A}$ and $\mathcal{C}$ are not d-connected given $\mathcal{B}$, then they are called *d-separated given* $\mathcal{B}$.

Theorem 4.2.3 (D-connection and d-separation)**.** Consider a set of n random variables $\underline{x}_1, \ldots, \underline{x}_n$ whose JPDF is represented by G. Furthermore, let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be mutually exclusive sets with $\mathcal{A}, \mathcal{C} \neq \emptyset$. Then, the sets $\mathcal{A}$ and $\mathcal{C}$ are *d-separated given* $\mathcal{B}$ if and only if

$$\mathcal{A} \perp \mathcal{C} | \mathcal{B}. \tag{4.3}$$

Part II

Statistical Inference

Chapter 5

Prediction

5.1 Problem statement

Predict (guess) the value of $\underline{\mathbf{y}}$ given an *observation*[1] of another (related) random variable $\underline{\mathbf{x}} = \mathbf{x}$. It is assumed that the JPDF $f(\mathbf{x}, \mathbf{y})$ is known.

A prediction can be "good" or "bad" with respect to some *criterion* or *cost*. The objective "*best prediction*" will refer to the *minimum mean squared error* prediction.

Note that in an estimation problem, the JPDF $f(x, y)$ is not known, but observations on $\underline{x} = x$ and $\underline{y} = y$ are available.

Furthermore, note that in the above problem statement.

There are two approaches to the prediction problem:

1. **Point prediction**: The objective is to identify a function $\mathbf{r}(\cdot)$ so that $\mathbf{r}(\underline{\mathbf{x}})$ provides a "good" (in some defined sense) estimate of $\underline{y}$.
2. **Interval prediction**: Identify two function $\mathbf{r}_1(\cdot)$ and $\mathbf{r}_2(\cdot)$ such that

$$P\left(\mathbf{r}_1(\underline{\mathbf{x}}) < \underline{\mathbf{y}} < \mathbf{r}_2(\underline{\mathbf{x}})\right) = \gamma, \tag{5.1}$$

   where $\gamma \in [0, 1]$ is a *confidence coefficient*.

In this document, we will be concerned with point prediction.

Note that the function $r(\cdot)$ is determined before any observation is made! The goal is to *guess* $r(\cdot)$ to be good over *all* possible observations of $\underline{x}$ and the associated true value of $\underline{y}$.

Terminology

- *Estimate of* $\underline{\mathbf{y}}$: the deterministic value of $\mathbf{r}(\mathbf{x})$ obtained by applying $\mathbf{r}(\cdot)$ on the observed value of $\underline{\mathbf{x}}$, $\mathbf{x}$.

[1] Or multiple observations.

- *Predictor* or *estimator* of $\underline{\mathbf{y}}$: the random random variable $\mathbf{r}(\underline{\mathbf{x}})$ obtained by applying $\mathbf{r}(\cdot)$ on the random variable $\underline{\mathbf{x}}$.
- Notation to be used:

$$\mathbf{r}(\underline{\mathbf{x}}) \to \hat{\underline{\mathbf{y}}}. \tag{5.2}$$

5.2 MMSE Prediction

Definition 5.2.1 (*Mean-Squared Predictor*)**.** The *mean-squared predictor* of $\underline{\mathbf{y}}$ from $\underline{\mathbf{x}}$, is the function $r(\cdot)$ that minimizes the mean-square error (MSE):

$$\mathbb{E}\left[\left\|\underline{\mathbf{y}} - \mathbf{r}(\underline{\mathbf{x}})\right\|^2\right] \mathbb{E}\left[\left(\underline{\mathbf{y}} - \mathbf{r}(\underline{\mathbf{x}})\right)^{\mathsf{T}}\left(\underline{\mathbf{y}} - \mathbf{r}(\underline{\mathbf{x}})\right)\right] \tag{5.3}$$

Theorem 5.2.1 (Constant-value predictor)**.** The mean square predictor of $\mathbf{y}$ *by a constant* $\mathbf{r}(\underline{\mathbf{x}}) = \mathbf{c}$ is given by

$$\mathbf{c} = \mathbb{E}\left[\underline{\mathbf{y}}\right]. \tag{5.4}$$

Proof. Let $\mathbf{e}(\mathbf{c}) := \mathbb{E}\left[\left\|\underline{\mathbf{y}} - \mathbf{r}(\underline{\mathbf{x}})\right\|^2\right]$. Then, the derivative of $\mathbf{e}$ w.r.t. $\mathbf{c}$ (the design variable) is

$$\frac{\mathrm{d}\mathbf{e}(\mathbf{c})}{\mathrm{d}\mathbf{c}} = \frac{\mathrm{d}}{\mathrm{d}\mathbf{c}}\mathbb{E}\left[\left(\underline{\mathbf{y}} - \mathbf{c}\right)^2)\right] \tag{5.5}$$

$$= -2\mathbf{c}^{\mathsf{T}}\mathbb{E}\left[\left(\underline{\mathbf{y}} - \mathbf{c}\right))\right]. \tag{5.6}$$

Setting $\frac{\mathrm{d}\mathbf{e}(\mathbf{c})}{\mathrm{d}\mathbf{c}} = \mathbf{0}$, gives the relation

$$\mathbf{c}^{\mathsf{T}}\mathbb{E}\left[\underline{\mathbf{y}}\right] = \mathbf{c}^{\mathsf{T}}\mathbf{c}. \tag{5.7}$$

A nontrivial solution is

$$\mathbf{c} = \mathbb{E}\left[\underline{\mathbf{y}}\right]. \tag{5.8}$$

□

Theorem 5.2.2 (Mean-squared predictor)**.** The mean-squared predictor of $\underline{\mathbf{y}}$ given $\underline{\mathbf{x}}$ is

$$\mathbf{r}(\underline{\mathbf{x}}) = \mathbb{E}_{\mathbf{y}|\mathbf{x}}\left[\underline{\mathbf{y}}\middle|\,\underline{\mathbf{x}}\right]. \tag{5.9}$$

Proof. MSE of $\mathbf{r}(\mathbf{x})$. □

Remark 5.2.1. Some remarks on MMSE prediction.

1. If $\underline{\mathbf{y}} = \mathbf{h}(\underline{\mathbf{x}})$ then $\mathbb{E}_{\mathbf{y}}\left[\underline{\mathbf{y}} \middle| \underline{\mathbf{x}}\right] = \mathbf{h}\left(\underline{\mathbf{x}}\right)$), and the resulting MSE is $\mathbf{0}$.
2. If $\underline{\mathbf{y}}$ and $\underline{\mathbf{x}}$ are independent then $\mathbb{E}_{\mathbf{y}}\left[\underline{\mathbf{y}} \middle| \underline{\mathbf{x}}\right] = \mathbb{E}\left[\underline{\mathbf{y}}\right] = \text{constant}$.

Theorem 5.2.3. If $\underline{\mathbf{y}}, \underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_N$ are jointly normal with zero mean, the linear and non-linear predictors of $\underline{\mathbf{y}}$ from $\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_N$.

Chapter 6

Parameter Estimation

6.1 Motivation

Let $\underline{x}$ be a random variable whose CDF, $F(x)$, *is unknown*. The goal is estimate $F(x)$ from n observations of $\underline{x}$, $x_1, \ldots, x_n$.

Definition 6.1.1 (*Empirical CDF*)**.** The empirical CDF of $\underline{x}$ from the observations $x_1, \ldots, x_n$ is

$$\hat{F}(x) = \frac{1}{n} \times (\#\text{ of observations } x_i \leq x)\,. \tag{6.1}$$

The empirical CDF $\hat{F}(x)$ is the CDF of a discrete random variable with PMF

$$\hat{p}_n(x) = \begin{cases} \frac{1}{n}, & x = x_i, \quad i = 1, \ldots, n, \\ 0 & \text{Otherwise.} \end{cases} \tag{6.2}$$

The PDF is then given by

$$\hat{f}_n(x) = \sum_{i=1}^{n} \frac{1}{n} \delta(x - x_i). \tag{6.3}$$

The empirical PDF $\hat{f}(x)$ in (6.3) assigns constant weights to every observation. The empirical PDF can be generalized by giving different weights to different observations. Specifically,

$$\hat{f}_n(x) = \sum_{i=1}^{n} \frac{1}{n} k(x - x_i), \tag{6.4}$$

where $k(\cdot)$ is the kernel function such that it is

1. symmetric around 0, and
2. $\int_{-\infty}^{\infty} k(x)\mathrm{d}x = 1$.

A common choice is the Gaussian kernel. That is, a PDF of $\mathcal{N}(0, 1)$.

6.2 Problem statement

Let $\underline{\mathbf{x}}$ be a random variable. Furthermore, let $f(\mathbf{x}; \boldsymbol{\theta})$ be the PDF of $\underline{\mathbf{x}}$ of *known form* (*e.g.*, Gaussian distribution) but depending on *unknown* parameters $\boldsymbol{\theta}$ (*e.g.*, mean and covariance of a Gaussian distribution).

Example 6.2.1. A Gaussian random variable has a PDF parameterized w.r.t. to the mean μ and the variance σ^2. Thus, the parameters could be $\boldsymbol{\theta} = \left[\mu, \sigma^2\right]$.

The goal is to estimate $\boldsymbol{\theta}$ from n observations of $\underline{\mathbf{x}}$: $\mathbf{x}_1, \ldots, \mathbf{x}_n$.

Define the data vector

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}. \tag{6.5}$$

The data vector $\mathbf{X}$ is a realization of the random vector

$$\underline{\mathbf{X}} = \begin{bmatrix} \underline{\mathbf{x}}_1 \\ \vdots \\ \underline{\mathbf{x}}_n \end{bmatrix}, \tag{6.6}$$

where $\underline{\mathbf{x}}_i$, $i = 1, \ldots, n$ are i.i.d.

Approaches

There are two approaches to parameter estimation.

1. *Point estimation*: identify a function $g(\cdot)$ of the sample such that $g(\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_n)$ is a good (in some sense) approximation of $\boldsymbol{\theta}$.
2. *Interval estimation*: identify two functions $g_1(\cdot)$ and $g_2(\cdot)$ of the sample, such that

$$P\left(g_1(\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_n) < \boldsymbol{\theta} < g_2(\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_n)\right) = \gamma, \tag{6.7}$$

   where γ is the confidence interval.

6.3 Definitions and terminology

Definition 6.3.1 (*Estimator*)**.** The estimator of $\boldsymbol{\theta}$ is given in the form

$$\underline{\hat{\boldsymbol{\theta}}} = g(\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_n). \tag{6.8}$$

The estimator models the behaviour of the estimation method over all possible samples.

Definition 6.3.2 (*Estimate of* $\boldsymbol{\theta}$)**.** The *deterministic* quantity $g(\mathbf{x}_1, \ldots, \mathbf{x}_n)$. That is, the the output of $\underline{\hat{\boldsymbol{\theta}}}$ for the realizations $\mathbf{x}_1, \ldots, \mathbf{x}_n$.

Definition 6.3.3 (*Statistic*)**.** Any function of $\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_n$.

Definition 6.3.4 (*Bias*)**.** The estimator $g(\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_n)$ is called

- *unbiased* if

$$\mathbb{E}\left[g(\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_n)\right] = \boldsymbol{\theta}. \tag{6.9}$$

- Otherwise, it's called *biased* with bias

$$\mathbb{E}\left[g(\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_n)\right] - \boldsymbol{\theta}. \tag{6.10}$$

Definition 6.3.5 (*Asymptic unbias*)**.** A biased estimator for which

$$\mathbb{E}\left[g(\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_n)\right] \overset{N\to\infty}{\to} \boldsymbol{\theta} \tag{6.11}$$

is called *asymptotically unbiased*.

Definition 6.3.6 (*Mean-squared best estimator*)**.** For a fixed n, the estimator that minimizes the mean-squared (MS) error

$$\mathbf{e}_n = \mathbb{E}\left[\left|g(\underline{\mathbf{x}}_1, \ldots \underline{\mathbf{x}}_n)^{\mathsf{T}} g(\underline{\mathbf{x}}_1, \ldots \underline{\mathbf{x}}_n)\right|\right] \tag{6.12}$$

is called the *best* estimator (in the mean-squared sense).

Definition 6.3.7 (*Consistency*)**.** The estimator $g(\underline{\mathbf{x}}_1, \ldots \underline{\mathbf{x}}_n)$ is called *consistent* if

$$\lim_{n\to\infty} P\left(|g(\underline{\mathbf{x}}_1, \ldots \underline{\mathbf{x}}_n)| > \boldsymbol{\epsilon}\right) = 0, \forall \boldsymbol{\epsilon} > \mathbf{0}, \tag{6.13}$$

or

$$\lim_{n\to\infty} P\left(|g(\underline{\mathbf{x}}_1, \ldots \underline{\mathbf{x}}_n)| < \boldsymbol{\epsilon}\right) = 1, \forall \boldsymbol{\epsilon} > \mathbf{0}. \tag{6.14}$$

Theorem 6.3.1 (Sufficient condition for consistency)**.** Let $g(\underline{\mathbf{x}}_1, \ldots \underline{\mathbf{x}}_n)$ be an estimator of $\boldsymbol{\theta}$. Then, if

$$\lim_{n\to\infty} \mathbb{E}\left[\left|g(\underline{\mathbf{x}}_1, \ldots \underline{\mathbf{x}}_n)^{\mathsf{T}} g(\underline{\mathbf{x}}_1, \ldots \underline{\mathbf{x}}_n)\right|\right] = \mathbf{0}, \tag{6.15}$$

then $g(\underline{\mathbf{x}}_1, \ldots \underline{\mathbf{x}}_n)$ is a *consistent estimator* of $\boldsymbol{\theta}$.

The converse is not true in general. That is, this is a sufficient but not necessary condition.

Theorem 6.3.2 (Sample mean estimator)**.** Let $\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_n$ be a random sample of size n and let $\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_n$ be *uncorrelated*. The *sample mean estimator*

$$\underline{\hat{\boldsymbol{\mu}}}_n = \frac{1}{n}\sum_{i=1}^{n} \underline{\mathbf{x}}_i \tag{6.16}$$

is

- an *unbiased estimator* of $\boldsymbol{\mu}$;
- a *consistent estimator* of $\boldsymbol{\mu}$.

Theorem 6.3.3 (Variance estimaton)**.** Consider the sample average estimators of the mean and variance of Gaussian random variable $\underline{x}$ from a random sample $\underline{x}_1, \ldots, \underline{x}_n$ of size n:

$$\underline{\hat{\mu}}_n = \frac{1}{n}\sum_{i=1}^{n} \underline{x}_i, \tag{6.17}$$

$$\underline{\hat{\sigma}}_n^2 = \frac{1}{n}\sum_{i=1}^{n} \left(\underline{x}_i - \underline{\hat{\mu}}_n\right)^2. \tag{6.18}$$

The random variables $\underline{\hat{\mu}}_n$ and $\underline{\hat{sigma}}_n^2$ are independent.

6.4 Method of moments

Recall that moments are defined in Def. 1.6.4 and Def. 2.4.1. The m-th moment estimator is given by

$$\hat{m}_m(\mathbf{X}) = \frac{1}{n}\sum_{i}^{n} x_i^m. \tag{6.19}$$

Let $\boldsymbol{\theta} \in \mathbb{R}^k$ be of size k and let

$$\mathbf{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{6.20}$$

be the samples of the random variable $\underline{x}$. Then, the idea of method of moments is to estimate the first k moments, thus getting k equations, and then solving the k equations for $\boldsymbol{\theta}$.

Example 6.4.1. Estimate the mean μ and variance σ^2 of a random variable $\underline{x}$ from a random sample $\underline{\mathbf{X}} = \begin{bmatrix} \underline{x}_1 & \cdots & \underline{x}_n \end{bmatrix}^\mathsf{T}$.

The moments are given by

$$m_1(\mu, \sigma^2) = \mathbb{E}[\underline{x}] \quad = \mu, \tag{6.21}$$
$$m_2(\mu, \sigma^2) = \mathbb{E}\left[\underline{x}^2\right] = \mu^2 + \sigma^2. \tag{6.22}$$

The sample average estimates of the moments are given by

$$\hat{m}_1(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^{n} x_i, \tag{6.23}$$
$$\hat{x}_2(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^{n} x_i^2. \tag{6.24}$$

Thus, the parameter estimates can be computed by solving the system of equations

$$\mu = \hat{m}_1(\mathbf{X}) \tag{6.25}$$
$$\mu^2 + \sigma^2 = \hat{m}_2(\mathbf{X}). \tag{6.26}$$

The solution is given by

$$\hat{\mu} = \hat{m}_1(\mathbf{X}) \tag{6.27}$$
$$= \frac{1}{n}\sum_{i=1}^{n} x_i, \tag{6.28}$$
$$\hat{\sigma}^2 = \hat{m}_2(\mathbf{X}) - \hat{m}_1^2(\mathbf{X}) \tag{6.29}$$
$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2. \tag{6.30}$$

6.5 Maximum likelihood (ML) estimator

Definition 6.5.1 (*Likelihood*)**.** Given the observations $\underline{\mathbf{x}}_1 = \mathbf{x}_1', \ldots, \underline{\mathbf{x}}_n = \mathbf{x}_n'$, the *likelihood* of $\boldsymbol{\theta}$ is the joint PDF of $\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_n$ evaluated at the observed points

$$L(\boldsymbol{\theta}; \mathbf{x}_1', \ldots, \mathbf{x}_N') = f\left(\mathbf{x}_1', \ldots, \mathbf{x}_n'; \boldsymbol{\theta}\right) \tag{6.31}$$
$$= f_{\mathbf{x}}(\mathbf{x}_1'; \boldsymbol{\theta}) \cdots f_{\mathbf{x}}(\mathbf{x}_n'; \boldsymbol{\theta}). \tag{6.32}$$

Remark 6.5.1.

- The likelihood $f\left(\mathbf{x}_1', \ldots, \mathbf{x}_n'; \boldsymbol{\theta}\right)$ is treated as a function of $\boldsymbol{\theta}$.
- It is not a PDF. That is,

$$\int_{-\infty}^{\infty} f\left(\mathbf{x}_1', \ldots, \mathbf{x}_n'; \boldsymbol{\theta}\right) \mathrm{d}\boldsymbol{\theta} \neq 1. \tag{6.33}$$

Definition 6.5.2 (*Maximum-likelihood estimator*)**.** The maximum-likelihood (ML) estimate of $\boldsymbol{\theta}$, $\hat{\boldsymbol{\theta}}_{\mathrm{ML}}$, is the value of $\boldsymbol{\theta}$ that maximizes $f\left(\mathbf{x}'_1, \ldots, \mathbf{x}'_n; \boldsymbol{\theta}\right)$. That is,

$$\hat{\boldsymbol{\theta}}_{\mathrm{ML}} = \arg\max_{\boldsymbol{\theta} \in \mathbb{R}^k} L(\boldsymbol{\theta}; \mathbf{x}'_1, \ldots, \mathbf{x}'_N) \tag{6.34}$$

$$= \arg\max_{\boldsymbol{\theta} \in \mathbb{R}^k} f\left(\mathbf{x}'_1, \ldots, \mathbf{x}'_n; \boldsymbol{\theta}\right). \tag{6.35}$$

Remark 6.5.2. Some remarks regarding the ML estimator.

- Log-likelihood function

$$\ell(\mathbf{x}'_1, \ldots, \mathbf{x}'_n; \boldsymbol{\theta}) = \log L(\boldsymbol{\theta}; \mathbf{x}'_1, \ldots, \mathbf{x}'_N) \tag{6.36}$$

$$= \log f\left(\mathbf{x}'_1, \ldots, \mathbf{x}'_n; \boldsymbol{\theta}\right). \tag{6.37}$$

- The maximizer of the log-likelihood function is the same maximizer of the likelihood function. That is,

$$\arg\max_{\boldsymbol{\theta} \in \mathbb{R}^k} L(\boldsymbol{\theta}; \mathbf{x}'_1, \ldots, \mathbf{x}'_N) = \arg\max_{\boldsymbol{\theta} \in \mathbb{R}^k} \ell(\boldsymbol{\theta}; \mathbf{x}'_1, \ldots, \mathbf{x}'_N). \tag{6.38}$$

- The ML estimate is

$$\hat{\boldsymbol{\theta}}_{\mathrm{ML}} = \arg\max_{\boldsymbol{\theta} \in \mathbb{R}^k} f\left(\mathbf{x}'_1, \ldots, \mathbf{x}'_n; \boldsymbol{\theta}\right) \tag{6.39}$$

$$= \arg\max_{\boldsymbol{\theta} \in \mathbb{R}^k} \log f\left(\mathbf{x}'_1, \ldots, \mathbf{x}'_n; \boldsymbol{\theta}\right) \tag{6.40}$$

$$= \arg\max_{\boldsymbol{\theta} \in \mathbb{R}^k} L(\mathbf{x}'_1, \ldots, \mathbf{x}'_n; \boldsymbol{\theta}). \tag{6.41}$$

Remark 6.5.3. Some remarks on ML estimation.

1. Let $\underline{\hat{\boldsymbol{\theta}}}_{\mathrm{ML}}$ be the ML estimator of $\boldsymbol{\theta}$ and let $g(\boldsymbol{\theta})$ be a function of $\boldsymbol{\theta}$. Then, the ML estimator of $g(\boldsymbol{\theta})$ is $g(\underline{\hat{\boldsymbol{\theta}}}_{\mathrm{ML}})$.
2. Limitations of the ML estimation:
   2.1. The ML estimate may not exist.
   2.2. The ML estimate may not be unique.
3. Maximum likelihood estimate can be finding by minimizing the negative likelihood or negative log-likelihood.

Example 6.5.1. Let $\underline{x} \sim U[0, \theta]$ be a uniformly distributed random variable in the form

$$f(x;\theta) = \begin{cases} \frac{1}{\theta}, & x \in [0,\theta], \\ 0, & \text{otherwise}, \end{cases} \tag{6.42}$$

where θ is some unknown parameter. Find the ML estimator of θ from a random sample of size N.

1. First, get the joint PDF of all the random samples

$$f(x_1,\ldots,x_N;\theta) = f(x_1;\theta)\cdots f(x_N;\theta) \tag{6.43}$$

$$= \begin{cases} \frac{1}{\theta^N}, & x_i \in [0,\theta], i = 1,\ldots,N, \\ 0, & \text{otherwise}, \end{cases} \tag{6.44}$$

   where the fact that the random samples $\underline{x}_i$ for $i = 1,\ldots,N$ are i.i.d.

2. The likelihood of θ is then given as a function of the samples of $\underline{x}_i = x_i'$. Specifically, the likelihood is

$$L(\theta; x_1',\ldots,x_N') = f\left(\underline{x}_1 = x_1',\ldots,\underline{x}_N = x_N';\theta\right) \tag{6.45}$$

$$= \begin{cases} \frac{1}{\theta^N}, & x_i' \in [0,\theta], i = 1,\ldots,N, \\ 0, & \text{otherwise} \end{cases} \tag{6.46}$$

$$= \begin{cases} \frac{1}{\theta^N}, & \max_i(x_i) \le \theta, \\ 0, & \text{otherwise}. \end{cases} \tag{6.47}$$

   Note that it was not necessary to specify the lower bound $\min_i(x_i) \ge 0$ since the samples are drawn from a uniform distribution $U[0,\theta]$ which is bounded from below by 0. Thus, $P(\underline{x} < 0) = 0$. Therefore, it is impossible to have a sample $x_i' < 0$.

3. Maximize the likelihood function. Maximizing (6.47) is a *constrained* optimization problem that entails maximizing $\frac{1}{\theta^N}$.

   Maximizing $\frac{1}{\theta^N}$ is the same as minimizing θ. However, since θ is bounded from below by the constraint in (6.47), then the minimum value of θ is $\max_i(x_i)$. Thus,

$$\hat{\theta}_{\text{ML}} = \max_i(x_i). \tag{6.48}$$

△

6.6 Bayesian parameter estimation

In the previous sections, the approaches used were the *classical* or *frequentist* approaches. The classical approach

1. treats the unknown parameter as deterministic parameter, and

2. assumes no prior information on the unknown parameter was available.

For Bayesian approach on the other hand

1. assumes that unknown parameter is random with known pdf $f(\boldsymbol{\theta})$, called the *prior*;
2. the JPDF of random sample given $\underline{\boldsymbol{\theta}} = \theta$ (likelihood) is given by

$$f(\mathbf{x}_1, \ldots, \mathbf{x}_n | \boldsymbol{\theta}); \tag{6.49}$$

3. The estimation problem becomes a prediction problem; predict $\underline{\boldsymbol{\theta}}$ from a correlated (with $\underline{\boldsymbol{\theta}}$) sample $\underline{\mathbf{X}}$.

The mean-squared (or Bayes) estimate $\hat{\boldsymbol{\theta}}$ of $\underline{\boldsymbol{\theta}}$ given $\underline{\mathbf{X}} = \mathbf{X}$ is

$$\hat{\boldsymbol{\theta}} = \mathbb{E}[\underline{\boldsymbol{\theta}} | \mathbf{X}] \tag{6.50}$$

$$= \int_{-\infty}^{\infty} \boldsymbol{\theta} f(\boldsymbol{\theta} | \mathbf{X}) \, \mathrm{d}\boldsymbol{\theta}, \tag{6.51}$$

where

$$f(\boldsymbol{\theta} | \mathbf{X}) = \frac{f(\mathbf{X} | \boldsymbol{\theta}) f(\boldsymbol{\theta})}{f(\mathbf{X})} \tag{6.52}$$

$$= \frac{f(\mathbf{X} | \boldsymbol{\theta}) f(\boldsymbol{\theta})}{\int_{-\infty}^{\infty} f(\mathbf{X} | \boldsymbol{\theta}) f(\boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{\theta}}. \tag{6.53}$$

Terminology

- The mean-squared estimator is also referred to as the Bayes estimator.
- $f(\boldsymbol{\theta})$: *prior* pdf of $\underline{\boldsymbol{\theta}}$ (before observing the sample).
- $f(\boldsymbol{\theta} | \mathbf{X})$: *posterior* pdf of $\boldsymbol{\theta}$ (after observing the sample).
- In [5], the term $f(\mathbf{X} | \boldsymbol{\theta})$ is sometimes referred to as the likelihood of $\mathbf{X}$ given $\boldsymbol{\theta}$. This is not to be confused with the likelihood from the ML estimator. In estimation, the term $f(\mathbf{X} | \boldsymbol{\theta})$ can also be referred to as an *observation model*.

Remark 6.6.1. Some remarks about Bayesian estimation.

- $\mathbb{E}[\underline{\boldsymbol{\theta}} | \mathbf{X}]$ from (6.50) is difficult to solve for two reasons. First, the denominator of the posterior pdf involves evaluating an integral over the entire sample space of $\boldsymbol{\theta}$. Second, evaluating the expectation $\mathbb{E}[\cdot]$ requires *another* integral to be evaluated over the entire sample space.
- Maximum a-posteriori (MAP) estimate maximizes the *posterior* pdf $f(\boldsymbol{\theta} | \mathbf{X})$ with respect to $\boldsymbol{\theta}$.

Since the denominator of the prior is constant, then it can be omitted. That is,

$$\hat{\boldsymbol{\theta}}_{\text{MAP}} = \arg\max_{\boldsymbol{\theta}} f\left(\boldsymbol{\theta}|\, \mathbf{X}\right) \tag{6.54}$$

$$= \arg\max_{\boldsymbol{\theta}} \frac{f\left(\mathbf{X}|\, \boldsymbol{\theta}\right) f\left(\boldsymbol{\theta}\right)}{f\left(\mathbf{X}\right)} \tag{6.55}$$

$$= \arg\max_{\boldsymbol{\theta}} f\left(\mathbf{X}|\, \boldsymbol{\theta}\right) f\left(\boldsymbol{\theta}\right). \tag{6.56}$$

6.7 Maximum a-posteriori (MAP) estimator

This estimator is an *approximation* to the Bayes estimator discussed in the previous section. It is easier (less computationally intensive) to compute.

Definition 6.7.1 (*Maximum a-posteriori estimator*)**.** Maximum a-posteriori (MAP) estimate maximizes the *posterior* pdf $f\left(\boldsymbol{\theta}|\, \mathbf{X}\right)$ with respect to $\boldsymbol{\theta}$. Since the denominator of the prior is constant, then it can be omitted. That is,

$$\hat{\boldsymbol{\theta}}_{\text{MAP}} = \arg\max_{\boldsymbol{\theta}} f\left(\boldsymbol{\theta}|\, \mathbf{X}\right) \tag{6.57}$$

$$= \arg\max_{\boldsymbol{\theta}} \frac{f\left(\mathbf{X}|\, \boldsymbol{\theta}\right) f\left(\boldsymbol{\theta}\right)}{f\left(\mathbf{X}\right)} \tag{6.58}$$

$$= \arg\max_{\boldsymbol{\theta}} f\left(\mathbf{X}|\, \boldsymbol{\theta}\right) f\left(\boldsymbol{\theta}\right). \tag{6.59}$$

6.7.1 MAP estimator of a Markov normally distributed random variable

In this section, the MAP estimator for normally distributed variables that follow a Markov chain will be derived. For each problem, a process model is provided in the form

$$\underline{\mathbf{x}}_{k+1} = \underline{\mathbf{f}}(\mathbf{x}_k, \underline{\mathbf{u}}_k, \underline{\mathbf{w}}_k), \tag{6.60}$$

where $\underline{\mathbf{w}}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k)$ is the process noise and the covariance matrix $\mathbf{Q}_K$ is known. A prior is distributed according to $\hat{\mathbf{x}}_0 \sim \mathcal{N}\left(\hat{\mathbf{x}}_1, \check{\mathbf{P}}_0\right)$.

Furthermore, a measurement model is given in the form

$$\underline{\mathbf{y}}_k = \mathbf{g}(\underline{\mathbf{x}}_k, \underline{\mathbf{v}}_k), \tag{6.61}$$

where $\underline{\mathbf{v}}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)$ is the Gaussian measurement noise, where the covariance matrix $\mathbf{R}_k$ is known.

The goal is to estimate states' $\underline{\mathbf{x}}_{1:K}$ parameters using the realizations of the prior $\mathbf{x}_0$, the realizations of the interoceptive measurements $\mathbf{u}_k$ and realizations of the exteroceptive measurements $\mathbf{y}_k$. If the RVs are Gaussian, then they can be parametrized by a mean and a covariance (*i.e.*, $\underline{\mathbf{x}}_k \sim \mathcal{N}(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$).

Remark 6.7.1. The interoceptive measurement $\underline{\mathbf{u}}_k$ is a random variable that can be assumed to be dis-

tributed according to

$$\underline{\mathbf{u}}_k \sim \mathcal{N}\left(\mathbf{u}_k, \mathbf{Q}_k^{\mathbf{u}}\right). \tag{6.62}$$

The measurement is assumed to be random due to measurement noise. However, the measurement noise can be embedded into the process noise $\mathbf{Q}$. Therefore, it is possible to assume that the interoceptive measurements are not random, but the interoceptive measurement process noise is indeed random and will be embedded with the process noise $\underline{\mathbf{w}}_k$.

Remark 6.7.2. The states will be assumed to be normally distributed. That is, $\underline{\mathbf{x}}_k \sim \mathcal{N}\left(\mathbf{x}_k, \mathbf{P}_k\right)$. Therefore, the parameters to be estimated should

$$\tilde{\boldsymbol{\theta}} = \left\{\mathbf{x}_0, \ldots, \mathbf{x}_N, \mathbf{P}_0, \ldots, \mathbf{P}_N\right\}. \tag{6.63}$$

However, to make things easier, the covariances $\mathbf{P}_{0:N}$ will not be estimated (they will be computed after the MAP estimate is computed). Thus, the parameters that'll be estimated is given by

$$\boldsymbol{\theta} = \left\{\mathbf{x}_0, \ldots, \mathbf{x}_N\right\}. \tag{6.64}$$

The realizations are then

$$\mathbf{X} = \left\{\check{\mathbf{x}}_0, \mathbf{u}_{0:N-1}, \mathbf{y}_{1:N}\right\}. \tag{6.65}$$

For the MAP estimator, the PDF of interest is

$$\begin{aligned}
f\left(\boldsymbol{\theta} \middle|\, \mathbf{X}\right) &= f\left(\mathbf{x}_0, \ldots, \mathbf{x}_N \middle|\, \check{\mathbf{x}}_0, \mathbf{u}_{0:N-1}, \mathbf{y}_{1:N}\right) && (6.66)\\
&= \frac{f\left(\mathbf{y}_{1:N} \middle|\, \mathbf{x}_{0:N}, \check{\mathbf{x}}_0, \mathbf{u}_{0:N-1}\right) f\left(\mathbf{x}_{0:N} \middle|\, \check{\mathbf{x}}_0, \mathbf{u}_{0:N-1}\right)}{f\left(\mathbf{y}_{1:N} \middle|\, \check{\mathbf{x}}_0, \mathbf{u}_{0:N-1}\right)} && (6.67)\\
&= \eta f\left(\mathbf{y}_1 \middle|\, \mathbf{x}_1\right) \cdots f\left(\mathbf{y}_N \middle|\, \mathbf{x}_N\right) f\left(\mathbf{x}_N \middle|\, \mathbf{x}_{N-1}, \mathbf{u}_{N-1}\right) \cdots f\left(\mathbf{x}_1 \middle|\, \mathbf{x}_0, \mathbf{u}_0\right) f\left(\mathbf{x}_0 \middle|\, \check{\mathbf{x}}_0\right) && (6.68)\\
&= \eta \prod_{k=1}^{N} f\left(\mathbf{y}_k \middle|\, \mathbf{x}_k\right) \prod_{k=1}^{N} f\left(\mathbf{x}_k \middle|\, \mathbf{x}_{k-1}, \mathbf{u}_{k-1}\right) f\left(\mathbf{x}_0, \middle|\, \check{\mathbf{x}}_0\right), && (6.69)
\end{aligned}$$

where η is a normalizing parameter that is independent of the design variables ($\mathbf{x}_{0:N}$) and the following assumptions were used

1. The measurements $\mathbf{y}_k$ depends only on $\mathbf{x}_k$ (and measurement noise $\underline{\mathbf{n}}_k$)as shown in the measurement model (6.61). Thus, the random variable $\underline{\mathbf{y}}_k$ is independent of all other random variables.
2. The state $\underline{\mathbf{x}}_k$ depends only on $\underline{\mathbf{x}}_{k-1}$ and $\mathbf{u}_{k-1}$ (and process noise $\underline{\mathbf{w}}_{k-1}$) as shown in (6.60). Therefore, the random variables $\mathbf{x}_{0:N}$ follow the **Markov sequence**: the future states $\underline{\mathbf{x}}_{k+1:N}$ are independent of the past states $\underline{\mathbf{x}}_{0:k-1}$ given the present state $\underline{\mathbf{x}}_k$. The Markov sequence is dictated by the process model (6.60).

If $\underline{\mathbf{x}}_{0:N}$ are assumed to be [marginally] Gaussian, then it is easier to minimize the negative-log of (6.66) than maximize (6.66) directly.

Taking the negative-log of (6.69) gives

$$-\log f\left(\mathbf{x}_{0:N}|\,\check{\mathbf{x}}_0, \mathbf{u}_{0:N-1}, \mathbf{y}_{1:N}\right) = \frac{1}{2}\left\|\mathbf{x}_0 - \check{\mathbf{x}}_0\right\|^2_{\check{\mathbf{P}}_0} + \frac{1}{2}\sum_{k=1}^{N}\left\|\mathbf{y}_k - \mathbf{g}(\mathbf{x}_k, \mathbf{0})\right\|^2_{\mathbf{R}_k} + \frac{1}{2}\sum_{k=1}^{N}\left\|\mathbf{x}_k - \mathbf{f}(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{0})\right\| + \boldsymbol{\gamma}, \tag{6.70}$$

where

$$\|\mathbf{z}\|^2_{\boldsymbol{\Sigma}} \triangleq \mathbf{z}^{\mathsf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{z}, \tag{6.71}$$

and $\boldsymbol{\gamma}$ are constant terms that are independent of the design variables thus they will not affect the optimization.

Therefore, the MAP estimate of $\hat{\mathbf{x}}^{\text{MAP}}_{0:N}$ is the solution to

$$\hat{\mathbf{x}}^{\text{MAP}}_{0:N} = \underset{\mathbf{x}_{0:N}\in\mathbb{R}^n}{\arg\min}\, \frac{1}{2}\left\|\mathbf{x}_0 - \check{\mathbf{x}}_0\right\|^2_{\check{\mathbf{P}}_0} + \frac{1}{2}\sum_{k=1}^{N}\left\|\mathbf{y}_k - \mathbf{g}(\mathbf{x}_k, \mathbf{0})\right\|^2_{\mathbf{R}_k} + \frac{1}{2}\sum_{k=1}^{N}\left\|\mathbf{x}_k - \mathbf{f}(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{0})\right\|, \quad = \underset{\mathbf{x}_{0:N}\in\mathbb{R}^n}{\arg\min}\, J \tag{6.72}$$

which is a (nonlinear) weighted least squares problem.

Remark 6.7.3. In (6.70), it was assumed that $\mathbf{w}_k = \mathbf{n}_k = \mathbf{0}$. I'm not not sure why this assumption is done. The way I think of it is that the noise $\mathbf{w}_k$ is embedded into the interoceptive measurement $\mathbf{u}_k$. Similarly, the noise in $\mathbf{n}_k$ is embedded in the exteroceptive measurement $\mathbf{y}_k$.

6.7.2 Expressing the nonlinear least squares problem in matrix form

The objective function can be expressed as

$$J(\mathbf{x}_{0:N}) = \frac{1}{2}\left\|\mathbf{x}_0 - \check{\mathbf{x}}_0\right\|^2_{\check{\mathbf{P}}_0} + \frac{1}{2}\sum_{k=1}^{N}\left\|\mathbf{x}_k - \mathbf{f}(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{0})\right\| + \frac{1}{2}\sum_{k=1}^{N}\left\|\mathbf{y}_k - \mathbf{g}(\mathbf{x}_k, \mathbf{0})\right\|^2_{\mathbf{R}_k} \tag{6.73}$$

$$= \frac{1}{2}\mathbf{e}(\mathbf{x}_{0:N})^{\mathsf{T}}\mathbf{W}\mathbf{e}(\mathbf{x}_{0:N}), \tag{6.74}$$

where

$$\mathbf{e}(\mathbf{x}_{0:N}) = \begin{bmatrix} \mathbf{x}_0 - \check{\mathbf{x}}_0 \\ \hline \mathbf{x}_1 - \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}_0, \mathbf{0}\right) \\ \vdots \\ \mathbf{x}_N - \mathbf{f}\left(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}, \mathbf{0}\right) \\ \hline \mathbf{y}_1 - \mathbf{g}(\mathbf{x}_1, \mathbf{0}) \\ \vdots \\ \mathbf{y}_N - \mathbf{g}(\mathbf{x}_N, \mathbf{0}) \end{bmatrix} \tag{6.75}$$

is the *error function*, and

$$\mathbf{W}^{-1} = \begin{bmatrix} \check{\mathbf{P}}_0 & & & & & & \\ & \mathbf{Q}_0 & & & & & \\ & & \ddots & & & & \\ & & & \mathbf{Q}_N & & & \\ & & & & \mathbf{R}_1 & & \\ & & & & & \ddots & \\ & & & & & & \mathbf{R}_N \end{bmatrix} \tag{6.76}$$

is the *weight matrix*.

6.7.3 Covariance on MAP estimate

In the optimization problem above, the covariance was not estimated. However, it can be estimated by using the MAP state estimates $\hat{\mathbf{x}}_{0:N}^{\text{MAP}}$. The *joint* covariance can be approximated by

$$\text{Cov}\left[\hat{\mathbf{x}}_{0:N}^{\text{MAP}}\right] = \left(\mathbf{J}\left(\hat{\mathbf{x}}_{0:N}^{\text{MAP}}\right)^{\mathsf{T}} \mathbf{W} \mathbf{J}\left(\hat{\mathbf{x}}_{0:N}^{\text{MAP}}\right)\right)^{-1}, \tag{6.77}$$

where

$$\mathbf{J}\left(\hat{\mathbf{x}}_{0:N}^{\text{MAP}}\right) = \left.\frac{\mathrm{d}\mathbf{e}\left(\mathbf{x}_{0:N}\right)}{\mathrm{d}\mathbf{x}_{0:N}}\right|_{\mathbf{x}_{0:N}=\hat{\mathbf{x}}_{0:N}^{\text{MAP}}}. \tag{6.78}$$

is the Jacobian of the error function (6.75) w.r.t. the design variables, evaluated at the MAP estimate $\hat{\mathbf{x}}_{0:N}^{\text{MAP}}$.

Chapter 7

Hypothesis Testing

7.1 Problem statement

Hypothesis testing, also known as *detection*[1] is a specific parameter estimation problem where the unknown parameter takes one of M possible values.

Say there's an unknown parameter η to be determined. Furthermore, say there are some measurements of a random variable that is correlated with η. Furthermore, say there are different disjoint sets $\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_{M-1}$ that η can belong to, then the problem is to determine the most likelihood set to which η belongs.

> Construct a detector that "maps back" the observed point to the hypothesis that produced it. That is, the "correct" hypothesis.

Example 7.1.1. Consider a digital transmitter that outputs bits that can be either 1 or 0. Then, there are $M = 2$ possible outputs. The hypothesis to be tested are

1. Hypothesis H_0 : Transmitted bit is $\underline{b} = -1$, with probability $\pi_0 = 1/2$, and
2. Hypothesis H_1 : Transmitted bit is $\underline{b} = 1$, with probability $\pi_1 = 1/2$.

7.2 Terminology

Consider a *probabilistic transition mechanism*

$$\text{source output} \;\rightarrow\; \text{random point } x \in \mathcal{Z}. \tag{7.1}$$

- $\mathcal{Z}$: *Observation space*.
- Under hypothesis H_m : observed point follows $f\left(x|H_m\right)$.

[1] Detection is a term usually used in Engineering while hypothesis testing is the proper statistical term.

	H_0 **true**	H_1 **true**
		Type II error
Decide H_0	Miss	Missed Detection
		False Negative
	Type I error	
Decide H_1	False Alarm	Detection
	False Positive	

Table 7.1: Terminology in hypothesis testing.

Assumptions

1. Prior probabilities

$$\pi_m := P(H_m) \tag{7.2}$$

   are known.

2. Conditional PDFs $f(x|\ H_m)$ are known for all $m = 0, \ldots, M-1$.

The strategy is to partition the observation space $\mathcal{Z}$ into M subspaces $\mathcal{Z}_0, \ldots, \mathcal{Z}_{M-1}$. The decision (or detection) rule is

> "if the observed point x lies in $\mathcal{Z}_m$, then the detector should decide in favor of H_m".

7.3 Binary hypothesis testing

In this document, most of the work will be on $M = 2$ hypothesis. This is popular in engineering and they have special names.

1. H_0 : Null hypothesis.
2. H_1 : Alternate hypothesis.

Table 7.1 lists the terms used for hypothesis testing.

Example 7.3.1 (Additive Gaussian noise channel)**.** Consider the output (measurement)

$$\underline{x} = \underline{b} + \underline{n}, \tag{7.3}$$

where the noise $\underline{n} \sim \mathcal{N}(0, 1))$ is independent of the bit $\underline{b}$. The observation space is given by $\mathcal{Z} = \mathbb{R}$. Under hypothesis H_m, the observed point follows $f(x|H_m)$

- $f(x|H_0) = \mathcal{N}(-1, 1)$, and
- $f(x|H_1) = \mathcal{N}(1, 1)$, and .

Each decision has a cost. Specifically,

$$\text{decide } H_0 \text{ and } H_0 \text{ is true } \rightarrow \text{cost } C_{00} \tag{7.4}$$

$$\text{decide } H_1 \text{ and } H_0 \text{ is true } \rightarrow \text{cost } C_{10} \tag{7.5}$$

$$\text{decide } H_1 \text{ and } H_1 \text{ is true } \rightarrow \text{cost } C_{11} \tag{7.6}$$

$$\text{decide } H_0 \text{ and } H_1 \text{ is true } \rightarrow \text{cost } C_{01} \tag{7.7}$$

The assumption is that it is more costly to make a mistake. That is,

$$C_{10} > C_{00} \text{ and } C_{01} > C_{11}. \tag{7.8}$$

Let the goal be to find the detector that minimizes the average cost C given by

$$C = \sum_{i=0}^{1} \sum_{j=0}^{1} C_{ij} P\left(\text{decide } H_i | \ H_j \text{ is true}\right), \tag{7.9}$$

where

$$P\left(\text{decide } H_i | \ H_j \text{ is true}\right) = P\left(\text{decide } H_i | \ H_j \text{ is true}\right) \pi_j \tag{7.10}$$

$$= \pi_j \int_{\mathcal{Z}_i} f\left(x | \ H_j\right) \mathrm{d}x. \tag{7.11}$$

The region $\mathcal{Z}_0$ that minimizes the average cost is

$$\mathcal{Z}_0 = \{x \in \mathcal{Z} : \pi_0 \left(C_{10} - C_{00}\right) f\left(X \mid H_0\right) \geq \pi_1 \left(C_{01} - C_{11}\right) f\left(X \mid H_1\right))\} \tag{7.12}$$

$$= \{x \in \mathcal{Z} : \overbrace{\frac{\pi_0 \left(C_{10} - C_{00}\right)}{\pi_1 \left(C_{01} - C_{11}\right)}}^{\text{threshold } \eta} \geq \underbrace{\frac{f\left(X \mid H_1\right)}{f\left(X \mid H_0\right)}}_{\text{Likelihood ratio } \Lambda(x)} \} \tag{7.13}$$

The decision rule is therefore

$$\underbrace{\frac{\pi_0 \left(C_{10} - C_{00}\right)}{\pi_1 \left(C_{01} - C_{11}\right)}}_{\eta} \quad \begin{matrix} H_0 \\ \geq \\ < \\ H_1 \end{matrix} \quad \underbrace{\frac{f\left(x \mid H_1\right)}{f\left(x \mid H_0\right)}}_{\Lambda(x)}. \tag{7.14}$$

If the goal is to minimize the error, then the problem is known as *minimum probability of error decision rule*, where the costs are set to

$$C_{01} = C_{10} = 1, \tag{7.15}$$

$$C_{00} = C_{11} = 0. \tag{7.16}$$

7.4 Neyman-Pearson hypothesis testing

In practice, the prior probabilities π_0 and π_1 may not be known. Thus, Neyman-Pearson have a different approach:

- Probability of detection:

$$P_d := P\left(\text{decide } H_1 \mid H_1 \text{ is true}\right). \tag{7.17}$$

- Probability of False Alarm:

$$\begin{aligned} P_{\text{fa}} &:= P\left(\text{decide } H_1 \mid H_0 \text{ is true}\right) && (7.18)\\ &= \alpha. && (7.19)\end{aligned}$$

Neyman-Pearson criterion is given by

$$\begin{aligned} &\max P_d && (7.20)\\ &\text{s.t. } P_{\text{fa}} \leq \alpha. && (7.21)\end{aligned}$$

The term α is referred to as *significance* [6]. The solution is to use the Likelihood ratio test with threshold η such that

$$P\left(\Lambda(\underline{x}) < \eta \mid H_0\right) = \alpha. \tag{7.22}$$

Example 7.4.1. Consider a (possibly unfair) coin with a probability of heads equal to p, where p is a realization of a RV $\underline{p}$ that is uniform on the interval $[0, 1]$. The goal is to decide among the following hypotheses:

$$\begin{aligned} H_0 &: \underline{p} \leq 1/3 && (7.23)\\ H_1 &: \underline{p} > 1/3 && (7.24)\end{aligned}$$

What is the minimum probability of error decision rule if N independent flips are observed?

The average cost is given by C

$$\begin{aligned} C = {} & C_{00} P\left(\text{decide } H_0, H_0 \text{ is true}\right) + C_{10} P\left(\text{decide } H_1, H_0 \text{ is true}\right) + \\ & C_{01} P\left(\text{decide } H_0, H_1 \text{ is true}\right) + C_{11} P\left(\text{decide } H_1, H_1 \text{ is true}\right) && (7.25)\\ = {} & C_{00}\pi_0 P\left(H_0|H_0\right) + C_{10}\pi_0 P\left(H_1|H_0\right) + C_{01}\pi_1 P\left(H_0 \mid H_1\right) + C_{11}\pi_1 P\left(H_1 \mid H_1\right). && (7.26)\end{aligned}$$

Let $C_{00} = C_{11} = 0$ and $C_{10} = C_{01} = 1$ for minimum average cost detector. Then, the average cost is given by

$$\begin{aligned} C &= \pi_0 P\left(H_1 \mid H_0\right) + \pi_1 P\left(H_0 \mid H_1\right) && (7.27)\\ &= \pi_0 \sum_{x \in \mathcal{Z}_1} p\left(x \mid H_0\right) + \pi_1 \sum_{x \in \mathcal{Z}_0} p\left(x \mid H_1\right) && (7.28)\\ &= \pi_0 \sum_{x \in \mathcal{Z} - \mathcal{Z}_0} p\left(x \mid H_0\right) + \pi_1 \sum_{x \in \mathcal{Z}_0} p\left(x \mid H_1\right) && (7.29)\end{aligned}$$

$$= \pi_0 \left(1 - \sum_{x \in \mathcal{Z}_0} p\left(x \mid H_0\right) \right) + \pi_1 \sum_{x \in \mathcal{Z}_0} p\left(x \mid H_1\right) \tag{7.30}$$

$$= \pi_0 + \sum_{x \in \mathcal{Z}_0} \pi_1 p\left(x \mid H_1\right) - \pi_0 p\left(x \mid H_0\right). \tag{7.31}$$

The region $\mathcal{Z}_0$ that minimizes the above summation is given by

$$\mathcal{Z}_0 = \left\{ x \in \mathcal{Z} : \pi_0 p\left(x \mid H_0\right) \geq \pi_1 p\left(x \mid H_1\right) \right\}. \tag{7.32}$$

The decision rule can be written as

$$\underbrace{\frac{\pi_0}{\pi_1}}_{\eta} \underset{H_1}{\overset{H_0}{\gtrless}} \underbrace{\frac{p\left(x \mid H_1\right)}{p\left(x \mid H_0\right)}}_{\Lambda(x)} \tag{7.33}$$

where the PMF $p\left(x \mid H_0\right)$ is given by

$$p\left(x \mid H_0\right) = P(\underline{x} = x \mid \underline{p} \leq 1/3) \tag{7.34}$$

$$= \frac{1}{P\left(\underline{p} \leq 1/3\right)} P\left(\underline{x} = x, \underline{p} \leq 1/3\right) \tag{7.35}$$

$$= \frac{1}{\pi_0} \int_0^{1/3} p\left(x \mid p\right) f\left(p\right) \mathrm{d}p. \tag{7.36}$$

And the PMF $p\left(x \mid H_1\right)$ is given by

$$p\left(x \mid H_1\right) = P(\underline{x} = x \mid \underline{p} > 1/3) \tag{7.37}$$

$$= \frac{1}{P\left(\underline{p} > 1/3\right)} P\left(\underline{x} = x, \underline{p} > 1/3\right) \tag{7.38}$$

$$= \frac{1}{\pi_1} \int_{1/3}^{1} p\left(x \mid p\right) f\left(p\right) \mathrm{d}p. \tag{7.39}$$

The PDF $f\left(p\right) = U([0,1])$ is a uniform distribution and $p\left(x \mid p\right)$ is the Binomial PMF $\underline{x} \sim B(N,p)$ given by

$$p\left(x \mid p\right) = \begin{cases} \binom{N}{x} p^x (1-p)^{N-x}, & x = 0, 1, \ldots, N, \\ 0, & \text{otherwise.} \end{cases} \tag{7.40}$$

η is evaluated as

$$\eta = \frac{\pi_0 (C_{10} - C_{00})}{\pi_1 (C_{01} - C_{11})} \tag{7.41}$$

$$= \frac{1}{2}. \tag{7.42}$$

Therefore the minimum probability of error decision rule is

$$\eta \underset{H_1}{\overset{H_0}{\gtrless}} \frac{p\left(x \mid H_1\right)}{p\left(x \mid H_0\right)} \tag{7.43}$$

$$\frac{1}{2} \underset{H_1}{\overset{H_0}{\gtrless}} \frac{\pi_0 \int_{1/3}^{1} p\left(x \mid p\right) f\left(p\right) \mathrm{d}p}{\pi_1 \int_{0}^{\frac{1}{3}} p\left(x \mid p\right) f\left(p\right) \mathrm{d}p} \tag{7.44}$$

$$1 \underset{H_1}{\overset{H_0}{\gtrless}} \frac{\int_{1/3}^{1} p\left(x \mid p\right) f\left(p\right) \mathrm{d}p}{\int_{0}^{\frac{1}{3}} p\left(x \mid p\right) f\left(p\right) \mathrm{d}p} \tag{7.45}$$

$$\int_{0}^{\frac{1}{3}} p\left(x \mid p\right) f\left(p\right) \mathrm{d}p \underset{H_1}{\overset{H_0}{\gtrless}} \int_{1/3}^{1} p\left(x \mid p\right) f\left(p\right) \mathrm{d}p. \tag{7.46}$$

△

Part III

Random Processes and Sequences

Chapter 8

Definitions and Terminology

8.1 Introduction

Recall:

- A random variable (Def. 1.4.1) $\underline{x}$ maps outcomes to number.
- A random vector (Def. 2.1.1) $\underline{\mathbf{x}}$ maps outcomes to vectors.

Definition 8.1.1 (*Random process*)**.** A random process (RP) $\underline{x}(t)$, $t \in T$, maps outcomes to *functions* of t.

Remark 8.1.1. Some remarks on random processes.

- A RP is a family of RVs indexed by t.
- The *range* I of $\underline{x}(t)$ is called the *state space* of the process.
- A realization $x(t) = \underline{x}(s_0; t)$ (for a fixed s_0) of a RP $\underline{x}(t)$ is also called a *sample path* or a *sample function* of the process.
- If I is a subset of $\mathbb{R}$ then the process is called a real RP. If I is a subset of $\mathbb{C}$ then the process is called a complex RP
- Two stochastic processes $\underline{x}_1(t)$ ad $\underline{x}_2(t)$ are called *equal* if their sample functions are always equal. That is,

$$\underline{x}_1(s; t) = \underline{x}_2(s; t), \qquad \forall t \in T, \forall s \in S. \tag{8.1}$$

8.2 Types of random processes

	T **discrete**	T **continuous**
I **discrete**	Discrete-time random chain, or discrete state random sequence	Continuous-time stochastic chain, or continuous-time discrete random process
I **continuous**	Continuous-state random sequence	Continuous-time continuous-state process

Table 8.1: Types of random processes.

Definition 8.2.1 (*First order distribution of a RP*)**.** The *first order* distribution of the RV $\underline{x}(t)$ is

$$F(x;t) = \underbrace{P(\underline{x}(t) \leq x))}_{\text{CDF of } \underline{x}(t)} \tag{8.2}$$

Definition 8.2.2 (*First order statistic of a RP*)**.** There are two cases.

1. If $\underline{x}(t)$ is a continuous RV, then the *first order density* of the RP $\underline{x}(t)$ is

$$f(x;t) = \frac{\mathrm{d}}{\mathrm{d}x} F(x;t), \qquad \forall t \in T. \tag{8.3}$$

2. If $\underline{x}(t)$ is a discrete RV, then the *first order probability mass function* is

$$p(x;t) = P(\underline{x}(t) = x), \qquad \forall t \in T. \tag{8.4}$$

Definition 8.2.3 (*Second order distribution of a RP*)**.** The *second order distribution of a RP* $\underline{x}(t)$ is

$$F(x_1, x_2; t_1, t_2) = \underbrace{P(\underline{x}(t_1) \leq x_1, \underline{x}(t_2) \leq x_2)}_{\text{JCDF of } \underline{x}(t_1), \underline{x}(t_2)} \tag{8.5}$$

Definition 8.2.4 (*Second order density of a RP*)**.** There are two cases.

1. If $\underline{x}(t_1)$ and $\underline{x}(t_2)$ are continuous in their domain. Then, the *second order density* of a RP $\underline{x}(t)$ is

$$f(x_1, x_2; t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2; t_1, t_2), \qquad \forall t \in T. \tag{8.6}$$

2. If $\underline{x}(t_1)$ and $\underline{x}(t_2)$ are discrete, then the *second order probability mass function* of the RP $\underline{x}(t)$ is

$$p(x_1, x_2; t_1, t_2) = P(\underline{x}(t_1) = x_1, \underline{x}(t_2) = x_2), \qquad \forall t \in T. \tag{8.7}$$

Definition 8.2.5 (*Higher order statistics of a RP*)**.** For the n RVs $\underline{x}(t_1), \ldots, \underline{x}(t_n)$,

1. the nth order distribution of $\underline{x}(t)$ is

$$F\left(x_1, \ldots, x_n; t_1, \ldots, t_n\right), \tag{8.8}$$

2. the nth order density of $\underline{x}(t)$ is

$$f\left(x_1, \ldots, x_n; t_1, \ldots, t_n\right), \tag{8.9}$$

3. the nth order probability mass function of $\underline{x}(t)$ is

$$p\left(x_1, \ldots, x_n; t_1, \ldots, t_n\right). \tag{8.10}$$

8.3 Mean, autocorrelation, and autocovariance

Definition 8.3.1 (*Mean of a random process*)**.** The *mean* $\eta(t)$ of a RP $\underline{x}(t)$ is

$$\eta(t) = \mathbb{E}\left[\underline{x}(t)\right] \tag{8.11}$$

$$= \int x f\left(x; t\right) \mathrm{d}x. \tag{8.12}$$

Note that the mean is a function of time.

Definition 8.3.2 (*Autocorrelation*)**.** The *autocorrelation* $R\left(t_1, t_2\right)$ of a random process $\underline{x}(t)$ is

$$R\left(t_1, t_2\right) = \mathbb{E}\left[\underline{x}(t_1)\underline{x}(t_2)\right] \tag{8.13}$$

$$= \int\int x_1 x_2 f\left(x_1, x_2; t_1, t_2\right) \mathrm{d}x_1 \mathrm{d}x_2. \tag{8.14}$$

Definition 8.3.3 (*Autocovariance*)**.** The *autocovariance* $C\left(t_1, t_2\right)$ of a random process $\underline{x}(t)$ is

$$C\left(t_1, t_2\right) = \mathbb{E}\left[\left(\underline{x}(t_1) - \eta(t_1)\right)\left(\underline{x}(t_2) - \eta(t_2)\right)\right] \tag{8.15}$$

$$= R\left(t_1, t_2\right) - \eta(t_1)\eta(t_2). \tag{8.16}$$

For $t_1 = t_2$,

$$C\left(t, t\right) = \mathrm{Var}\left[\underline{x}(t)\right]. \tag{8.17}$$

8.4 Properties of random processes

Definition 8.4.1 (*Independent increments*)**.** A RP $\underline{x}(t)$ is said to have *independent increments* if for any

k and any choice of $t_1 < t_2 < \ldots < t_k$, the RVs

$$\underline{x}(t_2) - \underline{x}(t_1), \ldots, \underline{x}(t_k) - \underline{x}(t_{k-1}) \tag{8.18}$$

are independent RVs.

Definition 8.4.2 (*Markov RP*)**.** A RP $\underline{x}(t)$ is said to be *Markov* if the future of the process given the present is independent of the past. That is, for any k and choice of

$$\underbrace{t_1 < \ldots < t_{k-1}}_{\text{past}} < \underbrace{t_k}_{\text{present}} < \underbrace{t_{k+1}}_{\text{future}}, \tag{8.19}$$

the following relation holds

$$f_{x(t_{k+1})}\left(x_{k+1} \mid \underline{x}(t_k) = x_k, \underline{x}(t_{k-1}) = x_{k-1}, \ldots, \underline{x}(t_1) = x_1\right) \tag{8.20}$$

$$= f_{x(t_{k+1})}\left(x_{k+1} \mid \underline{x}(t_k) = x_k\right). \tag{8.21}$$

Definition 8.4.3 (*Independent RPs*)**.** Two RPs $\underline{x}(t)$ and $\underline{y}(t)$ are called *independent* if the sets $(\underline{x}(t_1), \ldots, \underline{x}(t_n))$ and $\left(\underline{y}(t'_1), \ldots, \underline{y}(t'_n)\right))$ are mutually independent for any n, n' and choice of pointes $t_1, \ldots, t_n$ and $t'_1, \ldots, t'_{n'}$.

Definition 8.4.4 (*White noise*)**.** A RP $\underline{x}(t)$ is called *white noise* if its values $\underline{x}(t_1)$ and $\underline{x}(t_2)$ are uncorrelated for any $t_1 \neq t_2$. That is,

$$C\left(t_1, t_2\right) = 0, \qquad \forall t_1 \neq t_2. \tag{8.22}$$

Remark 8.4.1. The autocovariance of a non-trivial white-noise process must be of the form

$$C\left(t_1, t_2\right) = q(t_1)\delta(t_1 - t_2), \qquad q(t) \geq 0. \tag{8.23}$$

Definition 8.4.5 (*Strictly white noise*)**.** A RP $\underline{x}(t)$ is called *strictly white noise* if its values $\underline{x}(t_1)$ and $\underline{x}(t_2)$ are *independent* for all $t_1 \neq t_2$.

Definition 8.4.6 (*Normal (Gaussian) RPs*)**.** A RP $\underline{x}(t)$ is called *Normal (Gaussian)* if for any k and any choice of $t_1, \ldots, t_k$, the RVs $\underline{x}(t_1), \ldots, \underline{x}(t_k)$ are jointly normal.

Remark 8.4.2 (Gaussian RPs)**.** The statistics of a Normal RP are completely determined by its mean $\eta(t)$ and autocovariance $C\left(t_1, t_2\right)$.

Chapter 9

Stochastic Convergence

9.1 Problem statement

How can convergence of a random process $\underline{x}$ or a random sequence (RS) $\underline{x}_n$ be defined?

The case is not as straightforward as in the deterministic case. Therefore, there are different types of convergence in for random processes (sequences). Specifically,

1. convergence *everywhere* (*e*) or *surely* (*s*),
2. convergence *almost everywhere* (*a.e.*) or *almost surely* (*a.s.*), or *with probability 1* (*w.p. 1*)
3. convergence in *probability* (*p*),
4. convergence in *mean square sense* (*m.s.*), and
5. convergence in *distribution*.

The focus in this chapter will be on random sequences but the concepts generalize to random processes.

Notation: Let $\underline{x}_n$ be a random sequence. Then a realization of $\underline{x}_n$ is given by

$$x_n(s), \qquad s \in S, \tag{9.1}$$

or simply x_n.

9.2 Definitions

Definition 9.2.1 (*Surely convergence*)**.** A random sequence $\underline{x}_n$ is said to converge *surely* if *all* of its realizations converge. That is, for all $s \in S$, there exist a value $\chi(s)$ such that

$$\lim_{n\to\infty} x_n(s) = \chi(s). \tag{9.2}$$

This convergence is also known as *convergence everywhere*.

Remark 9.2.1 (On surely convergant random sequences)**.** The mapping $s \in S \to \chi(s) \in \mathbb{R}$ defines a new random variable $\underline{\chi}$ which is called the *limit of* $\underline{x}_n$.

Notation

- $\underline{x}_n \to \chi(s)$ (e) or (s) or simply $\underline{x}_n \to \underline{\chi}$.
- If $P\left(\underline{\chi} = c\right) = 1$, then the limit is written as

$$\underline{x}_n \to c. \tag{9.3}$$

Example 9.2.1 (Surely convergent sequence)**.** Let $\underline{z} \sim U[0,1]$ and

1. let $\underline{x}_n = \underline{z}/n$. Then, the random sequence $\underline{x}_n \to 0$ everywhere (converges surely).
2. Let $\underline{x}_n = \underline{z}(1 - 1/n)$, then any realization $x_n(s) = z(s)(1 - 1/n)$ converges to $z(s)$. Hence, $\underline{x}_n$ converges *everywhere* (converges surely).
3. Let $\underline{x}_n = \underline{z}e^n$. Since there is only one realization that converges (for $\underline{z} = 0$), then $\underline{x}_n$ does *not* converge everywhere.

Definition 9.2.2 (*Almost surely convergence*)**.** A random sequence $\underline{x}_n$ is said to converge *almost surely* or *almost everywhere* if the set of outcomes mapped to realizations that converge has probability 1. That is,

$$P\left(S'\right) = 1, \tag{9.4}$$

where

$$S' = \left\{s \in S : \exists \chi'(s) \text{s. t. } x_n(s) \to \chi'(s)\right\}. \tag{9.5}$$

Remark 9.2.2 (On almost surely convergent sequences)**.** The mapping $s \in S \to \chi(s) \in \mathbb{R}$ defined by

$$\chi(s) = \begin{cases} \chi'(s), & s \in S', \\ \text{arbitrary point}, & s \notin S' \end{cases} \tag{9.6}$$

defines a new random variable $\underline{chi}$, which is called the *limit of* $\underline{x}_n$.

Notation

- $\underline{x}_n \to \chi$ (a.s.) or (a.e.) or (w.p.1)[a] or $P\left(\underline{x}_n \to \underline{chi}\right) = 1$.
- If $P\left(\chi = c\right) = 1$, then the limit is written as $\underline{x}_n \to c$ (a.s.)/(a.e.)/(w.p.1)/$P\left(\underline{x}_n \to c\right)$.

[a] With probability 1.

Example 9.2.2. Let $\underline{z} \sim U[0,1]$ and define the sequence

$$\underline{x}_n = e^{-n(n\underline{z}-1)} \tag{9.7}$$

$$= e^{-n\underline{z}^2+n} \tag{9.8}$$

$$= \begin{cases} e^n, & \underline{z} = 0, \\ e^{-n^2+n}, & \underline{z} \neq 0. \end{cases} \tag{9.9}$$

- There is one realization of $\underline{x}_n$ that diverges (the one for $\underline{z} = 0$).
- All other realizations (for which $\underline{z} \neq 0$) converge to 0.
- Since $P(\underline{z} \neq 0) = 1$, then the random sequence $\underline{x}_n$ converges to 0 almost surely.

Definition 9.2.3 (*Mean square convergence*)**.** The random sequence $\underline{x}_n$ converges to *in the mean square sense* to $\underline{\chi}$ if

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\underline{x}_n - \underline{\chi}\right|^2\right] = 0. \tag{9.10}$$

Remark 9.2.3. The notation is given by

- l.i.m (limit in the mean sense) $\underline{x}_n = \underline{\chi}$.
- $\underline{x}_n \to \underline{\chi}$ (m.s.).

Example 9.2.3. Let $\underline{z} \sim U[0,1]$ and define the random sequence $\underline{x}_n = \underline{z}(1 - 1/n)$. From Example 9.2.1, the sequence $\underline{x}_n \to \underline{z}$ (a.s.). Now we want to check whether the sequence converges to the same limit in the mean squared sense. Thus,

$$\mathbb{E}\left[\left|\underline{x}_n - \underline{z}\right|^2\right] = \mathbb{E}\left[\left|\underline{z}(1-1/n) - \underline{z}\right|^2\right] \tag{9.11}$$

$$= \mathbb{E}\left[\left|-\underline{z}/n\right|^2\right] \tag{9.12}$$

$$= \mathbb{E}\left[(\underline{z}/n)^2\right] \tag{9.13}$$

$$= \int_0^1 (z/n)^2 \mathrm{d}z \tag{9.14}$$

$$= \frac{1}{n^2}\int_0^1 z^2 \mathrm{d}z \tag{9.15}$$

$$= \frac{1}{3n^2}. \tag{9.16}$$

Thus,

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\underline{x}_n - \underline{z}\right|^2\right] = \lim_{n\to\infty} \frac{1}{3n^2} \tag{9.17}$$

$$= 0. \tag{9.18}$$

Therefore, $\underline{x}_n$ converges to $\underline{z}$ in the mean square sense.

Definition 9.2.4 (*Convergence in probability*)**.** The random sequence $\underline{x}_n$ *converges in probability* to $\underline{\chi}$ if

$$P\left(\left|\underline{x}_n - \underline{\chi}\right| \geq \epsilon\right) \to 0, \quad \forall \epsilon > 0. \tag{9.19}$$

Or equivalently,

$$P\left(\left|\underline{x}_n - \underline{\chi}\right| < \epsilon\right) \to 1, \quad \forall \epsilon > 0. \tag{9.20}$$

Notation: $\underline{x}_n \xrightarrow{p} \underline{\chi}$ or $\underline{x}_n \to \underline{\chi}$ (p).

Example 9.2.4. Let $\underline{u}_1, \ldots, \underline{u}_n, \ldots$, be i.i.d. random variables that are uniformly distributed over $[0, 1)$. Define the sequence

$$\underline{x}_n = \max(\underline{u}_1, \ldots, \underline{u}_n). \tag{9.21}$$

Then the sequence $\underline{x}_n$ converges to 1 in the probability sense.

Proof. The CDF of $\underline{x}_n$ is

$$F_n\left(\left(\right) x\right) = \begin{cases} 0, & x < 0, \\ x^n, & 0 \leq x \leq 1, \\ 1, & x > 1. \end{cases} \tag{9.22}$$

Let $0 < \epsilon < 1$ (justified by the fact that $P\left(|\underline{x}_n - 1| > 1\right) = 0$). Then,

$$\begin{aligned} P\left(|\underline{x}_n - 1| \geq \epsilon\right) &= P\left(\underline{x}_n - 1 \leq \epsilon\right) && (9.23)\\ &= P\left(\underline{x}_n < 1 - \epsilon\right) && (9.24)\\ &= F_n\left(1 - \epsilon\right) && (9.25)\\ &= (1 - \epsilon)^n \to 0. && (9.26) \end{aligned}$$

Note that the first equality holds since $\underline{x}_n - 1 \leq 0$. □

Definition 9.2.5 (*Convergence in distribution*)**.** A random sequence $\underline{x}_n$ is said to *converge in distribution* to a random variable $\underline{\chi}$ if the CDF $F_{x_n}(\lambda)$ of $\underline{x}_n$ converges to $F_\chi(\lambda)$ at all points of continuity of $F_\chi(\lambda)$.

Example 9.2.5. Let $\underline{z} \sim \mathcal{N}(0, 1)$ and define $\underline{x}_n = (-1)^n \underline{z}$. Then, only *one* realization converges (the one for $\underline{z} = 0$). But, $\underline{x}_n \sim \mathcal{N}(0, 1)$.

Note that $\mathbb{E}\left[\underline{x}_n\right] = 0$ and $\mathrm{Var}\left[\underline{x}_n\right] = 1$.

Remark 9.2.4. Convergence in distribution is *weaker* than other convergence.

Notes

- Convergence *surely* and *almost surely* are
  - concerned with the behavior of the sequence as a whole, and
  - deal with the probability of the limit.
- Convergence in the *mean square* and *in probability*
  - focus on the behaviour of a single element of the sequence, and
  - deal with the limit of probability (or mean square distance).
- Convergence in distribution is very different and less strict than the other four types of convergence. It only deals with the limit of the PDF, not the limit itself (which may note exist!).

9.3 Theorems

Theorem 9.3.1 (Strong law of large numbers)**.** Let $\underline{x}_1, \ldots, \underline{x}_n, \ldots$ be sequence of i.i.d. random variables with $\mathbb{E}\left[|\underline{x}_i|\right] < \infty$. Then

$$\frac{1}{n}\sum_{i=1}^{n}\underline{x}_i \to \mu \quad \text{(a.s.)}, \tag{9.27}$$

where $\mu = \mathbb{E}\left[\underline{x}_i\right]$.

Theorem 9.3.2 (Weak law of large numbers)**.** Let $\underline{x}_1, \ldots, \underline{x}_n, \ldots$ be a sequence of i.i.d. random variables such that $\mathbb{E}\left[|\underline{x}_i|\right] < \infty$. Then

$$\frac{1}{n}\sum_{i=1}^{n}\underline{x}_i \xrightarrow{p} \mu, \tag{9.28}$$

where $\mu = \mathbb{E}\left[\underline{x}_i\right]$.

Theorem 9.3.3 (Central limit theorem)**.** Given n i.i.d. random variables $\underline{x}_i$ with mean μ and variance σ^2. Define the sum

$$\underline{y}_n = \sum_{i=1}^{n}\underline{x}_i. \tag{9.29}$$

Then, the random variable $\underline{y}_n$ has mean $n\mu$ and variance $n\sigma^2$. Furthermore, let

$$\underline{z}_n = \frac{\underline{y}_n - n\mu}{\sqrt{n\sigma^2}}. \tag{9.30}$$

Then, $\mathbb{E}[\underline{z}_n] = 0$ and $\mathrm{Var}[\underline{z}_n] = 1$. The central limit theorem (CLT) says that the random sequence $\underline{z}_1, \ldots, \underline{z}_n, \ldots$ converges *in distribution* to the distribution $\mathcal{N}(0, 1)$.

Remark 9.3.1 (On the central limit theorem)**.** Note that there was *no assumption* on the distribution of the random variable $\underline{x}_i$, yet the random sequence $\underline{z}_1, \ldots, \underline{z}_n, \ldots$ converges (in distribution) to a standard normal distribution.

This theorem is often used to justify the assumption that noise of a signal is normally distributed (even though it may not really be normally distributed) [7]. But the caveat in the theorem is that the sequence converges *as* $n \to \infty$. So what should one do when there are few realizations (say less than 20)? Well, I'm not quite sure about this yet, but I think that the sequence $\underline{z}_1, \ldots, \underline{z}_n, \ldots$ would then converge to a Student t distribution. This distribution converges to normal distribution when the number of samples (also known as degrees of freedom) are large (say larger than 20 samples).

Theorem 9.3.4. Convergence surely implies convergence almost surely.

Theorem 9.3.5. Convergence almost surely implies convergence in probability.

Theorem 9.3.6. Convergence in the mean square sense implies convergence in probability.

Theorem 9.3.7. Convergence in probability implies convergence in distribution.

Chapter 10

Stationary Processes

10.1 Definitions

Definition 10.1.1 (*Strict sense stationary*)**.** A random process $\underline{x}(t)$ is called *strict sense stationary* if its statistical properties are invariant to a shift in time. That is,

$$f\left(x_1, \ldots, x_n; t_1, \ldots, t_n\right) = f\left(x_1, \ldots, x_n; t_{1+\tau}, \ldots, t_{n+\tau}\right) \tag{10.1}$$

for any n, $t_1, \ldots, t_n$, and τ.

Definition 10.1.2 (*Jointly SSS*)**.** Two processes $\underline{x}(t)$ and $\underline{y}(t)$ are called *jointly stationary* if the joint statistics of

$$\underline{x}(t_1), \ldots, \underline{x}(t_n), \underline{y}(t'_1), \ldots, \underline{y}(t'_m) \tag{10.2}$$

are the same as the joint statistics of

$$\underline{x}(t_1 + \tau), \ldots, \underline{x}(t_n + \tau), \underline{y}(t'_1 + \tau), \ldots, \underline{y}(t'_m + \tau) \tag{10.3}$$

for any $t_1, \ldots, t_n$, n, $t'_1, \ldots, t'_m$, m, and τ.

Definition 10.1.3 (*Complex stationary*)**.** A complex process $\underline{z}(t) = \underline{x}(t) + j\underline{y}(t)$ is called *stationary* if the processes $\underline{x}(t)$ and $\underline{y}(t)$ are jointly stationary.

Definition 10.1.4 (*Wide sense stationary*)**.** A random process $\underline{x}(t)$ is called *wide sense stationary (WSS)* if

1. its mean is constant

$$\mathbb{E}\left[\underline{x}(t)\right] = \eta, \tag{10.4}$$

and

2. its autocorrelation function $R(t_1, t_2)$ depends only on $t_1 - t_2$. That is,

$$\mathbb{E}\left[\underline{x}(t_1)\underline{x}(t_2)^*\right] = R(t_1 - t_2), \tag{10.5}$$

or equivalently

$$\mathbb{E}\left[\underline{x}(t+\tau)\underline{x}^*\right] = R(\tau). \tag{10.6}$$

Remark 10.1.1. A SSS process is also WSS. However, the inverse is true only for Gaussian process.

Definition 10.1.5 (*Crosscorrelation*)**.** Let $\underline{x}(t)$ and $\underline{y}(t)$ be two random processes. The *cross-correlation* $R_{xy}(t_1, t_2)$ of $\underline{x}(t)$ and $\underline{y}(t)$ is

$$R_{xy}(t_1, t_2) = \mathbb{E}\left[\underline{x}(t_1)\underline{y}(t_2)^*\right]. \tag{10.7}$$

Definition 10.1.6 (*Joinly WSS*)**.** Two processes $\underline{x}(t)$ and $\underline{y}(t)$ are called jointly WSS if

1. each one of them is WSS, and
2. their cross-correlation function $R_{xy}(t_1, t_2)$ depends only on $t_1 - t_2$. That is,

$$\mathbb{E}\left[\underline{x}(t_1)\underline{y}(t_2)^*\right] = R_{xy}(t_1 - t_2), \tag{10.8}$$

or, equivalently,

$$R_{xy}(t+\tau, t) = \mathbb{E}\left[\underline{x}(t+\tau)\underline{y}(t)^*\right] \tag{10.9}$$
$$= R_{xy}(\tau). \tag{10.10}$$

Remark 10.1.2. Some properties of WSS processes.

- For $\tau = 0$,

$$\mathbb{E}\left[\left|\underline{x}(t)\right|^2\right] = R(0). \tag{10.11}$$

- $R(\tau) = R(-\tau)^*$.
- For a real WSS random processes, the autocorrelation function is an even function of τ.
- The autocorrelation function of a real random process is maximized at $\tau = 0$. That is,

$$R(0) \geq R(\tau). \tag{10.12}$$

Furthermore,

$$R(0) \geq |R(\tau)|. \tag{10.13}$$

Definition 10.1.7 (*Power spectral density (PSD)*)**.** The *power spectrum* or *power spectral density* (PSD) of a WSS process $\underline{x}(t)$ (real or complex) is the Fourier transform $S_{xx}(\omega)$ of its autocorrelation $R_{xx}(\tau) = \mathbb{E}[\underline{x}(t+\tau)\underline{x}(t)^*]$. That is,

$$S_{xx}(\omega) = \mathtt{F}\{R_{xx}(\tau)\} \tag{10.14}$$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-\jmath\omega\tau} \mathrm{d}\tau. \tag{10.15}$$

The autocorrelation function can be obtained from the PSD using the inverse Fourier transform. Specifically,

$$R_{xx}(\tau) = \mathtt{F}^{-1} S_{xx}(\omega) \tag{10.16}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{\jmath\omega\tau} \mathrm{d}\omega. \tag{10.17}$$

Remark 10.1.3. Some properties of the power spectral density.

- Since $R_{xx}(\tau)^* = R_{xx}(-\tau)$, then

$$S_{xx}(\omega)^* = S_{xx}(\omega). \tag{10.18}$$

  That is, the PSD is always a real function of ω.

If $\underline{x}(t)$ is a real process, then the PSD is a *real* and *even* function of ω.

Definition 10.1.8 (*Cross PSD*)**.** The *cross power spectrum* or *cross power spectral density* of the jointly WSS processes $\underline{x}(t)$ and $\underline{y}(t)$ is the Fourier transform, $S_{xy}(\omega)$ of their cross-correlation. That is,

$$S_{xy}(\omega) = \mathtt{F}\{R_{xy}(\tau)\} \tag{10.19}$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-\jmath\omega\tau} \mathrm{d}\tau. \tag{10.20}$$

The cross-correlation function can be computed from the cross PSD using the inverse Fourier transform. That is,

$$R_{xy}(\tau) = \mathtt{F}^{-1} S_{xy}(\omega) \tag{10.21}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega) e^{\jmath\omega\tau} \mathrm{d}\omega. \tag{10.22}$$

In general, the cross PSD is a complex function of ω with the property

$$S_{xy}(\omega) = S_{yx}(\omega)^*. \tag{10.23}$$

10.2 Theorems

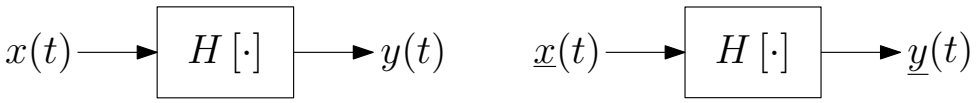
Theorem 10.2.1 (Wiener-Kinchin)**.** For any random process $\underline{x}(t)$,

$$S_{xx}(\omega) \geq 0. \tag{10.24}$$

Theorem 10.2.2 (Existence of PSD)**.** Given any arbitrary finite[a] positive function $S(\omega)$, there exists a complex process $\underline{x}(t)$ with PSD $S_{xx}(\omega)$ equal to $S(\omega)$.

[a] I.e., $\int S(\omega)\,\mathrm{d}\omega < \infty$.

10.3 Deterministic systems



(a) Deterministic input (b) Random input

Figure 10.1: Deterministic system schematic.

Definition 10.3.1 (*System*)**.** A system is an object, or a set of functions, that take an input function $x(t)$ and outputs an output function $y(t)$.

The function can be deterministic as seen in Figure 10.1a, or stochastic as seen in Figure 10.1b.

Remark 10.3.1. The notation

$$\underline{y}(t) = H[\underline{x}(t)]] \tag{10.25}$$

reads "system H applied on input $\underline{x}(t)$ to give the output $\underline{y}(t)$."

Definition 10.3.2 (*Deterministic systems*)**.** A system is called *deterministic* if, identical realizations of the input process $\underline{x}(t)$ yield identical realizations of the output process $\underline{y}(t)$.

Definition 10.3.3 (*Stochastic system*)**.** A system that is not deterministic is called *stochastic*.

Definition 10.3.4 (*Linear system*)**.** A system L is called *linear* if it satisfies the scaling and superposition

property. That is, for any random scalers $\underline{a}_1$, $\underline{a}_2$ and random processes $\underline{x}_1(t)$, $\underline{x}_2(t)$, the following holds

$$L\left[\underline{a}_1\underline{x}_1(t) + \underline{a}_2\underline{x}_2\right] = \underline{a}_1 L\left[\underline{x}_1(t)\right] + \underline{a}_2 L\left[\underline{x}_2(t)\right]. \tag{10.26}$$

Theorem 10.3.1 (Fundamental theorem)**.** For any linear system L the following property holds.

$$\mathbb{E}\left[L\left[\underline{x}(t)\right]\right] = L\left[\mathbb{E}\left[\underline{x}(t)\right]\right]. \tag{10.27}$$

10.3.1 Input-output cross-correlation

The input-output cross-correlation is given by

$$R_{xy}(t_1, t_2) = \mathbb{E}\left[\underline{x}(t_1)\underline{y}(t_2)\right]. \tag{10.28}$$

The output at t_2 is denoted by

$$\begin{aligned} \underline{y}(t_2) &= L\left[\underline{x}(t)\right]|_{t=t_2} && (10.29)\\ &= L_{t_2}\left[\underline{x}(t_2)\right]. && (10.30) \end{aligned}$$

The subscript t_2 on L denotes that the linear system operates on functions of t_2. Any other functions are treated as scalars. Thus,

$$\begin{aligned} \underline{x}(t_2)\underline{y}(t_2) &= \underline{x}(t_1) L_{t_2}\left[\underline{x}(t_2)\right] && (10.31)\\ &= L_{t_2}\left[\underline{x}(t_1)\underline{x}(t_2)\right], && (10.32) \end{aligned}$$

where $\underline{x}(t_1)$ was treated as a scaler in L_{t_2}.

Therefore,

$$\begin{aligned} R_{xy}(t_1, t_2) &= \mathbb{E}\left[\underline{x}(t_1)\underline{y}(t_2)\right] && (10.33)\\ &= \mathbb{E}\left[L_{t_2}\left[\underline{x}(t_1)\underline{x}(t_2)\right]\right] && (10.34)\\ &= L_{t_2}\left[\mathbb{E}\left[\underline{x}(t_1)\underline{x}(t_2)\right]\right] && (10.35)\\ &= L_{t_2}\left[R_{xx}(t_1, t_2)\right]. && (10.36) \end{aligned}$$

10.3.2 Output autocorrelation

THe output autocorrelation

$$R_{yy}(t_1, t_2) = \mathbb{E}\left[\underline{y}(t_1)\underline{y}(t_2)\right] \tag{10.37}$$

is given as follows.

The output at t_1 is given by

$$\underline{y}(t_1) = L_{t_1}\left[\underline{x}(t_1)\right]. \tag{10.38}$$

Therefore,

$$\begin{aligned} \underline{y}(t_1)\underline{y}(t_2) &= L_{t_1}\left[\underline{x}(t_1)\right]\underline{y}(t_2) && (10.39)\\ &= L_{t_1}\left[\underline{x}(t_1)\underline{y}(t_2)\right]. && (10.40) \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}\left[\underline{y}(t_1)\underline{y}(t_2)\right] &= \mathbb{E}\left[L_{t_1}\left[\underline{x}(t_1)\underline{y}(t_2)\right]\right] && (10.41)\\ &= L_{t_1}\left[\mathbb{E}\left[\underline{x}(t_1)\underline{y}(t_2)\right]\right] && (10.42)\\ &= L_{t_1}\left[R_{xy}\left(t_1,t_2\right)\right] && (10.43)\\ &= L_{t_1}\left[L_{t_2}\left[R_{xx}\left(t_1,t_2\right)\right]\right]. && (10.44) \end{aligned}$$

That is,

$$R_{yy}\left(t_1,t_2\right) = L_{t_1}\left[L_{t_2}\left[R_{xx}\left(t_1,t_2\right)\right]\right]. \tag{10.45}$$

Example 10.3.1. Consider the linear system

$$\begin{aligned} \underline{y}(t) &= L\left[\underline{x}(t)\right] && (10.46)\\ &= \underline{x}(t) - \underline{x}(t-1). && (10.47) \end{aligned}$$

Find

1. the mean of $\underline{y}(t)$,
2. the input-output cross-correlation, and
3. the output autocorrelation.

Solution The mean of $\underline{y}(t)$ is given by

$$\begin{aligned} \mathbb{E}\left[\underline{y}(t)\right] &= \mathbb{E}\left[L\left[\underline{x}(t)\right]\right] && (10.48)\\ &= L\left[\mathbb{E}\left[\underline{x}(t)\right]\right] && (10.49)\\ &= \mathbb{E}\left[\underline{x}(t)\right] - \mathbb{E}\left[\underline{x}(t-1)\right]. && (10.50) \end{aligned}$$

The input-output cross-correlation is given by

$$R_{xy}\left(t_1,t_2\right) = L_{t_2}\left[R_{xx}\left(t_1,t_2\right)\right] - R_{xx}\left(t_1,t_2\right). \tag{10.51}$$

Note that inside the system L_{t_2}, the function $R_{xx}\left(t_1,t_2\right)$ is treated as a function of a single variable (t_2).

That is, it's as if a function $g(t_2) :- R_{xx}(t_1, t_2)$ is defined and passed through the system $L_{t_2}[g(t_2)]$.

Alternatively,

$$\begin{aligned} R_{xy}(t_1, t_2) &= \mathbb{E}\left[\underline{x}(t_1)\underline{y}(t_2)\right] && (10.52)\\ &= \mathbb{E}\left[\underline{x}(t_1)\left[\underline{x}(t_2) - \underline{x}(t_2 - 1)\right]\right] && (10.53)\\ &= \mathbb{E}\left[\underline{x}(t_1)\underline{x}(t_2) - \underline{x}(t_1)\underline{x}(t_2 - 1)\right] && (10.54)\\ &= R_{xx}(t_1, t_2) - R_{xx}(t_1, t_2 - 1). && (10.55) \end{aligned}$$

The output autocorrelation is given by

$$\begin{aligned} R_{yy}(t_1, t_2) &= L_{t_1}\left[R_{xy}(t_1, t_2)\right] && (10.56)\\ &= R_{xy}(t_1, t_2) - R_{xy}(t_1 - 1, t_2) && (10.57)\\ &= R_{xx}(t_1, t_2) - R_{xx}(t_1 - 1, t_2) \\ &\quad - R_{xx}(t_1, t_2 - 1) + R_{xx}(t_1 - 1, t_2 - 1). && (10.58) \end{aligned}$$

Example 10.3.2 (Differentiator)**.** Consider the differentiator system L

$$\begin{aligned} \underline{y}(t) &= L\left[\underline{x}(t)\right] && (10.59)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\underline{x}(t). && (10.60) \end{aligned}$$

Find

1. the mean of $\underline{y}(t)$, and
2. the input-output cross-correlation.

Assumption: all realizations of $\underline{x}(t)$ are differentiable.

Solution. The mean of the derivative of $\underline{x}(t)$ is given by

$$\begin{aligned} \mathbb{E}\left[\underline{y}(t)\right] &= \mathbb{E}\left[L\left[\underline{x}(t)\right]\right] && (10.61)\\ &= L\left[\mathbb{E}\left[\underline{x}(t)\right]\right] && (10.62)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[\underline{x}(t)\right]. && (10.63) \end{aligned}$$

The input-output cross-correlation is given by

$$\begin{aligned} R_{xy}(t_1, t_2) &= L_{t_2}\left[R_{xx}(t_1, t_2)\right] && (10.64)\\ &= \frac{\partial}{\partial t_2}R_{xx}(t_1, t_2). && (10.65) \end{aligned}$$

10.3.3 Linear time-invariant systems

Linear time-invariant (LTI) systems are

- characterized by impulse response $h(t)$.
- The output can be expressed as a convolution with the impulse response. That is,

$$\underline{y}(t) = \underline{x}(t) * h(t) \tag{10.66}$$
$$= \int_{-\infty}^{\infty} \underline{x}(t-\alpha)h(\alpha)\mathrm{d}\alpha. \tag{10.67}$$

- The input-output cross-correlation is given by

$$R_{xy}(t_1, t_2) = L_{t_2}\left[R_{xx}(t_1, t_2)\right] \tag{10.68}$$
$$= R_{xx}(t_1, t_2) * h(t_2) \tag{10.69}$$
$$= \int_{-\infty}^{\infty} R_{xx}(t_1, t_2 - \beta)\, h(\beta)\mathrm{d}\beta. \tag{10.70}$$

- The output autocorrelation is given by

$$R_{yy}(t_1, t_2) = L_{t_1}\left[R_{xy}(t_1, t_2)\right] \tag{10.71}$$
$$= R_{xy}(t_1, t_2) * h(t_1) \tag{10.72}$$
$$= \int_{-\infty}^{\infty} R_{xy}(t_1 - \alpha, t_2)\, h(\alpha)\mathrm{d}\alpha \tag{10.73}$$
$$= \int_{\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(t_1 - \alpha, t_2 - \beta)\, h(\alpha)h(\beta)\mathrm{d}\alpha\mathrm{d}\beta. \tag{10.74}$$

Theorem 10.3.2. Consider a LTI system with impulse response $h(\tau)$ such tat $\int_{\infty}^{\infty} h(\tau)\mathrm{d}\tau < \infty$. If the input process $\underline{x}(t)$ is WSS then $\underline{x}(t)$ and the output process $\underline{y}(t)$ are jointly WSS with

$$R_{xy}(\tau) = R_{xx}(\tau) * h(-\tau), \tag{10.75}$$

and

$$R_{yy}(\tau) = R_{xy}(\tau) * h(\tau) \tag{10.76}$$
$$= R_{xx}(\tau) * h(\tau) * h(-\tau). \tag{10.77}$$

10.3.4 Power spectra

Consider a LTI system with the impulse response $h(t)$ such the input $\underline{x}(t)$ is WSS process and the output is $\underline{y}(t)$. Then,

$$R_{xy}(\tau) = R_{xx}(\tau) * h(-\tau)^* \tag{10.78}$$
$$R_{yy}(\tau) = R_{xy}(\tau) * h(\tau). \tag{10.79}$$

Let $H(\omega)$ be the transfer function[1] of the impulse response. Then,

$$S_{xy}(\omega) = S_{xx}(\omega) H^*(\omega), \tag{10.80}$$

$$S_{yy}(\omega) = S_{xy}(\omega) H(\omega) \tag{10.81}$$

$$= S_{xx}(\omega) |H(\omega)|^2 . \tag{10.82}$$

Note that since both $S_{xx}(\omega)$ and $S_{yy}(\omega)$ are positive and real, then it makes sense that $|H(\omega)|^2$, the term multiplying $S_{xx}(\omega)$, is positive and real as well.

[1] The Fourier transform of $h(t)$.

Chapter 11

Point and Counting Processes

11.1 Definitions

Definition 11.1.1 (*Point process*)**.** A *point process* is a collection of random points $\underline{\tau}_i$ on the (positive) time axis such that

$$\underline{\tau}_1 \leq \underline{\tau}_2 \leq \ldots \leq \underline{\tau}_n \leq \ldots. \tag{11.1}$$

Definition 11.1.2 (*Counting random process*)**.** To every point process we can associate a counting-time discrete-state *counting random process* $\underline{x}(t), t \geq 0$ where

$$\underline{x}(t) = \#\text{ of points } \underline{\tau}_i \in (0, t]. \tag{11.2}$$

Definition 11.1.3 (*Renewal process*)**.** To every point process it is possible to associate a continuous-state random sequence $\underline{y}_n$ called a *renewal process* such that

$$\underline{y}_1 = \underline{\tau}_1, \underline{y}_2 = \underline{\tau}_2 - \underline{\tau}_1, \ldots, \underline{y}_n = \underline{\tau}_n - \underline{\tau}_{n-1}. \tag{11.3}$$

In other words,

$$\underline{y}_n = \begin{cases} \underline{\tau}_1, & n = 1, \\ \underline{\tau}_n - \underline{\tau}_{n-1}, & n > 1. \end{cases} \tag{11.4}$$

11.2 Poisson counting processes

Definition 11.2.1 (*Poisson point process*)**.** A *Poisson point process* is a point process with the following two properties:

1. The number of points in any non-overlapping time intervals are independent random variables;
2. The number $\underline{n}(t_1, t_2)$ of points $\underline{\tau}_i$ in an interval with endpoints t_1, t_2, and length $t = t_2 - t_1$ is a Poisson random variable with parameter λt. That is,

$$P\left(\underline{n}(t_1, t_2) = k\right) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}. \tag{11.5}$$

Note that there parameter λ is the *density* of the points, or the average number of points per unit interval.

Definition 11.2.2 (*Poisson process*)**.** A *Poisson process* is a counting process corresponding to a Poisson point process. That is, for a given Poisson point process $(\underline{\tau}_i)$, the Poisson process $\underline{x}(t)$ is

$$\underline{x}(t) = \underline{n}(0, t) \tag{11.6}$$
$$= \#\text{ of points } \underline{\tau}_i in (0, t]. \tag{11.7}$$

The Poisson process $\underline{x}(t)$ is a counting process with increments that are

1. independent, and
2. Poisson distributed. That is, $\underline{x}(t_n) - \underline{x}(t_{n-1})$ is a Poisson random variable with parameter $\lambda(t_n - t_{n-1})$, where λ is the *rate* of the process.

Remark 11.2.1 (Poisson process)**.** Note that the parameter λ is "rate", or the density of the point process.

For a fixed t, $\underline{x}(t)$ is

- Poisson random variable with parameter λt,
- $\mathbb{E}\left[\underline{x}(t)\right] = \eta(t) = \lambda t$,
- $\text{Var}\left[\underline{x}(t)\right] = \lambda t$.

Theorem 11.2.1 (Counting random process)**.** Let $\underline{x}(t)$ be a counting random process with the properties:

1. The following expression holds

$$P\left(n \text{ points in } (t_1, t_1 + \delta)\right) = P\left(n \text{ points in } (t_2, t_2 + \delta)\right) \tag{11.8}$$

for all n, t_1, t_2, and δ;

2. The probability

$$P\left(n \text{ points in } (t_1, t_1 + \delta)\right) \tag{11.9}$$

does not depend on the number and locations of points in $(0, t]$;

3. $\lim_{\delta \to 0} P\left(\underline{x}(t+\delta) - \underline{x}(t) > 1\right) = 0$;
4. $0 < P\left(\underline{x}(t) = 0\right) < 1$ for all $t > 0$;
5. $\underline{x}(0) = 0$.

Then $\underline{x}(t)$ is a Poisson random process.

Theorem 11.2.2 (Aucorrelation of Poisson process)**.** The *autocorrelation* function of a Poisson process with rate λ is given by

$$R\left(t_1, t_2\right) = \lambda \min(t_1, t_2) + \lambda^2 t_1 t_2. \tag{11.10}$$

Theorem 11.2.3 (Autocovariance of Poisson process)**.** The *autocovariance* of a Poisson process is given by

$$\begin{aligned} C\left(t_1, t_2\right) &= R\left(t_1, t_2\right) - \eta(t_1)\eta(t_2) && (11.11) \\ &= \lambda \min(t_1, t_2). && (11.12) \end{aligned}$$

Theorem 11.2.4. If $\underline{x}_1(t)$ and $\underline{x}_2(t)$ are two independent Poisson random processes with rates λ_1 and λ_2, respectively, then

1. the process $\underline{x}_1 + \underline{x}_2(t)$ is also a Poisson random process with rate $\lambda_1 + \lambda_2$, and
2. the underlying points process is a Poisson point process with density $\lambda_1 + \lambda_2$.

Theorem 11.2.5 (Random selection of Poisson points)**.** Consider a Poisson points process with density λ and suppose that each point is independently tagged with probability p. Let $\underline{y}(t)$ be the number of *tagged* points in $(0, t]$. That is,

$$\underline{y}(t) = \text{ \# of tagged points in } (0, t]. \tag{11.13}$$

Then,

1. $\underline{y}(t)$ is a Poisson random process with rate λp, and
2. the underlying point process is a Poisson point process with density λp.

Property 11.2.1 (Memoryless property)**.** By definition, the number of arrivals in $(t, t+\delta]$ is independent of the number of arrivals in $(0, t]$. That is also true for any $\delta > 0$.

Essentially, the probability of an arrival at any instant is independent of the past history of the process.

Definition 11.2.3 (*Interarrival time sequence*)**.** Assume a Point point process $(\underline{\tau}_i)$ and the associated renewal process $\underline{y}_1, \ldots, \underline{y}_n, \ldots$. Then, the process $\underline{y}_n$ is called the *interarrival time* sequence.

Theorem 11.2.6. For a Poisson point process with density λ, the interarrival times sequence $\underline{y}_1, \ldots, \underline{y}_n, \ldots$ consists of i.i.d. random variables distributed according to the exponential distribution. That is,

$$f(y_n) = \begin{cases} \lambda e^{-\lambda y_n}, & y_n \geq 0, \\ 0, y_n < 0. \end{cases} \tag{11.14}$$

11.3 Random walks

Consider a random sequence $\underline{x}_i$, where $\underline{x}_i$ are i.i.d., and

$$\underline{x}_i = \begin{cases} 1, & \text{w.p. } p, \\ -1, & \text{w.p. } q = 1 - p. \end{cases} \tag{11.15}$$

Definition 11.3.1 (*Random walk*)**.** A *random walk* is a random process defined as

$$\underline{w}_0 = 0, \tag{11.16}$$

$$\underline{w}_n = \sum_{i=1}^{n} \underline{x}_i. \tag{11.17}$$

Remark 11.3.1. Terminology:

- Symmetric walk if $p = q$,
- unsymmetrical walk if $p \neq q$.

Chapter 12

Markov processes

12.1 Definitions

Definition 12.1.1 (*Markov process*)**.** A random process $\underline{x}(t)$ is a *Markov process* if "the future of the process given the present is independent of the past". That is,

$$F\left(x_{k+1}; t_{k+1} \mid x_k, \ldots, x_1; t_k, \ldots, t_1\right) = F\left(x_{k+1}; t_{k+1} \mid x_k; t_k\right). \tag{12.1}$$

Remark 12.1.1. Terminology:

- *Markov sequence* is a discrete time Markov process.
- *Markov chain* is a discrete-value Markov process.

Definition 12.1.2 (*Markov chain*)**.** A discrete-time Markov chain $\underline{x}_n$, $n = 0, \ldots$ is a discrete time discrete-value random sequence such that

$$P\left(\underline{x}_{n+1} = j \mid \underline{x}_n = i, \underline{x}_{n-1} = i_{n-1}, \ldots, \underline{x}_0 = i_0\right) = P\left(\underline{x}_{n+1} = j \mid \underline{x}_n = i\right). \tag{12.2}$$

Remark 12.1.2 (Terminology)**.**

- The set of states or state space $\mathcal{X}$ is a set of values that $\underline{x}_n$ can take. Assumption, $\mathcal{X} \subset \mathcal{Z}$.
- The state at time n is denoted by $\underline{x}_n$.
- If $\underline{x}_n = i$ the it is said that at time n the chain is at state i.
- $p_{ij}(n) = p_{i \to j}(n) = P\left(\underline{x}_n = j \mid \underline{x}_{n-1} = i\right)$ is the *one-step transition probability* from state i to state j at time n.
- Homogeneous transition probabilities do not change with n. That is,

$$p_{ij}(n) = p_{ij}. \tag{12.3}$$

- From the axioms of probability,

$$p_{ij} \geq 0, \tag{12.4}$$

$$\sum_j p_{ij} = 1. \tag{12.5}$$

- *State diagram*: a graph with nodes representing the different states of the chain and directional arcs for all pairs of states (i, j) with $p_{ij} > 0$.
- *Finite Markov chain*: number of states is finite. That is,

$$\mathcal{X} = \{0, \ldots, M\}. \tag{12.6}$$

- *Transition matrix* **P**

$$\mathbf{P} = \begin{bmatrix} p_{00} & \cdots & p_{0M} \\ \vdots & \ddots & \vdots \\ p_{M0} & \cdots & p_{MM}. \end{bmatrix} \tag{12.7}$$

  Clearly, the rows of **P** must sum to 1.

Definition 12.1.3 (*Right stochastic matrix*)**.** A square matrix with non-negative elements is called

- right stochastic if its rows sum up to 1,
- left stochastic if its columns sum up to 1,
- doubly stochastic if it is both left and right stochastic.

Property 12.1.1 (Transition matrix)**.** The transition matrix **P** is a right stochastic matrix. That is,

$$\begin{bmatrix} p_{00} & \cdots & p_{0M} \\ \vdots & \ddots & \vdots \\ p_{M0} & \cdots & p_{MM}. \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{12.8}$$

Theorem 12.1.1 (Eigenvalues of right stochastic matrix)**.** A right stochastic matrix always has an eigenvalue equal to 1. For any other eigenvalue λ, it satisfies

$$|\lambda| \leq 1. \tag{12.9}$$

It should be noted that a stochastic matrix can have complex eigenvalues.

Definition 12.1.4 (*n-step transition probaiblity*)**.** The n-step transition probability $p_{ij}^{(n)}$ is the probability that the state will reach state j in n steps beginning from state i. That is,

$$p_{ij}^{(n)} = P\left(\underline{x}_n \mid \underline{x}_0 = i\right). \tag{12.10}$$

Remark 12.1.3. It should be noted that the n-step transition probabilities are homogeneous. Therefore,

$$\begin{aligned} p_{ij}^{(n)} &= P\left(\underline{x}_n \mid \underline{x}_0 = i\right) && (12.11)\\ &= P\left(\underline{x}_{n+m} \mid \underline{x}_m = i\right). && (12.12) \end{aligned}$$

Theorem 12.1.2 (Chapman-Kologorov)**.** For any states i, j, and n, m¿0,

$$p_{ij}^{(n+m)} = \sum_{k\in\mathcal{X}} p_{ik}^{(n)} p_{kj}^{(m)}. \tag{12.13}$$

Remark 12.1.4. The nth transition probability is given by

$$\mathbf{P}^{(n)} = \mathbf{P}^n. \tag{12.14}$$

Definition 12.1.5 (*State probability*)**.** The jth *state probability* $\pi_j(n)$ at time n is defined as

$$\pi_j(n) = P\left(\underline{x}_n = j\right). \tag{12.15}$$

The following holds.

$$\begin{aligned} \pi_j(n) &= \sum_{i\in\mathcal{X}} P\left(\underline{x}_n = j, \underline{x}_{n-1} = i\right) && (12.16)\\ &= \sum_{i\in\mathcal{X}} \underbrace{P\left(\underline{x}_n = j \mid \underline{x}_{n-1} = i\right)}_{P_{ij}} \underbrace{P\left(\underline{x}_{n-1} = i\right)}_{\pi_i(n-1)} && (12.17)\\ &= \sum_i \pi_i(n-1) P_{ij} && (12.18) \end{aligned}$$

For a finite state Markov chain,

- the nth state probability vector (row matrix really) is given by

$$\Pi(n) = \begin{bmatrix} \pi_0(n) & \dots & \pi_M(n) \end{bmatrix}. \tag{12.19}$$

- The following expression holds

$$\begin{aligned} \Pi(n) &= \Pi(n-1)\mathbf{P} && (12.20)\\ &= \Pi(0)\mathbf{P}^n. && (12.21) \end{aligned}$$

Definition 12.1.6 (*Steady state probability*)**.** The limiting (or steady) state probability $\bar{\pi}_j$ is defined as

$$\bar{\pi}_j = \lim_{n\to\infty} \pi_j(n) \tag{12.22}$$

if the limit exists.

In many cases, it is of interest to analyze the behavior of the chain after a very long period of time. For finite state Markov chains, the limiting state probability vector is

$$\bar{\Pi} = \begin{bmatrix} \bar{\pi}_0 & \dots & \bar{\pi}_M \end{bmatrix}. \tag{12.23}$$

Note that the limit may or may not exist and if they exist, they may or may not depend on the initial state probabilities.

Chapter 13

Ergodicity

Suppose that the mean μ_n of a random sequence $\underline{x}_n$ is to be estimated. Then, the process usually involves

1. obtaining M realizations $x_n^{(1)}, \ldots, x_n^{(M)}$ of $\underline{x}_n$, $n = 1, \ldots$, and then
2. estimate the mean using "ensemble" averaging

$$\hat{\mu}_n = \frac{1}{M} \sum_{m=1}^{M} x_n^{(m)}. \tag{13.1}$$

Suppose however, that there is access to only a *single* realization of the random sequence. Then it is only possible to perform time averaging; it is no longer possible to perform ensemble averaging. The question that ergodicity tries to answer is "is it possible to substitute ensemble averages with time averages?"

Case 1

- Random sequence $\underline{x}_n$ consists of i.i.d. random variables such that $\mathbb{E}[\underline{x}_n] = \mu$.
- Single realizations $x_1, \ldots, x_N$.
- The mean can then be estimated using $\hat{\mu} = \frac{1}{N} \sum_{n=1}^{N} x_n$.
- The corresponding estimator converges to μ in the mean squared sense.

Case 2

- The random sequence $\underline{x}_n = \underline{a}$ with $\underline{x}_n = \mathbb{E}[\underline{a}] = \mu$.
- Single realization $x_1 = a$, $x_2 = a, \ldots, x_N = a$.
- Estimate of the mean is $\hat{\mu} = \frac{1}{N} \sum_{n=1}^{N} x_n = a$.
- In this case, time averaging will **not** work.
- MSE of the corresponding estimator is

$$\mathbb{E}\left[(\underline{a} - \mu)^2\right] = \mathrm{Var}[\underline{a}] \tag{13.2}$$

$$\neq 0. \tag{13.3}$$

Definition 13.0.1 (*Time average estimator*)**.** The *time average estimator* of the mean of a random process $\underline{x}(t), t \in \mathbb{R}$ is

$$\hat{\mu}_T = \frac{1}{2T} \int_{-T}^{T} \underline{x}(t) \mathrm{d}t. \tag{13.4}$$

It is assumed that all realizations of $\underline{x}(t)$ are integrable.

Definition 13.0.2 (*Ergodic*)**.** A WSS process $\underline{x}(t)$ is called *ergodic* in the mean if the time average of $\underline{x}(t)$ converges (in the mean square sense) to the mean μ of the process. That is,

$$\lim_{T \to \infty} \hat{\mu}_T = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \underline{x}(t) \mathrm{d}t \tag{13.5}$$

$$= \mu. \tag{13.6}$$

Remark 13.0.1. For an ergodic process, the expectation of the ergodic mean is given by

$$\mathbb{E}\left[\underline{\hat{\mu}}_T\right] = \frac{1}{2T} \int_{-T}^{T} \mathbb{E}\left[\underline{x}(t)\right] \mathrm{d}t \tag{13.7}$$

$$= \mu. \tag{13.8}$$

The estimator $\underline{\hat{\mu}}_T$ is an unbiased estimator of μ. Furthermore,

$$\mathrm{Var}\left[\underline{\hat{\mu}}_T\right] = \mathbb{E}\left[\left|\underline{\hat{\mu}}_T - \mu\right|^2\right] \tag{13.9}$$

$$\overset{T \to \infty}{\to} 0. \tag{13.10}$$

Finally,

$$\mathrm{Var}\left[\underline{\hat{\mu}}_T\right] = \frac{1}{2T} \int_{-2T}^{2T} C_{xx}(\tau) \left(1 - \frac{|\tau|}{2T}\right) \mathrm{d}\tau. \tag{13.11}$$

Theorem 13.0.1. A WSS process $\underline{x}(t)$ is ergodic in the mean if and only if

$$\frac{1}{2T} \int_{-2T}^{2T} C_{xx}(\tau) \left(1 - \frac{|\tau|}{2T}\right) \mathrm{d}\tau \overset{T \to \infty}{\to} 0. \tag{13.12}$$

Equivalently, if and only if

$$\frac{1}{2T}\int_{-2T}^{2T} C_{xx}(\tau)\,\mathrm{d}\tau \overset{T\to\infty}{\to} 0. \tag{13.13}$$

And one more time, if and only if

$$\int_{-\infty}^{\infty} |C_{xx}(\tau)|\,\mathrm{d}\tau < \infty. \tag{13.14}$$

Theorem 13.0.2. A WSS process $\underline{x}(t)$ is ergodic in the mean if

$$C_{xx}(\tau) \overset{\tau\to\infty}{\to} 0, \tag{13.15}$$

or

$$R_{xx}(\tau) \overset{\tau\to\infty}{\to} \mu^2. \tag{13.16}$$

Appendices

Appendix A

Linear Algebra and Calculus

A.1 Schur complement

Theorem A.1.1 (Schur's complement)**.** Let

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12} & \mathbf{A}_{22} \end{bmatrix} \tag{A.1}$$

be a square matrix. If $\mathbf{A}_{11}$ is nonsingular, then the Schur complement of the block $\mathbf{A}_{11}$ of the matrix $\mathbf{A}$ is the matrix

$$\mathbf{A}/\mathbf{A}_{11} := \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}. \tag{A.2}$$

Similarly, if $\mathbf{A}_{22}$ is nonsingular, then the Schur complement of the block $\mathbf{A}_{22}$ of the matrix $\mathbf{A}$ is the matrix

$$\mathbf{A}/\mathbf{A}_{22} := \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}. \tag{A.3}$$

△

A.1.1 Conditioning using covariance matrix

Given a the random variable

$$\begin{bmatrix} \underline{\mathbf{x}}_1 \\ \underline{\mathbf{x}}_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \bar{\mathbf{x}}_1 \\ \bar{\mathbf{x}}_2 \end{bmatrix}, \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{12}^{\mathsf{T}} & \mathbf{\Sigma}_{22} \end{bmatrix} \right). \tag{A.4}$$

Then, the marginal covariance of $\underline{\mathbf{x}}_1$ is $\mathbf{\Sigma}_{11}$. However, the covariance of $\underline{\mathbf{x}}_1$ *given* $\underline{\mathbf{x}}_2 = \hat{\mathbf{x}}_2$ is given by the Schur complement. Specifically,

$$\mathbb{E}\left[\underline{\mathbf{x}}_1 | \underline{\mathbf{x}}_2 = \hat{\mathbf{x}}_2\right] = \bar{\mathbf{x}}_1 + \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\left(\hat{\mathbf{x}}_2 - \bar{\mathbf{x}}_2\right), \tag{A.5}$$

$$\mathrm{Cov}\left[\underline{\mathbf{x}}_1 | \underline{\mathbf{x}}_2 = \hat{\mathbf{x}}_2\right] = \mathbf{\Sigma}/\mathbf{\Sigma}_{22} \tag{A.6}$$

$$= \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{12}^{\mathsf{T}}. \tag{A.7}$$

Note that this is equivalent to solving the linear system of equations

$$\begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{12}^{\mathsf{T}} & \mathbf{\Sigma}_{22} \end{bmatrix} \begin{bmatrix} \mathbb{E}\left[\underline{\mathbf{x}}_1 | \underline{\mathbf{x}}_2 = \hat{\mathbf{x}}_2\right] \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{x}}_1 \\ \bar{\mathbf{x}}_2 - \hat{\mathbf{x}}_2 \end{bmatrix}. \tag{A.8}$$

I'm not sure what's the relation between (A.8) and (A.5).

A.1.2 Conditioning using information matrix

On the other hand, if we are given the same random variable

$$\begin{bmatrix} \underline{\mathbf{x}}_1 \\ \underline{\mathbf{x}}_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \bar{\mathbf{x}}_1 \\ \bar{\mathbf{x}}_2 \end{bmatrix}, \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^{\mathsf{T}} & \mathbf{A}_{22} \end{bmatrix}^{-1} \right), \tag{A.9}$$

where

$$\mathbf{A} = \mathbf{\Sigma}^{-1} \tag{A.10}$$

is the *information matrix*. Then, conditioning is easily done by partitioning the system. That is,

$$\operatorname{Cov}\left[\underline{\mathbf{x}}_1 | \underline{\mathbf{x}}_2 = \hat{\mathbf{x}}_2\right] = \mathbf{A}_{11}^{-1}. \tag{A.11}$$

On the other hand, marginalization is more "difficult" and is given by the Schur complement. Specifically,

$$\operatorname{Cov}\left[\underline{\mathbf{x}}_1\right] = (\mathbf{A}/\mathbf{A}_{22})^{-1} \tag{A.12}$$

$$= \left(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{12}^{\mathsf{T}}\right)^{-1}. \tag{A.13}$$

A.2 Liebniz rule for integration

Definition A.2.1 (Liebniz rule)**.**

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{a(x)}^{b(x)} f(x,t)\mathrm{d}t\right) = f(x,b(x)) \cdot \frac{\mathrm{d}}{\mathrm{d}x} b(x) - f(x,a(x)) \cdot \frac{\mathrm{d}}{\mathrm{d}x} a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\mathrm{d}t. \tag{A.14}$$

Appendix B

Numerically sampling from a normal distribution

Say we want to sample realizations of a normal random variable $\underline{\mathbf{x}} \sim \mathcal{N}(\boldsymbol{\mu}_x, \boldsymbol{\Sigma})$, given the mean $\boldsymbol{\mu}_x$ and covariance $\boldsymbol{\Sigma}_x$. How can this be done using MATLAB's `randn` function?

B.1 MATLAB's `randn` function

The `randn` function produces a realization of a normal random variable with zero mean and unit variance. How to compute the variance of such variable? Well, that depends on how the realizations are stored in MATLAB. That is, the realizations can be stored as columns or as rows.

Say the random variable has n degrees of freedom and we want m realizations. Thus, if the realizations are stored as columns, then the matrix is of size $n \times m$. On the other hand, if the realizations are stored as rows, then the matrix of size $m \times n$. Specifically,

$$\mathbf{x}_{\text{unit,col}} = \texttt{randn(n,m)} \in \mathbb{R}^{n \times m}, \tag{B.1}$$

$$\mathbf{x}_{\text{unit,row}} = \texttt{randn(m,n)} \in \mathbb{R}^{m \times n}. \tag{B.2}$$

The covariance of $\mathbf{x}_{\text{unit}}$ can be approximated numerically as

$$\begin{aligned} \text{Cov}\left[\mathbf{x}_{\text{unit}}\right] &= \boldsymbol{\Sigma}_{\text{unit}} && \text{(B.3)} \\ &= \mathbf{1}_{n \times n} && \text{(B.4)} \\ &\approx \frac{1}{m-1} \mathbf{x}_{\text{unit,col}} \mathbf{x}_{\text{unit,col}}^{\mathsf{T}} && \text{(B.5)} \\ &= \frac{1}{m-1} \mathbf{x}_{\text{row,unit}}^{\mathsf{T}} \mathbf{x}_{\text{row,unit}}. && \text{(B.6)} \end{aligned}$$

Well, what if we want a realization of a normal random variable with zero mean but non-unit variance $\boldsymbol{\Sigma}_x$? The covariance is then

$$\text{Cov}\left[\mathbf{x}\right] = \boldsymbol{\Sigma}_x \tag{B.7}$$

$$= \mathbf{L} \underbrace{\boldsymbol{\Sigma}_{\text{unit}}}_{\mathbf{1}} \mathbf{L}^{\mathsf{T}} \tag{B.8}$$

$$= \mathbf{R}^{\mathsf{T}} \underbrace{\boldsymbol{\Sigma}_{\text{unit}}}_{\mathbf{1}} \mathbf{R}, \tag{B.9}$$

where $\mathbf{L}$ and $\mathbf{R}$ are the lower- and upper-triangular matrices obtained from the Cholesky factorization. So which of these two should we use? Well, the answer depends on how the realizations of $\underline{\mathbf{x}}$ are stored.

If the realizations are stored in columns, then from (B.5),

$$\begin{aligned}
\text{Cov}\left[\mathbf{x}\right] &= \boldsymbol{\Sigma}_x && \text{(B.10)}\\
&= \mathbf{L}\boldsymbol{\Sigma}_{\text{unit}}\mathbf{L}^{\mathsf{T}} && \text{(B.11)}\\
&\approx \mathbf{L}\mathbf{x}_{\text{unit,col}}\mathbf{x}_{\text{unit,col}}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}} && \text{(B.12)}\\
&= \underbrace{\left(\mathbf{L}\mathbf{x}_{\text{unit,col}}\right)}_{\mathbf{x}_{\text{col}}} \left(\mathbf{L}\mathbf{x}_{\text{unit,col}}\right)^{\mathsf{T}} && \text{(B.13)}\\
&= \mathbf{x}_{\text{col}}\mathbf{x}_{\text{col}}^{\mathsf{T}}, && \text{(B.14)}
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{x}_{\text{col}} &= \mathbf{L}\mathbf{x}_{\text{unit,col}} && \text{(B.15)}\\
&= \mathbf{L}\texttt{randn(n,m)}. && \text{(B.16)}
\end{aligned}$$

It should be noted that if $\mathbf{R}$ is used in place of $\mathbf{L}$, then the results would not be the same in general. The upper triangular matrix $\mathbf{R}$ can still be used by setting

$$\mathbf{L} = \mathbf{R}^{\mathsf{T}}. \tag{B.17}$$

Specifically,

$$\begin{aligned}
\text{Cov}\left[\mathbf{x}\right] &= \boldsymbol{\Sigma}_x && \text{(B.18)}\\
&= \mathbf{R}^{\mathsf{T}}\boldsymbol{\Sigma}_{\text{unit}}\mathbf{R} && \text{(B.19)}\\
&\approx \mathbf{R}^{\mathsf{T}}\mathbf{x}_{\text{unit,row}}^{\mathsf{T}}\mathbf{x}_{\text{unit,row}}\mathbf{R} && \text{(B.20)}\\
&= \left(\mathbf{x}_{\text{unit,row}}\mathbf{R}\right)^{\mathsf{T}} \underbrace{\left(\mathbf{x}_{\text{unit,row}}\mathbf{R}\right)}_{\mathbf{x}_{\text{row}}} && \text{(B.21)}\\
&= \mathbf{x}_{\text{row}}^{\mathsf{T}}\mathbf{x}_{\text{row}}, && \text{(B.22)}
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{x}_{\text{row}} &= \mathbf{x}_{\text{unit,row}}\mathbf{R} && \text{(B.23)}\\
&= \texttt{randn(m,n)}\,\mathbf{R}. && \text{(B.24)}
\end{aligned}$$

Summary:

$$\begin{aligned}
\boldsymbol{\Sigma}_x &= \mathbf{R}^\mathsf{T}\mathbf{R} && \text{(B.25)}\\
&= \mathbf{L}\mathbf{L}^\mathsf{T}, && \text{(B.26)}\\
\mathbf{x}_{\text{unit,col}} &= \texttt{randn(n,m)} && \text{(B.27)}\\
&= \mathbf{x}_{\text{unit,row}}^\mathsf{T}, && \text{(B.28)}\\
\mathbf{x}_{\text{col}} &= \mathbf{L}\mathbf{x}_{\text{unit,col}}, && \text{(B.29)}\\
&= \mathbf{x}_{\text{row}}^\mathsf{T}, && \text{(B.30)}\\
\mathbf{x}_{\text{row}} &= \mathbf{x}_{\text{unit,row}}\mathbf{R}. && \text{(B.31)}
\end{aligned}$$

B.2 Sampling (another derivation)

Assume we have N realizations $\mathbf{x} \in \mathbb{R}^{n\times N}$ of the random variable $\underline{\mathbf{x}} \in \mathbb{R}^n$. For simplicity, we'll assume the random variable has zero mean. Let's assume that there exists a matrix $\mathbf{R}$ such that

$$\underline{\mathbf{x}} = \mathbf{R}\underline{\mathbf{x}}_0, \tag{B.32}$$

where $\underline{\mathbf{x}}_0 \sim N(\mathbf{0}, \mathbf{1})$. Then,

$$\begin{aligned}
\boldsymbol{\Sigma} &= \mathbb{E}\left[\underline{\mathbf{x}}\underline{\mathbf{x}}^\mathsf{T}\right] && \text{(B.33)}\\
&= \mathbf{R}\mathbb{E}\left[\underline{\mathbf{x}}_0\underline{\mathbf{x}}_0^\mathsf{T}\right]\mathbf{R}^\mathsf{T} && \text{(B.34)}\\
&= \mathbf{R}\mathbf{R}^\mathsf{T}. && \text{(B.35)}
\end{aligned}$$

well, how can we construct $\mathbf{R}$? Decompose the covariance matrix using eigendecomposition. Specifically,

$$\begin{aligned}
\boldsymbol{\Sigma} &= \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\mathsf{T} && \text{(B.36)}\\
&= \underbrace{\mathbf{U}\sqrt{\boldsymbol{\Lambda}}}_{\mathbf{R}}\mathbf{1}\underbrace{\sqrt{\boldsymbol{\Lambda}}\mathbf{U}^\mathsf{T}}_{\mathbf{R}^\mathsf{T}} && \text{(B.37)}\\
&= \mathbf{R}\mathbf{R}^\mathsf{T}, && \text{(B.38)}
\end{aligned}$$

where $\mathbf{U}$ and $\boldsymbol{\Lambda}$ are the eigenvector and the eigenvalue matrices, respectively. Thus,

$$\mathbf{R} = \mathbf{U}\sqrt{\boldsymbol{\Lambda}}. \tag{B.39}$$

Therefore, the random variable $\underline{\mathbf{x}}$ can be sampled by sampling from the unit normal random variable $\underline{\mathbf{x}}_0$ and scaling it using

$$\mathbf{x} = \mathbf{U}\sqrt{\boldsymbol{\Lambda}}\mathbf{x}_0. \tag{B.40}$$

B.3 Covariance ellipses

To plot covariance ellipses (in 2D), eigendecomposition is used. Specifically, given a covariance matrix $\mathbf{\Sigma}$, then decompose

$$\mathbf{\Sigma} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathsf{T}}, \tag{B.41}$$

where $\mathbf{U}^{-1} = \mathbf{U}^{\mathsf{T}}$ because $\mathbf{\Sigma} = \mathbf{\Sigma}^{\mathsf{T}}$ is symmetric. The standard deviation ellipses[1] are the computed as

$$\boldsymbol{\sigma} = \mathbf{U}\sqrt{\mathbf{\Lambda}} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \qquad \theta \in [0, 2\pi). \tag{B.42}$$

Figure B.1 shows an example of such ellipses.

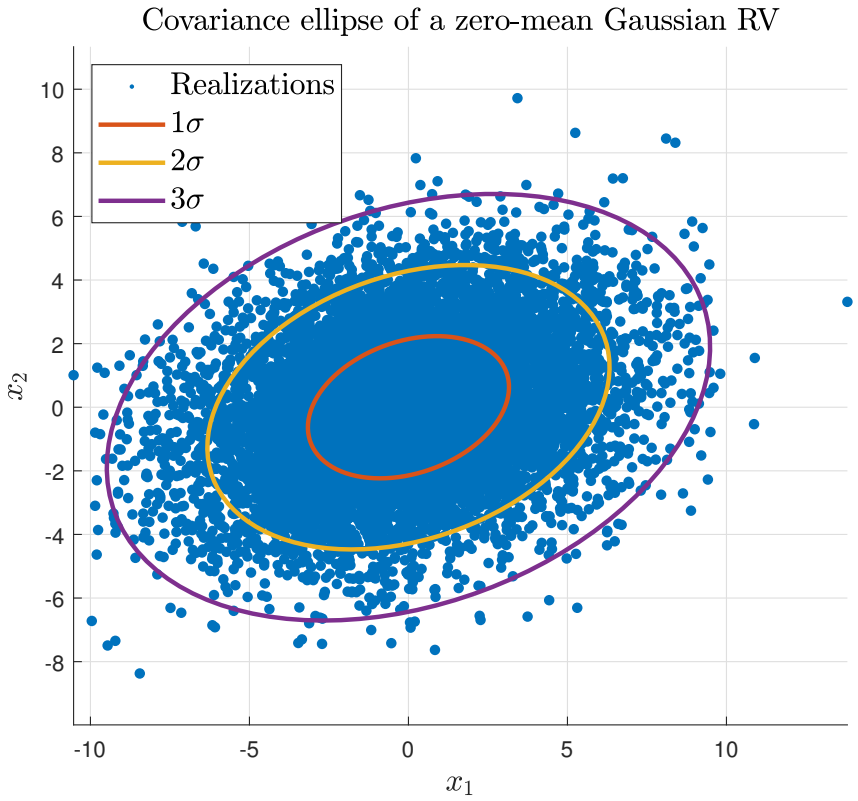


Figure B.1: Covariance ellipses (actually, standard deviation ellipses) around $N = 10^4$ samples.

B.4 Generating positive definite matrices

Symmetric positive definite (SPD) matrices are a subset of Matrix Lie Groups [8]. The Lie algebra is the set of all symmetric matrices. Thus, taking the exponential map of a symmetric matrix gives a symmetric positive definite matrix [8, Prop. 3.4].

Proposition B.4.1 (Exponential map of symmatric matrices)**.** From [8, Prop. 3.4]. Let $\mathbf{S} \in \mathrm{Sym}(n)$ be a

[1] I'm not sure about the naming here. But the ellipse computed represents the 1σ bounds.

symmetric matrix. Then,

$$\exp(\mathbf{S}) \in \mathrm{Sym}_\star^+(n), \tag{B.43}$$

where $\mathrm{Sym}_\star^+$ is the set of symmetric positive definite matrices.

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