COM1002: Foundations of Computer Science
Problem Sheet 9: Stochastic Processes

1. Consider the difference equation $Av_{k-1} = v_k$ where

$$A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Find v_7.

ANSWER:

Note that A has eigenvalues 1 and 2.

Let $(x, y)^T$ be a 1-eigenvector. Then

$$\begin{aligned} x &= x \\ 3x + 2y &= y \end{aligned}$$

that is to say $y = -3x$. So $(1, -3)^T$ is a 1-eigenvector.

Let $(x, y)^T$ be a 2-eigenvector. Then

$$\begin{aligned} x &= 2x \\ 3x + 2y &= 2y \end{aligned}$$

that is to say $x = 0$. So $(0, 1)^T$ is a 2-eigenvector.

Let

$$P = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then $P^{-1}AP = D$, so $A = PDP^{-1}$, and $A^7 = PD^7P^{-1}$.

Now by the usual methods,

$$P^{-1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \qquad D^7 = \begin{pmatrix} 1 & 0 \\ 0 & 128 \end{pmatrix}$$

so

$$A^7 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 128 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 381 & 128 \end{pmatrix}$$

and

$$v_7 = A^7 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 509 \end{pmatrix}.$$

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2. Find the eigenvalues of the matrix

$$M := \begin{pmatrix} 0.87 & 0.19 \\ 0.13 & 0.81 \end{pmatrix}.$$

ANSWER:

The matrix M is stochastic, so 1 is an eigenvalue. Hence $t-1$ is a factor of the characteristic polynomial, which is

$$\chi_M(t) = \begin{vmatrix} 0.87-t & 0.19 \\ 0.13 & 0.81-t \end{vmatrix} = t^2 - 1.68t + 0.7047 - 0.0247 = (t-1)(t-0.68).$$

So M has eigenvalues 1 and 0.68.

3. (a) Find the eigenvalues of the matrix

$$M = \begin{pmatrix} \frac{95}{100} & \frac{3}{100} \\ \frac{5}{100} & \frac{97}{100} \end{pmatrix}$$

(b) For each eigenvalue, find a corresponding eigenvector.

(c) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}MP = D$.

ANSWER:

(a) The matrix M is stochastic, and so has 1 as an eigenvalue. The characteristic polynomial of M is

$$\chi_M(t) = \begin{vmatrix} \frac{95}{100}-t & \frac{3}{100} \\ \frac{5}{100} & \frac{97}{100}-t \end{vmatrix} = \left(t - \tfrac{95}{100}\right)\left(t - \tfrac{97}{100}\right) - \left(\tfrac{3}{100}\right)\left(\tfrac{5}{100}\right) = t^2 - \tfrac{192}{100}t + \tfrac{95.97-15}{10000}$$
$$= t^2 - \tfrac{192}{100}t + \tfrac{92}{100} = (t-1)\left(t - \tfrac{92}{100}\right).$$

Thus the eigenvalues of M are 1 and $\frac{92}{100}$.

(b) To find an eigenvector of M corresponding to the eigenvalue $\lambda_1 := 1$, we need to solve the system of linear equations $(M - I_2)\begin{pmatrix} x & y \end{pmatrix}^T = 0$. This has augmented matrix

$$\left(M - I_2 | 0\right) = \left(\begin{array}{cc|c} -\frac{5}{100} & \frac{3}{100} & 0 \\ \frac{5}{100} & -\frac{3}{100} & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -\frac{5}{100} & \frac{3}{100} & 0 \\ 0 & 0 & 0 \end{array}\right),$$

and so $w_1 := (3,5)^T$ is an eigenvector of M corresponding to the eigenvalue 1.

To find an eigenvector of M corresponding to the eigenvalue $\lambda_2 := \frac{92}{100}$, we need to solve the system of linear equations $\left(M - \frac{92}{100}I_2\right)\begin{pmatrix} x & y \end{pmatrix}^T = 0$. This has augmented matrix

$$\left(M - \tfrac{92}{100}I_2 | 0\right) = \left(\begin{array}{cc|c} \frac{3}{100} & \frac{3}{100} & 0 \\ \frac{5}{100} & \frac{5}{100} & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} \frac{3}{100} & \frac{3}{100} & 0 \\ 0 & 0 & 0 \end{array}\right),$$

and so $w_2 := (1,-1)^T$ is an eigenvector of M corresponding to the eigenvalue $\frac{92}{100}$.

(c) Set

$$P = \begin{pmatrix} 3 & 1 \\ 5 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{92}{100} \end{pmatrix}$$

Then $PMP^{-1} = D$.

4. Over a period of 5 minutes, in a typical COM1002 lecture, 90% of students who are awake at the beginning of the 5-minute period will still be so at the end of it (but the other 10% will fall asleep) and 90% of students who are asleep at the beginning of the 5-minute period will still be so at the end of it (and the other 10% will wake up). If all the students are awake at the beginning of the lecture, what percentage will be awake 50 minutes later?

ANSWER:

For each $k = 0, \ldots, 10$, let a_k and b_k be the proportions of students who are awake and who are asleep after $5k$ minutes of the lecture, respectively, and set $v_k = (\ a_k \quad b_k\ )^T$. We have

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a_{k+1} \\ b_{k+1} \end{pmatrix} = \begin{pmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{9}{10} \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix}$$

for $k = 0, \ldots, 9$. Set

$$A = \begin{pmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{9}{10} \end{pmatrix}.$$

We are thus considering the difference equation $v_{k+1} = Av_k$, so that $v_k = A^k v_0$ for $k = 0, \ldots, 10$, and we wish to find $v_{10} = A^{10} v_0$.

The matrix A is stochastic, and so has 1 as an eigenvalue. The characteristic polynomial of A is

$$\chi_A(t) = \begin{vmatrix} \frac{9}{10} - t & \frac{1}{10} \\ \frac{1}{10} & \frac{9}{10} - t \end{vmatrix} = \left(t - \tfrac{9}{10}\right)^2 - \left(\tfrac{1}{10}\right)^2 = (t-1)\left(t - \tfrac{8}{10}\right).$$

Thus the eigenvalues of A are 1 and $\frac{8}{10}$.

To find an eigenvector of A corresponding to the eigenvalue $\lambda_1 := 1$, we need to solve the system of linear equations $(A - I_2)(\ x \quad y\ )^T = 0$. This has augmented matrix

$$\left(A - I_2 | 0\right) = \left(\begin{array}{cc|c} -\frac{1}{10} & \frac{1}{10} & 0 \\ \frac{1}{10} & -\frac{1}{10} & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -\frac{1}{10} & \frac{1}{10} & 0 \\ 0 & 0 & 0 \end{array}\right),$$

and so $w_1 := (1,1)^T$ is an eigenvector of A corresponding to the eigenvalue 1.

To find an eigenvector of A corresponding to the eigenvalue $\lambda_2 := \frac{8}{10}$, we need to solve the system of linear equations $\left(A - \frac{8}{10} I_2\right)(\ x \quad y\ )^T = 0$. This has augmented matrix

$$\left(A - \tfrac{8}{10} I_2 | 0\right) = \left(\begin{array}{cc|c} \frac{1}{10} & \frac{1}{10} & 0 \\ \frac{1}{10} & \frac{1}{10} & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} \frac{1}{10} & \frac{1}{10} & 0 \\ 0 & 0 & 0 \end{array}\right),$$

and so $w_2 := (1, -1)^T$ is an eigenvector of A corresponding to the eigenvalue $\frac{8}{10}$.
Now, let

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{8}{10} \end{pmatrix}.$$

Then $A = PDP^{-1}$, and by the usual methods

$$P^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

We have

$$A^{10} = PD^{10}P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.8^{10} \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix} = 0.5 \begin{pmatrix} 1 + 0.8^{10} & 1 - 0.8^{10} \\ 1 - 0.8^{10} & 1 + 0.8^{10} \end{pmatrix}$$

We have

$$v_{10} = A^{10} v_0 = 0.5 \begin{pmatrix} 1 + 0.8^{10} & 1 - 0.8^{10} \\ 1 - 0.8^{10} & 1 + 0.8^{10} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5(1 + 0.8^{10}) \\ 0.5(1 - 0.8^{10}) \end{pmatrix}$$

Since $(0.8)^{10} \approx 0.107374$, we conclude that approximately 55.37% of students are awake after 50 minutes.

5. Let

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Find the eigenvalues of A, and for each eigenvalue, find a corresponding eigenvector of A. Define recursively a sequence of vectors $\{\begin{pmatrix} u_n \\ v_n \end{pmatrix}\}$ as follows: $u_0 = 1$, $v_0 = 0$ and for all $n > 0$

$$\begin{aligned} u_n &= u_{n-1} + v_{n-1} \\ v_n &= 2u_{n-1} + v_{n-1}. \end{aligned}$$

Use your eigenvectors of A to find expressions for u_n and v_n (for a general positive integer n).

ANSWER:

We have

$$\chi_A(t) = \begin{vmatrix} 1 - t & 1 \\ 2 & 1 - t \end{vmatrix} = (t-1)^2 - 2 = t^2 - 2t - 1 = \left(t - 1 - \sqrt{2}\right)\left(t - 1 + \sqrt{2}\right).$$

We thus see that the eigenvalues of A are $\lambda_1 = 1 + \sqrt{2}$ and $\lambda_2 = 1 - \sqrt{2}$. Notice that $(1 + \sqrt{2})(1 - \sqrt{2}) = -1$ and $(1 + \sqrt{2}) + (1 - \sqrt{2}) = 2$.

To find an eigenvector corresponding to λ_1, we consider $(A - \lambda_1 I_2)\begin{pmatrix} x & y \end{pmatrix}^T = 0$:

$$\begin{aligned}(A - \lambda_1 I_2|0) &= \left(\begin{array}{cc|c} 1-(1+\sqrt{2}) & 1 & 0 \\ 2 & 1-(1+\sqrt{2}) & 0 \end{array}\right) = \left(\begin{array}{cc|c} -\sqrt{2} & 1 & 0 \\ 2 & -\sqrt{2} & 0 \end{array}\right) \\ &\sim \left(\begin{array}{cc|c} -\sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right),\end{aligned}$$

so that $w_1 := \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$ is an eigenvector of A corresponding to λ_1.

To find an eigenvector corresponding to λ_2, we consider $(A - \lambda_2 I_2)\begin{pmatrix} x & y \end{pmatrix}^T = 0$:

$$\begin{aligned}(A - \lambda_2 I_2|0) &= \left(\begin{array}{cc|c} 1-(1-\sqrt{2}) & 1 & 0 \\ 2 & 1-(1-\sqrt{2}) & 0 \end{array}\right) = \left(\begin{array}{cc|c} \sqrt{2} & 1 & 0 \\ 2 & \sqrt{2} & 0 \end{array}\right) \\ &\sim \left(\begin{array}{cc|c} \sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right),\end{aligned}$$

so that $w_2 := \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$ is an eigenvector of A corresponding to λ_2.

We have

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} u_{n-1} \\ v_{n-1} \end{pmatrix} = A\begin{pmatrix} u_{n-1} \\ v_{n-1} \end{pmatrix} \quad \text{for } n > 0.$$

Therefore

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{for } n > 0.$$

We calculate this by using the above eigenvectors w_1 and w_2 of A.

Observe:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} = \tfrac{1}{2}w_1 + \tfrac{1}{2}w_2.$$

Therefore, for all $n > 0$,

$$\begin{aligned}\begin{pmatrix} u_n \\ v_n \end{pmatrix} &= A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = A^n\left(\tfrac{1}{2}w_1 + \tfrac{1}{2}w_2\right) \\ &= \tfrac{1}{2}A^n w_1 + \tfrac{1}{2}A^n w_2 = \tfrac{1}{2}\lambda_1^n w_1 + \tfrac{1}{2}\lambda_2^n w_2 \\ &= \tfrac{1}{2}\left(1+\sqrt{2}\right)^n \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + \tfrac{1}{2}\left(1-\sqrt{2}\right)^n \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} \tfrac{1}{2}\left(\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n\right) \\ \tfrac{\sqrt{2}}{2}\left(\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n\right) \end{pmatrix}.\end{aligned}$$

Thus

$$u_n = \tfrac{1}{2}\left((1+\sqrt{2})^n + (1-\sqrt{2})^n\right) \qquad \text{and} \qquad v_n = \tfrac{1}{\sqrt{2}}\left((1+\sqrt{2})^n - (1-\sqrt{2})^n\right)$$

for all $n > 0$.

6. (a) 60% of the residents in a particular city live within the city boundary and the other 40% in the suburbs. It is expected that, during each year, 5% of those people who were living in the city at the beginning of the year will move to the suburbs, while the other 95% will remain in the city. It is also expected that, during each year, 3% of those people in $\mathcal{P}$ who were living in the suburbs at the beginning of year will move to the city, while the other 97% will remain in the suburbs.

   Estimate the percentage of people that will be living in the suburbs in 7 years' time.

   (b) Estimate the percentage of people that will be living in the suburbs in many years time.

ANSWER:

(a) The changes in population over time can be represented using a transition matrix:

$$M = \begin{pmatrix} \frac{95}{100} & \frac{3}{100} \\ \frac{5}{100} & \frac{97}{100} \end{pmatrix}$$

(Notice that this is the same matrix as question 3a for which we have already done most of the work required to invert it.)

We need $v^7 = M^7 v_0$, and we have $M = PDP^{-1}$, where

$$P = \begin{pmatrix} 3 & 1 \\ 5 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{92}{100} \end{pmatrix}$$

By the usual methods

$$P^{-1} = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{5}{8} & -\frac{3}{8} \end{pmatrix}$$

so

$$A^7 = PD^7P^{-1} = \begin{pmatrix} 3 & 1 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.92^7 \end{pmatrix} \begin{pmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{5}{8} & -\frac{3}{8} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 3 + 5 \times 0.92^7 & 3 - 3 \times 0.92^7 \\ 5 - 5 \times 0.92^7 & 5 + 3 \times 0.92^7 \end{pmatrix}$$

Thus

$$v_7 = \frac{1}{8} \begin{pmatrix} 3 + 5 \times 0.92^7 & 3 - 3 \times 0.92^7 \\ 5 - 5 \times 0.92^7 & 5 + 3 \times 0.92^7 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.50 \\ 0.50 \end{pmatrix}$$

to two significant figures

Hence, in 7 years' time, 50% of people (to 2 significant figures) will be living in the suburbs.

(b) We have a 1-eigenvector $(3, 5)^T$. Other 1-eigenvectors are multiples. So a 1-eigenvector whose entries add up to 1 is $V = (3/8, 5/8)^T$. For large k, $v_k \approx v$, so for large k, we have 5/8, that is 57.5% of people living in the suburbs.