COM1002: Foundations of Computer Science
Problem Sheet 4:
Matrices and Systems of Linear Equations

1. Solve the system of linear equations

$$\begin{aligned} x + y + z &= 1 \\ x + 2y + 3z &= 6 \\ x - y - z &= 0 \end{aligned}$$

ANSWER:

In matrix form, the system is:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 6 \\ 1 & -1 & -1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & -2 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 2 & 9 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & \frac{9}{2} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & -\frac{7}{2} \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & \frac{9}{2} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & \frac{9}{2} \end{array}\right)$$

From this, we can read off the solutions

$$x = \frac{1}{2} \quad y = -4 \quad z = \frac{9}{2}.$$

2. Solve the system of linear equations

$$\begin{aligned} 5x + 3y + 2z &= 19 \\ x + y + z &= 4 \\ 3x + 2y + z &= 12 \end{aligned}$$

ANSWER:

In matrix form, the system is:

$$\left(\begin{array}{ccc|c} 5 & 3 & 2 & 19 \\ 1 & 1 & 1 & 4 \\ 3 & 2 & 1 & 12 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 5 & 3 & 2 & 19 \\ 3 & 2 & 1 & 12 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & -3 & -1 \\ 0 & -1 & -2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

From this, we can read off the solutions

$$x = 3 \quad y = 2 \quad z = -1.$$

3. Let

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 2 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

(a) Transform A by elementary row operations to reduced row echelon form.

(b) Find the general solution to the system of linear equations

$$\begin{aligned} w + y &= 2 \\ w + x + y + z &= 1 \\ x + z &= -1 \\ 2w + x + y + z &= 0 \end{aligned}$$

ANSWER:

(a) We have

$$A \sim \begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 & -4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) From the above, we have the equations

$$\begin{array}{rcl} w &=& -1 \\ x+z &=& -1 \\ y &=& 3 \\ 0 &=& 0 \end{array}$$

The last one does nothing (there was no point in writing it down). So we solve for the variables starting with z.
Set $z = \alpha \in \mathbb{R}$. Then

$$w = -1 \quad x = -1-\alpha \quad y = 3 \quad z = \alpha.$$

4. Let

$$B = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 & 1 & 4 \\ 1 & 1 & 1 & 0 & 0 & 6 \end{pmatrix}.$$

(a) Transform B by elementary row operations to reduced row echelon form.

(b) Find the general solution to the system of linear equations

$$\begin{array}{rcl} v+2w+x+2y+z &=& 0 \\ v+w+x+y+z &=& 2 \\ 2v+2w+x+y+z &=& 4 \\ v+w+x &=& 6 \end{array}$$

(c) Find the solution to the system of linear equations

$$\begin{array}{rcl} w+x+2y+z &=& 0 \\ w+x+y+z &=& 2 \\ 2w+x+y+z &=& 4 \\ w+x &=& 6 \end{array}$$

ANSWER:

(a) We have

$$B \sim \begin{pmatrix} 1 & 2 & 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & -2 & -1 & -3 & -1 & 4 \\ 0 & -1 & 0 & -2 & -1 & 6 \end{pmatrix}$$

$$B \sim \begin{pmatrix} 1 & 2 & 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 4 \end{pmatrix}$$

$$B \sim \begin{pmatrix} 1 & 2 & 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -4 \end{pmatrix}$$

$$B \sim \begin{pmatrix} 1 & 2 & 1 & 0 & -1 & 8 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 & -4 \end{pmatrix}$$

$$B \sim \begin{pmatrix} 1 & 2 & 0 & 0 & -1 & 4 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 & -4 \end{pmatrix}$$

$$B \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 & -4 \end{pmatrix}$$

$$B \sim \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 4 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 & -4 \end{pmatrix}$$

(b) From the above we have the equations

$$\begin{aligned} v - y &= 4 \\ w - z &= 2 \\ x &= 4 \\ y + z &= -4 \end{aligned}$$

So if we set $z = \alpha \in \mathbb{R}$, we have

$$\begin{aligned} v &= -\alpha \\ w &= \alpha + 2 \\ x &= 4 \\ y &= -(\alpha + 4) \\ z &= \alpha \end{aligned}$$

(c) This system has matrix

$$C = \begin{pmatrix} 1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 1 & 4 \\ 1 & 1 & 0 & 0 & 6 \end{pmatrix}$$

which is B with column 2 deleted. Hence we can use the above calculation, and have

$$C \sim \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & -4 \end{pmatrix}$$

$$C \sim \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$C \sim \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

$$C \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

From this we can read off the solutions

$$w = 2,\ x = 4,\ y = -2,\ z = -2$$

5. Find the general solution to the system of linear equations

$$\begin{aligned} w + x + 2y + z &= 1 \\ w + x - y + z &= -1 \\ x + y + z &= 0 \\ w + 3x + 2y + 3z &= 2 \end{aligned}$$

ANSWER:

We have associated matrix

$$
\begin{pmatrix}
1 & 1 & 2 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 \\
0 & 1 & 1 & 1 & 0 \\
1 & 3 & 2 & 3 & 2
\end{pmatrix}
$$

$$
\sim \begin{pmatrix}
1 & 1 & 2 & 1 & 1 \\
0 & 0 & -3 & 0 & -2 \\
0 & 1 & 1 & 1 & 0 \\
0 & 2 & 0 & 2 & 1
\end{pmatrix}
$$

$$
\sim \begin{pmatrix}
1 & 1 & 2 & 1 & 1 \\
0 & 0 & 1 & 0 & \frac{2}{3} \\
0 & 1 & 1 & 1 & 0 \\
0 & 2 & 0 & 2 & 1
\end{pmatrix}
$$

$$
\sim \begin{pmatrix}
1 & 1 & 2 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & \frac{2}{3} \\
0 & 2 & 0 & 2 & 1
\end{pmatrix}
$$

$$
\sim \begin{pmatrix}
1 & 1 & 2 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & \frac{2}{3} \\
0 & 0 & -2 & 0 & 1
\end{pmatrix}
$$

$$
\sim \begin{pmatrix}
1 & 1 & 2 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & \frac{2}{3} \\
0 & 0 & 0 & 0 & \frac{7}{3}
\end{pmatrix}
$$

At this stage the bottom equation is $0 = \frac{7}{3}$, which means that the system has no solution. (Note that it is possible to get a different bottom equation by applying a different set of EROs, but they all indicate that the system has no solution.)

6. Alice is now twice as old as Paul. Three years ago she was two years younger than three times Paul's age. How old are Alice and Paul now?

ANSWER:

Let x and y be Alice and Paul's ages now (respectively). Then, the sentences in the problem can be translated into these equivalences $x = 2y$ and $x - 3 = 3(y - 3) - 2$ which give rise to the following linear system:

$$\begin{aligned} x - 2y &= 0 \\ x - 3y &= -8 \end{aligned}$$

The solution to this system is $y = 8$ and $x = 2y$. Therefore Alice is 16 and Paul is 8.