COM1002: Foundations of Computer Science
Problem Sheet 5: Matrix Algebra

1. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

   (a) Calculate the products AB and BA.

   (b) Let $S, T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by writing $S(v) = Av$ and $T(v) = Bv$. Write down explicit formulae for the linear transformations $S \circ T$ and $T \circ S$.

2. Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \\ 2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

   Calculate AB. What about BA?

3. Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}, B = \begin{pmatrix} -1 & 2 & 5 \\ 1 & 0 & -1 \\ -1 & 2 & 3 \end{pmatrix} D = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

   Which of the products AB, AD, DA, ABD is well-defined? Compute those products that are well-defined.

4. Let

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

   Calculate the inverse matrices A^{-1}, B^{-1} and C^{-1}. Check that $A^{-1}A = AA^{-1} = I$, $B^{-1}B = BB^{-1} = I$, and $C^{-1}C = CC^{-1} = I$.

5. Calculate the determinants of each of the matrices in the above question.

6. Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 5 & 7 & 2 & 0 & 0 \\ -8 & -4 & 3 & -1 & 0 \\ 20 & 7 & 4 & 2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 6 & 5 & -2 & -3 \\ 0 & 8 & 3 & 1 & \frac{1}{3} \\ 0 & 0 & 13 & 3 & 5 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix}$$

   (a) Calculate $\det(A)$ and $\det(B)$.

   (b) Calculate AB.

   (c) Calculate $\det(AB)$.

7. Let
$$A := \begin{pmatrix} 1 & -3 & 4 & 8 \\ -1 & 2 & 3 & 4 \\ 2 & 7 & -3 & -5 \\ 3 & -1 & 0 & 6 \end{pmatrix}.$$

Calculate the determinant $\det A$.

8. Let a be a real number. Calculate $\det A$, where
$$A := \begin{pmatrix} a & 1 & 1 & 0 \\ 0 & 1 & a & a \\ a & 2a & a & a \\ a & 0 & 0 & a \end{pmatrix}.$$

Use your answer to determine the value(s) of a for which A is not invertible.

9. Let a be a real number different from 1. Show that the matrix
$$\begin{pmatrix} 1 & a \\ a & 2a-1 \end{pmatrix},$$

is invertible, and find its inverse.

10. Show that the matrix
$$A := \begin{pmatrix} 3 & -5 & 5 \\ 2 & -4 & 5 \\ 2 & -2 & 3 \end{pmatrix}$$

is invertible, and calculate its inverse.

11. Determine whether the following square matrices are invertible, and find the inverses of those that are.

   (a)
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 9 & 4 & 1 \end{pmatrix}.$$

   (b)
$$B = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 13 & 4 & 1 \end{pmatrix}.$$

   (c)
$$C = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & -1 & -2 & 0 \\ -6 & 1 & 4 & 0 \\ 6 & 3 & 4 & 4 \end{pmatrix}.$$