COM1002: Foundations of Computer Science
Problem Sheet 9: Stochastic Processes

1. Consider the difference equation $Av_{k-1} = v_k$ where

$$A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

   Find v_7.

2. Find the eigenvalues of the matrix

$$M := \begin{pmatrix} 0.87 & 0.19 \\ 0.13 & 0.81 \end{pmatrix}.$$

3. (a) Find the eigenvalues of the matrix

$$M = \begin{pmatrix} \frac{95}{100} & \frac{3}{100} \\ \frac{5}{100} & \frac{97}{100} \end{pmatrix}$$

   (b) For each eigenvalue, find a corresponding eigenvector.

   (c) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}MP = D$.

4. Over a period of 5 minutes, in a typical COM1002 lecture, 90% of students who are awake at the beginning of the 5-minute period will still be so at the end of it (but the other 10% will fall asleep) and 90% of students who are asleep at the beginning of the 5-minute period will still be so at the end of it (and the other 10% will wake up). If all the students are awake at the beginning of the lecture, what percentage will be awake 50 minutes later?

5. Let

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

   Find the eigenvalues of A, and for each eigenvalue, find a corresponding eigenvector of A.

   Define recursively a sequence of vectors $\{\begin{pmatrix} u_n \\ v_n \end{pmatrix}\}$ as follows: $u_0 = 1$, $v_0 = 0$ and for all $n > 0$

$$\begin{aligned} u_n &= u_{n-1} + v_{n-1} \\ v_n &= 2u_{n-1} + v_{n-1}. \end{aligned}$$

   Use your eigenvectors of A to find expressions for u_n and v_n (for a general positive integer n).

6. (a) 60% of the residents in a particular city live within the city boundary and the other 40% in the suburbs. It is expected that, during each year, 5% of those people who were living in the city at the beginning of the year will move to the suburbs, while the other 95% will remain in the city. It is also expected that, during each year, 3% of those people in $\mathcal{P}$ who were living in the suburbs at the beginning of year will move to the city, while the other 97% will remain in the suburbs.

   Estimate the percentage of people that will be living in the suburbs in 7 years' time.

   (b) Estimate the percentage of people that will be living in the suburbs in many years time.