COM1002: Foundations of Computer Science
Problem Sheet 8: Random Variables

1. Consider tossing a fair coin three times. Let X be the number of heads.

   (a) Write down the probability mass function p_X.

   (b) Calculate the expectation $E(X)$.

   (c) Calculate the standard deviation $sd(X)$.

ANSWER:

(a) The chance of getting k heads is

$$p_X(k) = \frac{C(3,k)}{8} = \frac{3!}{8k!(3-k)}$$

(b)

$$E(X) = 0p_X(0) + 1p_X(1) + 2p_X(2) + 3p_X(3) = \frac{3}{8} + \frac{2 \times 3}{8} + \frac{3}{8} = \frac{12}{8} = \frac{3}{2}.$$

(c)

$$var(X) = E(X^2) - E(X)^2 = \frac{3}{8} + \frac{2^2 \times 3}{8} + \frac{3^2}{8} - \left(\frac{3}{2}\right)^2 = \frac{3}{4}.$$

So $sd(X) = \frac{\sqrt{3}}{2}$.

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2. If X is a random variable with the Benoulli distribution with parameter p, show that $E(X) = p$ and $var(X) = p(1-p)$.

ANSWER:

X has probability mass function

$$p_X(k) = \begin{cases} 1-p & k=0 \\ p & k=1 \\ 0 & \text{otherwise} \end{cases}$$

We have

$$E(X) = 0(1-p) + 1p = p$$

To compute $var(X)$ we first compute $E(X^2)$

$$E(X^2) = 0^2(1-p) + 1^2 p = p$$

Now

$$var(X) = E(X^2) - E(X)^2 = p - p^2 = p(1-p).$$

3. If X has the binomial distribution with parameters (n, p), find $E(X)$ and $var(X)$. Relate this calculation to question 1, above.

ANSWER:

X is a sum of n independent random variables, $Y_1, \ldots, Y_n$, each with Bernoulli distribution with parameter p.

Hence

$$E(X) = E(Y_1) + \cdots + E(Y_n) = p + \cdots + p = np$$

and

$$var(X) = var(Y_1) + \cdots + var(Y_n) = np(1-p)$$

In question 1, X has a binomial distribution with $n = 3$, $p = \frac{1}{2}$. Hence, by the above, $E(X) = \frac{3}{2}$, and $var(X) = \frac{3}{4}$, which agrees with our earlier answer.

4. An early application of the Poisson distribution was to model the numbers of deaths by horse kicks in the Prussian army. Assume that the average number of deaths per year caused by horse kicks is 1.5. Use the Poisson distribution to compute the probability that there will be 4 deaths in a particular year.

ANSWER:

The Poisson distribution is

$$p_X(k) = \frac{1}{k!}\lambda^k e^{-\lambda}$$

In this case $\lambda = E(X) = 1.5$

So, $P(X = 4) = \frac{1.5^4}{4!}e^{-1.5} \approx 4.7\%$

5. If X has the Poisson distribution with parameter λ, show that $E(X) = \lambda$ and $var(X) = \lambda$.

Note: This problem is very difficult. It will help you to know the Taylor Series Expansion for the Exponential Function:

$$e^{\lambda} = \sum_{l=0}^{\infty} \frac{1}{l!}\lambda^{l}$$

ANSWER:

The probability mass function of the Poisson distribution is

$$p_X(k) = \frac{1}{k!}\lambda^{k}e^{-\lambda}$$

so

$$\begin{aligned}
E(X) &= \sum_{k=0}^{\infty} \frac{k}{k!}\lambda^{k}e^{-\lambda} \\
&= \sum_{k=1}^{\infty} \frac{k}{k!}\lambda^{k}e^{-\lambda} \quad \text{(since } k=0 \text{ doesn't contribute anything to the sum)} \\
&= \lambda e^{-\lambda}\sum_{k=1}^{\infty} \frac{k}{k!}\lambda^{(k-1)} \\
&= \lambda e^{-\lambda}\sum_{k=1}^{\infty} \frac{1}{(k-1)!}\lambda^{(k-1)} \\
&= \lambda e^{-\lambda}\sum_{k=0}^{\infty} \frac{1}{l!}\lambda^{l} \quad \text{(by substituting } l \text{ for } k-1\text{)} \\
&= \lambda e^{-\lambda}e^{\lambda} \quad \text{(by applying Taylor Series Expansion)} \\
&= \lambda e^{0} \\
&= \lambda
\end{aligned}$$

Now

$$var(X) = E(X^2) - E(X)^2.$$

By the above, $E(X)^2 = \lambda^2$. So

$$E(X^2) = \sum_{k=0}^{\infty} \frac{k^2}{k!}\lambda^{k}e^{-\lambda} = \sum_{k=0}^{\infty} \frac{k(k-1)}{k!}\lambda^{k}e^{-\lambda} + \sum_{k=0}^{\infty} \frac{k}{k!}\lambda^{k}e^{-\lambda}.$$

As above, the second of these sums is λ. The first of these sums is

$$\lambda^2\sum_{k=2}^{\infty} \frac{1}{(k-2)!}\lambda^{k-2}e^{-\lambda} = \lambda^2.$$

So

$$var(X) = \lambda^2 + \lambda - \lambda^2 = \lambda.$$