COM1002: Foundations of Computer Science
Problem Sheet 6: Eigenvectors and Diagonalisation

1. Find the characteristic polynomial, eigenvalues and all the corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{pmatrix}.$$

ANSWER:

We have characteristic polynomial:

$$\chi_A(t) = \det \begin{pmatrix} 1-t & 4 & 6 \\ 0 & 2-t & 5 \\ 0 & 0 & 3-t \end{pmatrix}$$

So A has eigenvalues 1, 2 and 3.

- Let $(x, y, z)^T$ be a 1-eigenvector. Then $A(x, y, z)^T = (x, y, z)^T$, and

$$\begin{aligned} x + 4y + 6z &= x \\ 2y + 5z &= y \\ 3z &= z \end{aligned}$$

that is

$$\begin{aligned} 4y + 6z &= 0 \\ y + 5z &= 0 \\ 2z &= 0 \end{aligned}$$

Hence $y = z = 0$, and $x = \alpha$ for some $\alpha \in \mathbb{R}$. So the 1-eigenevectors are

$$\begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}$$

where $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

- Let $(x, y, z)^T$ be a 2-eigenvector. Then $A(x, y, z)^T = 2(x, y, z)^T$, and

$$\begin{aligned} x + 4y + 6z &= 2x \\ 2y + 5z &= 2y \;. \\ 3z &= 2z \end{aligned}$$

that is

$$\begin{aligned} -x + 4y + 6z &= 0 \\ 5z &= 0 \\ z &= 0 \end{aligned}$$

Hence $z = 0$ and $x = 4y$. So the 2-eigenevectors are

$$\begin{pmatrix} 4\beta \\ \beta \\ 0 \end{pmatrix}$$

where $\beta \in \mathbb{R}$, $\beta \neq 0$.

- Let $(x, y, z)^T$ be a 3-eigenvector. Then $A(x, y, z)^T = 3(x, y, z)^T$, and

$$\begin{aligned} x + 4y + 6z &= 3x \\ 2y + 5z &= 3y \\ 3z &= 3z \end{aligned}$$

that is

$$\begin{aligned} -2x + 4y + 6z &= 0 \\ -y + 5z &= 0 \\ 3z &= 3z \end{aligned}$$

Hence

$$\begin{aligned} -2x + 4y + 6z &= 0 \\ -y + 5z &= 0 \end{aligned}$$

Set $z = \gamma$. Then $y = 5\gamma$ and

$$x = 2y + 3z = 10\gamma + 3\gamma = 13\gamma.$$

So the 3-eigenvectors are

$$\begin{pmatrix} 13\gamma \\ 5\gamma \\ \gamma \end{pmatrix}$$

where $\gamma \in \mathbb{R}$, $\gamma \neq 0$.

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2. Find the characteristic polynomial, eigenvalues and all the corresponding eigenvectors of the matrix

$$B = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix}.$$

ANSWER:

We have characteristic polynomial:

$$\chi_B(t) = \det \begin{pmatrix} 3-t & 2 & 1 \\ 0 & 1-t & 2 \\ 0 & 1 & -1-t \end{pmatrix} = (3-t)\det \begin{pmatrix} 1-t & 2 \\ 1 & -1-t \end{pmatrix}$$

$$= (3-t)((1-t)(-1-t)-2) = (3-t)(t^2-3) = (3-t)(t+\sqrt{3})(t-\sqrt{3}).$$

So B has eigenvalues 3, $\sqrt{3}$ and $-\sqrt{3}$.

- Let $(x,y,z)^T$ be a 3-eigenvector. Then $B(x,y,z)^T = 3(x,y,z)^T$, and

$$\begin{array}{rcl} 3x+2y+z &=& 3x \\ y+2z &=& 3y \\ y-z &=& 3z \end{array}$$

  that is

$$\begin{array}{rcl} 2y+z &=& 0 \\ -2y+2z &=& 0 \\ y-4z &=& 0 \end{array}$$

  Hence $y=z=0$, and $x=\alpha$ for some $\alpha \in \mathbb{R}$. So the 3-eigenevectors are

$$\begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}$$

  where $\alpha \in \mathbb{R}$, $\alpha \neq 0$.
- Let $(x,y,z)^T$ be a $\sqrt{3}$-eigenvector. Then $B(x,y,z)^T = \sqrt{3}(x,y,z)^T$, and

$$\begin{array}{rcl} 3x+2y+z &=& \sqrt{3}x \\ y+2z &=& \sqrt{3}y \\ y-z &=& \sqrt{3}z \end{array}.$$

  that is

$$\begin{array}{rcl} (3-\sqrt{3})x+2y+z &=& 0 \\ (1-\sqrt{3})y+2z &=& 0 \\ y-(1+\sqrt{3})z &=& 0 \end{array}$$

  Now $(1+\sqrt{3})(1-\sqrt{3}) = -2$. If we multiply the bottom equation by $1-\sqrt{3}$, we get the second equation, so we solve:

$$\begin{array}{rcl} (3-\sqrt{3})x+2y+z &=& 0 \\ (1-\sqrt{3})y+2z &=& 0 \end{array}$$

  Let $y=\beta$. Then $z = \frac{1}{2}(1-\sqrt{3})\beta$ and

$$(3-\sqrt{3})x = -2\beta - \frac{1}{2}(1-\sqrt{3})\beta$$

Multiply by $3+\sqrt{3}$:

$$6x = (6+\sqrt{3})(-2-\frac{1}{2}+\frac{\sqrt{3}}{2})\beta$$

so

$$12x = (3+\sqrt{3})(-5+\sqrt{3})\beta = (-12-2\sqrt{3})\beta$$

and

$$x = (-1-\frac{1}{6}\sqrt{3})\beta$$

So the $\sqrt{3}$-eigenevectors are

$$\begin{pmatrix} (-1-\frac{1}{6}\sqrt{3})\beta \\ \beta \\ \frac{1}{2}(1-\sqrt{3})\beta \end{pmatrix}$$

where $\beta \in \mathbb{R}$, $\beta \neq 0$.

- Similarly to the above, the $-\sqrt{3}$-eigenevectors are

$$\begin{pmatrix} (-1+\frac{1}{2}\sqrt{3})\gamma \\ \gamma \\ -\frac{1}{2}(1+\sqrt{3})\gamma \end{pmatrix}$$

where $\gamma \in \mathbb{R}$, $\gamma \neq 0$.

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3. Show, directly from the definition of eigenvalue, that 0 is an eigenvalue of the matrix

$$N := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Show, also directly from the definition of eigenvalue, that an arbitrary non-zero number k is not an eigenvalue of N.

Find all the eigenvectors of N.

ANSWER:

Let $k \neq 0$. Suppose $N(w,x,y,z)^T = k(w,x,y,z)$. Then

$$\begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix} = \begin{pmatrix} kw \\ kx \\ ky \\ kz \end{pmatrix}.$$

Since $k \neq 0$, this tells us that $z = 0/k = 0$, $y = 0/k = 0$, $x = 0/k = 0$, and $w = 0/k = 0$.

So if $Nv = kv$, then $v = 0$. But 0 is not allowed as an eigenvector, so k cannot be an eigenvector.

On the other hand, if $N(w,x,y,z)^T = 0(w,x,y,z)^T$, then

$$\begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

so $x = y = z = 0$, and $w = \lambda$, where $\lambda \in \mathbb{R}$. Thus the set of 0-eigenvectors is

$$\left\{ \begin{pmatrix} \lambda \\ 0 \\ 0 \\ 0 \end{pmatrix} \mid \lambda \in \mathbb{R}, \lambda \neq 0 \right\}$$

4. Let

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

(a) Find the eigenvalues of A, and for each eigenvalue a corresponding eigenvector.

(b) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

ANSWER:

We have characteristic polynomial:

$$\chi_A(t) = \det \begin{pmatrix} 1-t & 2 \\ 1 & -t \end{pmatrix} = -t(1-t) - 2$$

$$= t^2 - t - 2 = (t-2)(t+1).$$

So A has eigenvalues 2 and -1.

- Let $(x,y)^T$ be a 2-eigenvector. Then $A(x,y)^T = 2(x,y)^T$, and

$$\begin{aligned} x + 2y &= 2x \\ x &= 2y \end{aligned}$$

  that is to say $x = 2y$. Take $y = 1$. So $(2,1)^T$ is a 2-eigenvector.

- Let $(x,y)^T$ be a -1-eigenvector. Then $A(x,y)^T = -(x,y)^T$, and

$$\begin{aligned} x + 2y &= -x \\ x &= -y \end{aligned}$$

  that is to say $x = -y$. Take $y = 1$. So $(-1,1)^T$ is a -1-eigenvector.

Let

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \qquad P = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then $P^{-1}AP = D$.

5. Let

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 1 \\ -3 & 0 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues of B, and for each eigenvalue a corresponding eigenvector.

(b) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}BP = D$.

ANSWER:

We have characteristic polynomial:

$$\chi_B(t) = \det \begin{pmatrix} 1-t & 0 & 0 \\ 3 & -1-t & 1 \\ -3 & 0 & 2-t \end{pmatrix} = (1-t)(-1-t)(2-t)$$

So B has eigenvalues 1, -1 and 2.

- Let $(x, y, z)^T$ be a 1-eigenvector. Then $B(x, y, z)^T = (x, y, z)^T$, and

$$\begin{aligned} x &= x \\ 3x - y + z &= y \\ -3x + 2z &= z \end{aligned}$$

and

$$\begin{aligned} 3x - 2y + z &= 0 \\ -3x + z &= 0 \end{aligned}$$

Take $z = 3$ (it could be any non-zero real number). Then $x = 1$ and $y = 3$. So $(1, 3, 3)^T$ is a 1-eigenvector.

- Let $(x, y, z)^T$ be a -1-eigenvector. Then $B(x, y, z)^T = -(x, y, z)^T$, and

$$\begin{aligned} x &= -x \\ 3x - y + z &= -y \\ -3x + 2z &= -z \end{aligned}$$

We see that $x = 0$ and $z = 0$. Take $y = 1$. So $(0, 1, 0)^T$ is a -1-eigenvector.

- Let $(x, y, z)^T$ be a 2-eigenvector. Then $B(x, y, z)^T = 2(x, y, z)^T$, and

$$\begin{aligned} x &= 2x \\ 3x - y + z &= 2y \\ -3x + 2z &= 2z \end{aligned}$$

that is to say $x = 0$ and $-3y + z = 0$. Take $y = 1$. Then $z = 3$ and $(0, 1, 3)^T$ is a 2-eigenvector.

Let

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 1 \\ 3 & 0 & 3 \end{pmatrix}.$$

Then $P^{-1}BP = D$.

6. For the matrices A and B in the previous two questions, work out A^6 and B^6.

ANSWER:

- With P and D as in question 4, $A = PDP^{-1}$, so

$$A^6 = PD^6P^{-1}.$$

Now

$$P = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \qquad P^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

and

$$D^6 = \begin{pmatrix} 2^6 & 0 \\ 0 & (-1)^6 \end{pmatrix} = \begin{pmatrix} 64 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that

$$A^6 = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 64 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 129 & 126 \\ 65 & 66 \end{pmatrix}$$

- With P and D as in question 5, $B = PDP^{-1}$, so

$$B^6 = PD^6P^{-1}.$$

Now

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 1 \\ 3 & 0 & 3 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and

$$D^6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 64 \end{pmatrix}.$$

To find P^{-1}:

$$(P|I) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 1 & 0 \\ 3 & 0 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 11 & -3 & 1 & 0 \\ 0 & 0 & 3 & -3 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & \frac{1}{3} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 & -1 & 0 & \frac{1}{3} \end{array}\right)$$

So

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & -\frac{1}{3} \\ -1 & 0 & \frac{1}{3} \end{pmatrix}.$$

So

$$B^6 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 1 \\ 3 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 64 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & -\frac{1}{3} \\ -1 & 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -63 & 1 & 21 \\ -189 & 0 & 64 \end{pmatrix}.$$