COM1002: Foundations of Computer Science
Problem Sheet 5: Matrix Algebra

1. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

(a) Calculate the products AB and BA.

ANSWER:

$$AB = \begin{pmatrix} 0 & 4 & 2 \\ 2 & 2 & 2 \\ 1 & 5 & 3 \end{pmatrix} \qquad BA = \begin{pmatrix} 3 & 4 & -1 \\ 4 & 3 & 0 \\ -1 & 1 & -1 \end{pmatrix}.$$

(b) Let $S, T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by writing $S(v) = Av$ and $T(v) = Bv$. Write down explicit formulae for the linear transformations $S \circ T$ and $T \circ S$.

ANSWER:

$S \circ T$ has matrix AB, so

$$S \circ T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 4 & 2 \\ 2 & 2 & 2 \\ 1 & 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4y + 2z \\ 2x + 2y + 2z \\ x + 5y + 3z \end{pmatrix}.$$

$T \circ S$ has matrix BA, so

$$T \circ S \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & 4 & -1 \\ 4 & 3 & 0 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x + 4y - z \\ 4x + 3y \\ -x + y - z \end{pmatrix}.$$

2. Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \\ 2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Calculate AB. What about BA?

ANSWER:

$$AB = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 1 & -1 & -2 \\ 2 & 2 & 6 & 4 \end{pmatrix}.$$

BA does not exist; the matrices have the wrong sizes to form this particular product.

3. Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}, B = \begin{pmatrix} -1 & 2 & 5 \\ 1 & 0 & -1 \\ -1 & 2 & 3 \end{pmatrix} D = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Which of the products AB, AD, DA, ABD is well-defined? Compute those products that are well-defined.

ANSWER:

AB, AD, and ABD are all well-defined.

$$AB = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$AD = \begin{pmatrix} 1 & 3 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

$$ABD = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix}.$$

4. Let

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

Calculate the inverse matrices A^{-1}, B^{-1} and C^{-1}. Check that $A^{-1}A = AA^{-1} = I$, $B^{-1}B = BB^{-1} = I$, and $C^{-1}C = CC^{-1} = I$.

ANSWER:

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$C^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

5. Calculate the determinants of each of the matrices in the above question.

ANSWER:

$$\det(A) = 1 - (-1) = 2.$$

Expanding along the bottom row

$$\det(B) = \det\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 2.$$

Expanding along the top row

$$\det(C) = \det\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} - \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= -\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} - \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 2 + 2 = 4.$$

6. Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 5 & 7 & 2 & 0 & 0 \\ -8 & -4 & 3 & -1 & 0 \\ 20 & 7 & 4 & 2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 6 & 5 & -2 & -3 \\ 0 & 8 & 3 & 1 & \frac{1}{3} \\ 0 & 0 & 13 & 3 & 5 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix}$$

(a) Calculate $\det(A)$ and $\det(B)$.

ANSWER:

The matrix A is lower triangular, so the determinant is the product of the diagonal entries, and $\det(A) = 2$.

The matrix B is upper triangular, so the determinant is the product of the diagonal entries, and $\det(B) = 0$.

(b) Calculate AB.

ANSWER:

$$AB = \begin{pmatrix} 5 & 6 & 5 & -7 & -3 \\ 15 & 10 & 12 & -7 & -\frac{28}{3} \\ 25 & -26 & 72 & 3 & -\frac{8}{3} \\ -40 & -72 & -11 & 21 & \frac{107}{3} \\ 100 & 176 & 173 & -21 & -\frac{113}{3} \end{pmatrix}.$$

(c) Calculate $\det(AB)$.

ANSWER:

$\det(AB) = \det(A)\det(B) = 2 \times 0 = 0.$

7. Let

$$A := \begin{pmatrix} 1 & -3 & 4 & 8 \\ -1 & 2 & 3 & 4 \\ 2 & 7 & -3 & -5 \\ 3 & -1 & 0 & 6 \end{pmatrix}.$$

Calculate the determinant $\det A$.

ANSWER:

We have

$$\det A = \det \begin{pmatrix} 1 & -3 & 4 & 8 \\ -1 & 2 & 3 & 4 \\ 2 & 7 & -3 & -5 \\ 3 & -1 & 0 & 6 \end{pmatrix}$$

$$= 1(-1)^{1+1}M_{11} + (-3)(-1)^{1+2}M_{12} + 4(-1)^{1+3}M_{13} + 8(-1)^{1+4}M_{14},$$

where

$$M_{11} = \det \begin{pmatrix} 2 & 3 & 4 \\ 7 & -3 & -5 \\ -1 & 0 & 6 \end{pmatrix}, \quad M_{12} = \det \begin{pmatrix} -1 & 3 & 4 \\ 2 & -3 & -5 \\ 3 & 0 & 6 \end{pmatrix}, \quad M_{13} = \det \begin{pmatrix} -1 & 2 & 4 \\ 2 & 7 & -5 \\ 3 & -1 & 6 \end{pmatrix},$$

$$M_{14} = \det \begin{pmatrix} -1 & 2 & 3 \\ 2 & 7 & -3 \\ 3 & -1 & 0 \end{pmatrix}.$$

Now

$$M_{11} = \det \begin{pmatrix} 2 & 3 & 4 \\ 7 & -3 & -5 \\ -1 & 0 & 6 \end{pmatrix} = 2\times(-18)-3\times(42-5)+4\times(-3) = -36-111-12 = -159,$$

$$M_{12} = \det \begin{pmatrix} -1 & 3 & 4 \\ 2 & -3 & -5 \\ 3 & 0 & 6 \end{pmatrix} = (-1)\times(-18)-3\times(12+15)+4\times 9 = 18-81+36 = -27,$$

$$M_{13} = \det \begin{pmatrix} -1 & 2 & 4 \\ 2 & 7 & -5 \\ 3 & -1 & 6 \end{pmatrix} = (-1)\times(42-5)-2\times(12+15)+4\times(-2-21) = -37-54-92 = -183,$$

$$M_{14} = \det \begin{pmatrix} -1 & 2 & 3 \\ 2 & 7 & -3 \\ 3 & -1 & 0 \end{pmatrix} = (-1)\times(-3)-2\times 9+3\times(-2-21) = 3-18-69 = -84.$$

Therefore

$$\begin{aligned} \det A &= M_{11} + 3M_{12} + 4M_{13} - 8M_{14} = -159 + 3 \times (-27) + 4 \times (-183) - 8 \times (-84) \\ &= -159 - 81 - 732 + 672 = -300. \end{aligned}$$

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8. Let a be a real number. Calculate $\det A$, where

$$A := \begin{pmatrix} a & 1 & 1 & 0 \\ 0 & 1 & a & a \\ a & 2a & a & a \\ a & 0 & 0 & a \end{pmatrix}.$$

Use your answer to determine the value(s) of a for which A is not invertible.

ANSWER:

We use several times the fact that subtraction of one row of a square matrix from another row has no effect on the determinant. We have

$$\det A = \begin{vmatrix} a & 1 & 1 & 0 \\ 0 & 1 & a & a \\ a & 2a & a & a \\ a & 0 & 0 & a \end{vmatrix}$$

$$= \begin{vmatrix} a & 1 & 1 & 0 \\ 0 & 1 & a & a \\ 0 & 2a-1 & a-1 & a \\ 0 & -1 & -1 & a \end{vmatrix} \quad \text{(on subtraction of the 1st row from the 3rd and 4th rows)}$$

$$= a \begin{vmatrix} 1 & a & a \\ 2a-1 & a-1 & a \\ -1 & -1 & a \end{vmatrix} \quad \text{(on expansion down the first column)}$$

$$= a \begin{vmatrix} 1 & a & a \\ 2a-2 & -1 & 0 \\ -2 & -a-1 & 0 \end{vmatrix} \quad \text{(on subtraction of the 1st row from the other two)}$$

$$= a^2 \begin{vmatrix} 2a-2 & -1 \\ -2 & -a-1 \end{vmatrix} \quad \text{(on expansion down the third column)}$$

$$= a^2\left(-(2a-2)(a+1)-2\right) = a^2(-2a^2+2-2) = -2a^4.$$

Now the matrix A is singular if and only if $\det A = 0$, that is, if and only if $-2a^4 = 0$. Hence A is singular if and only if $a = 0$.

9. Let a be a real number different from 1. Show that the matrix

$$\begin{pmatrix} 1 & a \\ a & 2a-1 \end{pmatrix},$$

is invertible, and find its inverse.

ANSWER:

Note that $a - 1 \neq 0$.

$$\left(\begin{array}{cc|cc} 1 & a & 1 & 0 \\ a & 2a-1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & a & 1 & 0 \\ 0 & 2a-1-a^2 & -a & 1 \end{array}\right) = \left(\begin{array}{cc|cc} 1 & a & 1 & 0 \\ 0 & -(a-1)^2 & -a & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 1 & a & 1 & 0 \\ 0 & 1 & \frac{a}{(a-1)^2} & -\frac{1}{(a-1)^2} \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1-\frac{a^2}{(a-1)^2} & \frac{a}{(a-1)^2} \\ 0 & 1 & \frac{a}{(a-1)^2} & -\frac{1}{(a-1)^2} \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} 1 & 0 & \frac{1-2a}{(a-1)^2} & \frac{a}{(a-1)^2} \\ 0 & 1 & \frac{a}{(a-1)^2} & -\frac{1}{(a-1)^2} \end{array}\right).$$

Hence, $\begin{pmatrix} 1 & a \\ a & 2a-1 \end{pmatrix}$ is invertible, and

$$\begin{pmatrix} 1 & a \\ a & 2a-1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1-2a}{(a-1)^2} & \frac{a}{(a-1)^2} \\ \frac{a}{(a-1)^2} & -\frac{1}{(a-1)^2} \end{pmatrix}.$$

10. Show that the matrix

$$A := \begin{pmatrix} 3 & -5 & 5 \\ 2 & -4 & 5 \\ 2 & -2 & 3 \end{pmatrix}$$

is invertible, and calculate its inverse.

ANSWER:

We have

$$\begin{aligned}
(A|I_3) &= \left(\begin{array}{ccc|ccc} 3 & -5 & 5 & 1 & 0 & 0 \\ 2 & -4 & 5 & 0 & 1 & 0 \\ 2 & -2 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 2 & -4 & 5 & 0 & 1 & 0 \\ 0 & 2 & -2 & 0 & -1 & 1 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & -2 & 5 & -2 & 3 & 0 \\ 0 & 2 & -2 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & -2 & 5 & -2 & 3 & 0 \\ 0 & 0 & 3 & -2 & 2 & 1 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -\frac{5}{2} & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{5}{2} & 2 & -\frac{5}{2} & 0 \\ 0 & 1 & -\frac{5}{2} & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{5}{6} & \frac{5}{6} \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{6} & \frac{5}{6} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{array}\right).
\end{aligned}$$

Therefore A is invertible and

$$A^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{5}{6} & \frac{5}{6} \\ -\frac{2}{3} & \frac{1}{6} & \frac{5}{6} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

11. Determine whether the following square matrices are invertible, and find the inverses of those that are.

(a)

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 9 & 4 & 1 \end{pmatrix}.$$

ANSWER:

e have

$$\left(A|I_3\right) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 9 & 4 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -3 & 1 & 0 \\ 0 & -5 & -8 & -9 & 0 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 \\ 0 & 0 & 2 & 6 & -5 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -3 & 4 & -1 \\ 0 & 0 & 1 & 3 & -\frac{5}{2} & \frac{1}{2} \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -3 & 4 & -1 \\ 0 & 0 & 1 & 3 & -\frac{5}{2} & \frac{1}{2} \end{array}\right).$$

Therefore A is invertible and

$$A^{-1} = \begin{pmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ -3 & 4 & -1 \\ 3 & -\frac{5}{2} & \frac{1}{2} \end{pmatrix}.$$

(b)

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 13 & 4 & 1 \end{pmatrix}.$$

ANSWER:

We have

$$\left(B|I_3\right) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 13 & 4 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -5 & 1 & 0 \\ 0 & 4 & 1 & -13 & 0 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -5 & 1 & 0 \\ 0 & 0 & 1 & 7 & -4 & 1 \end{array}\right).$$

Therefore B is invertible and

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 7 & -4 & 1 \end{pmatrix}.$$

(c)

$$C = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & -1 & -2 & 0 \\ -6 & 1 & 4 & 0 \\ 6 & 3 & 4 & 4 \end{pmatrix}.$$

ANSWER:

We have

$$(C|I_4) = \left(\begin{array}{cccc|cccc} 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 & 0 & 1 & 0 & 0 \\ -6 & 1 & 4 & 0 & 0 & 0 & 1 & 0 \\ 6 & 3 & 4 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 6 & 3 & 4 & 4 & 0 & 0 & 0 & 1 \\ 0 & -1 & -2 & 0 & 0 & 1 & 0 & 0 \\ -6 & 1 & 4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{cccc|cccc} 6 & 3 & 4 & 4 & 0 & 0 & 0 & 1 \\ 0 & -1 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 8 & 4 & 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 6 & 3 & 4 & 4 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 8 & 4 & 0 & 0 & 1 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{cccc|cccc} 6 & 3 & 4 & 4 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 1 & 1 \end{array}\right) =: D.$$

Since the first four columns of D form a matrix in row echelon form having just 3 non-zero rows tells us that see that C is not invertible.