COM1002: Foundations of Computer Science
Problem Sheet 7: Combinatorics and Probability

1. In arranging people around a circular table, we take into account their seats relative to each other, not the actual position of any one person. Show that n people can be arranged around a circular table in $(n-1)!$ ways.

ANSWER:

- It doesn't matter where person 1 sits as we are considering relative positions.
- Person 2 has $n-1$ places to choose from.
- Person 3 has $n-2$ places to choose from.

and so on until

- Person n has 1 place to choose from.

So the total number is

$$(n-1)(n-2)\cdots 1 = (n-1)!$$

2. Prove that at least two people in Sheffield (population 534,500) have the same initials, assuming no one has more than four initials.

ANSWER:

A person has two, three or four initials.

- The number of possibilities for 2 initials is $26 \times 26 = 676$ (there are 26 possibilities for each letter).
- The number of possibilities for 3 initials is $26 \times 26 \times 26 = 17576$.
- The number of possibilities for 4 initials is $26 \times 26 \times 26 \times 26 = 456976$.

This gives us a total of

$$676 + 17576 + 456976 = 475228$$

This is lower than the population of Sheffield. So there must be some overlap, ie: more than one person with the same initials.

(N.B. This type of argument is sometimes called the 'pigeonhole principle'- if there are more letters present than pigeonholes in a mailbox, at least one mailbox must contain more than one letter.)

3. A poker hand consists of 5 cards from an ordinary deck of 52 cards.

(a) How many different poker hands are there?

ANSWER:

There are

$$C(52,5) = \frac{52 \times 51 \times 50 \times 49 \times 48}{5!} = 2598960$$

possible hands.

(b) A *flush* is a poker hand where all cards have the same suit. How many flushes are there?

ANSWER:

There are 13 cards in each suit, and 4 different suits.

For each suit, there are $C(13,5)$ different ways to choose all five cards from that suit. Thus there are

$$4 \times C(13,5) = 5148$$

different flushes.

(c) *Three of a kind* is a poker hand where three cards show the same number. How many three of a kind hands are there?

ANSWER:

First of all, note that there are 13 possible numbers for a three of a kind, and $C(4,3) = 4$ ways of drawing 3 cards showing a particular number.

There are $52 - 4 = 48$ cards in the which don't match the three of a kind. So the number of ways they can be drawn is $C(48,2)$, and the total number of such hands is:

$$13 \times 4 \times C(48,2) = 58656.$$

Note that this leaves open the possibility that there are three cards of one number, and two of another (a 'full house').

4. When drawing three cards from an ordinary 52 card deck in order, how many draws are there where the first card is an ace, the second card a two, and the third card a three?

ANSWER:

The number of draws is

$$4 \times 4 \times 4 = 64$$

as there are four cards of each number available.

5. When rolling an ordinary fair six-sided dice twice, work out the following probabilities:

   (a) The probability that the total of the two rolls is 8 or higher.
   (b) The probability that the first die roll is odd.
   (c) The probability that the total of the two rolls is 7 or lower.
   (d) The probability that the first die roll is odd, and the total of the two rolls is 8 or higher.

ANSWER:

Let us write (x, y) to denote the outcome where the first die roll is x, and the second die roll is y.

There are $6 \times 6 = 36$ different outcomes in total.

(a) The set of outcomes where the total is 8 or more is:

$$A = \left\{ \begin{array}{l} (2,6),(3,6),(4,6),(5,6),(6,6), \\ (3,5),(4,5),(5,5),(6,5), \\ (4,4),(5,4),(6,4), \\ (5,3),(6,3), \\ (6,2) \end{array} \right\}$$

Now $|A| = 15$, so

$$P(A) = \frac{|A|}{|\Omega|} = \frac{15}{36} = \frac{5}{12}.$$

(b) The probability is clearly $\frac{1}{2}$.

(c) The probability is

$$1 - P(A) = \frac{7}{12}.$$

(d) Note that the events are *not* independent- we cannot just multiply probabilities. Let B be the set of outcomes where the total is 8 or higher and the first die roll is odd. Then

$$B = \left\{ \begin{array}{l} (3,6),(5,6), \\ (3,5),(5,5), \\ (5,4), \\ (5,3) \end{array} \right\}$$

Now $|B| = 6$, so

$$P(B) = \frac{|B|}{|\Omega|} = \frac{6}{36} = \frac{1}{6}$$

6. Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a prize; behind the other two, no prize.

   You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which reveals no prize. He then says to you, "Do you want to pick door No. 2?"

   Is it to your advantage to switch your choice?

ANSWER:

This example is called the *Monty Haul problem*. It is quite famous, and worth looking up if what is below is unclear.

Anyway, there is a probability of $\frac{1}{3}$ that the prize is behind the door first picked. So if you don't switch doors, you have a probability of $\frac{1}{3}$ of winning the prize. Nothing changes this.

Switching doors means you win whenever the prize is not behind the first door you pick- the host will open the door that is not yours, and that the prize is not behind. So switching means the probability of winning the prize is $1 - \frac{1}{3} = \frac{2}{3}$. So you should switch!

7. What is the probability that in a group of n people there will be at least one pair with the same birthday?

   How large must n be so that this probability is at least $1/2$. Assume that all birthdays are equally likely, and that the birthdays of different people are independent (so no twins, triplets and quadruplets in the group!)

ANSWER:

We ignore leap years, so there are 365 days in the year. Let A be the event that nobody shares the same birthday. If $n > 365$, $P(A) = 0$. So let $n \leq 365$. Then

$$\begin{aligned} P(A) &= \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right)\cdots\left(1 - \frac{n-1}{365}\right) \\ &= \frac{364 \times 363 \times \cdots \times (365 - n + 1)}{365^n} \\ &= \frac{365!}{365^n (365-n)!}. \end{aligned}$$

Thus the probability of at least two people having the same birthday is:

$$P_n = 1 - \frac{365!}{365^n (365-n)!}$$

The probability P_n increases as n gets bigger. Playing with a calculator and putting different values in, we see that

$$P_{22} = 0.493 \qquad P_{23} = 0.507$$

to 3 decimal places. Thus we need $n = 23$ for the probability of a match to be 0.5 or more.

8. To play the national lottery, you pick 6 different numbers from 1 to 49. In the prize draw, six numbers are drawn from a set of individually numbered balls with numbers in the range 1 to 49. Balls, once drawn are not returned to the draw machine. Prizes are awarded to players who match at least three of the six drawn numbers with increasing prize value for matching more of the drawn numbers.

   (a) Calculate the probability of matching three numbers.

   (b) Calculate the probability of matching four numbers.

   (c) Calculate the probability of matching five numbers.

   (d) Calculate the probability of matching all six numbers.

   (e) There is also a bonus ball, which is drawn after the six main numbers. A player wins a special prize if their numbers match five of those drawn initially, along with the bonus ball. Calculate the probability of this.

ANSWER:

There are

$$C(49, 6) = 13983816$$

possible selections of 6 balls.

(a) We need 3 of our 6 balls to match our numbers. There are $C(6, 3) = 20$ ways of arranging this.

The remaining 3 balls won't match our numbers. There are $49-6 = 43$ to choose from. The number of ways this can occur is $C(43, 3) = 12341$.

So the probability is

$$\frac{20 \times 12341}{13983816} = 0.018$$

to two significant figures.

(b) We need 4 of our 6 balls to match our numbers. There are $C(6, 4) = 15$ ways of arranging this.

The remaining 2 balls won't match our numbers. There are $49-6 = 43$ to choose from. The number of ways this can occur is $C(43, 2) = 903$.

So the probability is

$$\frac{15 \times 903}{13983816} = 0.00097$$

to two significant figures.

(c) We need 5 of our 6 balls to match our numbers. There are $C(6,5) = 6$ ways of arranging this.

The remaining ball won't match our numbers. There are $49 - 6 = 43$ to choose from. The number of ways this can occur is $C(43,1) = 43$.

So the probability is

$$\frac{6 \times 43}{13983816} = 0.000018$$

to two significant figures.

(d) We need all 6 balls to match. There is only one way this can occur, so the probability is

$$\frac{1}{13983816} = 0.000000072$$

to two significant figures.

(e) There are $49 - 6 = 43$ possibilities for the bonus ball. So 1 out of every 43 combinations with 5 matching numbers have the final number matching the bonus ball. So the probability here is

$$\frac{6 \times 1}{13983816} = 0.00000043.$$

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9. A population consists of 53% men. The probability of colour blindness is 0.02 for a man and 0.001 for a woman. Find the probability that a person picked at random is colour blind.

ANSWER:

Let A be the event that a person picked at random is colour blind. Let B be the event that a person picked at random is a man. Then

$$\begin{aligned} P(A) &= P(A|B)P(B) + P(A|B^c)P(B^c) \\ &= 0.02 \times \tfrac{53}{100} + 0.001 \times \tfrac{47}{100} \\ &= 0.01107 \end{aligned}$$

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10. On a particular journey, the probability of a sober driver having an accident is 0.001, and the probability of a drunk driver having an accident is 0.1. The probability that a given driver is drunk is 0.08.

If a driver has an accident, how likely is it that they were drunk?

ANSWER:

Let A be the event of an accident, and D be the event the driver is drunk. Then

$$\begin{aligned} P(A) &= P(A|D)P(D) + P(A|D^c)P(D^c) \\ &= 0.1 \times 0.08 + 0.001 \times 0.92 \\ &= 0.00892 \end{aligned}$$

We want to know

$$P(D|A) = \frac{P(D \cap A)}{P(A)} = \frac{P(A \cap D)}{P(A)}.$$

We have

$$P(A \cap D) = P(A|D)P(D) = 0.1 \times 0.08 = 0.008.$$

So

$$P(D|A) = \frac{0.008}{0.00892} = 0.90.$$