1.1. Vectors (Continued)

1.1.1. Projection of a vector $\mathbf{A}$ onto an axis $\mathbf{u}$: $|\mathbf{u}| = 1$.

Definition: $\mathrm{Proj}_{\mathbf{u}}\mathbf{A} = (|\mathbf{A}|\cos\phi)\mathbf{u}$.

Here ϕ represents the angle between the two vectors $\mathbf{A}$ and $\mathbf{u}$.

Figure 1.1.7. Projections with acute and obtuse angles.

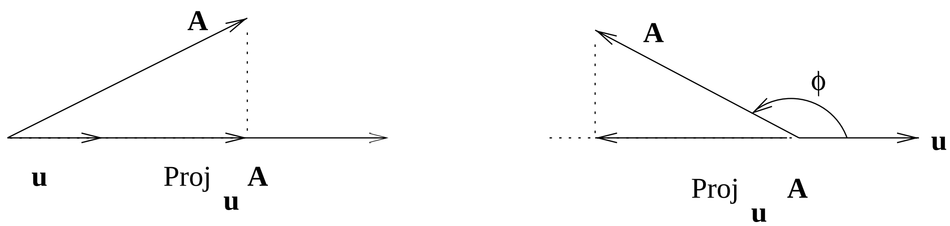

Figure 1.1.7. Projections with acute and obtuse angles.

If ϕ is between $-\pi/2$ and $\pi/2$, then the cos is positive and the projection is in the same direction as $\mathbf{u}$. If ϕ is between $\pi/2$ and $3\pi/2$, the projection is in the opposite direction.

We see that if $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal axes, the projection can be used to decompose $\mathbf{A}$:

$$\mathbf{A} = \mathrm{Proj}_{\mathbf{u}_1}\mathbf{A} + \mathrm{Proj}_{\mathbf{u}_2}\mathbf{A}.$$

For a general nonzero vector $\mathbf{B}$, the projection onto $\mathbf{B}$ is

$$\mathrm{Proj}_{\mathbf{B}}\mathbf{A} = (|\mathbf{A}|\cos\phi)\frac{\mathbf{B}}{|\mathbf{B}|}.$$

1.1.2. Inner product. (a.k.a. scalar product, dot product)

Example. Let $\mathbf{A} = (1, -1, 2), \quad \mathbf{B} = (2, 3, -5)$. Then

$$\mathbf{A} \cdot \mathbf{B} = 1 \cdot 2 + (-1) \cdot 3 + 2 \cdot (-5) = -11.$$

In general, for any two vectors $\mathbf{A} = a_1\mathbf{i}_1 + a_2\mathbf{i}_2 + a_3\mathbf{i}_3, \quad \mathbf{B} = b_1\mathbf{i}_1 + b_2\mathbf{i}_2 + b_3\mathbf{i}_3$, the inner product is given by

$$\mathbf{A} \cdot \mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3. \tag{1}$$

Properties: $\mathbf{A}\cdot\mathbf{B}=\mathbf{B}\cdot\mathbf{A}, \quad (2\mathbf{A})\cdot\mathbf{B}=2(\mathbf{A}\cdot\mathbf{B})$.

Distributive law: $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C})=\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$.

Now let us examine an example. Let

$$\mathbf{i}_1=(1,0,0), \quad \mathbf{i}_2=(0,1,0), \quad \mathbf{i}_3=(0,0,1).$$

Then,

$$\begin{aligned} &\mathbf{i}_1\cdot\mathbf{i}_1=1, \quad \mathbf{i}_2\cdot\mathbf{i}_2=1, \quad \mathbf{i}_3\cdot\mathbf{i}_3=1, \\ &\mathbf{i}_1\cdot\mathbf{i}_2=0, \quad \mathbf{i}_1\cdot\mathbf{i}_3=0, \quad \mathbf{i}_2\cdot\mathbf{i}_3=0. \end{aligned} \tag{2}$$

If a set of vectors satisfies (2), then the set is called *orthonormal*. Conditions (2) are called orthonormal conditions.

Recall the law of cosine from High School trig course:

$$c^2=a^2+b^2-2ab\cos\phi,$$

where ϕ is the angle that faces the side c in a triangle with side lengths a, b, c.

Now look at Figure 1.1.8: (Angle and inner product.)

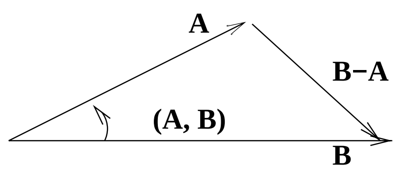

Figure 1.1.8. Angle and inner product.

Let us use the law of cosine for $a=|\mathbf{A}|, b=|\mathbf{B}|, c=|\mathbf{B}-\mathbf{A}|$ in Figure 1.1.8. Do the calculation

$$\begin{aligned} c^2 &= |\mathbf{B}-\mathbf{A}|^2=(\mathbf{B}-\mathbf{A})\cdot(\mathbf{B}-\mathbf{A}) \\ &= |\mathbf{B}|^2+|\mathbf{A}|^2-2\mathbf{B}\cdot\mathbf{A}. \end{aligned} \tag{3}$$

Then we see that

$$\mathbf{A}\cdot\mathbf{B}=|\mathbf{A}||\mathbf{B}|\cos(\mathbf{A},\mathbf{B}) \tag{4}$$

where $(\mathbf{A},\mathbf{B})$ is now used to denote the angle between $\mathbf{A}$ and $\mathbf{B}$. Although the inner product in (1) is simple and direct, formula (4) is also a popular, equivalent definition.

The inner product can be used to express the projection. Formula:

$$\mathrm{Proj}_{\mathbf{B}}\mathbf{A}=\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}|^2}\mathbf{B}. \tag{5}$$

We give a one line proof for the formula (5):

$$\text{Proj}_{\mathbf{B}}\mathbf{A} = (|\mathbf{A}|\cos\phi)\frac{\mathbf{B}}{|\mathbf{B}|} = (|\mathbf{A}||\mathbf{B}|\cos\phi)\frac{\mathbf{B}}{|\mathbf{B}|^2} = (\mathbf{A}\cdot\mathbf{B})\frac{\mathbf{B}}{|\mathbf{B}|^2}.$$

Now we see that $\mathbf{A}$ and $\mathbf{B}$ is orthogonal (defined as $\phi = \pm\pi/2$) if and only if $\mathbf{A}\cdot\mathbf{B} = 0$, and if and only if $\text{Proj}_{\mathbf{B}}\mathbf{A} = \mathbf{0}$.

1.1.3. Vector product (a.k.a cross product)

Given two vectors $\mathbf{A}$ and $\mathbf{B}$. We define the **vector product** of $\mathbf{A}$ and $\mathbf{B}$ to be a vector $\mathbf{C}$:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

where

1. The length of $\mathbf{C}$ is the area of the parallelogram spanned by $\mathbf{A}$ and $\mathbf{B}$:

$$|\mathbf{C}| = |\mathbf{A}||\mathbf{B}||\sin(\mathbf{A},\mathbf{B})|;$$

2. The direction of $\mathbf{C}$ is perpendicular to the plane formed by $\mathbf{A}$ and $\mathbf{B}$. And the three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ follow the *right-hand rule*. (It will follow the left-hand rule if the coordinate system is left-handed. We shall always use right-handed coordinate systems.)

The right-hand rule is: when the four fingers of the right hand turn from $\mathbf{A}$ to $\mathbf{B}$, the thumb points to the direction of $\mathbf{C}$.

(Figure 1.1.3.1. Definition of cross product.)

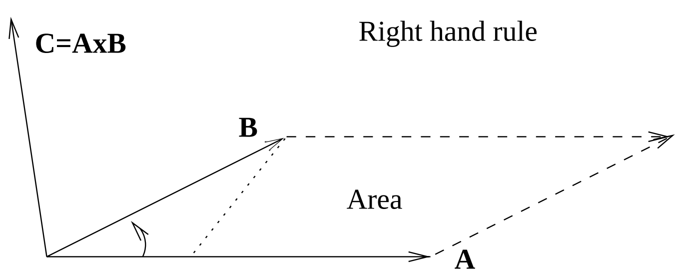

Figure 1.1.3.1. Definition of cross product.

Properties:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A};$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C};$$

$$\mathbf{A} \| \mathbf{B} \text{ is the same as } \mathbf{A} \times \mathbf{B} = \mathbf{0}$$

where the symbol $\|$ means parallel.

Example 1.1.3a: Recall the three vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$. They follow the right hand rule. By definition we can find that

$$\mathbf{i}_1 \times \mathbf{i}_1 = \mathbf{0}, \quad \mathbf{i}_2 \times \mathbf{i}_2 = \mathbf{0}, \quad \mathbf{i}_3 \times \mathbf{i}_3 = \mathbf{0};$$

and

$$\mathbf{i}_1 \times \mathbf{i}_2 = \mathbf{i}_3, \quad \mathbf{i}_2 \times \mathbf{i}_3 = \mathbf{i}_1, \quad \mathbf{i}_3 \times \mathbf{i}_1 = \mathbf{i}_2.$$

With Example 1.1.3a and the distributive property, we can find a formula for the product. Let

$$\mathbf{A} = a_1\mathbf{i}_1 + a_2\mathbf{i}_2 + a_3\mathbf{i}_3, \quad \mathbf{B} = b_1\mathbf{i}_1 + b_2\mathbf{i}_2 + b_3\mathbf{i}_3.$$

Then (please do it on your own. you can do it.)

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Example 1.1.3b. An electric charge e moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{H}$ experiences a force:

$$\mathbf{F} = \frac{e}{c}(\mathbf{v} \times \mathbf{H})$$

where c is the speed of light.

Example 1.1.3c. The moment $\mathbf{M}$ of a force $\mathbf{F}$ about a point O is

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}.$$

(Figure 1.1.3.2. Moment of a force.)

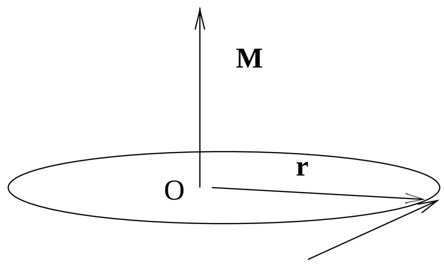

Figure 1.1.3.2. The moment of a force.

—End of Lecture 2 —