Math Refresher for DS

Lecture 1

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Today

1. Course overview
2. Linear Algebra
   - Core objects
   - Vector spaces
3. A bit of Analytic Geometry
   - Orthogonal projections
   - Hyperplanes
   - Normals

About this course

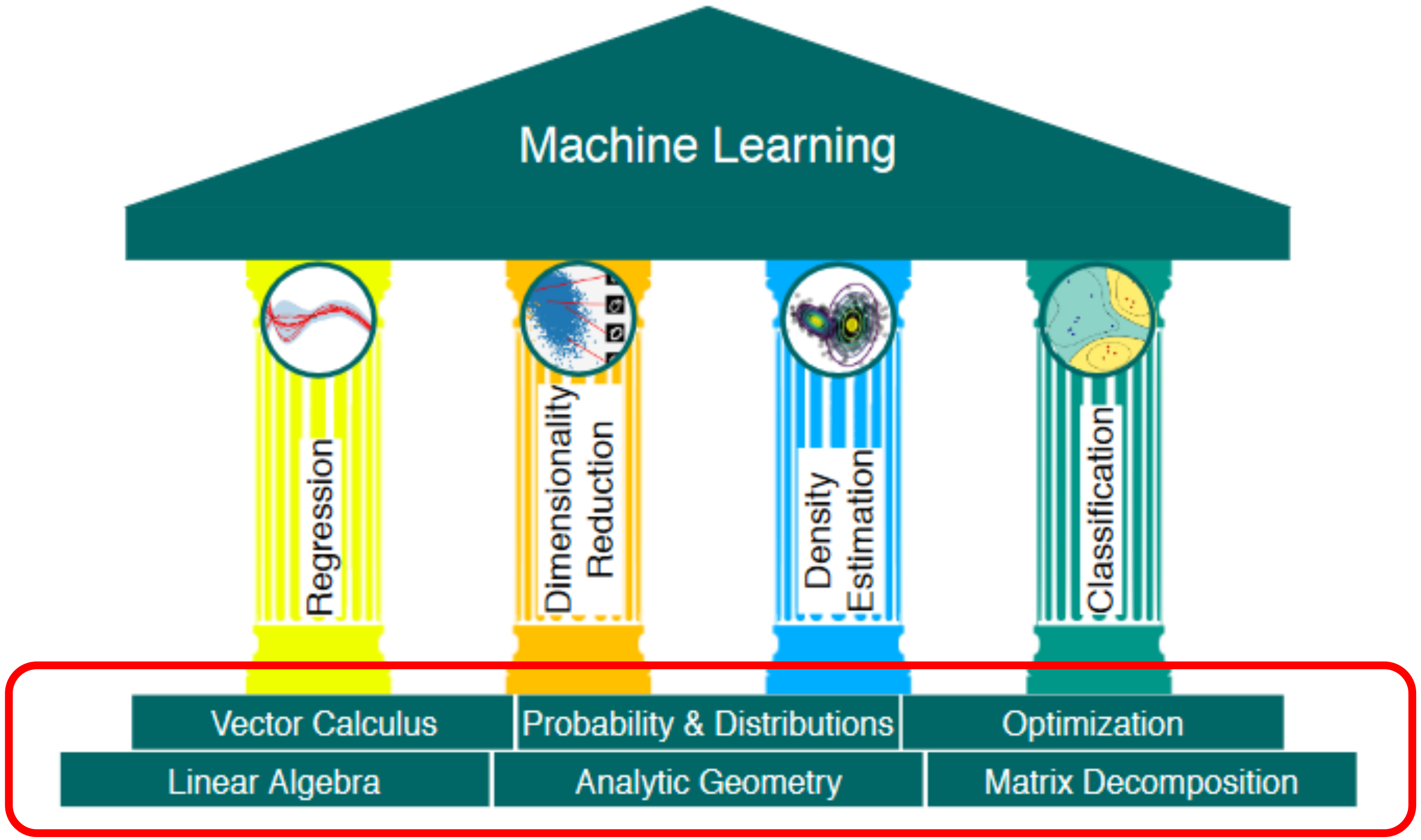

Image source: Mathematics for Machine Learning, p. 14 (https://mml-book.github.io/book/mml-book.pdf)

About this course

In this course:

1. Linear algebra
2. Calculus
3. (Basic) optimization

Prerequisites:

- basic knowledge of math;
- some Python.

About this course

Logistics

- Pre-recorded lectures
- Online practical sessions
  - → Tuesdays & Fridays
    19:00 Moscow time
- 5 graded assignments
- 2 exams
- Final grade:
  - 30% Linear Algebra exam
  - 30% Calculus & Optimization exam
  - 40% graded assignments

About me

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- PhD researcher KU LEUVEN
- Lecturer HARBOUR SPACE
- DS content lead JetBrains Academy
- *(Soon)* Data Scientists acmetric

Linear Algebra: the Basics

Linear Algebra: Core Objects

- $\alpha \in \mathbb{R}$ - a scalar *Example:* -2

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Linear Algebra: Core Objects

- $\alpha \in \mathbb{R}$ - a **scalar** *Example:* -2
- $x \in \mathbb{R}^n$ - a **vector** with n entries

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Example: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$, $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \in \mathbb{R}^5$

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- $A \in \mathbb{R}^{m \times n}$ - a **matrix** with m rows and n columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{2\times 3}$

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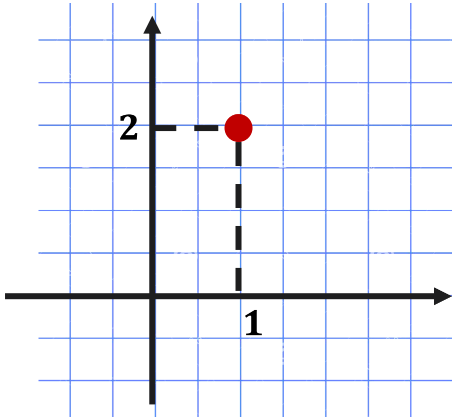

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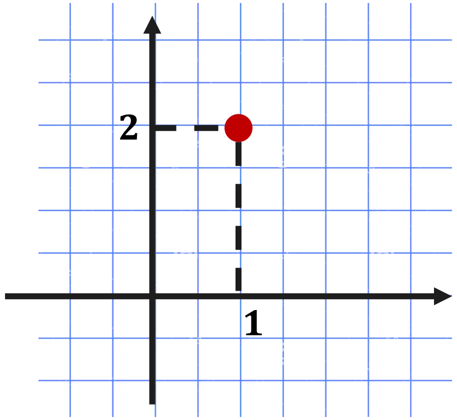
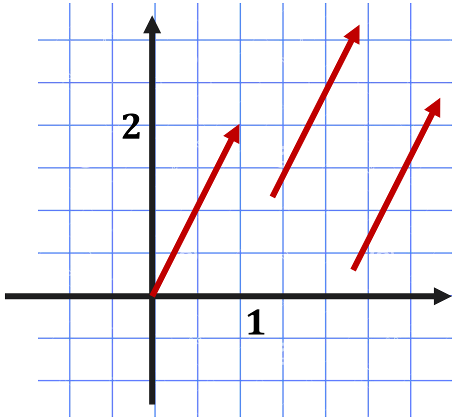


- Direction + length



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Vector Spaces

Vector Space: Definition

- A real-valued **vector space** $(V, +, \cdot)$ is a set of vectors V with two operations

$$(1)\ +: \ V \times V \to V, \qquad (2)\ \cdot: \ \mathbb{R} \times V \to V$$

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that satisfy the following properties (axioms):

	Property	Meaning
1.	**Associativity** of addition	$x + (y + z) = (x + y) + z$
2.	**Commutativity** of addition	$x + y = y + x$
3.	**Identity element** of addition	$\exists 0 \in V: \ \forall x \in V \quad 0 + x = x$
4.	**Identity element** of scalar multiplication	$\forall x \in V \quad 1 \cdot x = x$
5.	**Inverse element** of addition	$\forall x \in V \ \exists -x \in V: \ x + (-x) = 0$
6.	**Compatibility** of scalar multiplication	$\alpha(\beta x) = (\alpha\beta)x$
7.	**Distributivity**	$(\alpha + \beta)x = \alpha x + \beta x$
8.		$\alpha(x + y) = \alpha x + \alpha y$

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Let’s define vector operations!

Operations with Vectors

1. Sum of two vectors:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n, \qquad x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_2 \end{bmatrix} \in \mathbb{R}^n$$

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2. Multiplying by a scalar:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \qquad \alpha \in \mathbb{R}, \qquad \alpha x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix} \in \mathbb{R}^n$$

Operations with Vectors: Example

$x, y \in \mathbb{R}^3$:

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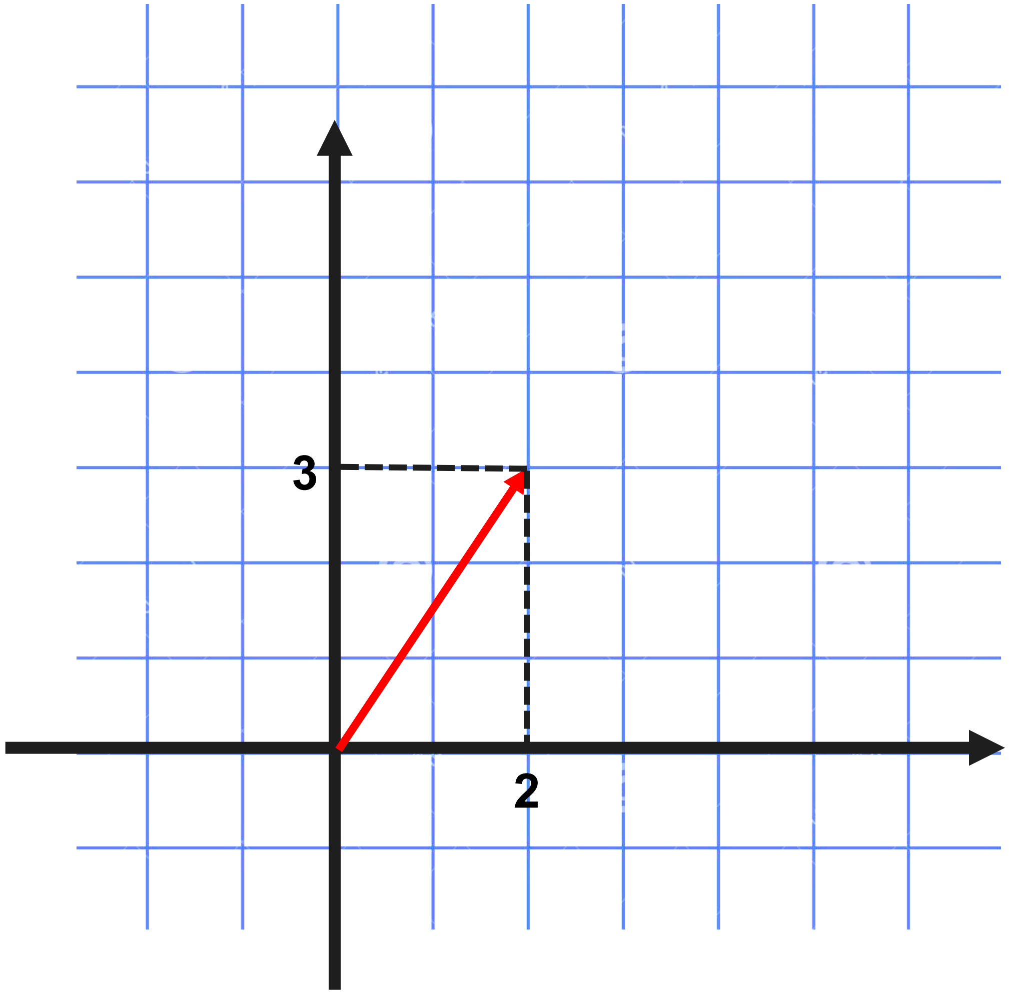
Sum:

$$x + y = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Vector Operations: Geometrical Interpretation

Vectors: Geometrical Interpretation

$\vec{a} = [2, 3]$



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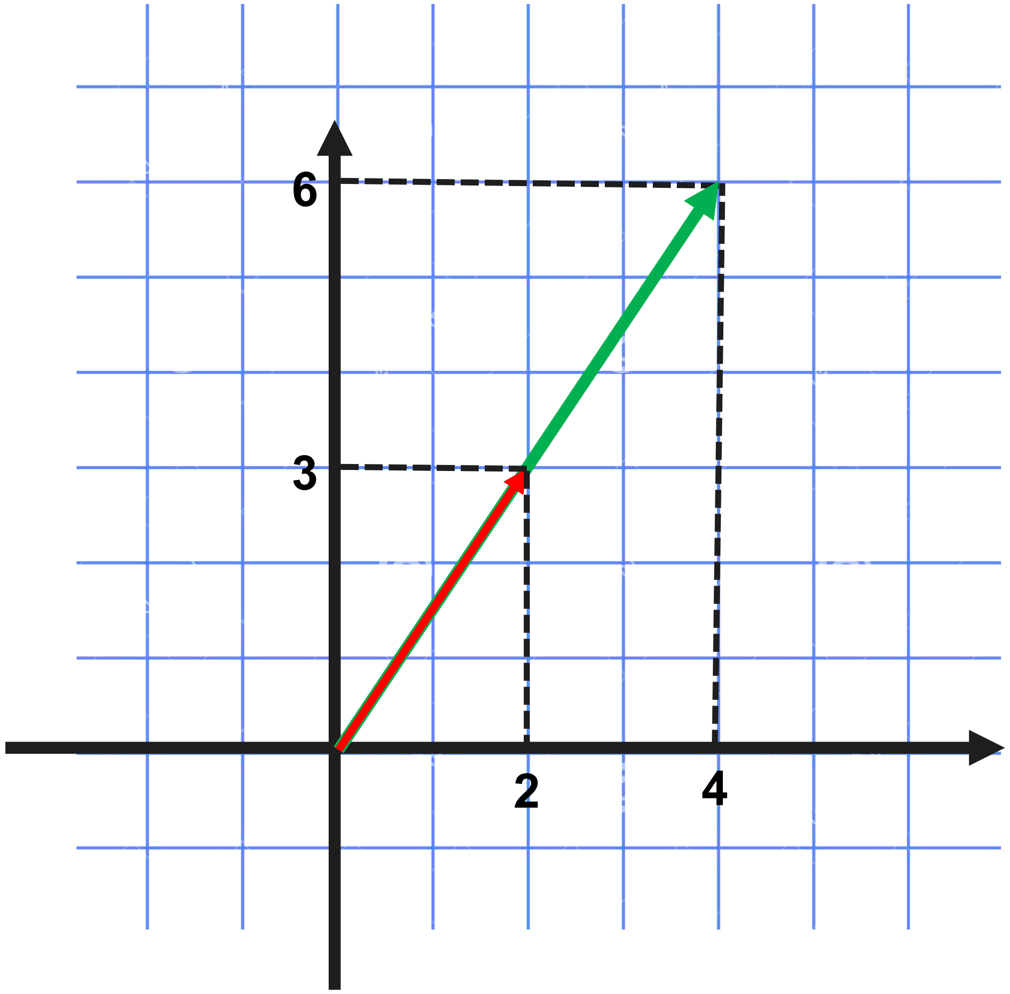

Vectors: Geometrical Interpretation

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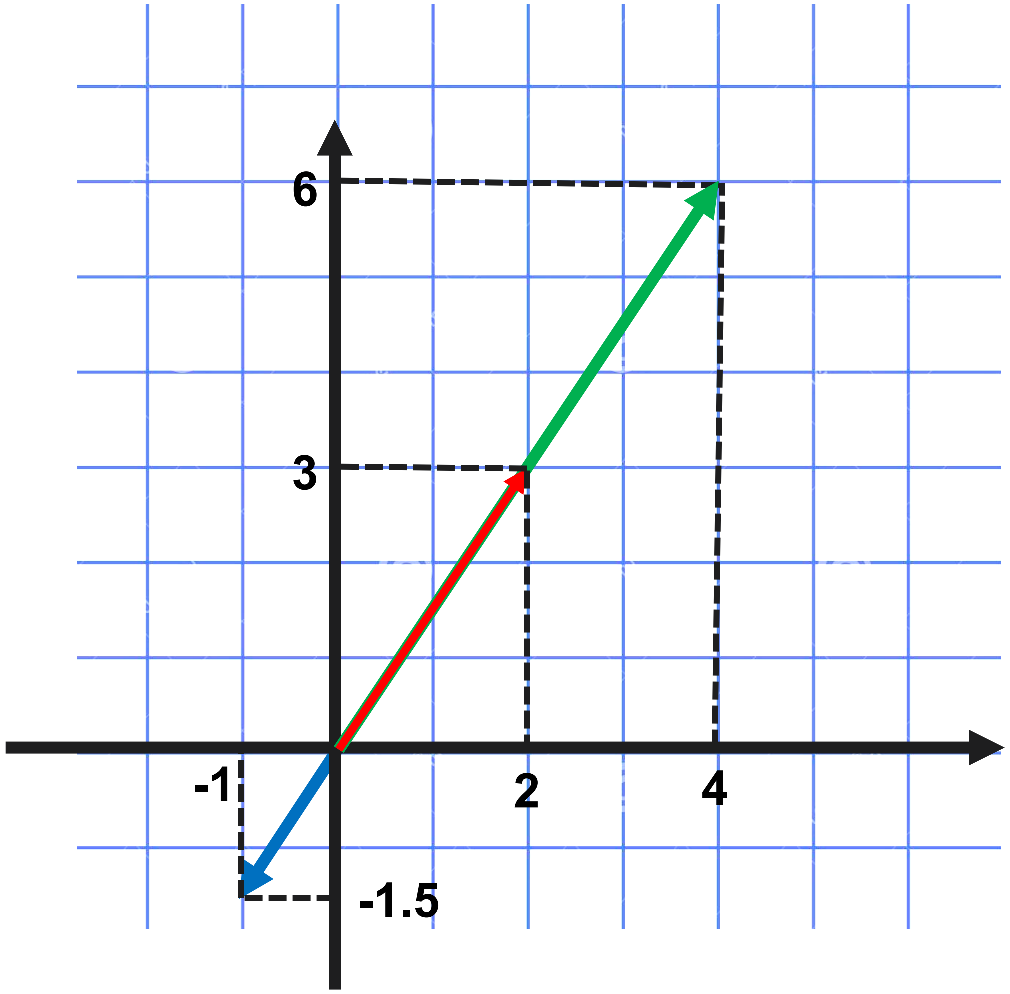
$$-0.5\vec{a} = [-1, -1.5]$$



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$\vec{a} = [2, 5]$

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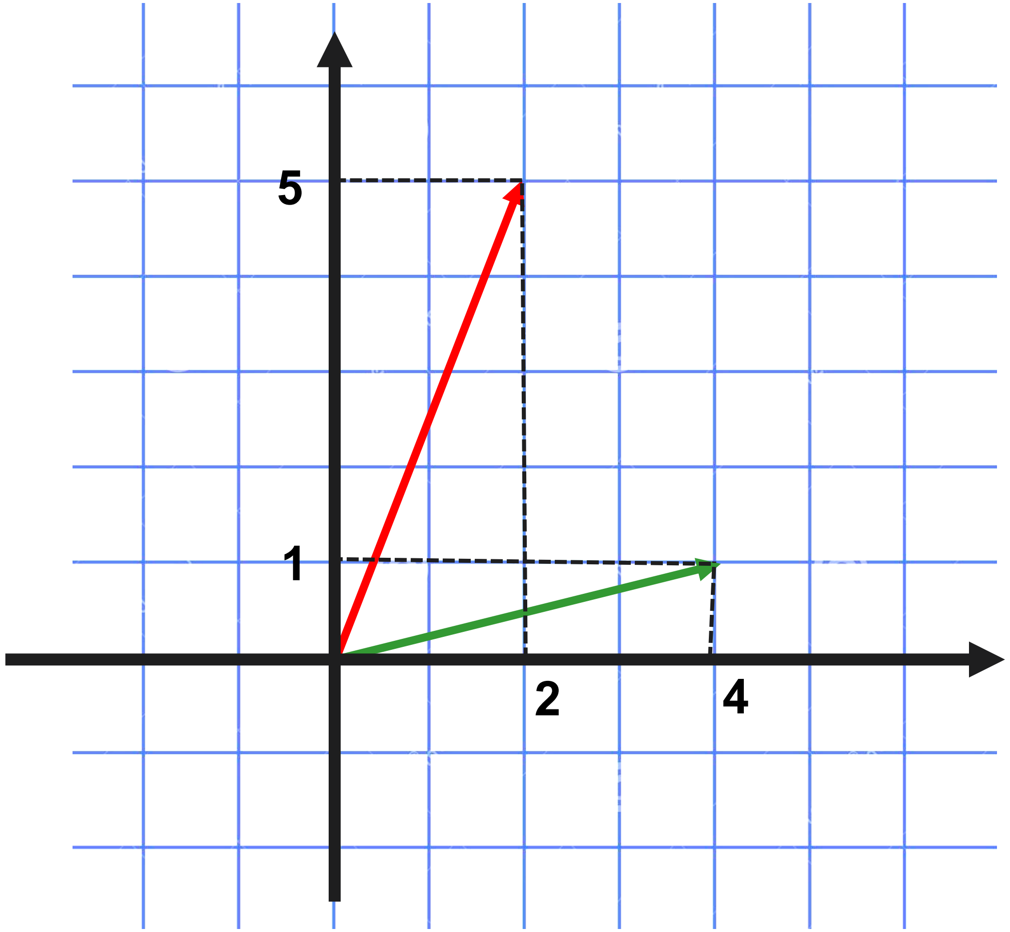

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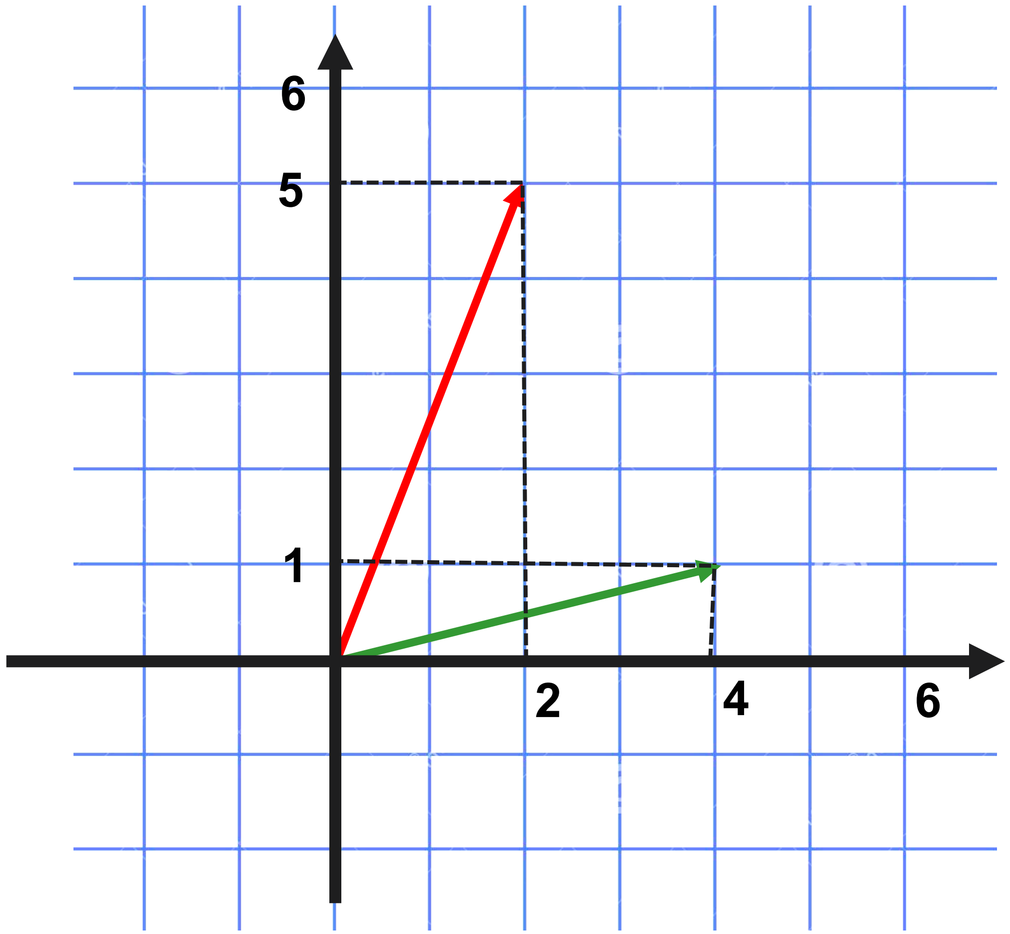

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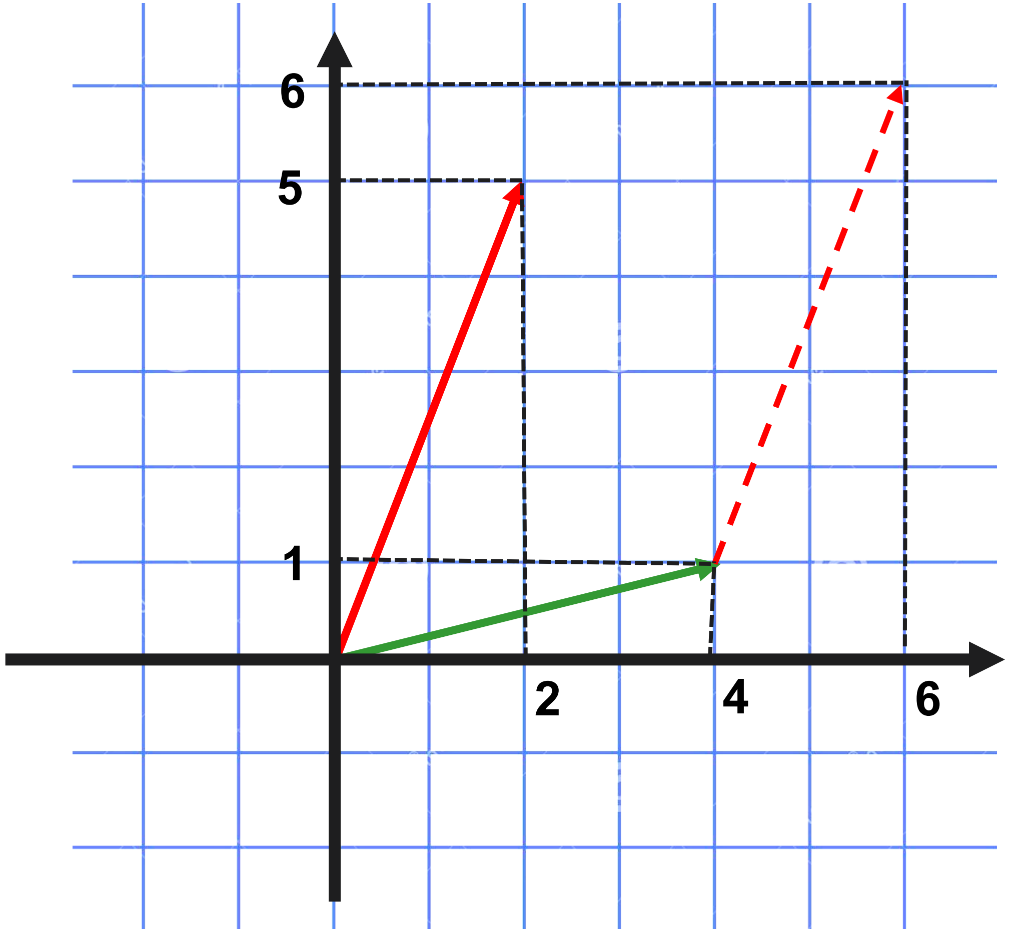

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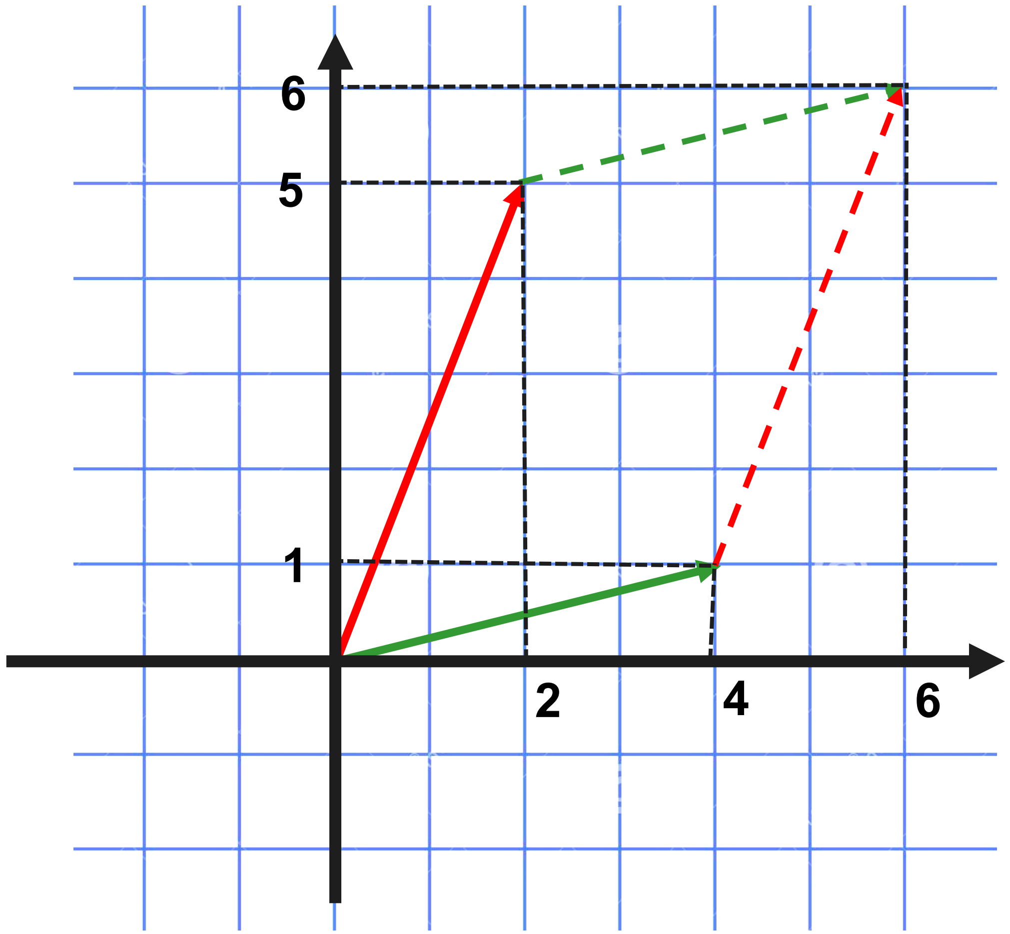

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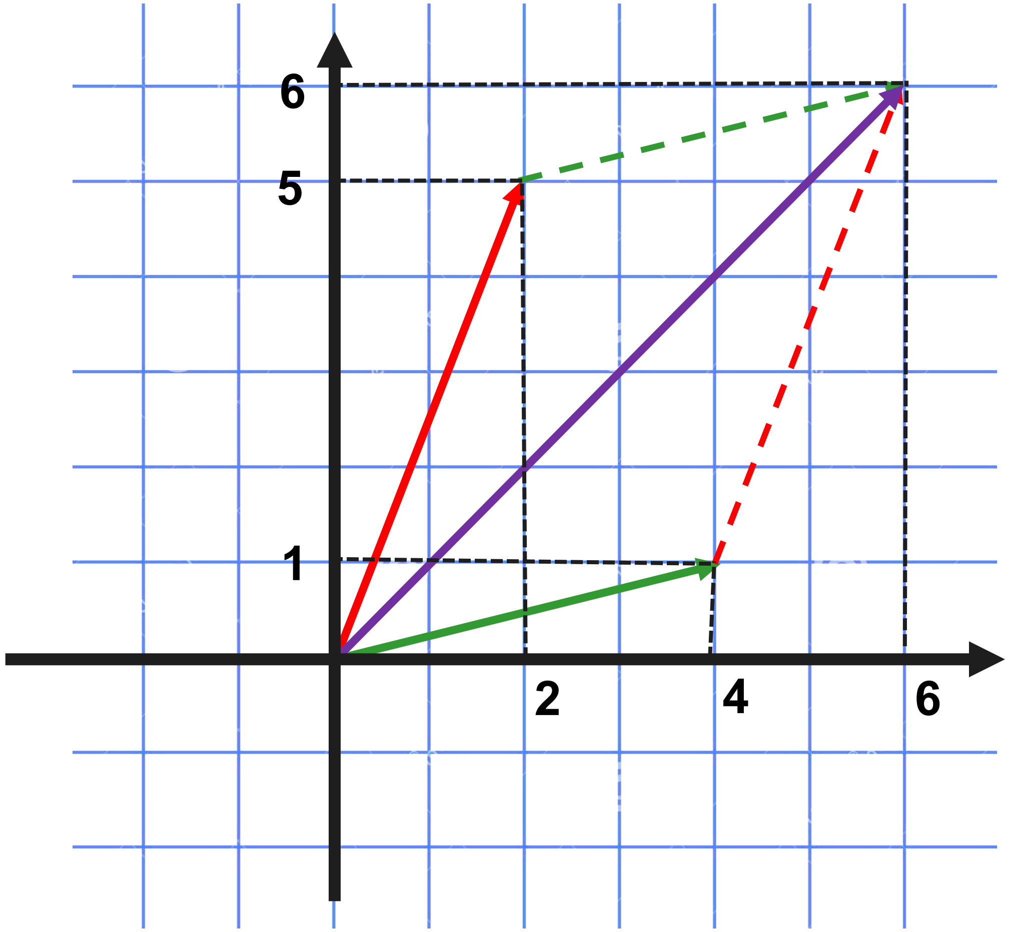

Back to Vector Spaces

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2. Multiplying by a scalar:

satisfy axioms (1) – (8)
(check it yourself)

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Vector Spaces

$(\mathbb{R}^n, +, \cdot), n \in \mathbb{N}$ - a vector space with operations

1. vector addition:

$$x + y = (x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

2. multiplication by a scalar:

$$\alpha x = \alpha(x_1, x_2, \dots, x_n) = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$$

Vector Spaces: Another Example

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- $\mathbb{P}^n$ - a set of polynomials of degree $\leq n$ with real coefficients
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- We can add up polynomials:

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- These operations satisfy axioms (1) – (8)!

$\rightarrow (\mathbb{P}^n, +, \cdot)$ is also a vector space!

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Inner Product

Inner Product

- Inner product is a function $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies the following properties:
    - *Symmetric:* $\forall x, y \in V \;\; \langle x, y \rangle = \langle y, x \rangle$
    - *Positive definite:* $\forall x \in V \setminus \{0\} \;\; \langle x, x \rangle > 0$ and $\langle x, 0 \rangle = 0$.

Dot Product

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- For $x = [x_1, \dots, x_n],\ y = [y_1, \dots, y_n] \in \mathbb{R}^n$

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- Example:

$$x = [1, 2, 3, 4], \qquad y = [-1, 0, 1, 2]$$

$$(x, y) = 1 \cdot (-1) + 2 \cdot 0 + 3 \cdot 1 + 4 \cdot 2 = -1 + 0 + 3 + 8 = 10$$

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- *Note: there're inner products different from dot product.*

Norms

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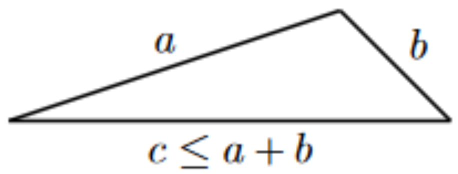
  - *Positive definite:* $\forall x \in \mathbb{V} \quad \|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$
  - *Triangle inequality:* $\forall x, y \in \mathbb{V} \ \|x + y\| \leq \|x\| + \|y\|$



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Examples of Norms

Manhattan Norm

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- Examples:

$$\|[1, 2, 3]\|_1 = 1 + 2 + 3 = 6$$

$$\|[1, 0]\|_1 = 1 + 0 = 1$$

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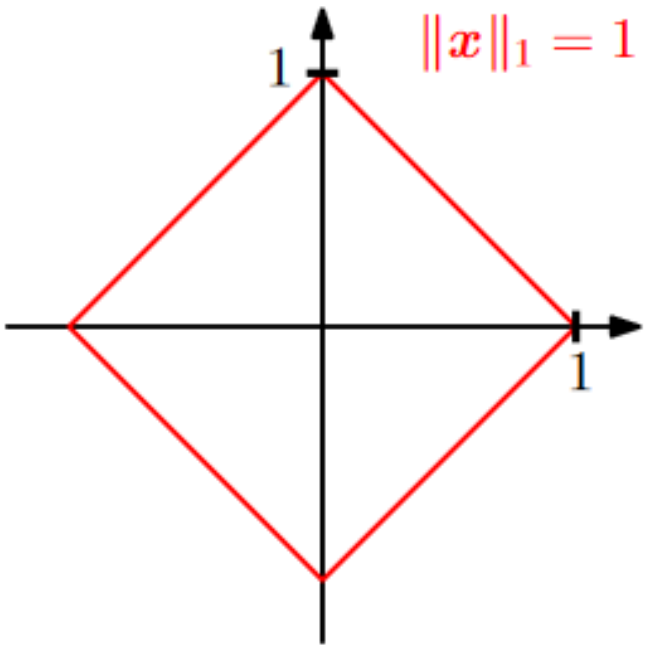

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- Examples:

$$\|[1,2,3]\|_2 = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

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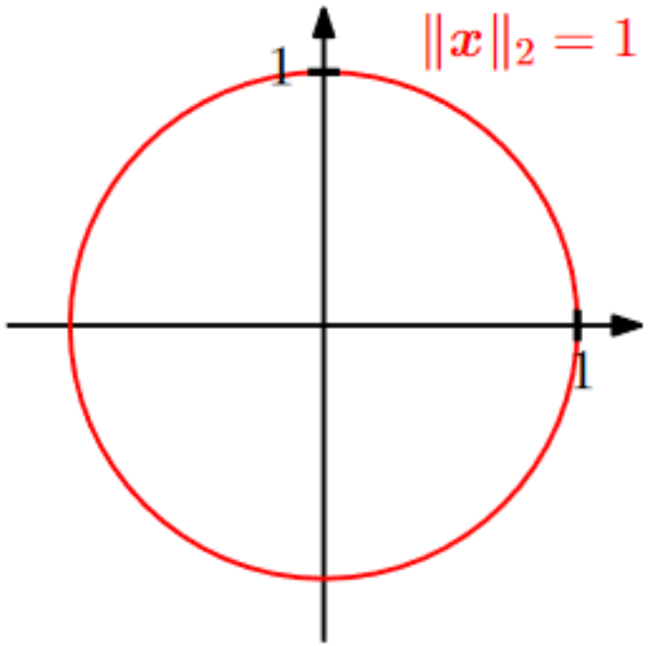

Other norms

- In general, for $x = [x_1, \dots, x_n] \in \mathbb{R}^n$ an $\boldsymbol{\ell_p}$-norm is defined as follows:

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  - ℓ_1 - Manhattan norm $\|\cdot\|_1$;
  - ℓ_2 - Euclidian norm $\|\cdot\|$ (default);
  - ℓ_∞: $\| x \|_\infty = \max_i |x_i|$

    Example: $\|[1,2,3]\|_\infty = 3,\quad \|[1,0]\|_\infty = 1,\quad \|[-1,0.5]\|_\infty = 1.$

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- Example: dot product induces Euclidian norm:

$$\sqrt{(x, x)} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \|x\|_2$$

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Check yourself that this is indeed a norm.

- Example: dot product induces Euclidian norm:

$$\sqrt{(x, x)} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \|x\|_2$$

- (!) Not every norm is induced by an inner product.
  Example: Manhattan norm.

Cauchy-Schwarz Inequality

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- For dot product and Euclidian norm, we get *Euclidian distance*:

$$d(x, y) = \|x - y\|_2 = \sqrt{(x - y, x - y)} =$$

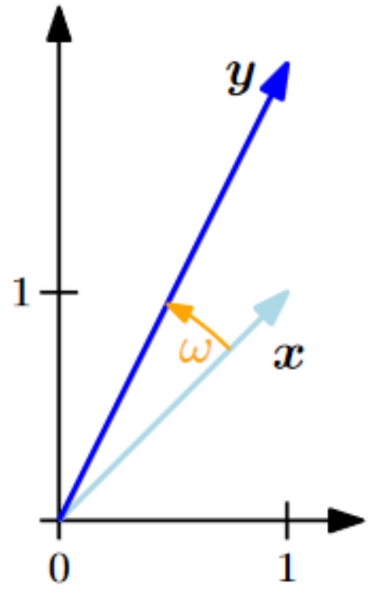
$$= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

Angles and Orthogonality

Angle between Two Vectors

- Inner product also captures the geometry of vector space by defining the angle between two vectors.

- Remember Cauchy-Schwarz inequality:



$$|(x, y)| \leq \|x\| \cdot \|y\|$$

$$-1 \leq \frac{(x, y)}{\|x\| \cdot \|y\|} \leq 1$$

$\omega\colon\ \cos\omega = \frac{(x,y)}{\|x\|\cdot\|y\|}$ - angle between x and y.

Angle between Two Vectors: Example

- What is the angle ω between $x = [5, 0]$ and $y = [1, 1]$?

$$\omega = \arccos \frac{(x, y)}{\|x\| \|y\|} = \arccos \frac{5 \cdot 1 + 0 \cdot 1}{\sqrt{5^2 + 0^2} \cdot \sqrt{1^2 + 1^2}} = \arccos \frac{5}{5\sqrt{2}} = \arccos \frac{\sqrt{2}}{4} = \frac{\pi}{4}.$$

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$$x = [1, 0], \qquad y = [0,1], \qquad (x, y) = 0, \qquad \|x\| = \|y\| = 1 \;\; \rightarrow$$

x and y are *orthonormal*.

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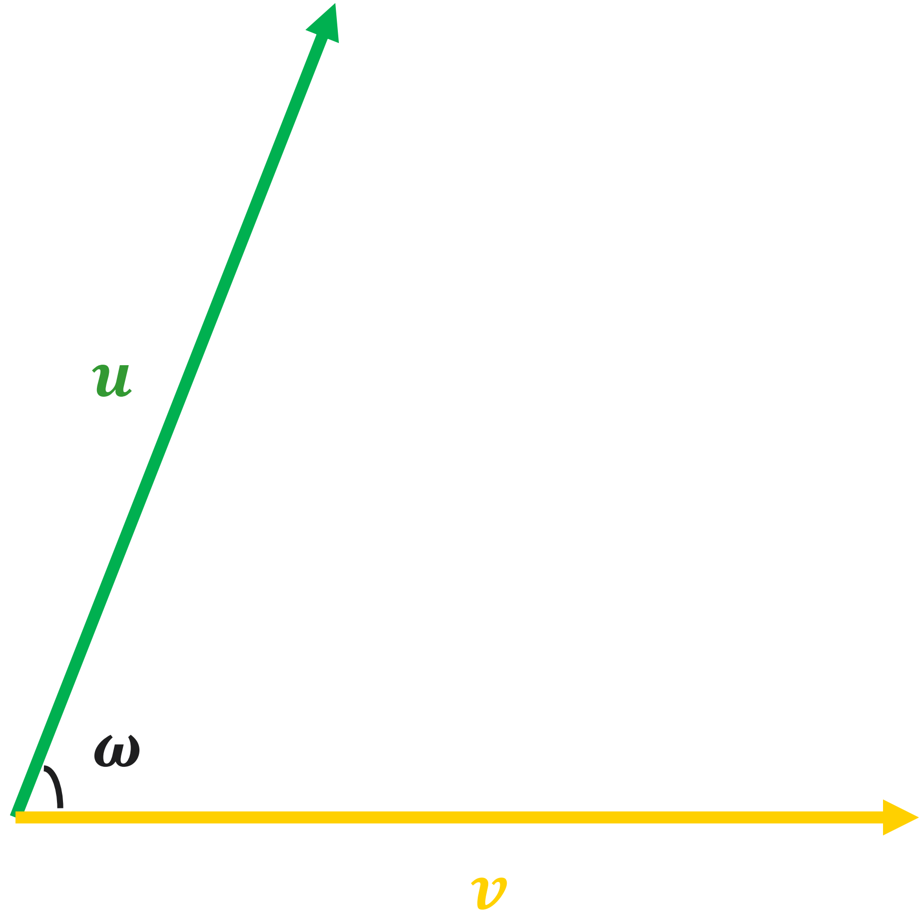
Orthogonal Projection

- Suppose we have two vectors u and v.



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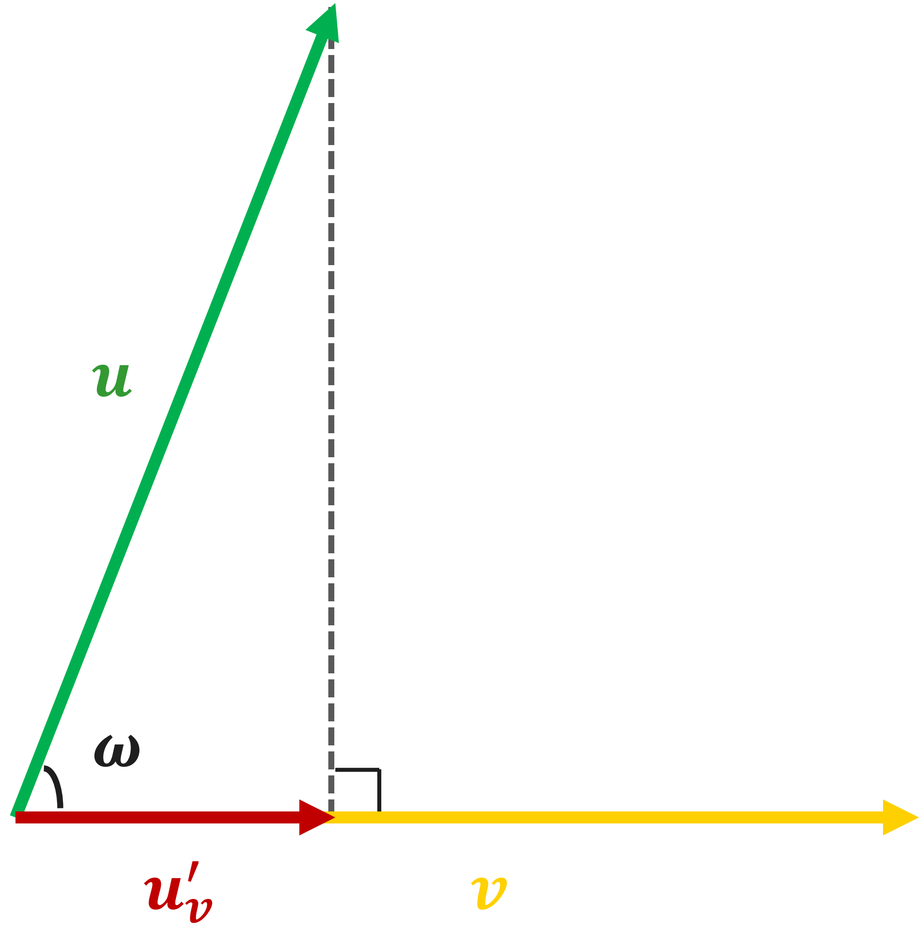

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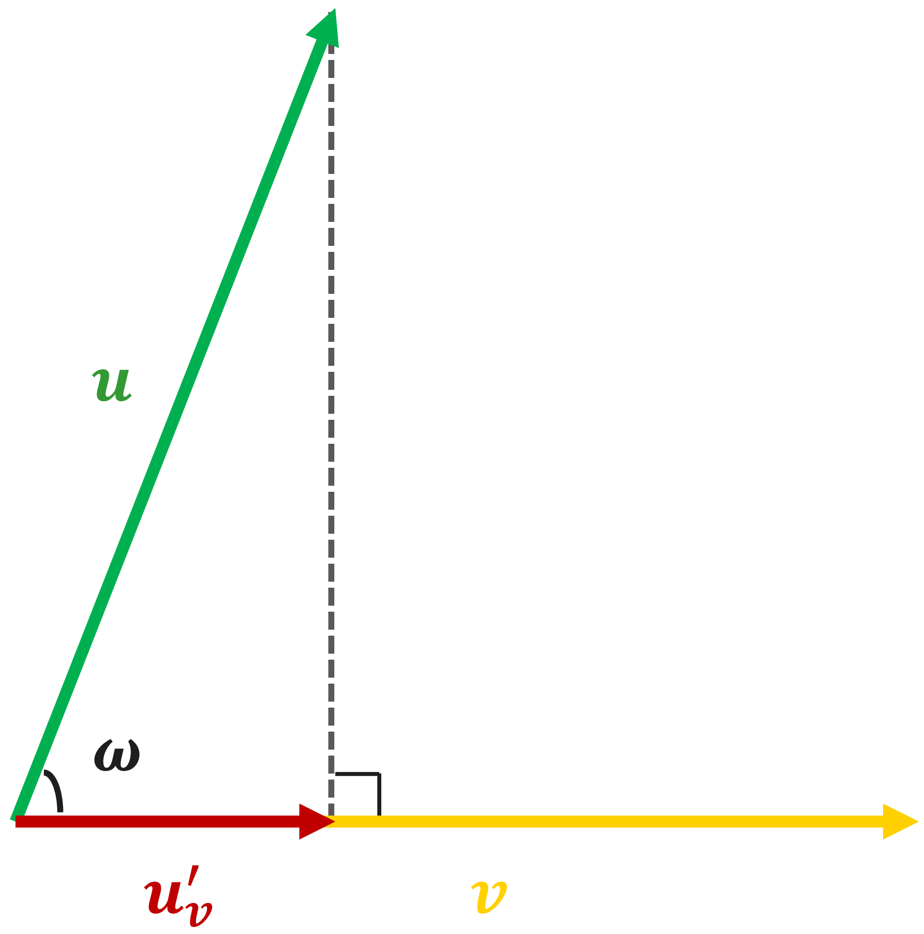

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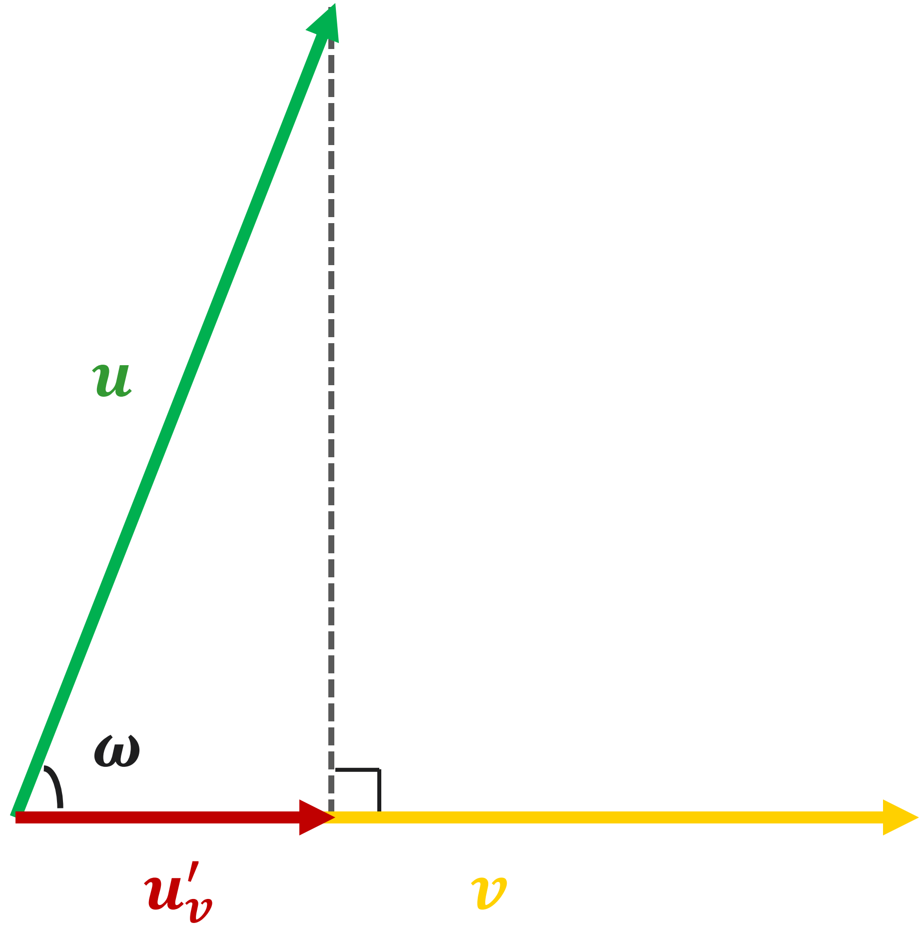
If $\mathbf{0} \leq \boldsymbol{\omega} \leq \mathbf{90}$

$$(u, v) = \|u\|\|v\| \cos \boldsymbol{\omega} = \|u\|\|v\| \frac{\|\boldsymbol{u'_v}\|}{\|\boldsymbol{u}\|} =$$

$$= \|u'_v\|\|v\|$$



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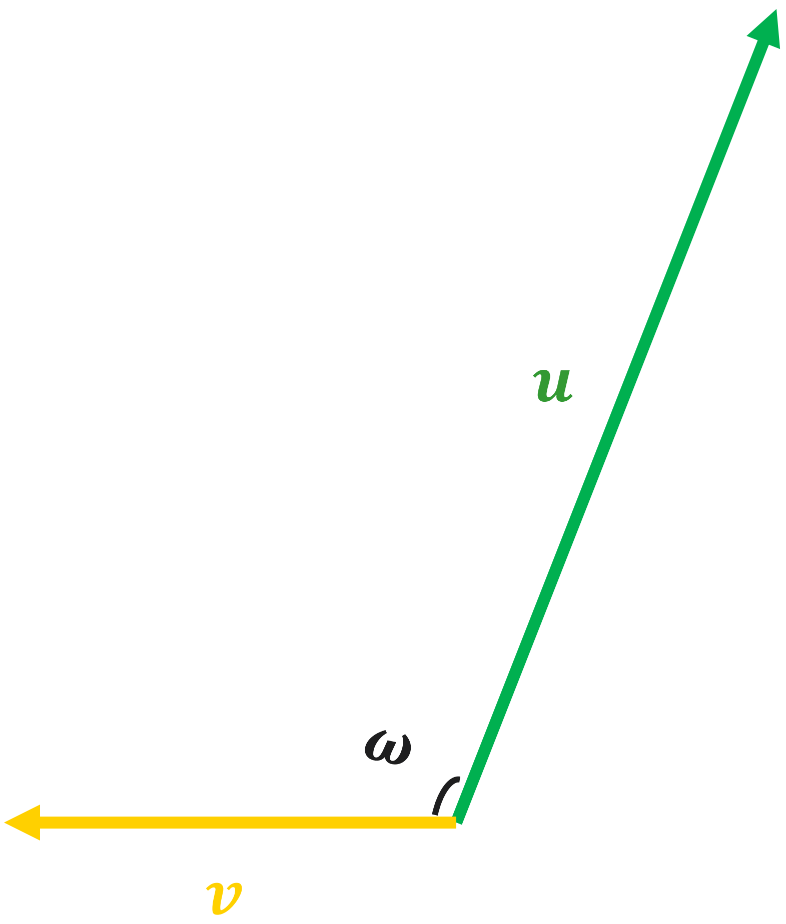

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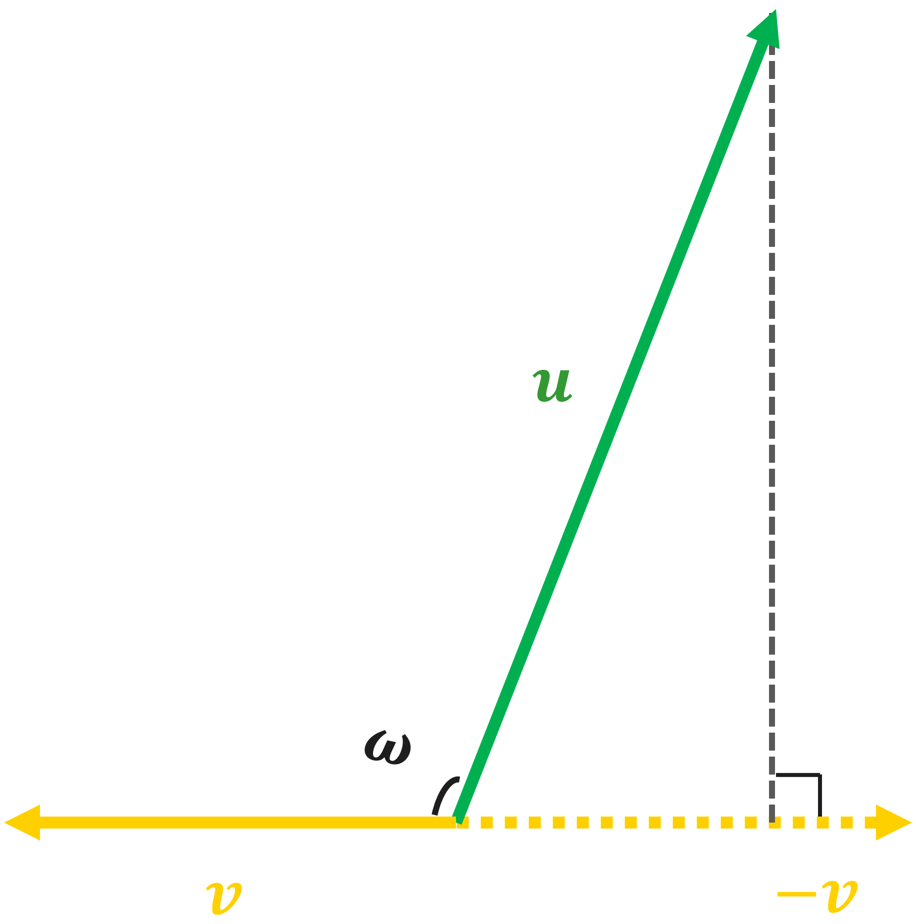

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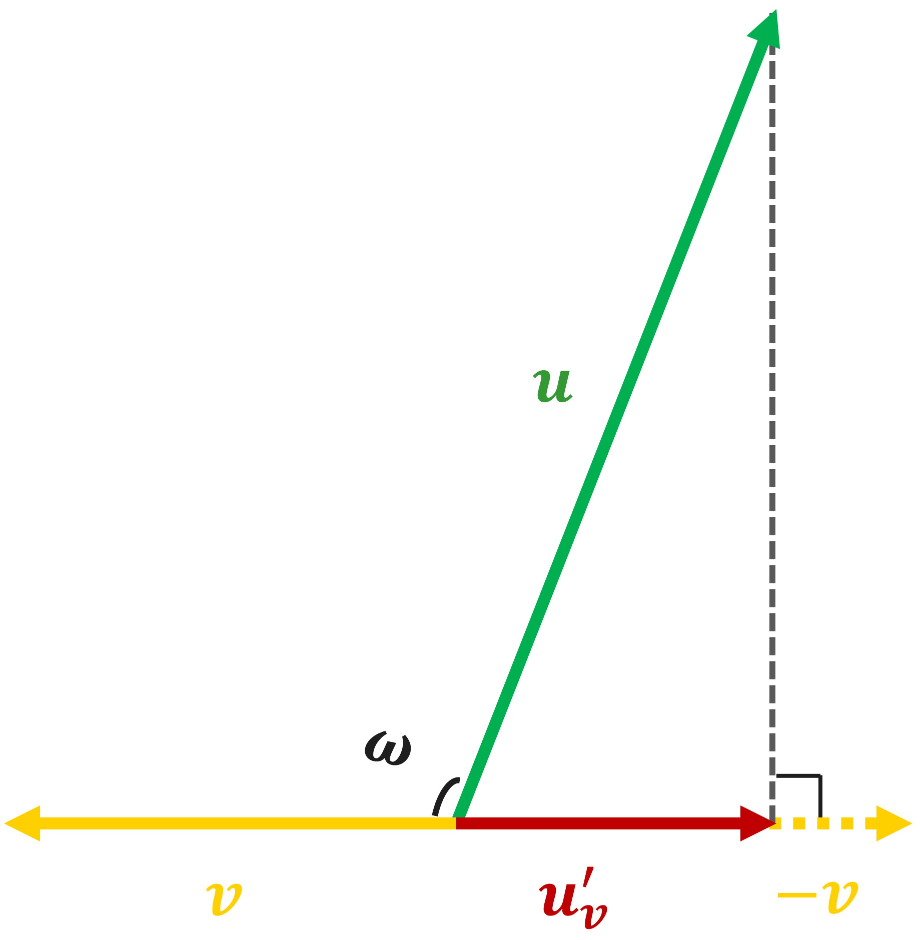

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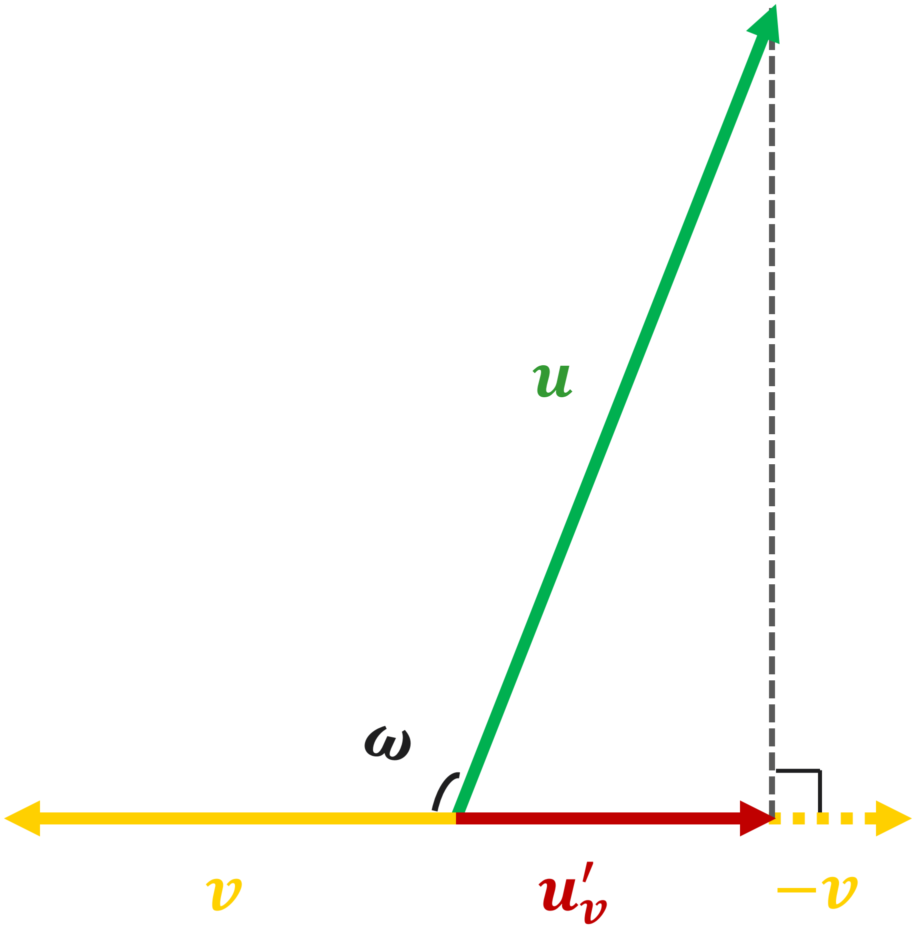

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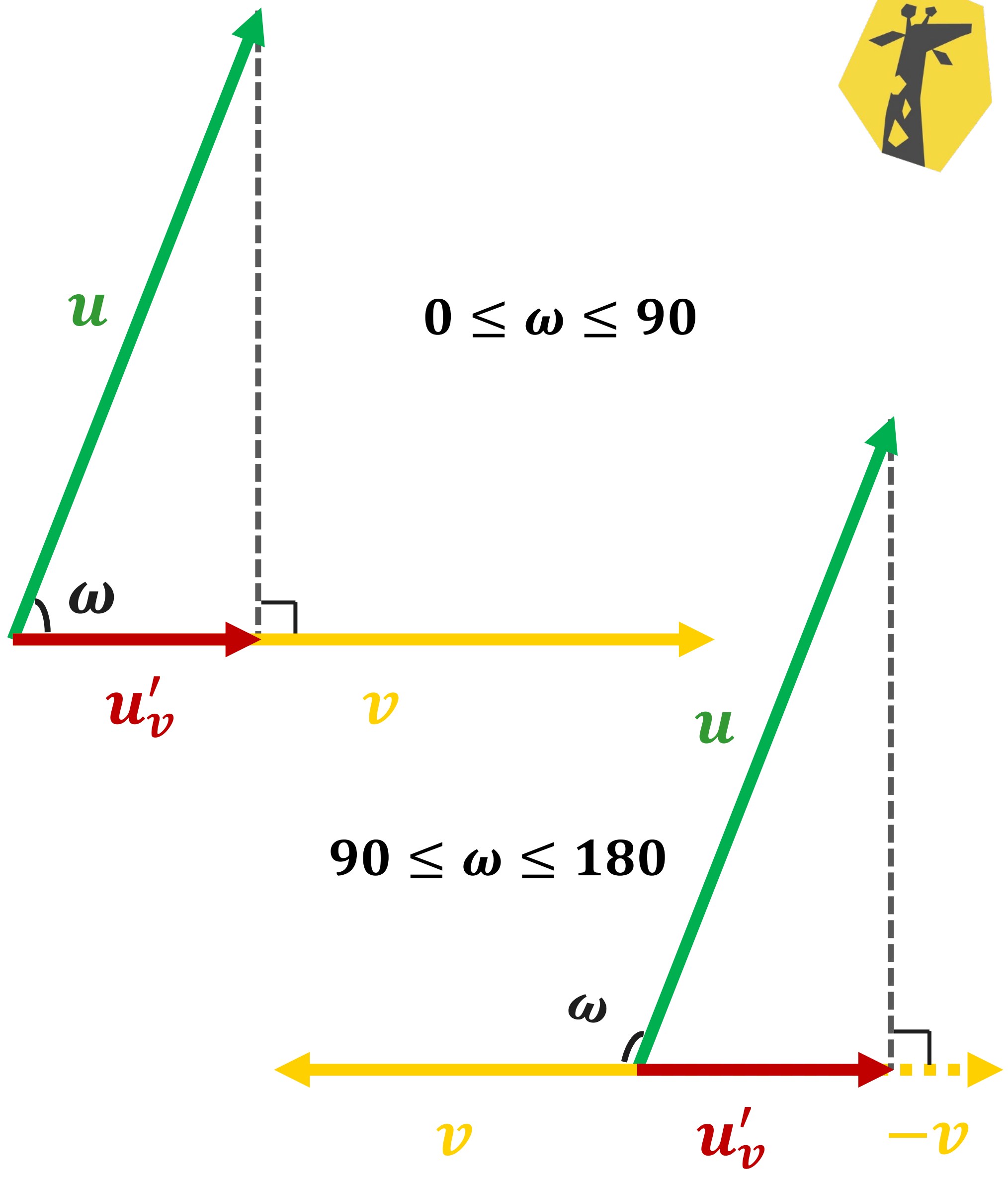

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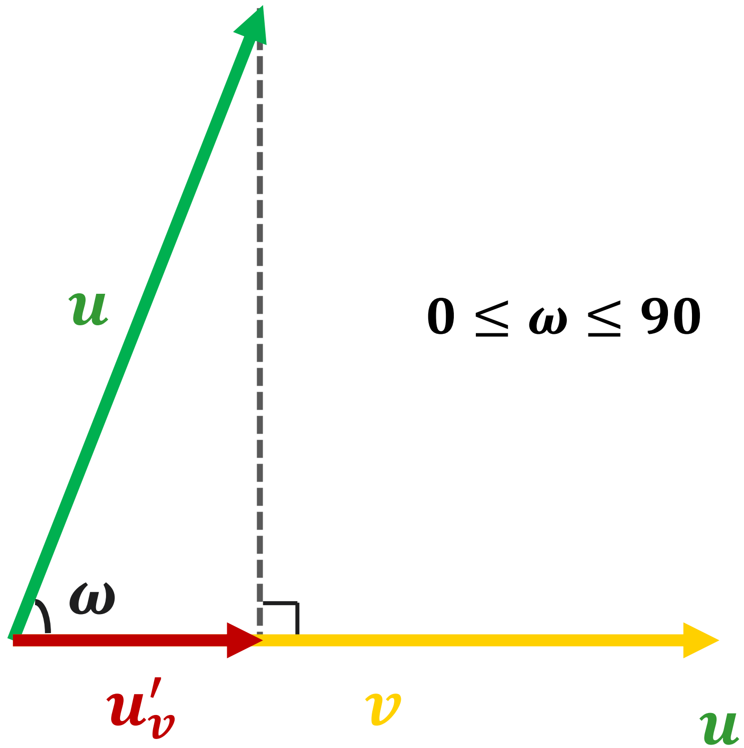
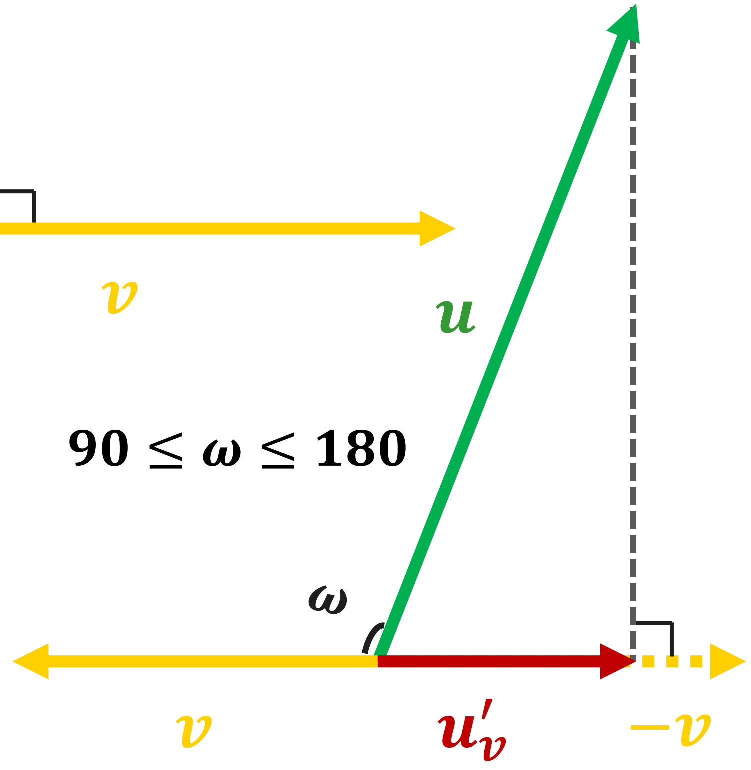
$$u'_v = \frac{(u, v)}{(v, v)} v.$$





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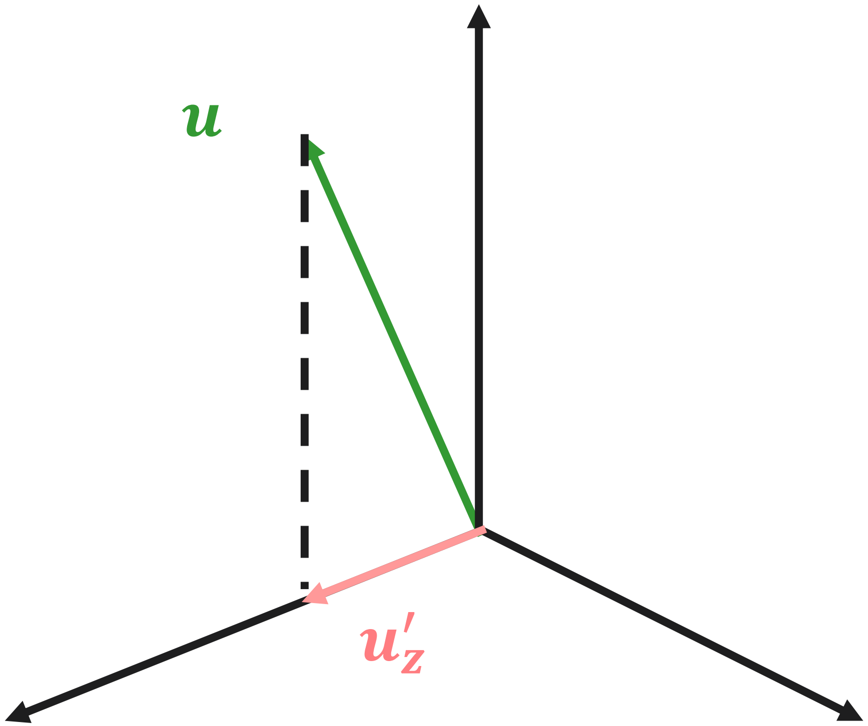
- What's projection of $u = [1, 3, 2]$ on $z = [0, 0, 1]$?



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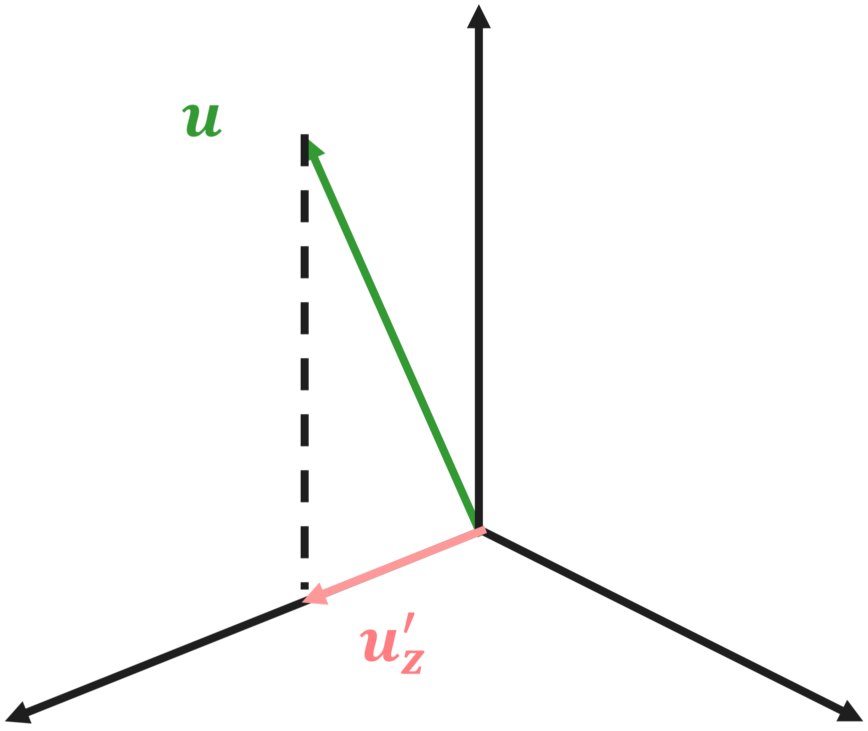
(Projection on the axis = drop other coordinates)



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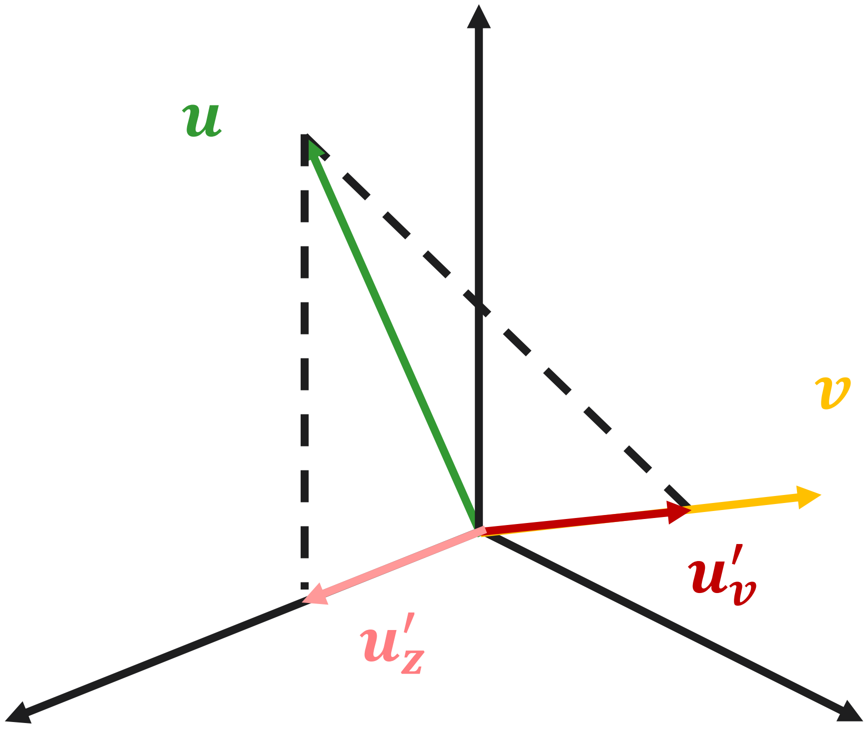

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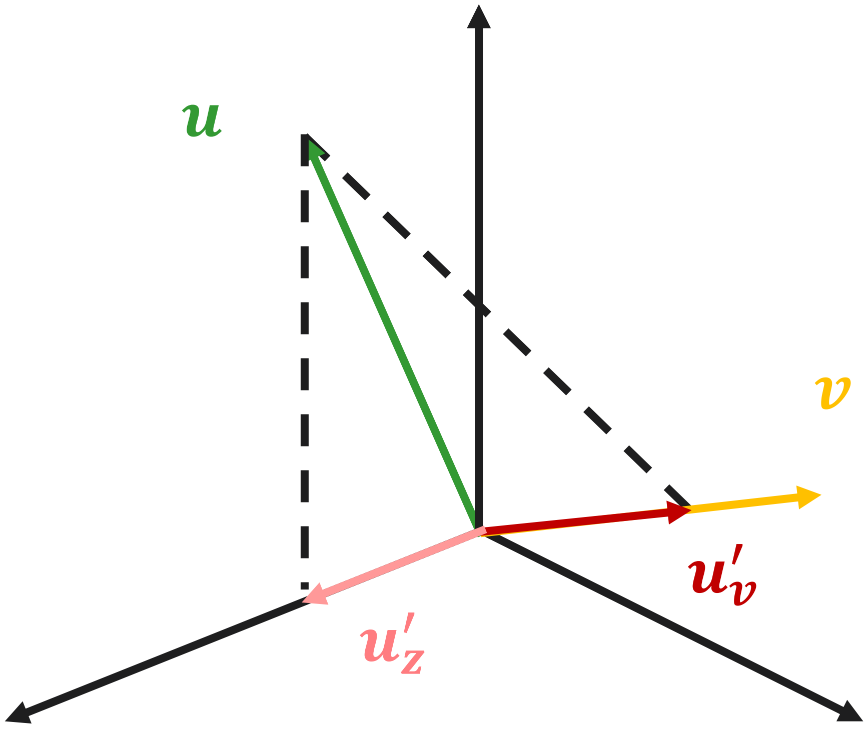
$$u'_v = \frac{(u, v)}{(v, v)} v = \frac{4 + 3 + 6}{16 + 1 + 9} v = \frac{1}{2} v = [2, 0.5, 1.5].$$



Hyperplanes

- A hyperplane is described by equation

$$w_1 x_1 + w_2 x_2 + \cdots + w_n x_n + b = 0$$

where at least one $w_i \neq 0$.

- A more compact notation:

$$(w, x) + b = 0, \qquad w = (w_1, \dots, w_n)$$

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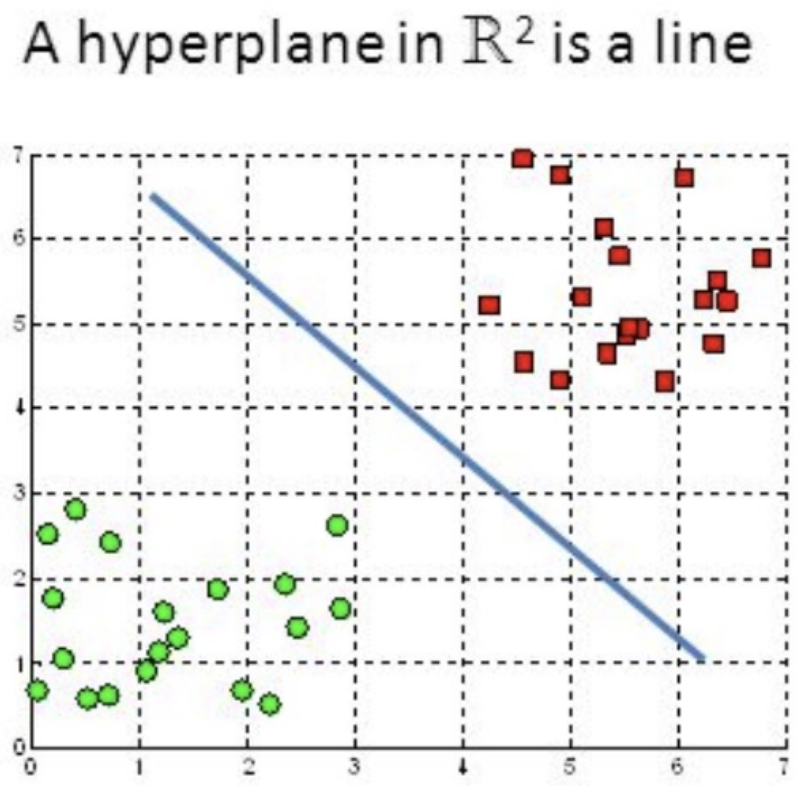
Hyperplanes



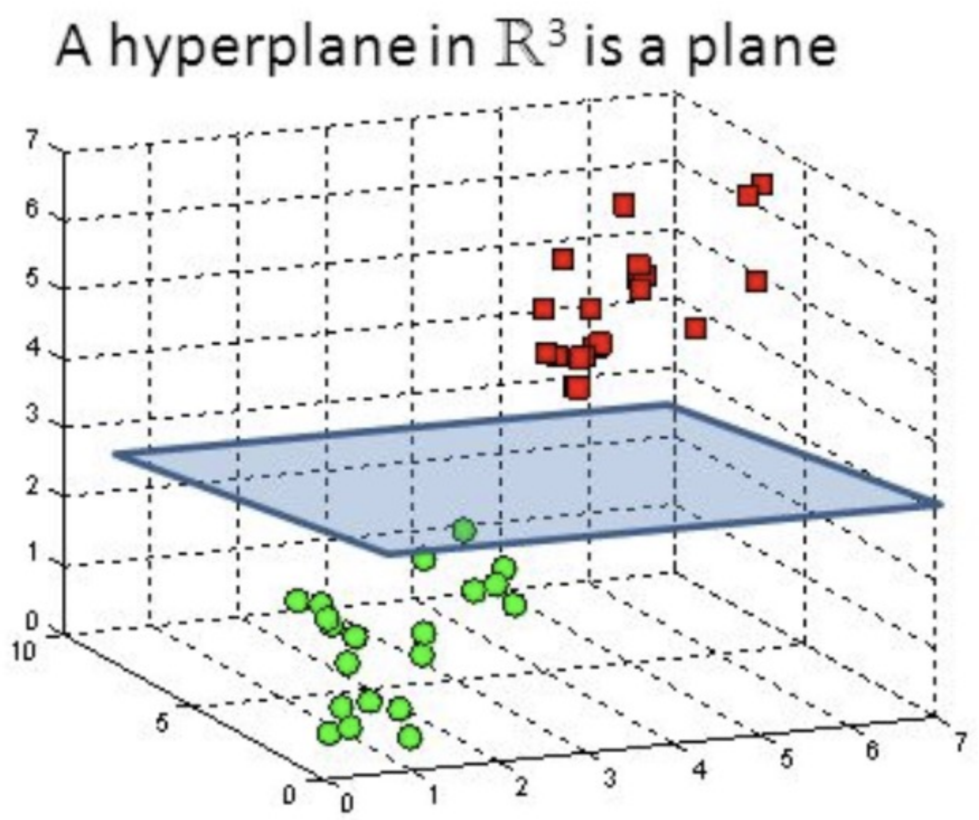


Image source: https://deepai.org/machine-learning-glossary-and-terms/hyperplane

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- Vector $w = (w_1, \dots, w_n)$ defines the hyperplane.

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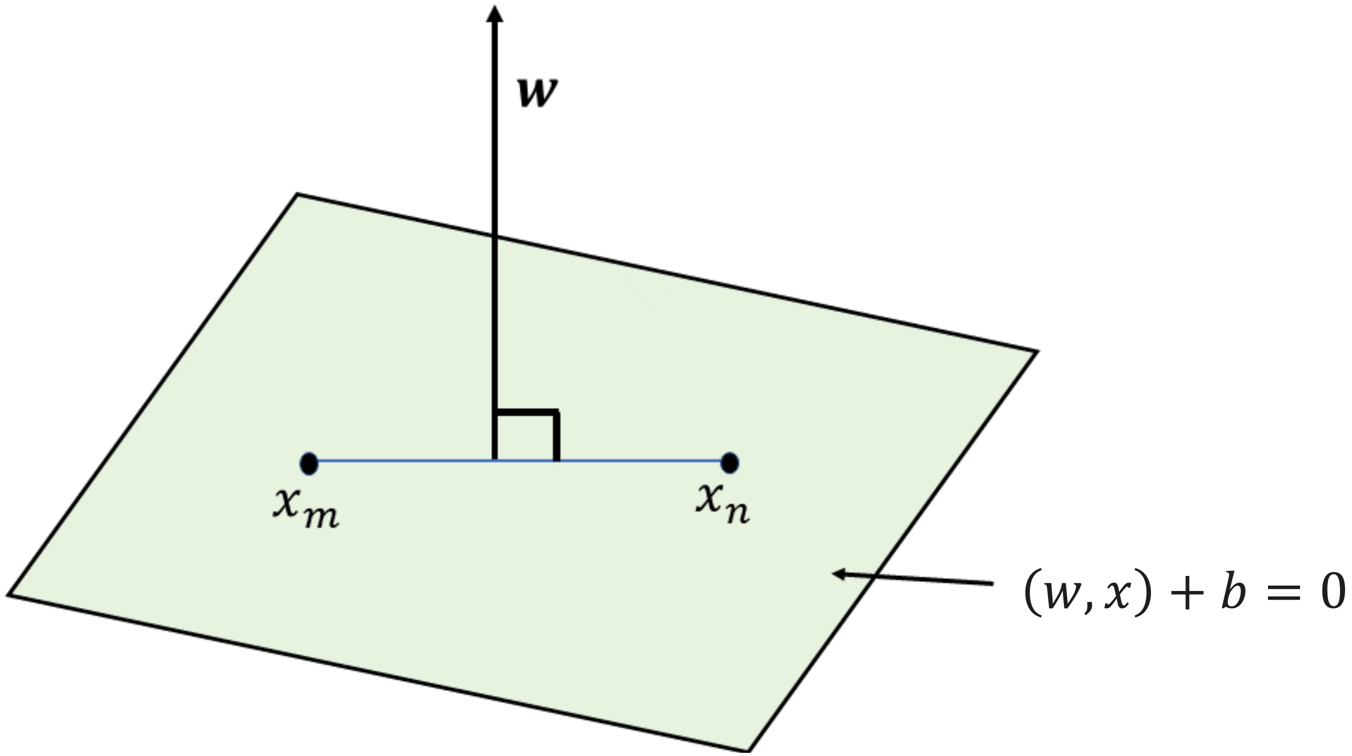
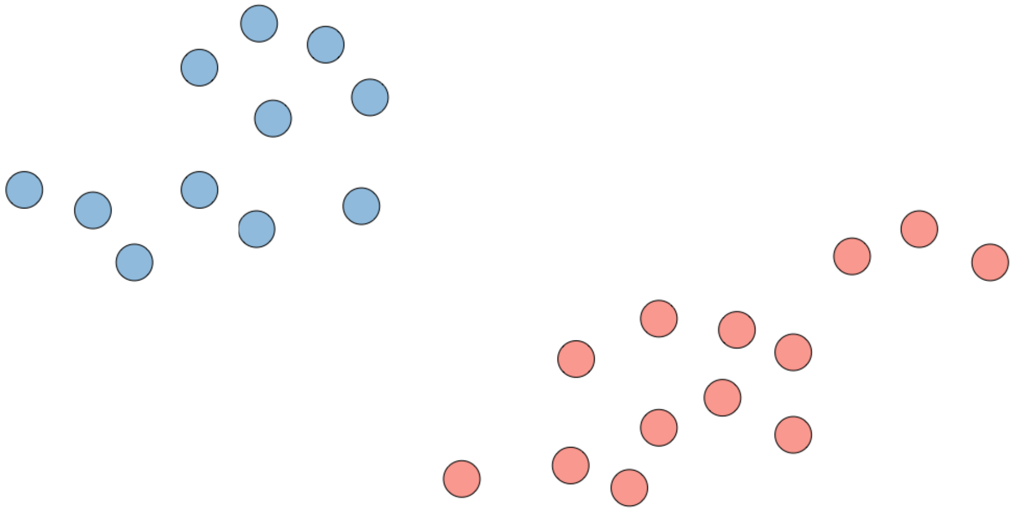
- w is a *normal vector* to this hyperplane: it's orthogonal to every vector on it.



ML Example: Linear Classifier

- Objects = 2D vectors
- Binary classification: classes +1 and -1.



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  - objects "above" are class +1
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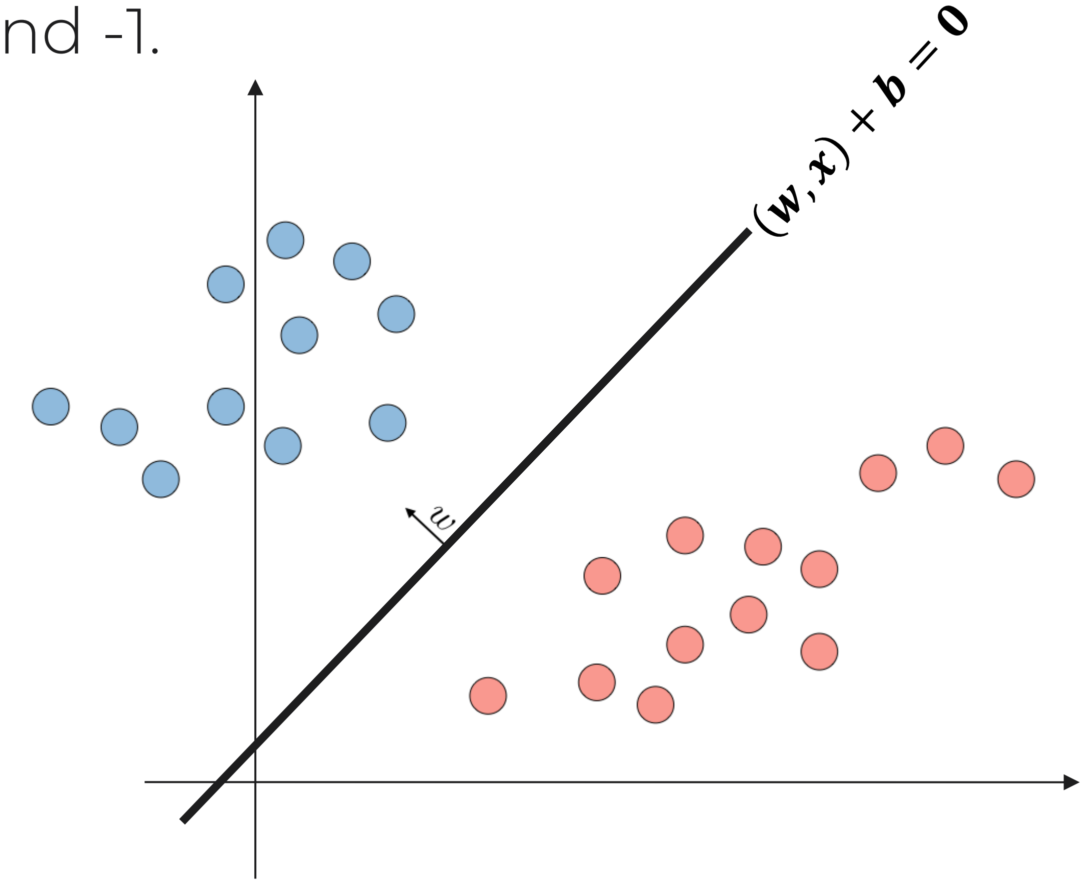

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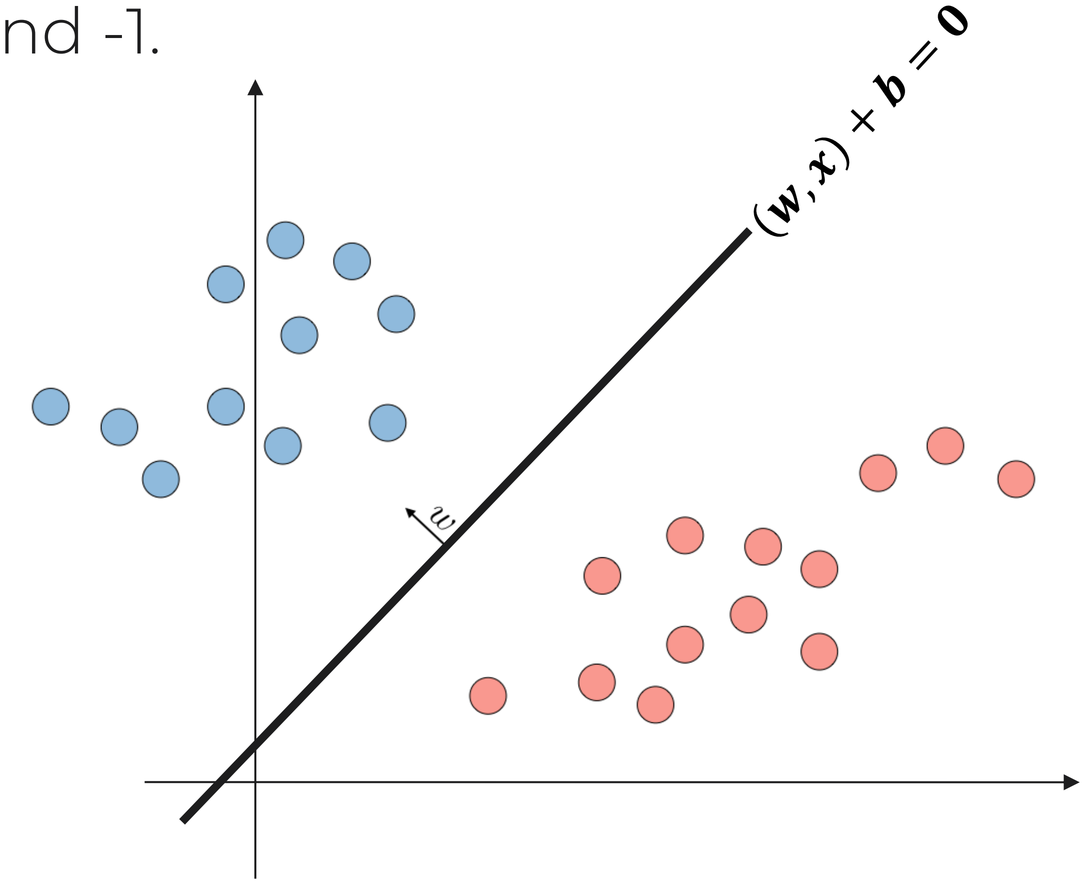

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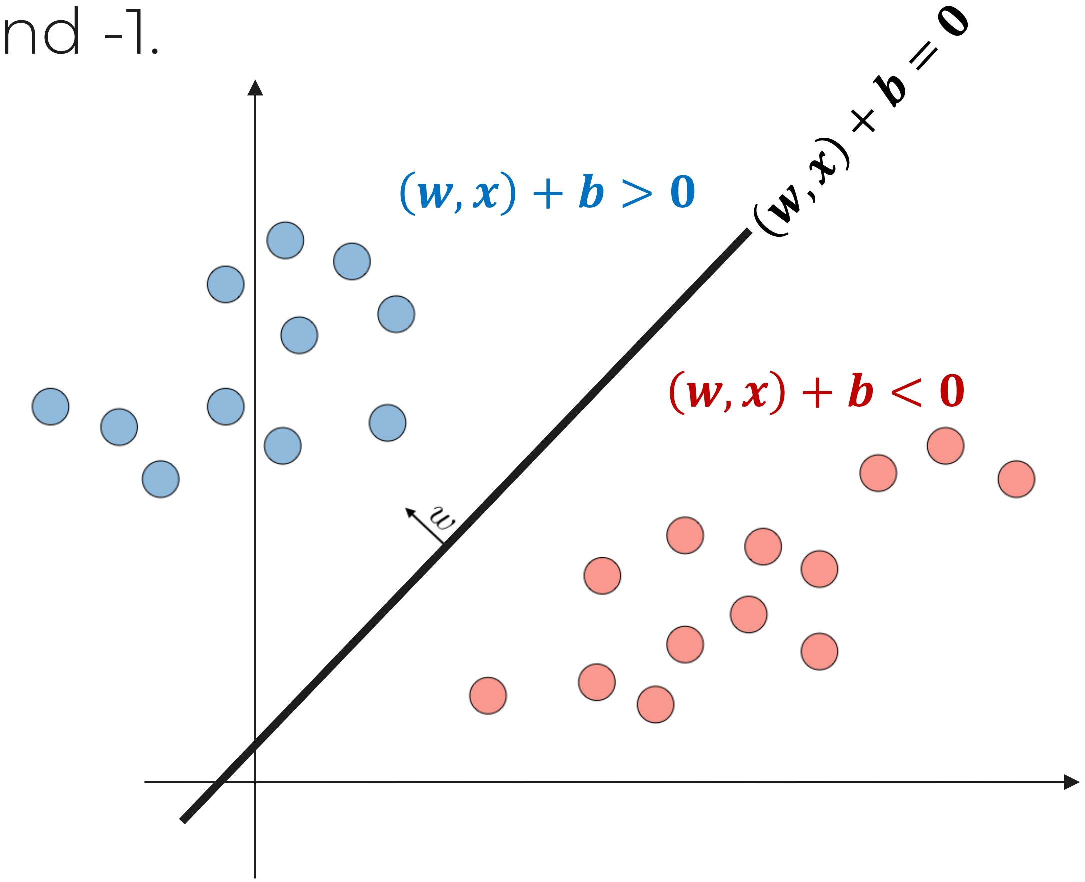
- How can we formalize this?
    - objects "above": $(w, x) + b > 0$
    - objects "below": $(w, x) + b < 0$



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To sum up

- Vectors
    - Vector spaces
    - Inner products
    - Lengths
    - Distances
    - Angles
- Analytic Geometry
    - Projections
    - Hyperplanes
    - Normal vector

Next Time

- More on vector spaces.