girafe ai

Math Refresher for DS

Practical Session 3

Plan for Today

- A short quiz
- SLE with no solutions
- Practice in Python

Short Quiz Lectures 1 - 3

Solving Systems of Linear Equations

- $Ax = b$ – a system of linear equations (SLE).
- A – $m \times n$ matrix (= m equations, n variables).

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    - $rank(A) = rank(A|b) = n \Leftrightarrow$ there is a unique solution $x = A^{-1}b$;
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How to find a reasonable approximate solution?

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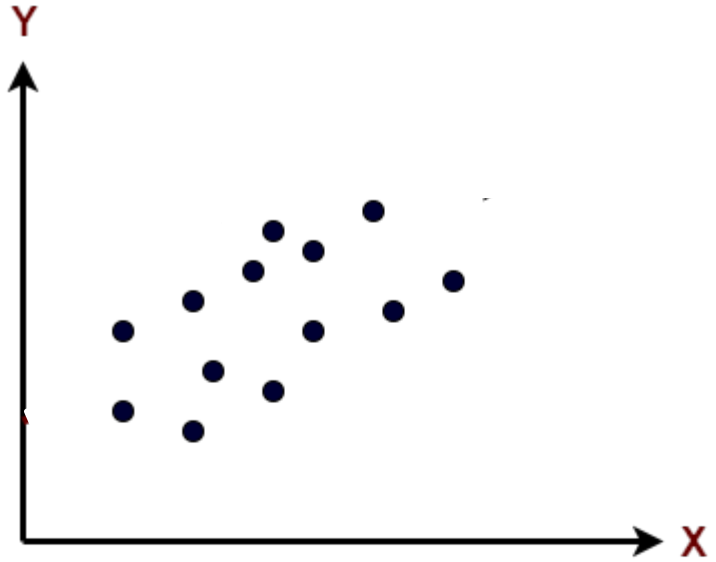
Least Squares

Motivating Example

- Imagine that you have m observations:

$$(x_1, y_1), \qquad (x_2, y_2), \qquad \dots, \qquad (x_m, y_m)$$



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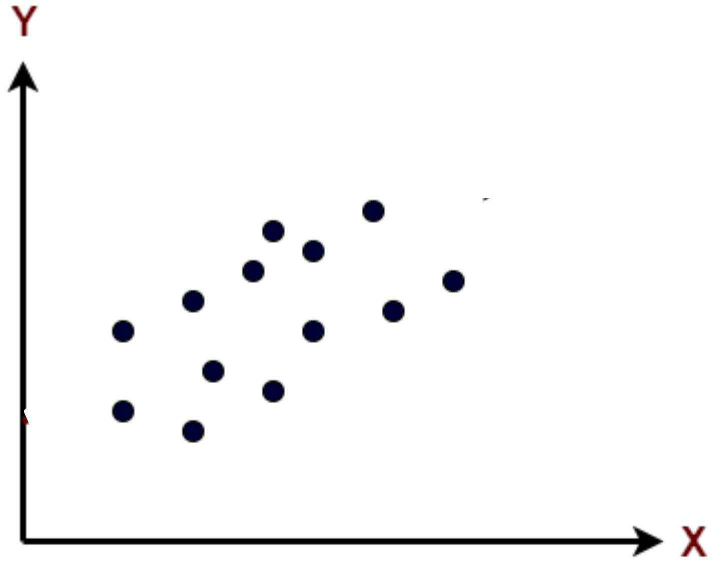

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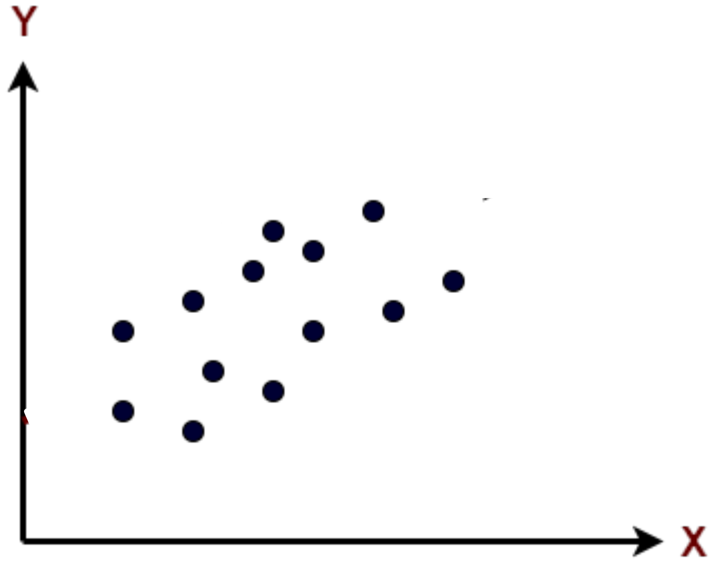

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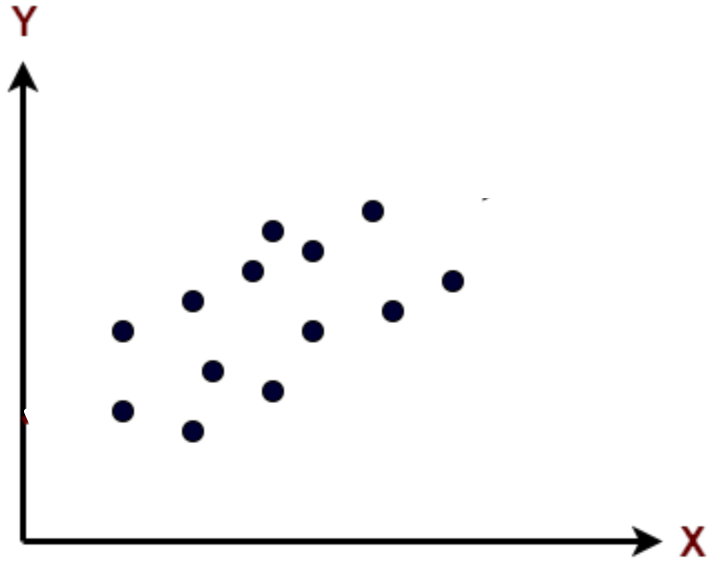

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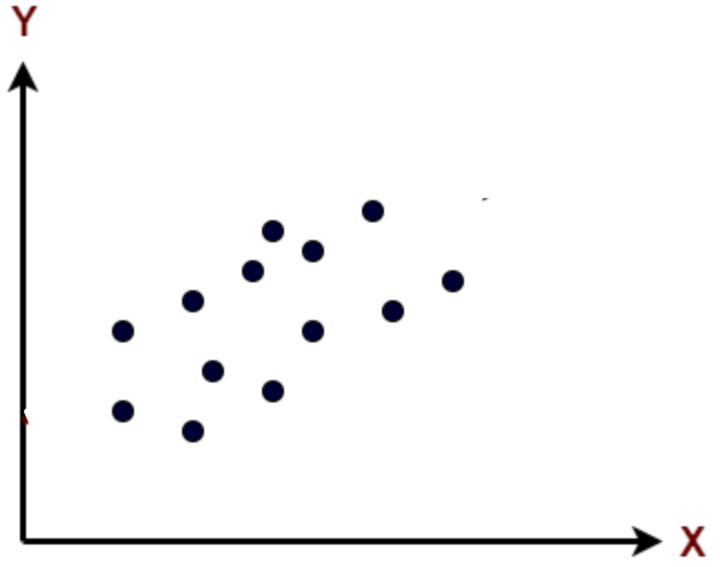

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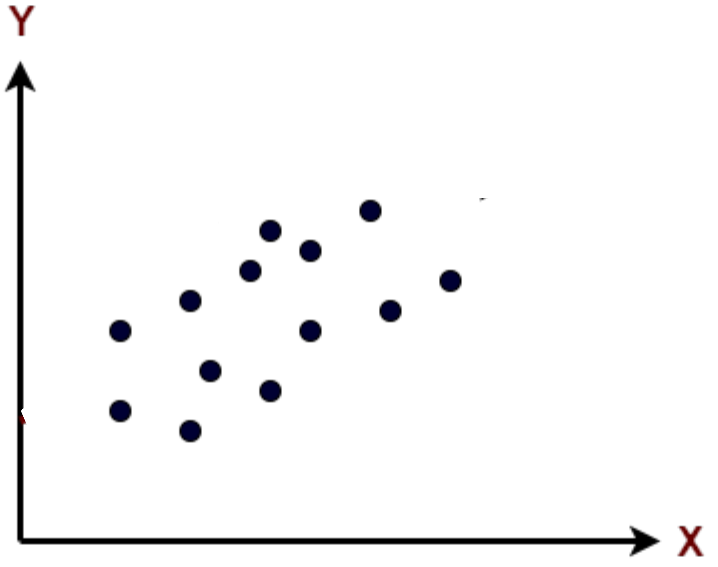

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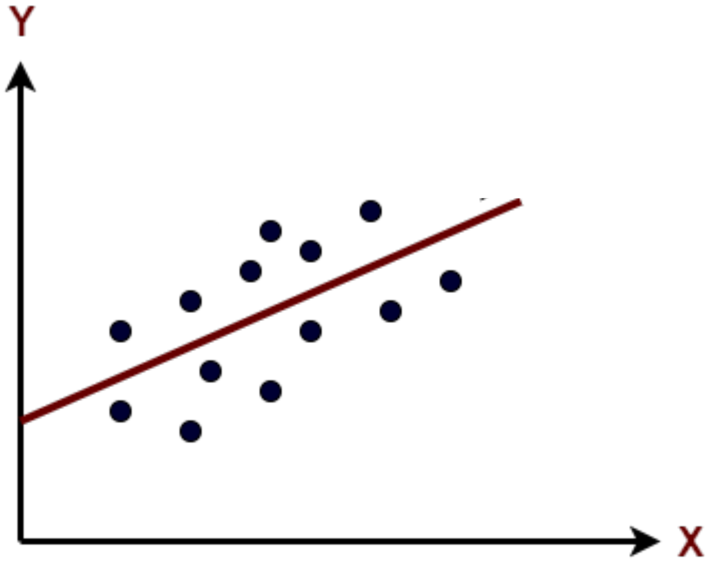

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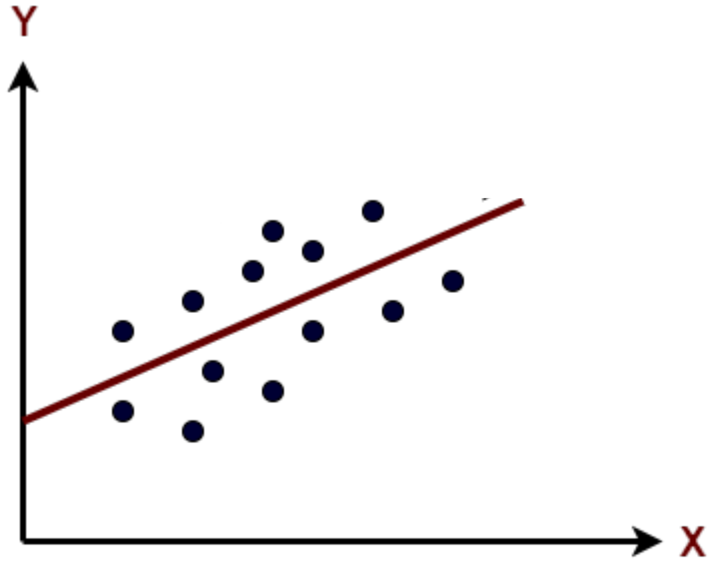

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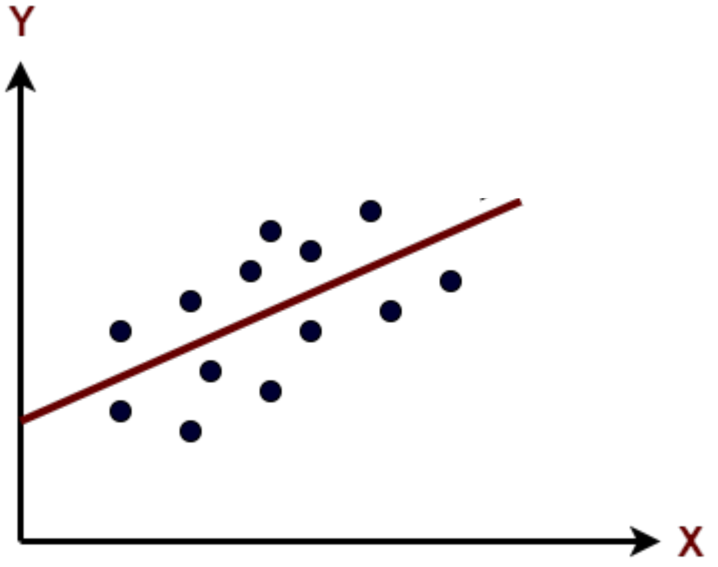

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- Let's look at it from the Linear Algebra perspective.

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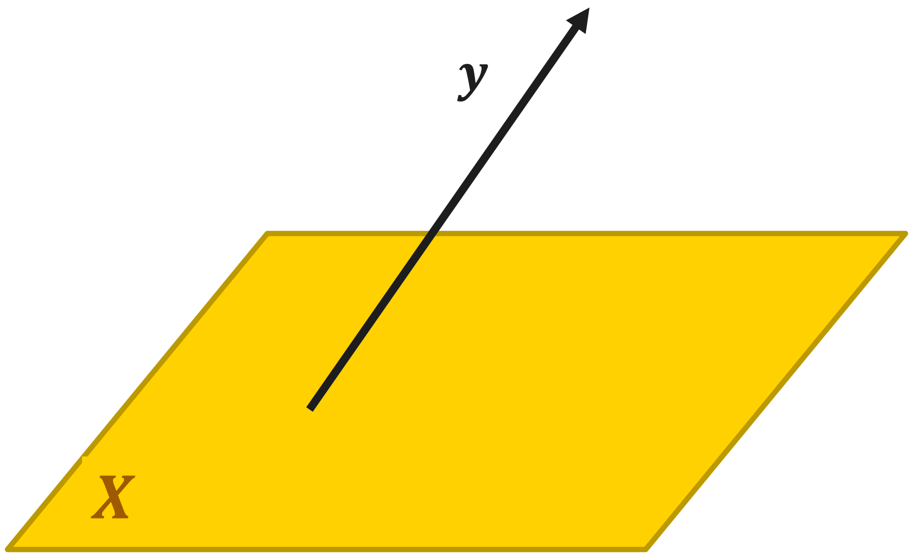
- $Xw = y$ has no solutions $\Leftrightarrow$ y is not in the column space of X.



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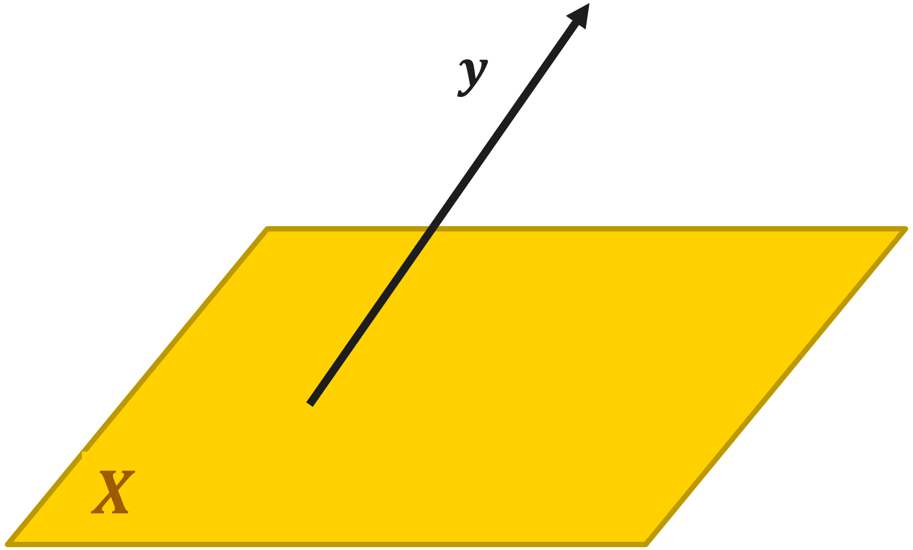

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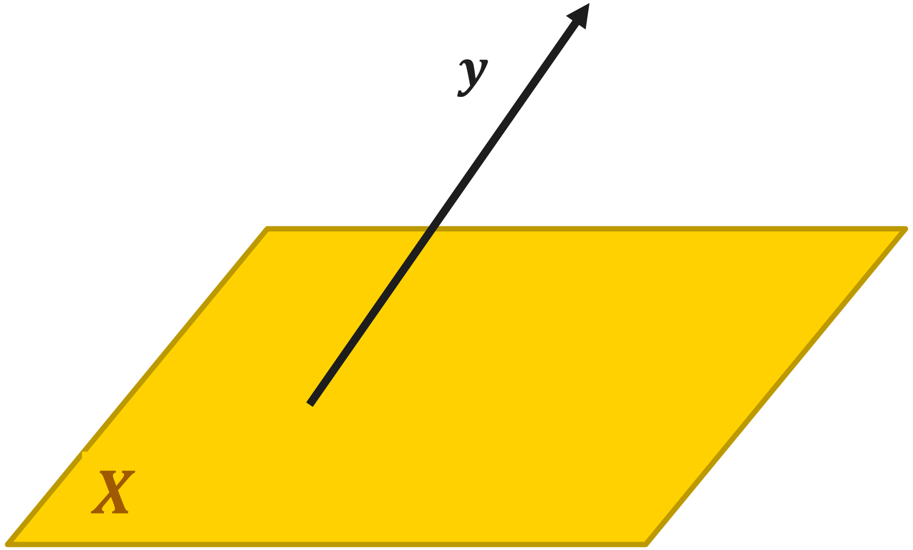

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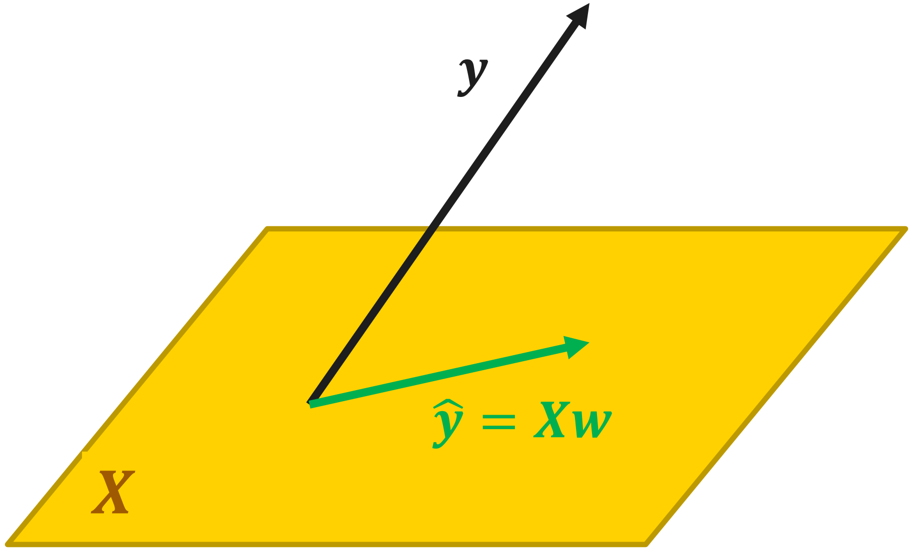

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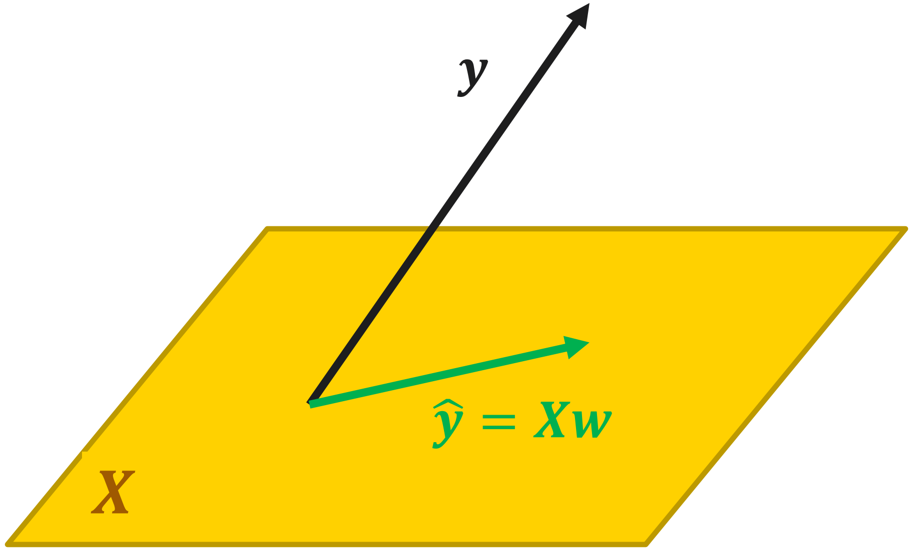

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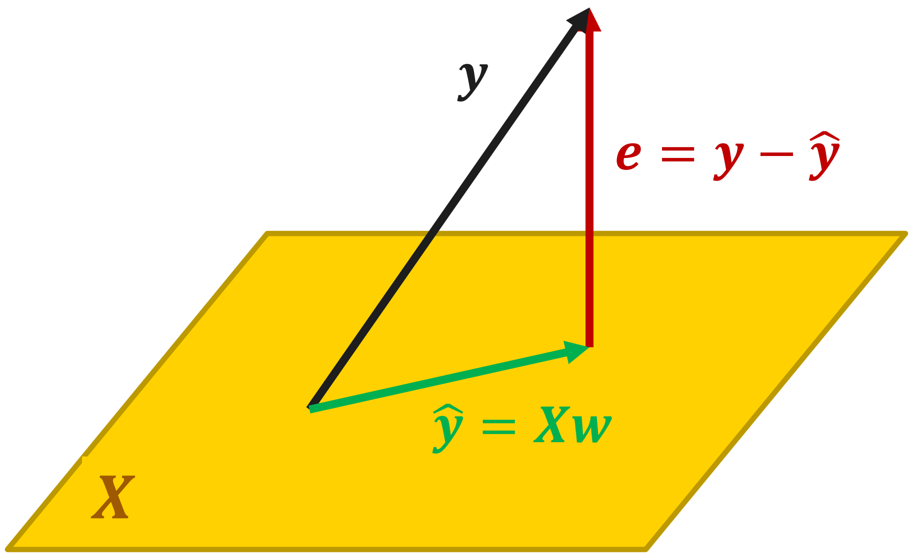

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- What do $\hat{y}$ and $\boldsymbol{e}$ look like?