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Math Refresher for DS

Lecture 3

Last Time

- Vector Spaces
  - Linear combinations
  - Spans
  - Bases
  - Change of coordinates
- Matrices

Today

- More on matrices
  - matrix operations;
  - rank;
  - determinant.
- Linear transformations
- Systems of linear equations

Matrices: a small review

A Matrix

- $A \in \mathbb{R}^{m \times n}$ - a **matrix** with m rows and n columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- *Examples:*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2 \times 2}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{2 \times 3}$$

Special Matrices

- Diagonal matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\left(a_{ii} \neq 0,\ a_{ij} = 0\ \forall i \neq j\right)$
- Identity matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\left(a_{ii} = 1,\ a_{ij} = 0\ \forall i \neq j\right)$
- Symmetric matrix: $\begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}$ $\left(a_{ij} = a_{ji}\right)$
- Triangular matrix: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ $\left(a_{ij} = 0\ \forall i > j \ or\ \ \forall i < j\right)$

Basic Operations with Matrices

- Addition:

$$A = \{a_{ij}\}_{i=1,\dots,m,j=1,\dots,n}, \qquad B = \{b_{ij}\}_{i=1,\dots,m,j=1,\dots,n}, \qquad A + B = \{a_{ij} + b_{ij}\}_{i=1,\dots,m,j=1,\dots,n}$$

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- Multiplication by a scalar:

$$A = \{a_{ij}\}_{i=1,\dots,m, j=1,\dots,n}, \qquad \lambda \in \mathbb{R}, \qquad \lambda A = \{\lambda a_{ij}\}_{i=1,\dots,m, j=1,\dots,n}$$

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Matrix Multiplication

- Matrix multiplication:

$$A = \{a_{ij}\}_{i=1,\dots,m,j=1,\dots,n}, \qquad B = \{b_{ij}\}_{i=1,\dots,n,j=1,\dots,k}$$

$$A \cdot B = \{(A_i, B^j)\}_{i=1,\dots,m,j=1,\dots,k} = \left\{\sum\nolimits_{l=1,\dots,n} a_{il} \cdot b_{lj}\right\}_{i=1,\dots,m,j=1,\dots,k}$$

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- Example $\mathbb{R}^{2\times 2}$:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Matrix Multiplication

- For numbers: $2 \times 3 = 3 \times 2 = 6$.

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- Example:

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 2 \\ 3 & 2 \end{bmatrix}, \quad AB = \begin{bmatrix} 30 & 14 \\ 12 & 6 \end{bmatrix}, \quad BA = \begin{bmatrix} 20 & 28 \\ 11 & 16 \end{bmatrix}$$

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Matrix Multiplication

- Multiplication by identity matrix E:

$$AE = EA = A$$

Matrix Multiplication

- Multiplication by identity matrix E:

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- Multiplication by zero matrix O:

$$AO = OA = O$$

Transposing a Matrix

- The transpose of a matrix results from “flipping” the rows and columns:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

Transposing a Matrix

- The following properties of transposes are easily verified:
  - A – symmetric matrix $\Longrightarrow A^T = A$
  - $(A^T)^T = A$
  - $(A + B)^T = A^T + B^T$
  - $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$

Linear Transforms

A more interesting way of looking at matrices.

Linear Transformation

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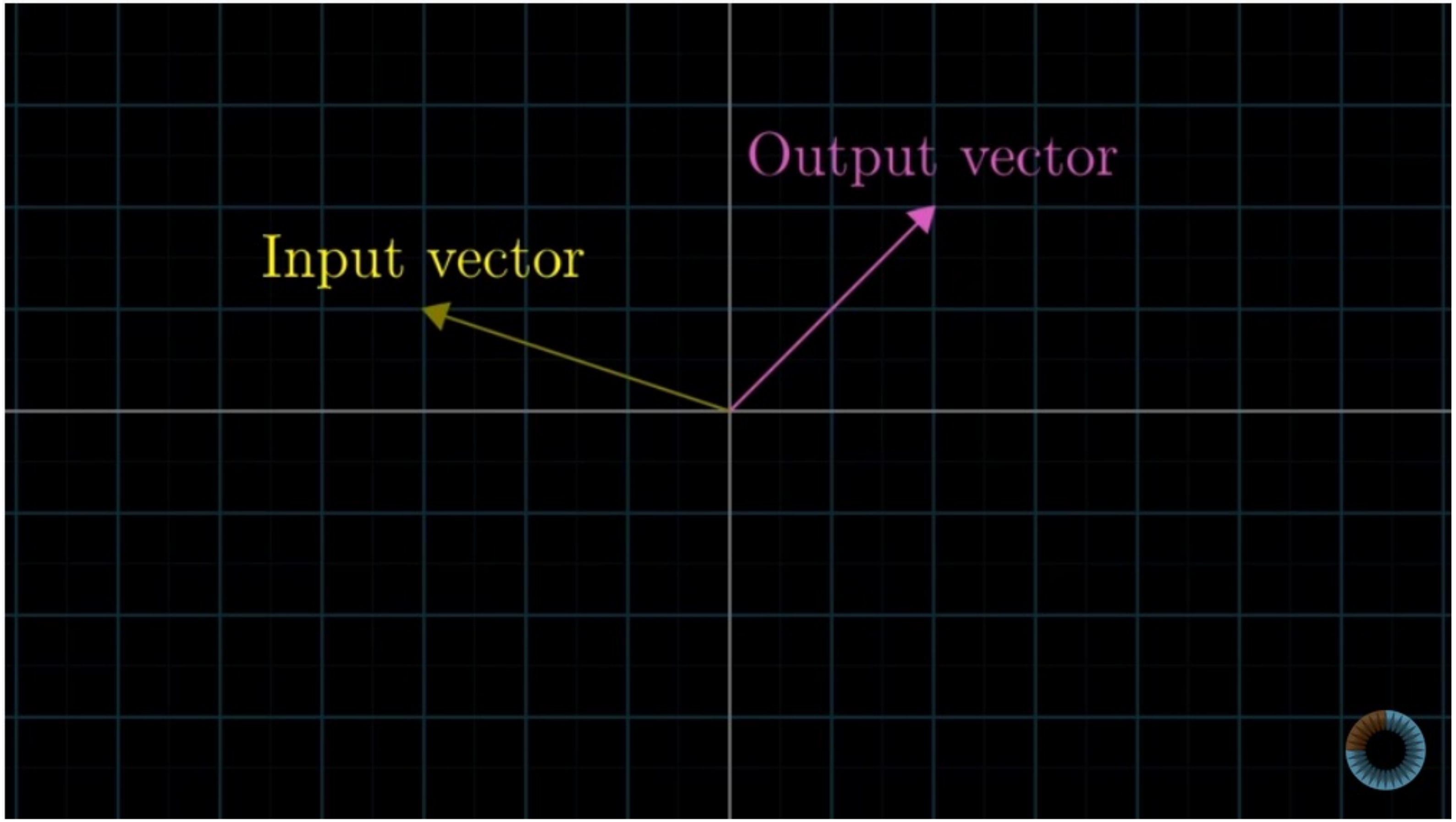
Linear Transformation

$$x_{input} \to A \to x_{output}$$

A – transformation

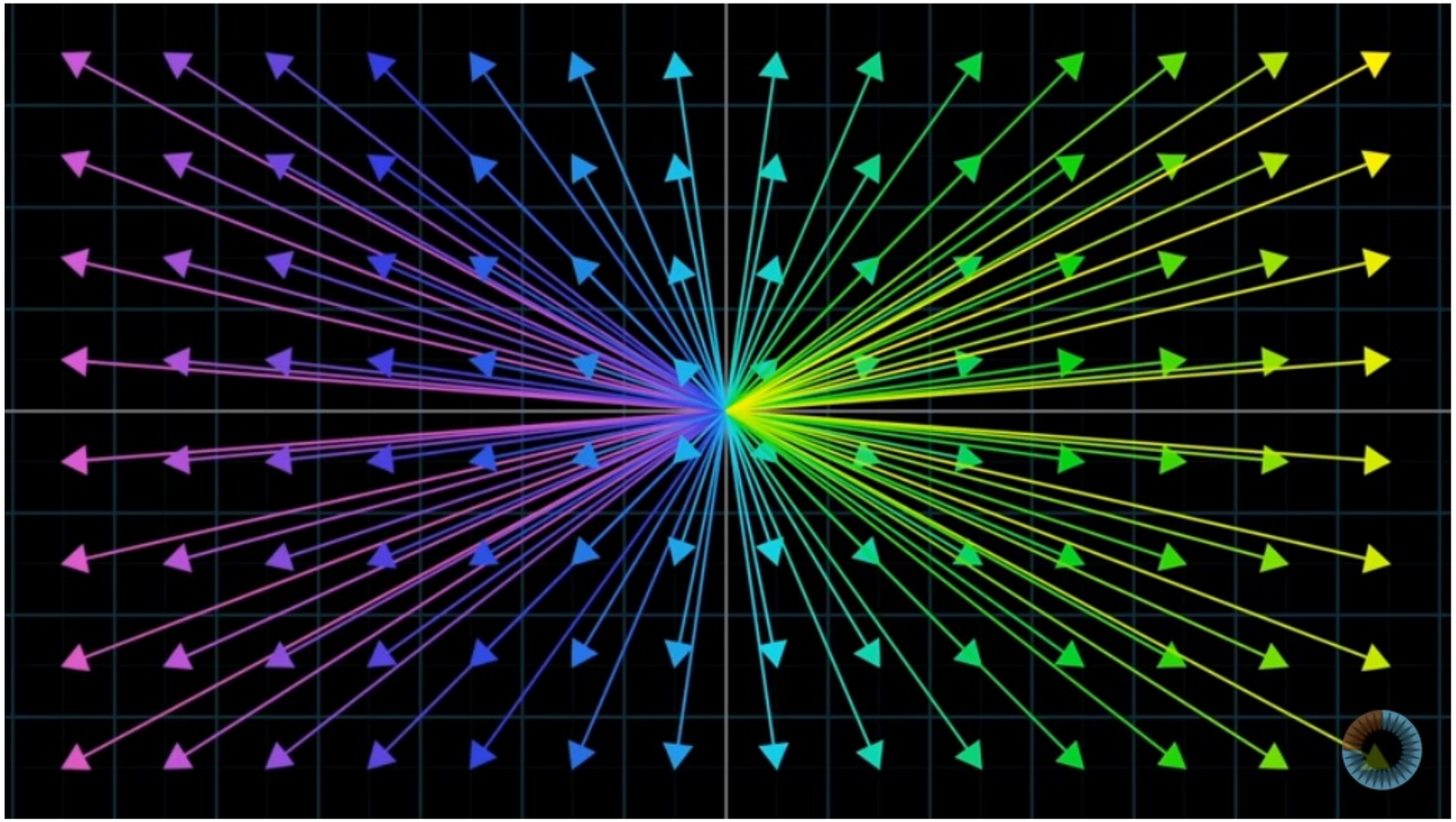
x_{input}, x_{output} – vectors

Transformation



Source: [Linear Transformations and Matrices]

Transformation



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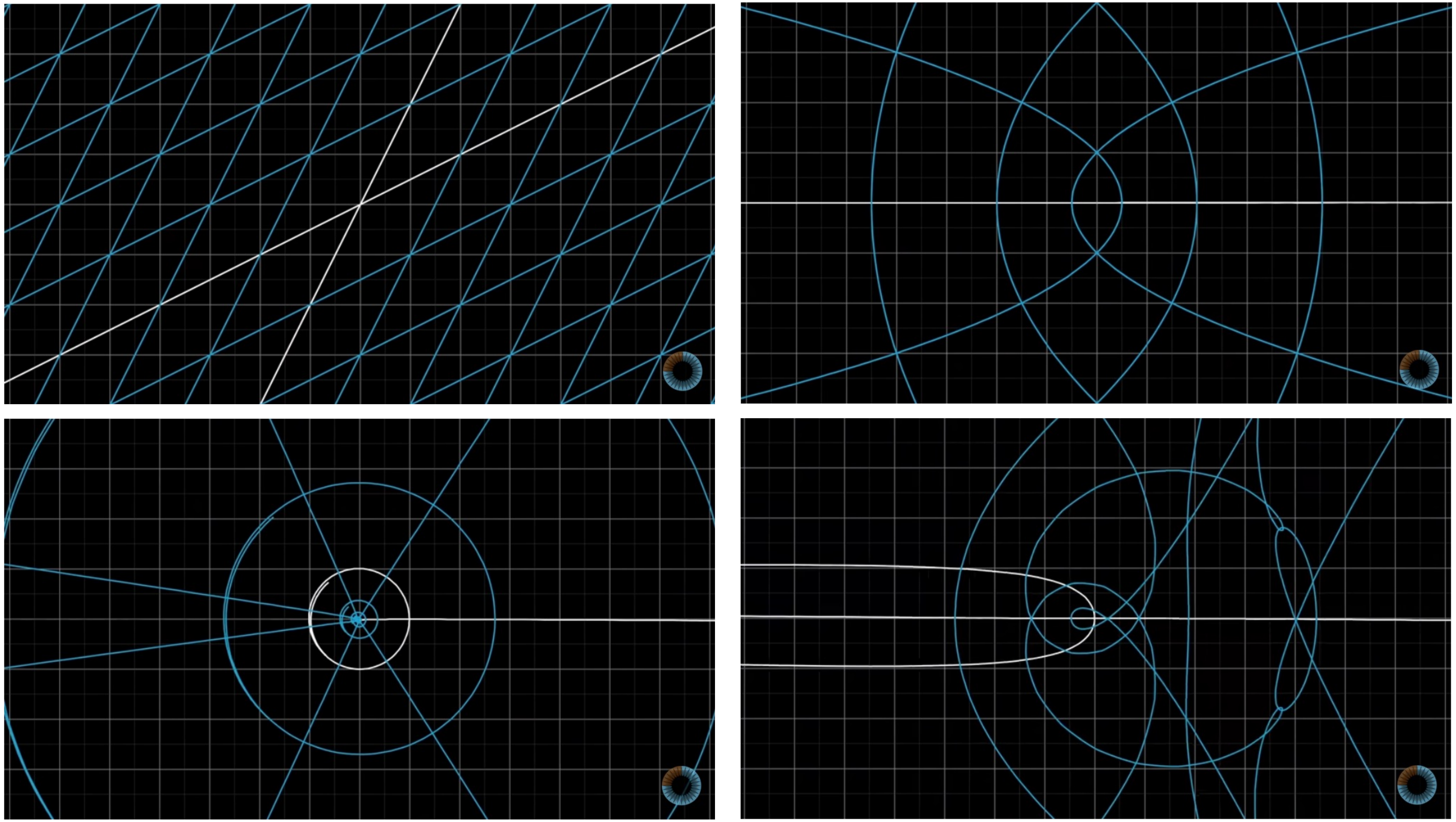
Transformation: Examples



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Linear Transformation

Linear Transformation

$$x_{input} \rightarrow A \rightarrow x_{output}$$

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A transformation that satisfies two properties:

1. $A(x+y) = A(x) + A(y)$
2. $A(\lambda x) = \lambda A x$

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$$\Longrightarrow x_{output} = A(x_{input}) = A \cdot x_{input}$$

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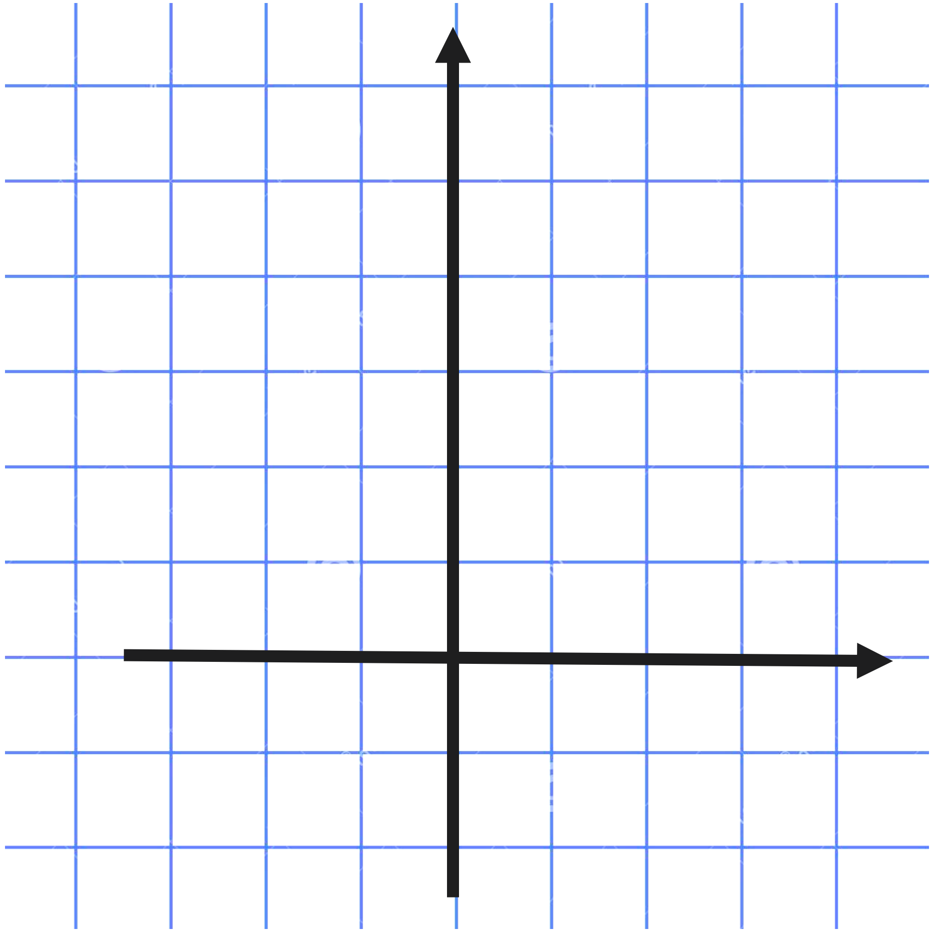
Example: Rotation

- Imagine that we want to rotate vectors in $\mathbb{R}^2$ 90° anti-clockwise.



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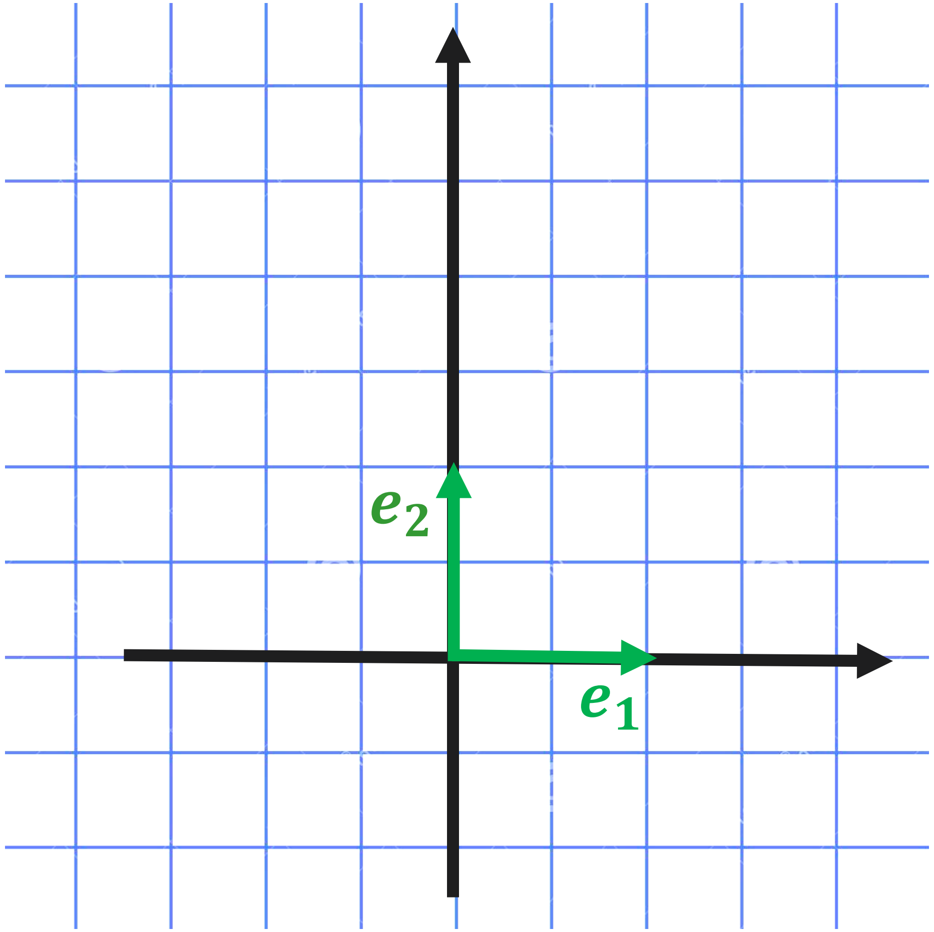

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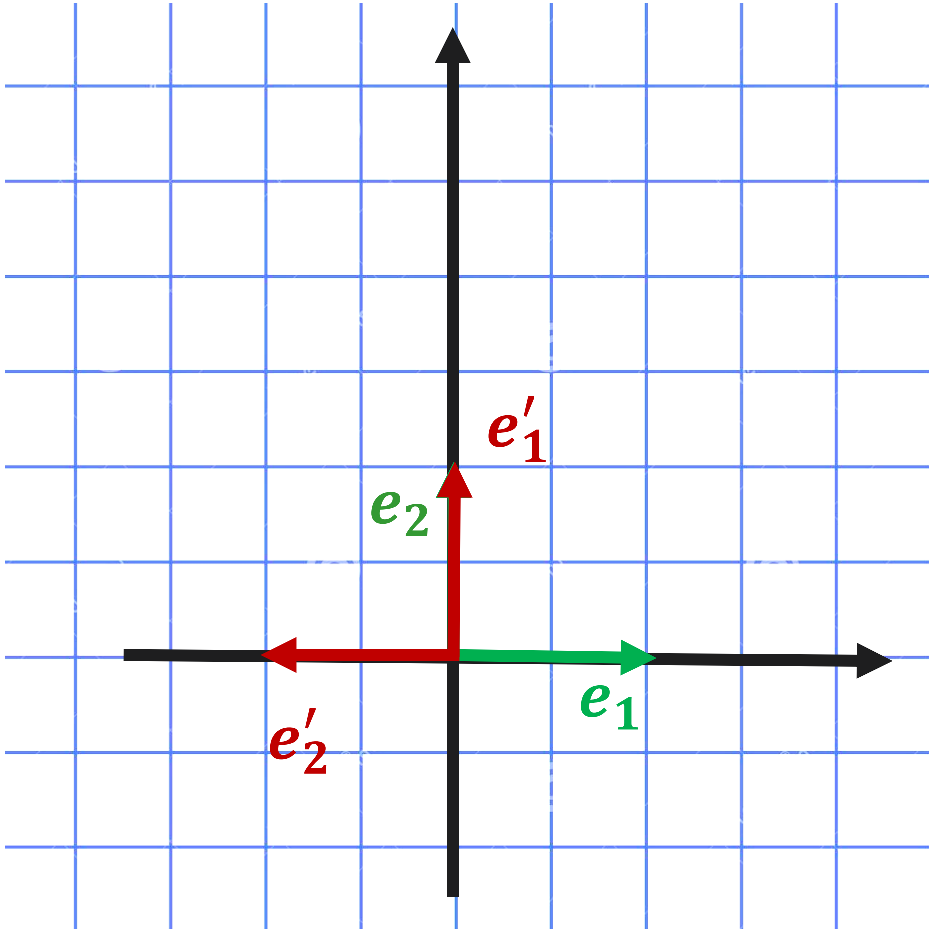

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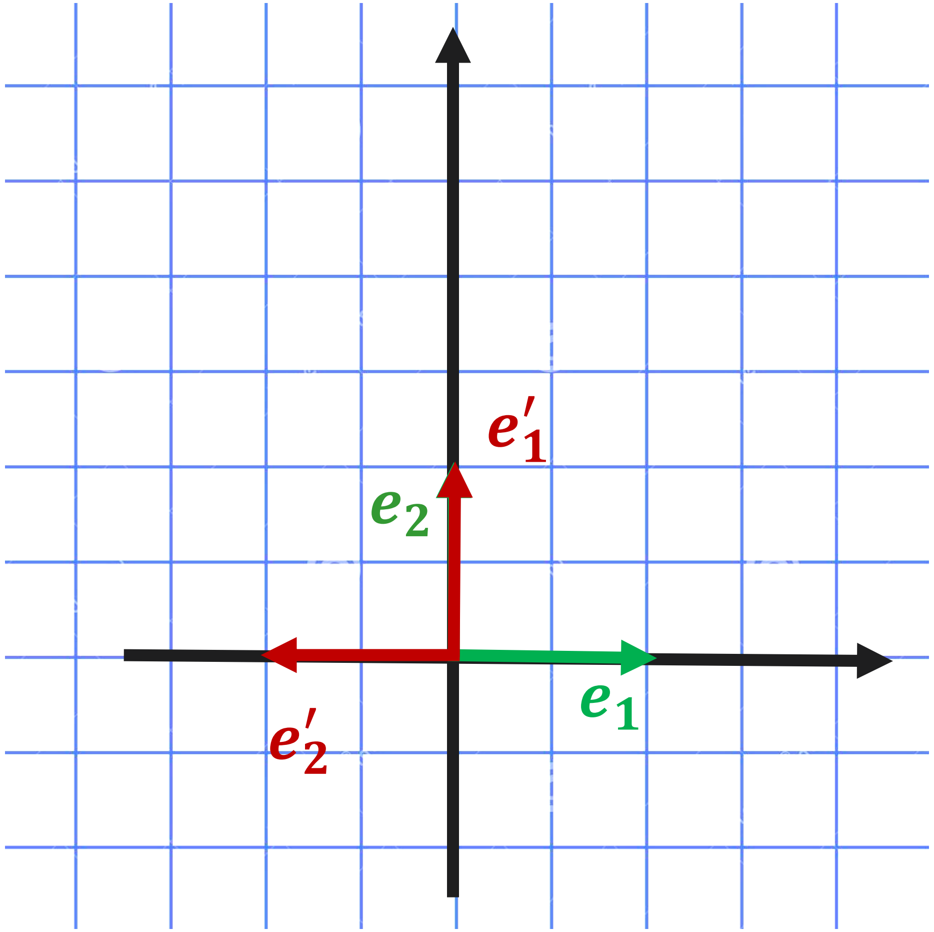

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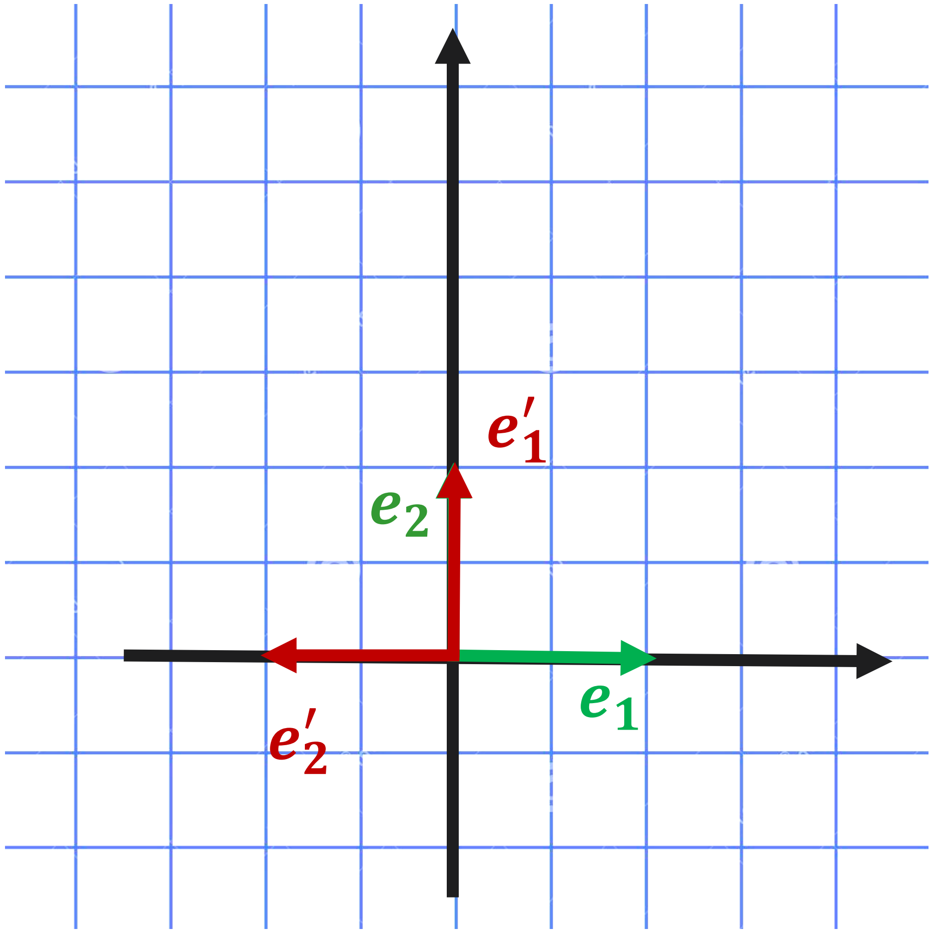

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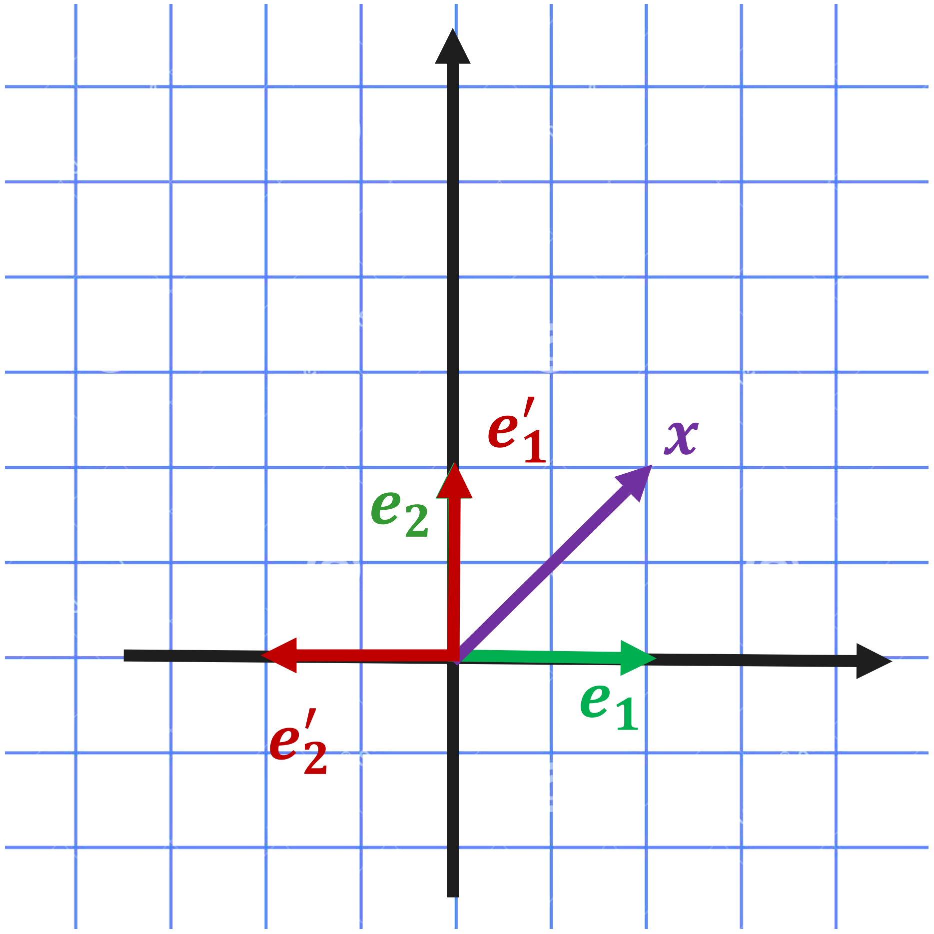

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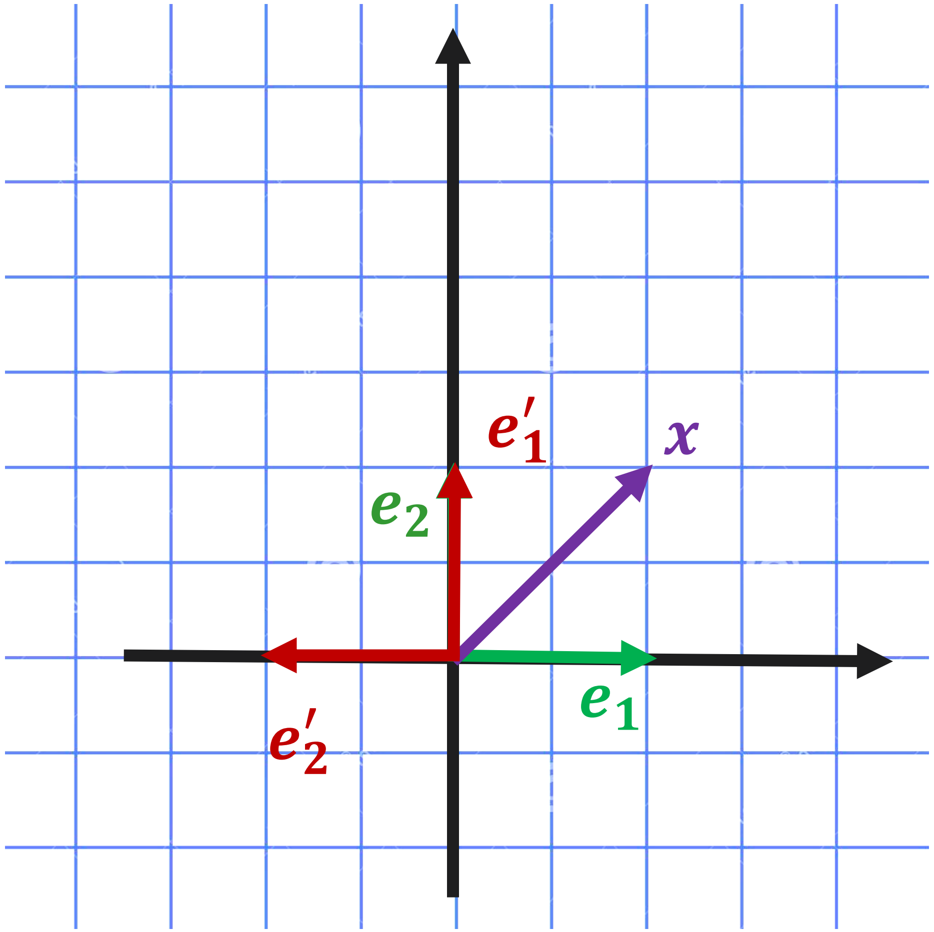

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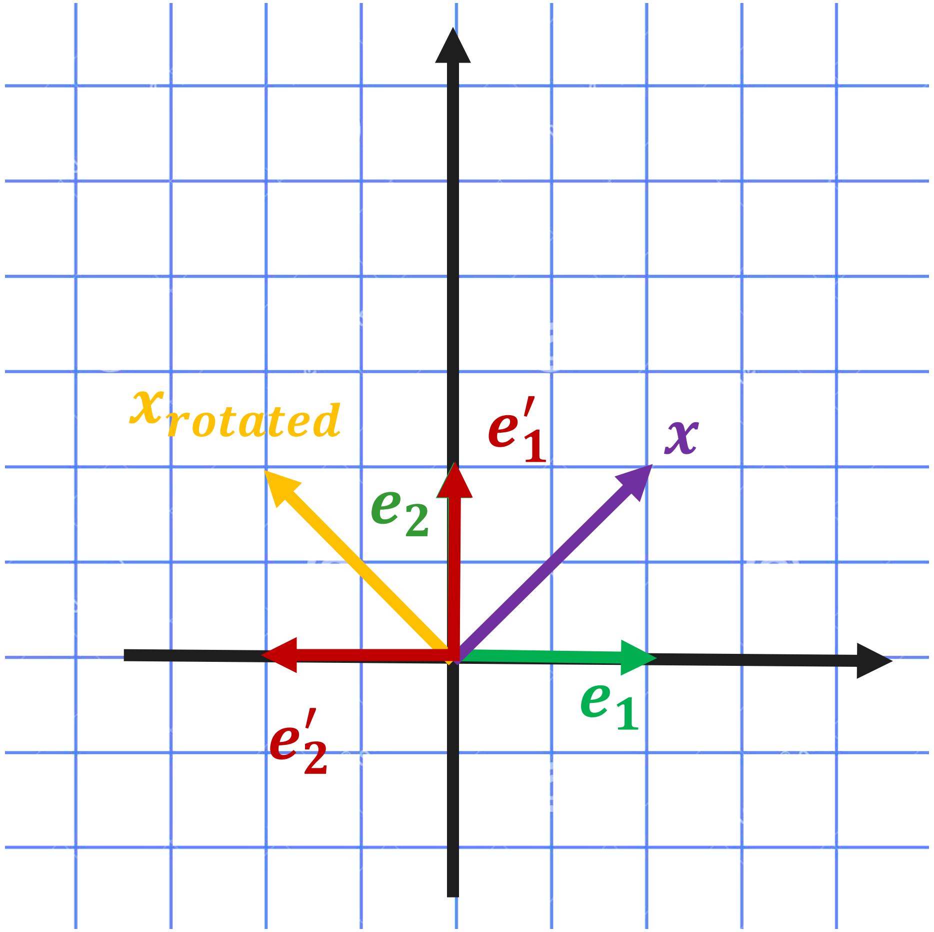

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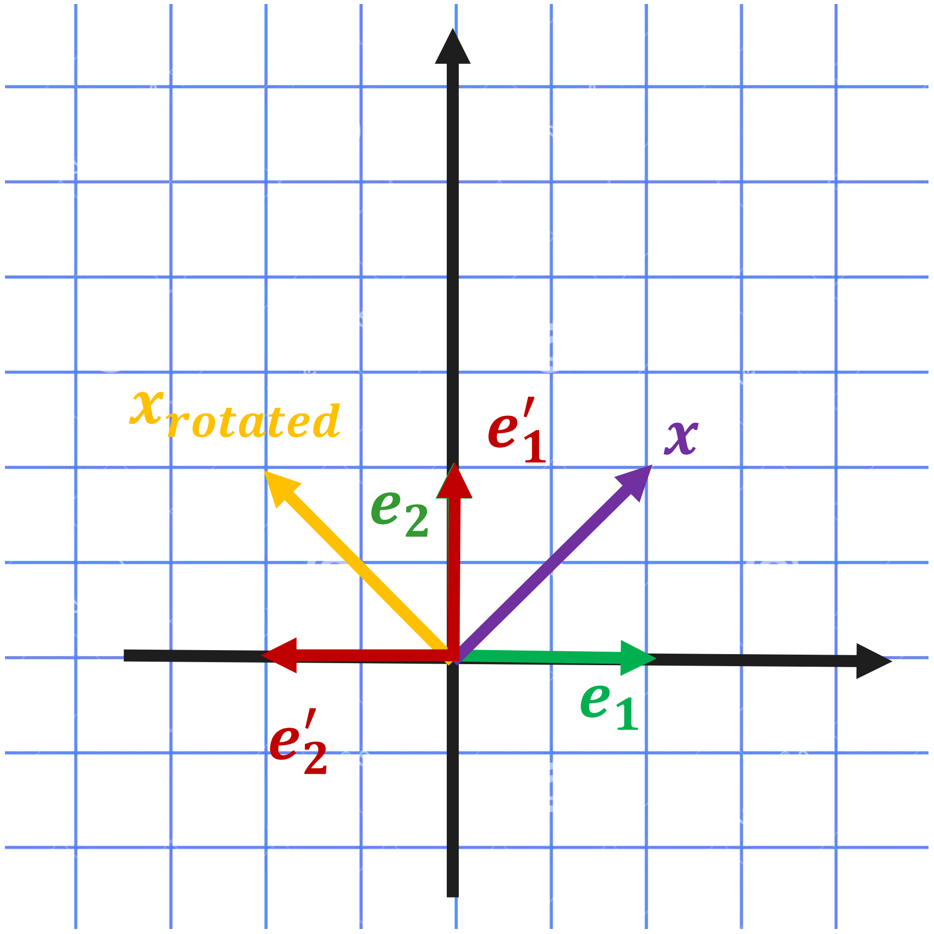

Linear Transformation

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  Columns = how this transformation changes the vectors in the selected basis.

- Vice versa: every square matrix defines some linear transformation.

Common Transforms

Identity Transformation

- Doesn't change anything.
- Transformation matrix E:

$$Ex = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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- Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Stretching / Squeezing

- Enlarge (compress) all distances in a particular direction by a constant factor.

- Transformation matrix:

$$Kx = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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- Example: stretch x-axis (x3) and squeeze y-axis (x 0.5):

$$\begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Projection on an Axis

- Consider $\mathbb{R}^3$. Project on the XY −plane.

- Transformation matrix:

$$Px = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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Rotation

- Rotating points anticlockwise by θ.
- Rotation matrix R_θ:

$$R_\theta\, x = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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- Example: rotate by 45° anticlockwise:

$$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$$

Combining Transforms

Composition

- Let A and B be two linear transforms.
  What if we first apply A and then B?

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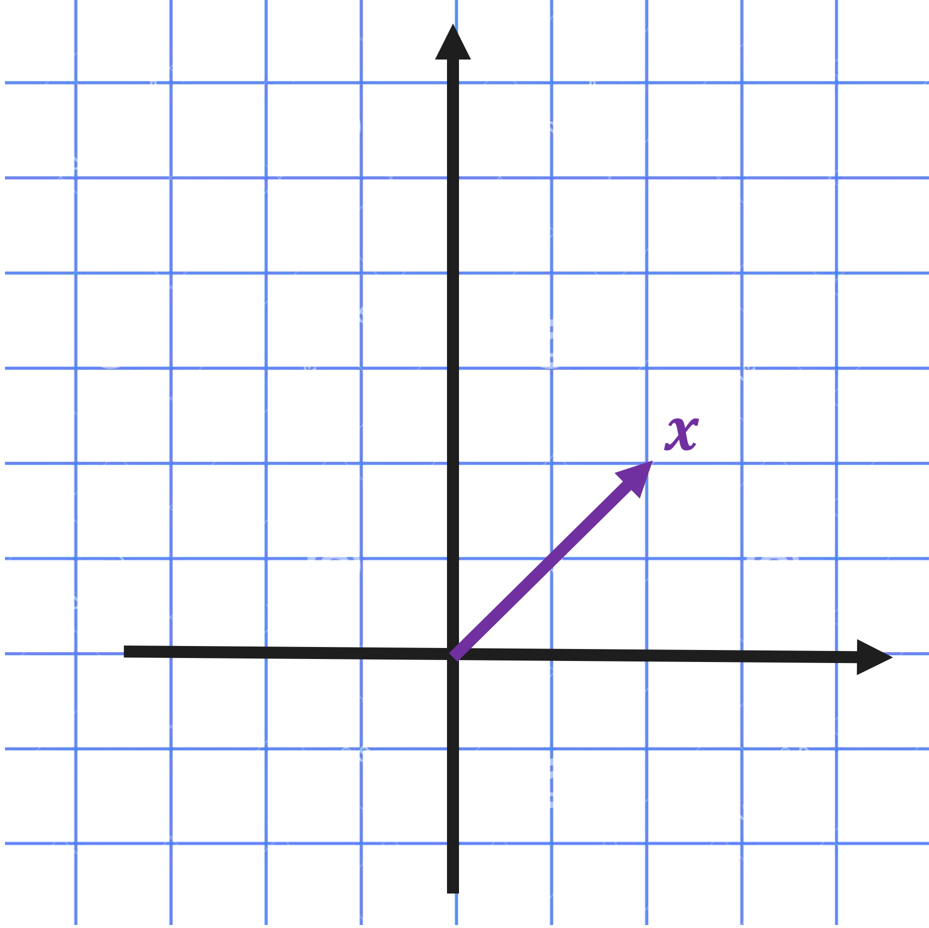

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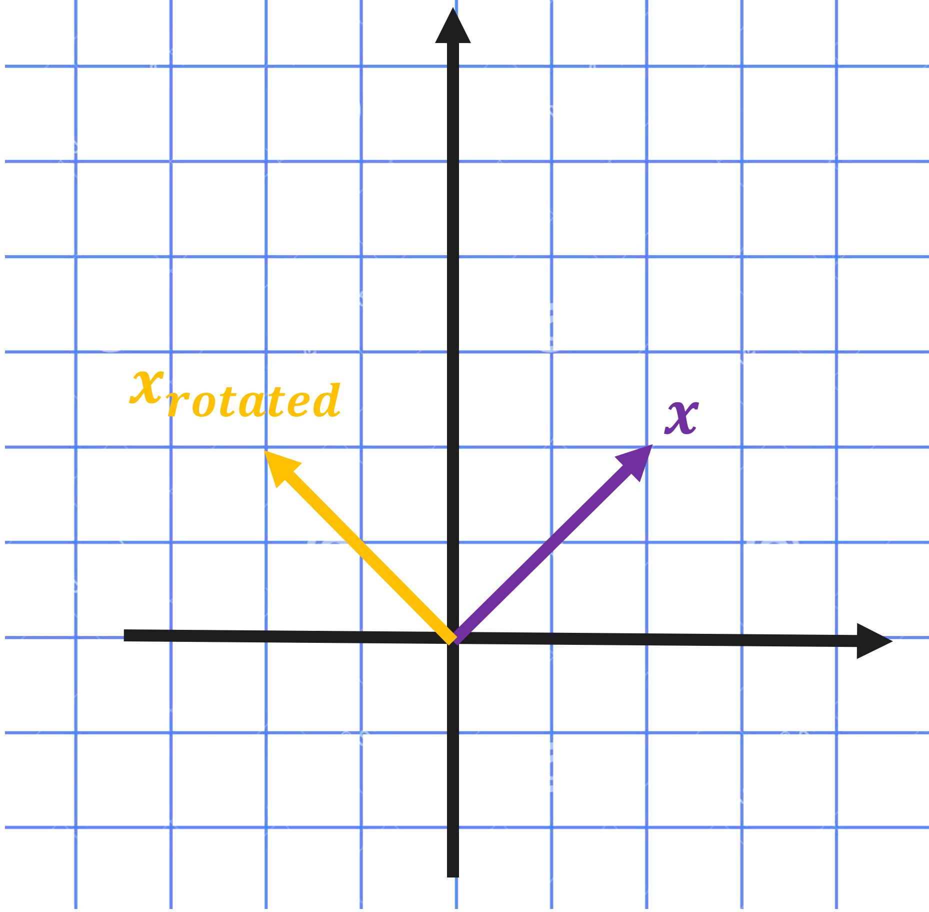

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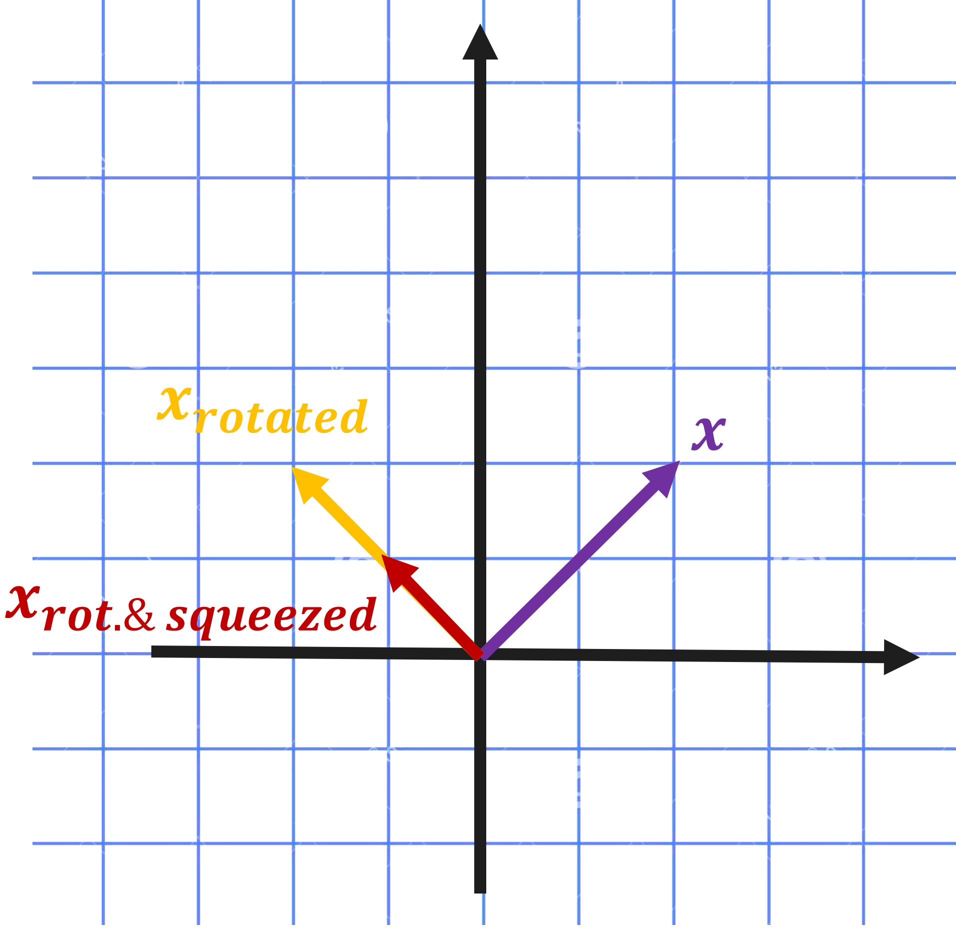

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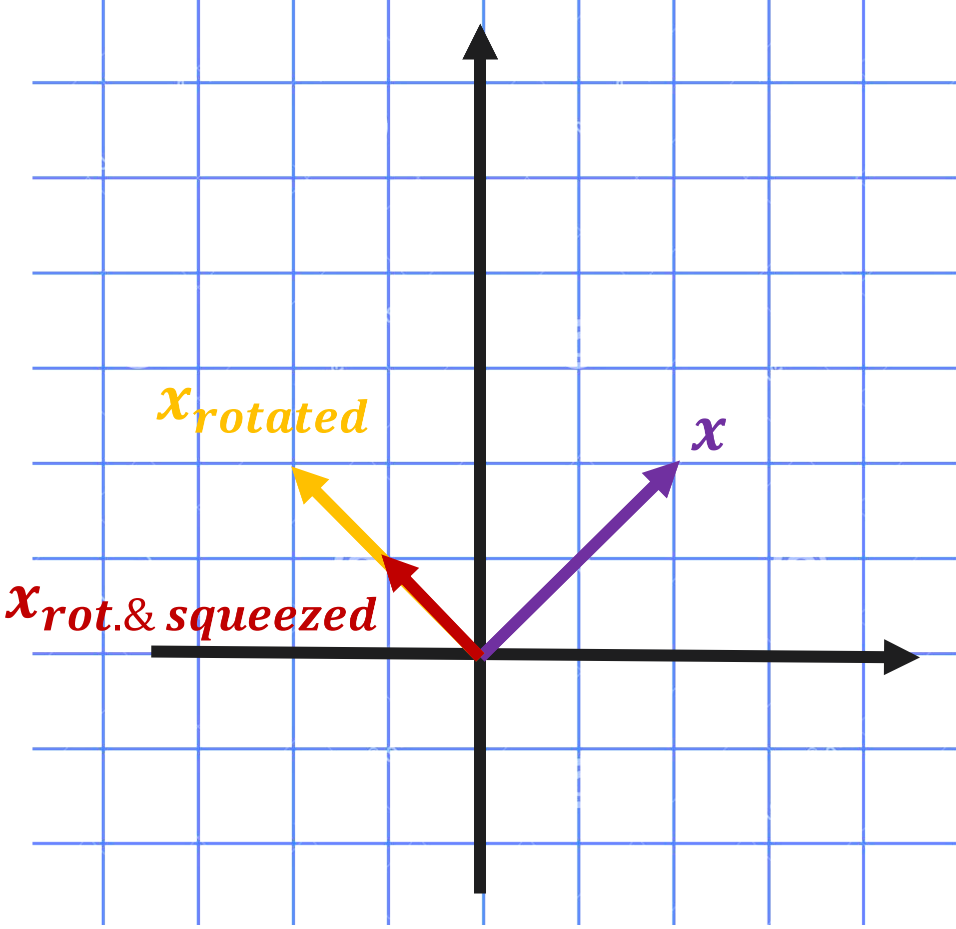

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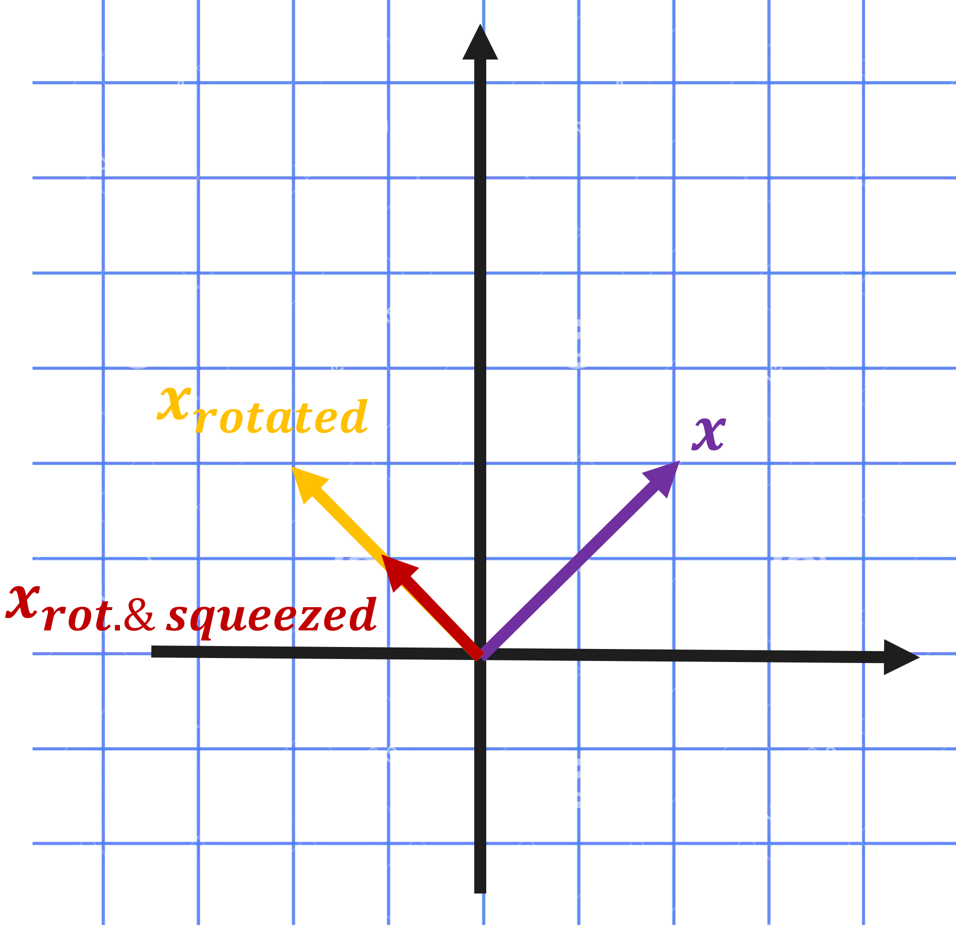

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$$C = BA = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix}$$

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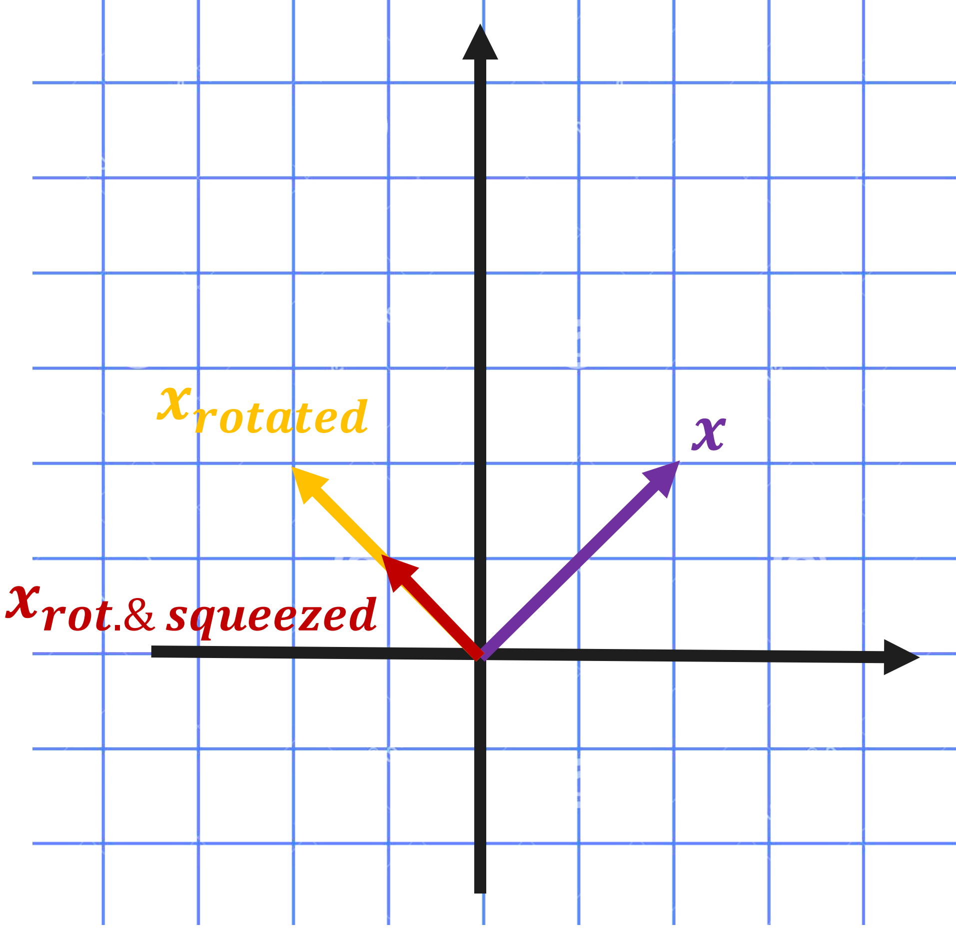

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= “rotate by 90° and squeeze”

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$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \text{rotation}, \quad B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} - \text{squeezing}$$

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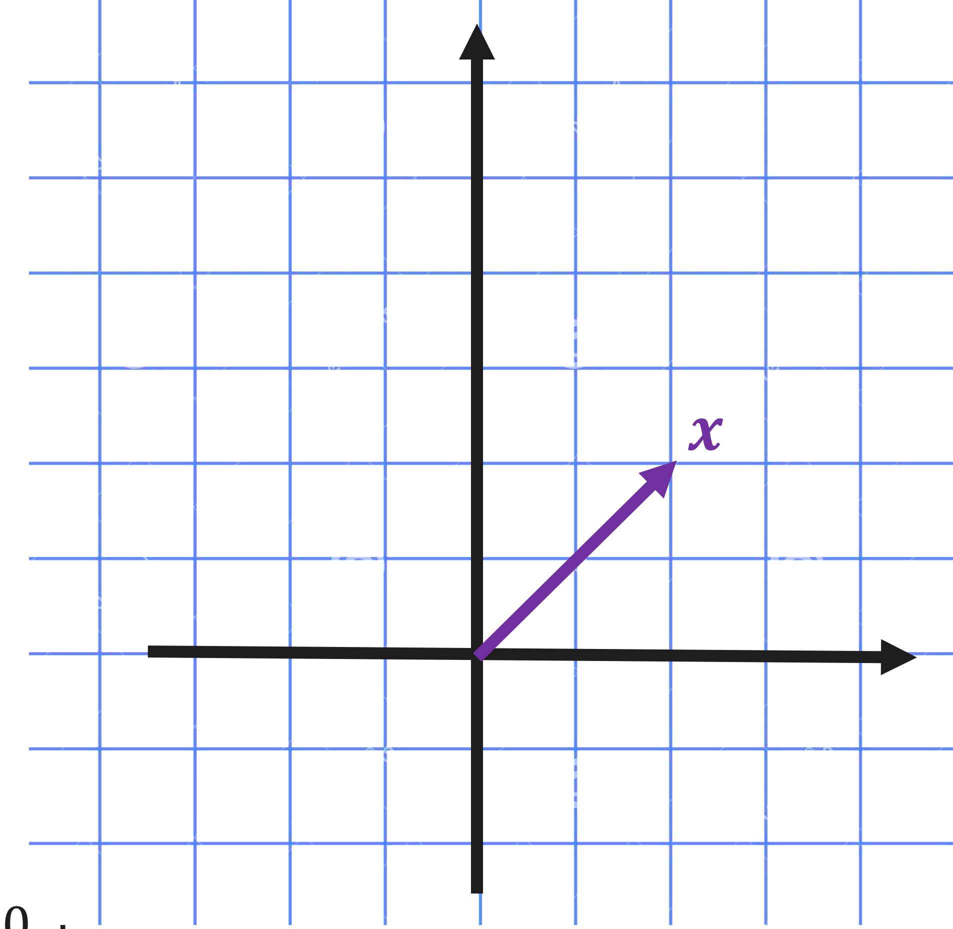

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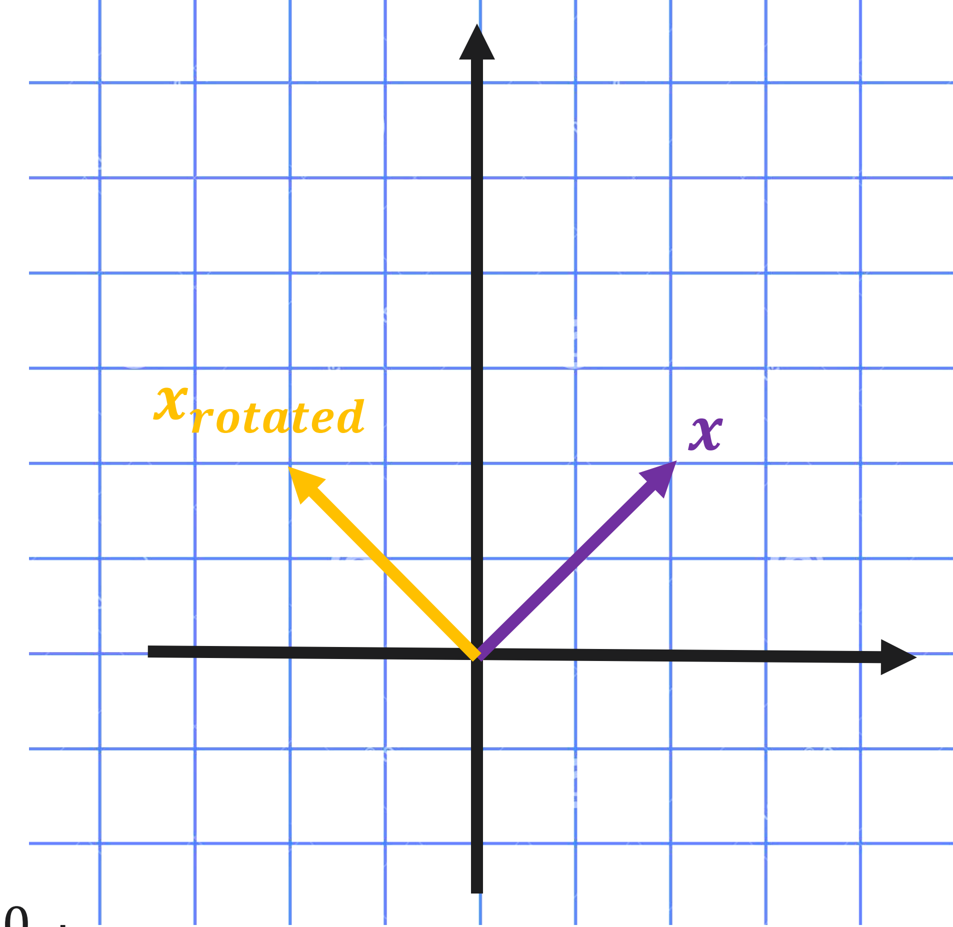

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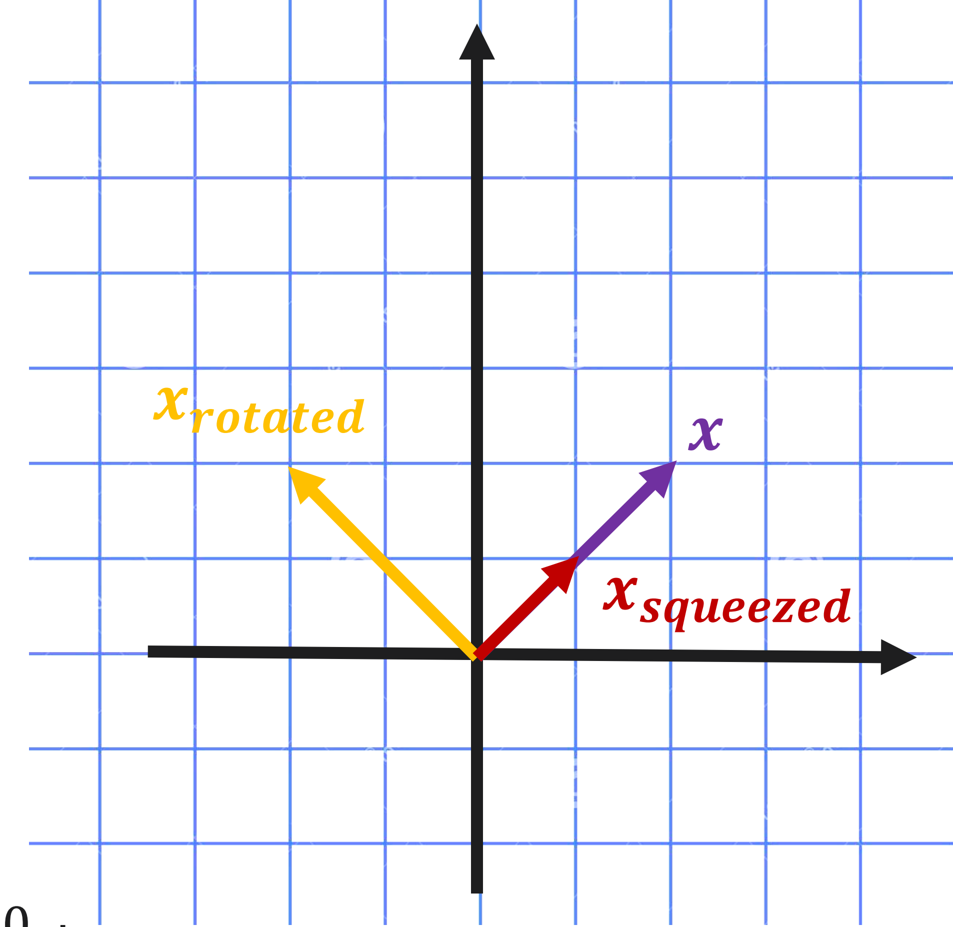

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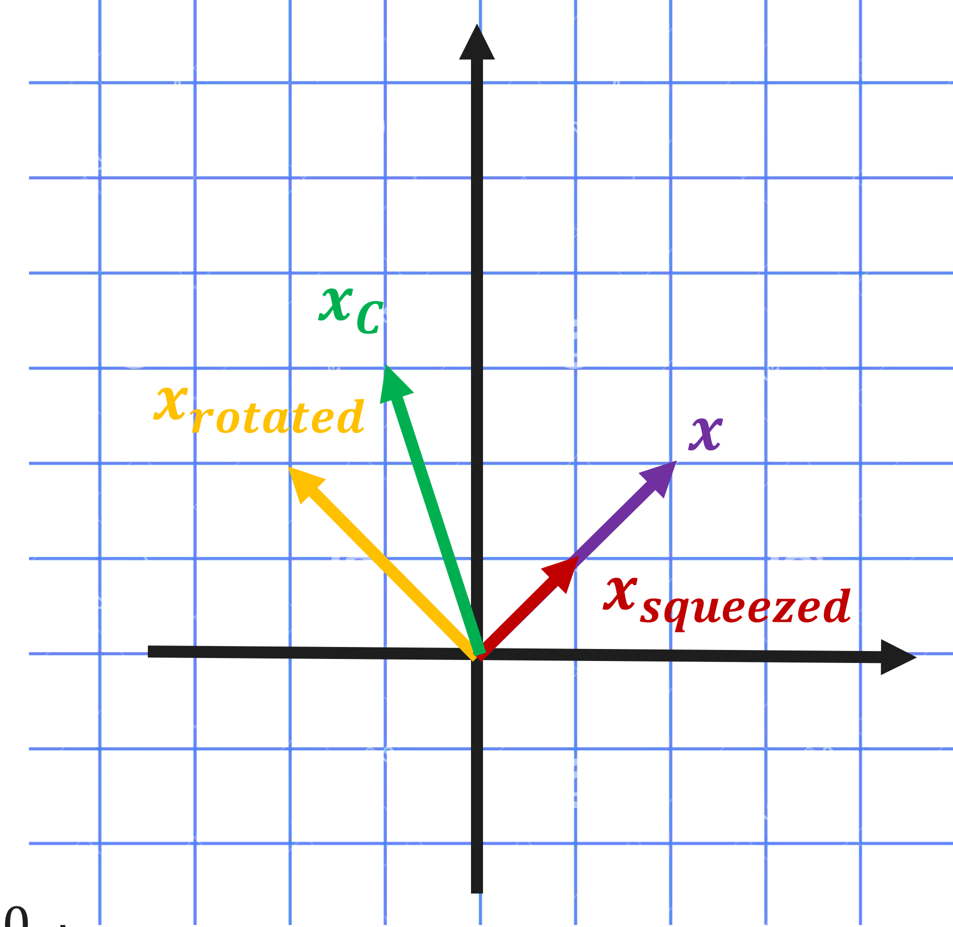

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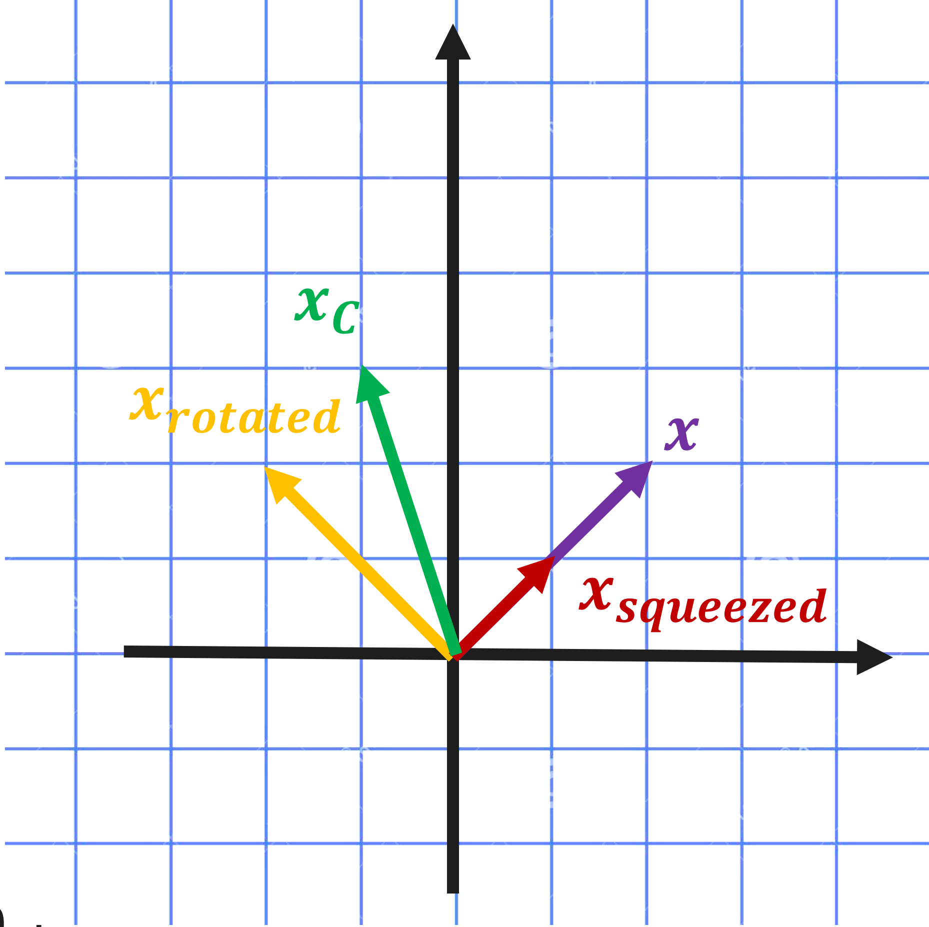

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Inverse of a Matrix

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- Which matrices have an inverse?

Determinant

Determinant

- A numerical way to characterize a linear transformation (and its matrix):
  - absolute value = how much area changes;
  - sign = change of orientation.

- More info on the interpretation: see video.

Determinant

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    - $\det A \neq 0$:
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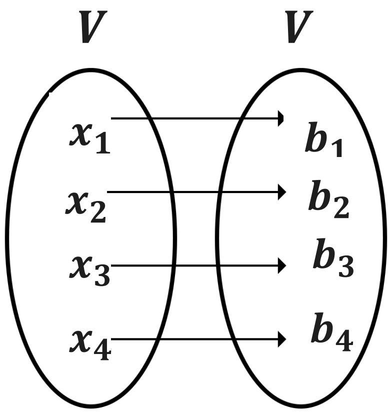

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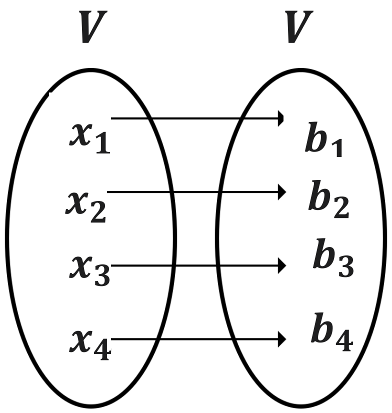
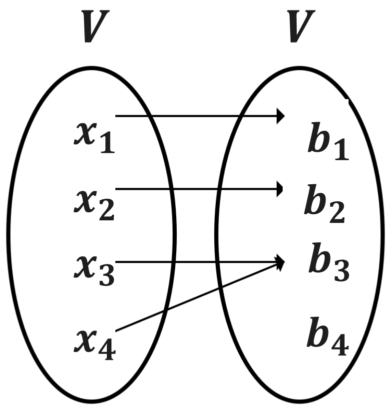


  - $\det A = 0$:
    - Several vectors are mapped onto the same vector $\Leftrightarrow$ A maps original vector space onto a lower-dimensional space.



Computing Determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

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- Example:

$$R_{90} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 0 - (-1) = 1 \Leftrightarrow$$

“there is a transform inverse to rotation by $\mathbf{90}^{\circ}$ anticlockwise”.

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Computing Determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

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- Example:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - 0 + 0 = 0 \Leftrightarrow$$

"there is no transpose inverse to projection onto XY-plane"

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- $A = \{a_{ij}\} - n{\times}n$ matrix.

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- Laplace expansion.

Some Properties of the Determinant

- $\det A^T = \det A$
- $\det AB = \det A \cdot \det B$
- $\det A^{-1} = \dfrac{1}{\det A}$

Finding Inverse of a Matrix

Gaussian Elimination

- $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
- $\det A \neq 0 \Longrightarrow$ there exists A^{-1}. Let's find it!

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$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

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- Perform **elementary row operations** and obtain identity matrix on the left. The inverse will be on the right!

Gaussian Elimination

- Elementary row operations:
    - swap rows;
    - multiply rows by some number;
    - add / subtract one row to / from another.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

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Gaussian Elimination

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \{(3)-(1)\} \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right) \to$$

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$$\rightarrow \{swap\ (2)\ and\ (3)\} \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & -2 \end{array}\right)$$

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$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} -1 & -1 & 2 \\ -1 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix}, \quad AA^{-1} = A^{-1}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rank

Column Space

- Consider a square matrix A.
- Its columns $A^1, \ldots, A^n$ can be seen as vectors.

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- $U = span\{A^1, \dots, A^n\}$ – column space of A.
  - All vectors that can be obtain by linearly combining columns of A.
  - $\Leftrightarrow$ image of linear transformation A (= all the vectors we can get by applying A).

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    - Full rank matrix: n columns, all linearly independent.

    - Lower-rank matrices: linearly dependent columns present.

Rank: Examples

- $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

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Column vs Row Rank

- Column space of A = span of A's columns.
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- Fundamental result: the column rank and the row rank are always equal.
  See proofs.

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$$X = [x_1 \mid x_2 \mid \dots \mid x_n] = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1m} \\ x_{21} & x_{22} & \dots & x_{2m} \\ \dots & \dots & \dots & \dots \\ x_{n1} & x_{n2} & \dots & x_{nm} \end{bmatrix}$$

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$$rank(X) \leq \min\{n, m\}$$

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- $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $rank(A) = 1 \iff \mathbb{R}^3$ is mapped onto a line

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  Infinitely many vectors are mapped into a zero vector.

- $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, $rank(A) = 2 \Leftrightarrow \mathbb{R}^3$is mapped onto a plane

  Infinitely many vectors are mapped into a zero vector.

- $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $rank(A) = 3 \Leftrightarrow \mathbb{R}^3$is mapped on itself (isomorphism)

  Only a zero vector is mapped into a zero vector.

Null space

- A set of vectors that are mapped to $\mathbf{0}$ by a linear transformation A.

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- Example: projection onto XY-plane:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow$$

Null space: $\left\{ v \in \mathbb{R}^3 \mid v = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix},\ z \in \mathbb{R} \right\}$

Systems of Linear Equations

What is a SLE?

$$\begin{cases} 2x_1 + 5x_2 + 3x_3 = -3 \\ 4x_1 + 0x_2 + 8x_3 = 0 \\ 1x_1 + 3x_2 + 0x_3 = 2 \end{cases}$$

Solutions to SLE

1. $$\begin{cases} x + y = 1 \\ x + y = 2 \end{cases}$$

2. $$\begin{cases} x + y = 1 \\ 2x + y = 2 \end{cases}$$

3. $$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$$

Solutions to SLE

1. $$\begin{cases} x + y = 1 \\ x + y = 2 \end{cases}$$

No solutions.

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A single solution: $x = 1,\ y = 0$

3. $$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$$

Infinitely many solutions.

SLE: Matrix Notation

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$$A = \begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$Ax = b$$

SLE: Matrix Notation

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$$Ax = b$$

"Find vector(s) x that are mapped into b by transform A"

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  So, there is a single solution.
- If $\mathbf{det}\,\boldsymbol{A} = \mathbf{0}$, it's not that simple:

Number of Solutions

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Solutions to SLE

1. $\begin{cases} x+y=1 \\ x+y=2 \end{cases}$

No solutions.

2. $\begin{cases} x+y=1 \\ 2x+y=2 \end{cases}$

A single solution: $x=1,\ y=0$

$$A=\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \det A \neq 0.$$

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How do we check that?

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- $Ax = b$ – SLE.
- Consider matrix $(A|b) = \begin{bmatrix} a_{11} & \dots & a_{1n} & b_1 \\ .. & .. & .. & \dots \\ a_{m1} & \dots & a_{mn} & b_n \end{bmatrix}$.

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  - $Ax = b$ has no solutions $\Leftrightarrow rank(A|b) > rank(A)$.

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Solutions to SLE

1. $\begin{cases} x + y = 1 \\ x + y = 2 \end{cases}$

No solutions.

$1 = rank(A) < rank(A|b) = 2$

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A single solution: $x = 1,\ y = 0$

$rank(A) = rank(A|b) = 2$

3. $\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$

Infinitely many solutions.

$rank(A) = rank(A|b) = 1 < 2$

Gaussian Elimination

Gaussian elimination

- $Ax = b$

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{bmatrix}, \qquad b = \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}$$

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- Elementary row operations:

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 2 & 4 & 3 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 0 & -2 & 7 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & -2 & 7 & -2 \end{array}\right]$$

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Unique solution.

Gaussian Elimination

- $Ax = b$ – SLE.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 5 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

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No solutions.

Gaussian Elimination

- $Ax = b$ – SLE.

$$A = \begin{bmatrix} -3 & -5 & 36 \\ -1 & 0 & 7 \\ 1 & 1 & -10 \end{bmatrix}, \qquad b = \begin{bmatrix} 10 \\ 5 \\ -4 \end{bmatrix}$$

-

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Infinitely many solutions.

Homogeneous SLE

Homogeneous SLE

$$\begin{cases} 2x_1 + 5x_2 + 3x_3 = \boxed{\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}} \\ 4x_1 + 0x_2 + 8x_3 = \\ 1x_1 + 3x_2 + 0x_3 = \end{cases}$$

Homogeneous SLE

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Solutions = null space of A.

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$\operatorname{rank} A = \#variables \rightarrow$ unique solution (0)

$\operatorname{rank} A < \#variables \rightarrow$ infinitely many solutions.

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- $Ax = 0$
- Let V be a set of solutions:

$$\forall v \in V \quad Av = 0$$

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    Indeed, $A(\lambda v) = \lambda A v = 0$
  - $\forall v_1, v_2 \in V \rightarrow (v_1 + v_2) \in V$.
    Indeed, $A(v_1 + v_2) = Av_1 + Av_2 = 0$

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$\mathbf{V}$ is a linear subspace!

To sum up

- Matrices as linear transforms
- Examples of common transforms
- Inverse
- Determinant
- Rank
- Solutions to SLE