Math Refresher for DS

Practical Session 5

girafe
ai

Coordinate Change for Linear Transforms

Change of Basis for Vectors

- V – a vector space.
- $B = \{b_1, \dots, b_n\}$ – current basis, $S = \{s_1, \dots, s_n\}$ – new basis.

- $x \in V$ – some vector.

- We know already that

Change of Basis for Vectors

- V – a vector space.
- $B = \{b_1, \dots, b_n\}$ – current basis, $S = \{s_1, \dots, s_n\}$ – new basis.

- $x \in V$ – some vector.

- We know already that

$$x_B = Mx_S,\ M = M_{B\to S} = \left[[s_1]_B \mid \dots \mid [s_n]_B\right] \text{ – transition matrix.}$$

girafe ai

Change of Basis for Vectors

- V – a vector space.
- $B = \{b_1, \dots, b_n\}$ – current basis, $S = \{s_1, \dots, s_n\}$ – new basis.

- $x \in V$ – some vector.

- We know already that

$$x_B = Mx_S,\ \ M = M_{B \to S} = \big[[s_1]_B \mid \dots \mid [s_n]_B\big] \text{ – transition matrix.}$$

- We also established that

$$\exists M^{-1} = M_{S \to B}, \qquad x_S = M^{-1} x_B$$

Change of Basis for Vectors

- V – a vector space.
- $B = \{b_1, \dots, b_n\}$ – current basis, $S = \{s_1, \dots, s_n\}$ – new basis.

- $x \in V$ – some vector.

- We know already that

$$x_B = Mx_S,\ \ M = M_{B\to S} = \big[[s_1]_B \mid \dots \mid [s_n]_B\big] \text{ – transition matrix.}$$

- We also established that

$$\exists M^{-1} = M_{S\to B}, \qquad x_S = M^{-1}x_B$$

- But vectors aren't the only things with coordinates…

Change of Basis for Linear Transforms

- Consider a linear transform A.
- It's defined by its matrix: columns = what happens to basis vectors.

- Example: rotation

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Change of Basis for Linear Transforms

- Consider a linear transform A.
- It's defined by its matrix: columns = what happens to basis vectors.

- Example: rotation

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- $S = \left\{ s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ – another basis.

How would A look like in this basis?

Change of Basis for Linear Transforms

- A – linear transform;
- $B = \{b_1, \dots, b_n\}$ – current basis, $S = \{s_1, \dots, s_n\}$ – new basis;
- $M = M_{B \to S}$ – transition matrix.

- x – some vector.

girafe ai

Change of Basis for Linear Transforms

- A – linear transform;
- $B = \{b_1, \dots, b_n\}$ – current basis, $S = \{s_1, \dots, s_n\}$ – new basis;
- $M = M_{B \to S}$ – transition matrix.

- x – some vector.

$$[x']_B = [A]_B \cdot [x]_B$$

girafe ai

Change of Basis for Linear Transforms

- A – linear transform;
- $B = \{b_1, \dots, b_n\}$ – current basis, $S = \{s_1, \dots, s_n\}$ – new basis;
- $M = M_{B \to S}$ – transition matrix.

- x – some vector.

$$[x']_B = [A]_B \cdot [x]_B$$

$$[x']_S = \quad [\boldsymbol{A}]_{\boldsymbol{S}} \quad \cdot [x]_S$$

Change of Basis for Linear Transforms

- A – linear transform;
- $B = \{b_1, \dots, b_n\}$ – current basis, $S = \{s_1, \dots, s_n\}$ – new basis;
- $M = M_{B \to S}$ – transition matrix.

- x – some vector.

$$[x']_B = [A]_B \cdot [x]_B$$

$$[x']_s = \quad \boldsymbol{[A]_S} \quad \cdot [x]_S$$

girafe ai

Change of Basis for Linear Transforms

- A – linear transform;
- $B = \{b_1, \ldots, b_n\}$ – current basis, $S = \{s_1, \ldots, s_n\}$ – new basis;
- $M = M_{B \to S}$ – transition matrix.

- x – some vector.

$$[x']_B = [A]_B \cdot [x]_B$$

$$[x']_s = \qquad\qquad\qquad \cdot [x]_S$$

Change of Basis for Linear Transforms

- A – linear transform;
- $B = \{b_1, \dots, b_n\}$ – current basis, $S = \{s_1, \dots, s_n\}$ – new basis;
- $M = M_{B \to S}$ – transition matrix.

- x – some vector.

$$[x']_B = [A]_B \cdot [x]_B$$

$$[x']_s = \qquad\qquad M \cdot [x]_S$$

girafe ai

Change of Basis for Linear Transforms

- A – linear transform;
- $B = \{b_1, \dots, b_n\}$ – current basis, $S = \{s_1, \dots, s_n\}$ – new basis;
- $M = M_{B \to S}$ – transition matrix.

- x – some vector.

$$[x']_B = [A]_B \cdot [x]_B$$

$$[x']_s = \qquad [A]_B M \cdot [x]_S$$

girafe ai

Change of Basis for Linear Transforms

- A – linear transform;
- $B = \{b_1, \ldots, b_n\}$ – current basis, $S = \{s_1, \ldots, s_n\}$ – new basis;
- $M = M_{B \to S}$ – transition matrix.

- x – some vector.

$$[x']_B = [A]_B \cdot [x]_B$$

$$[x']_s = M^{-1}[A]_B M \cdot [x]_S$$

girafe ai

Change of Basis for Linear Transforms

- A – linear transform;
- $B = \{b_1, \dots, b_n\}$ – current basis, $S = \{s_1, \dots, s_n\}$ – new basis;
- $M = M_{B \to S}$ – transition matrix.

- x – some vector.

$$[x']_B = [A]_B \cdot [x]_B$$

$$[x']_s = M^{-1}[A]_B M \cdot [x]_S$$

Change of Basis for Linear Transforms

- A – linear transform;
- $B = \{b_1, \dots, b_n\}$ – current basis, $S = \{s_1, \dots, s_n\}$ – new basis;
- $M = M_{B\to S}$ – transition matrix.

- x – some vector.

$$[x']_B = [A]_B \cdot [x]_B$$

$$[x']_S = M^{-1}[A]_B M \cdot [x]_S$$

$$\mathbf{[A]_S} =$$

Change of Basis for Linear Transforms

- A – linear transform;
- $B = \{b_1, \dots, b_n\}$ – current basis, $S = \{s_1, \dots, s_n\}$ – new basis;
- $M = M_{B \to S}$ – transition matrix.

- x – some vector.

$$[x']_B = [A]_B \cdot [x]_B$$

$$[x']_s = M^{-1}[A]_B M \cdot [x]_S$$

$$\mathbf{[A]_S = M^{-1}[A]_B M}$$

girafe ai

Change of Basis for Linear Transforms

- Back to our example:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad S\left\{s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} -$$ new basis.

$$[A]_S = M^{-1}AM =$$

Change of Basis for Linear Transforms

- Back to our example:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad S\left\{s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} - \text{ new basis.}$$

$$[A]_S = M^{-1}AM = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} =$$

Change of Basis for Linear Transforms

- Back to our example:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad S\left\{s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} -$$ new basis.

$$[A]_S = M^{-1}AM = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} =$$

girafe ai

Change of Basis for Linear Transforms

- Back to our example:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad S\left\{s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} - \text{ new basis.}$$

$$[A]_S = M^{-1}AM = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} =$$

Change of Basis for Linear Transforms

- Back to our example:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad S\left\{s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} - \text{ new basis.}$$

$$[A]_S = M^{-1}AM = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix}$$

Change of Basis for Linear Transforms

$[A]_E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $[A]_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix}$, $S = \left\{ s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ – new basis;

$$x_S = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad x'_S = [A]_S \cdot x_S =$$

Change of Basis for Linear Transforms

$[A]_E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $[A]_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix}$, $S = \left\{ s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ – new basis;

$$x_S = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad x'_S = [A]_S \cdot x_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$$

girafe ai

Change of Basis for Linear Transforms

$[A]_E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $[A]_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix}$, $S = \left\{ s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ – new basis;

$$x_S = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad x'_S = [A]_S \cdot x_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Change of Basis for Linear Transforms

$[A]_E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $[A]_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix}$, $S = \left\{ s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ – new basis;

$$x_S = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad x'_S = [A]_S \cdot x_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$x_E =$$

Change of Basis for Linear Transforms

$[A]_E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $[A]_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix}$, $S = \left\{ s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ – new basis;

$$x_S = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad x'_S = [A]_S \cdot x_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$x_E = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \qquad x'_E =$$

girafe ai

Change of Basis for Linear Transforms

$[A]_E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $[A]_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix}$, $S = \left\{ s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ – new basis;

$$x_S = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad x'_S = [A]_S \cdot x_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$x_E = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \qquad x'_E = [A]_E \cdot x_E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

Change of Basis for Linear Transforms

- Another example:

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, \qquad S = \left\{ s_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} -$$ new basis

$$[A]_S = M^{-1}AM =$$

Change of Basis for Linear Transforms

- Another example:

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, \qquad S = \left\{ s_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} -$$ new basis

$$[A]_S = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} =$$

Change of Basis for Linear Transforms

- Another example:

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, \qquad S = \left\{ s_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} -$$ new basis

$$[A]_S = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} =$$

Change of Basis for Linear Transforms

- Another example:

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, \qquad S = \left\{ s_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} -$$ new basis

$$[A]_S = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Change of Basis for Linear Transforms

- Another example:

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, \qquad S = \left\{ s_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} - \text{ new basis}$$

$$[A]_S = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

We get a diagonal matrix, it’s easier to work with it!

Coordinate Change for Linear Transforms

Linear Transformations

- Every $n{\times}n$ matrix A represents a linear transformation of $\mathbb{R}^n$.
- Columns of A = what happens to the basis vectors.

Linear Transformations

- Another example:

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}$$

Linear Transformations

- Another example:

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}$$

Every vector in $\mathbb{R}^2$ is mapped onto a line, a one-dimensional subspace of $\mathbb{R}^2$ (but we still stay in $\mathbb{R}^2$)

$$Ax = \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 \\ 0 \end{bmatrix}$$

Linear Transformations

- Another example:

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}$$

Every vector in $\mathbb{R}^2$ is mapped onto a line, a one-dimensional subspace of $\mathbb{R}^2$ (but we still stay in $\mathbb{R}^2$)

$$Ax = \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 \\ 0 \end{bmatrix}$$

Many vectors are mapped onto the same one (no inverse!):

Example: $\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \end{bmatrix}$

Linear Transformations

So far: only square matrices.

But about non-square ones?

Linear Transformations

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 5 & 9 \end{bmatrix} - 3\times2 \text{ matrix.}$$

Linear Transformations

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 5 & 9 \end{bmatrix} - 3\times2 \text{ matrix.}$$

$$x \in \mathbb{R}^2, \qquad Ax = \begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 5 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

girafe ai

Linear Transformations

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 5 & 9 \end{bmatrix} - 3\times 2 \text{ matrix.}$$

$$x \in \mathbb{R}^2, \qquad Ax = \begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 5 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 14 \end{bmatrix} \in \mathbb{R}^3$$

Linear Transformations

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 5 & 9 \end{bmatrix} - 3{\times}2 \text{ matrix.}$$

$$x \in \mathbb{R}^2, \qquad Ax = \begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 5 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 14 \end{bmatrix} \in \mathbb{R}^3$$

A is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$!

Linear Transformations

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 5 & 9 \end{bmatrix}$$ – 3×2 matrix.

$$x \in \mathbb{R}^2, \qquad Ax = \begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 5 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 14 \end{bmatrix} \in \mathbb{R}^3$$

A is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$!

$rank(A) = 2$: vectors that were <u>independent</u> in $\mathbb{R}^2$ will be mapped on <u>independent</u> vectors in $\mathbb{R}^3$.

Linear Transformations

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}$$ – 2×3 matrix.

Linear Transformations

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}$$ – 2×3 matrix.

$$x \in \mathbb{R}^3, \qquad Ax = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} =$$

girafe ai

Linear Transformations

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix} - 2\times 3 \text{ matrix.}$$

$$x \in \mathbb{R}^3, \qquad Ax = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \end{bmatrix} \in \mathbb{R}^2$$

Linear Transformations

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix} - 2\times 3 \text{ matrix.}$$

$$x \in \mathbb{R}^3, \qquad Ax = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \end{bmatrix} \in \mathbb{R}^2$$

A is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$!

Linear Transformations

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}$$ – 2×3 matrix.

$$x \in \mathbb{R}^3, \qquad Ax = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \end{bmatrix} \in \mathbb{R}^2$$

A is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$!

$rank(A) = 2$: vectors that were <u>independent</u> in $\mathbb{R}^3$ may be mapped on <u>dependent</u> vectors in $\mathbb{R}^2$.

Linear Transformations

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} - m \times n \text{ matrix.}$$

$$x \in \mathbb{R}^n, \qquad Ax = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \in \mathbb{R}^m$$

A is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$!

Linear Transformations From $\mathbb{R}^n$ to $\mathbb{R}^m$

Why this is useful?

Dimensionality reduction!

Dimensionality Reduction

Dimensionality Reduction

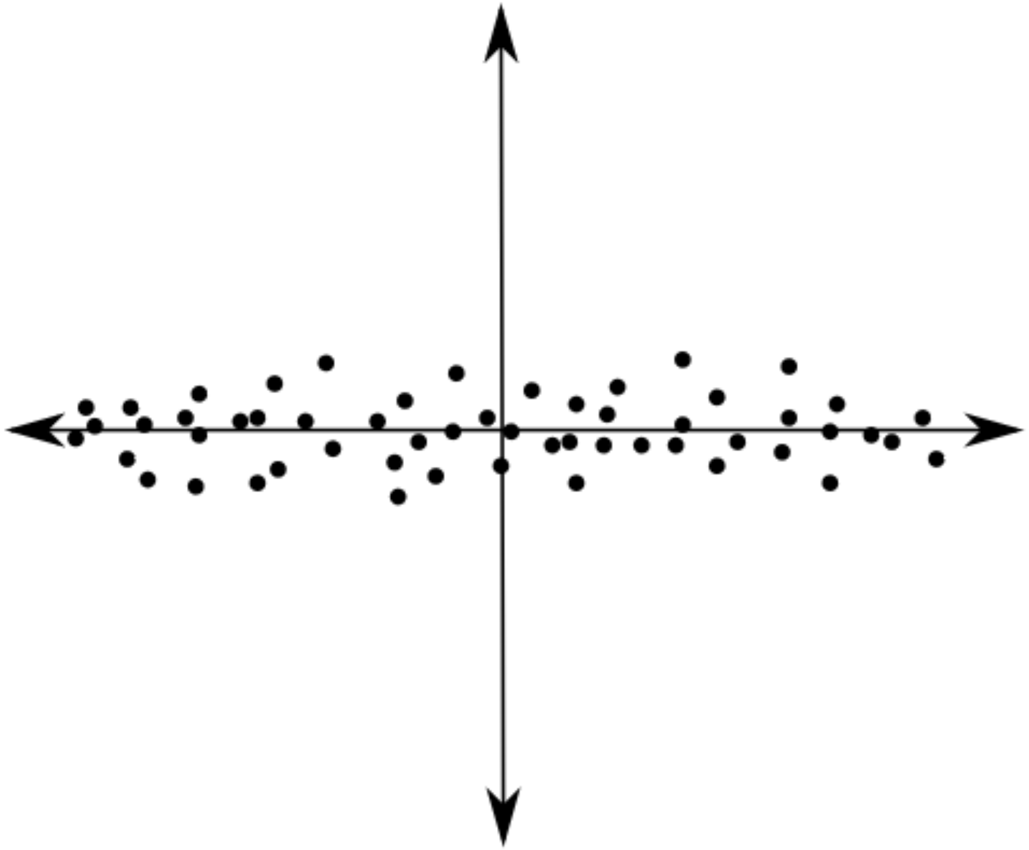
- Imagine that you have some data: m features, n examples.
- Each example $x = (x_1, \dots, x_m)$ – a point in $\mathbb{R}^m$.
- How to visualize this data?

Dimensionality Reduction

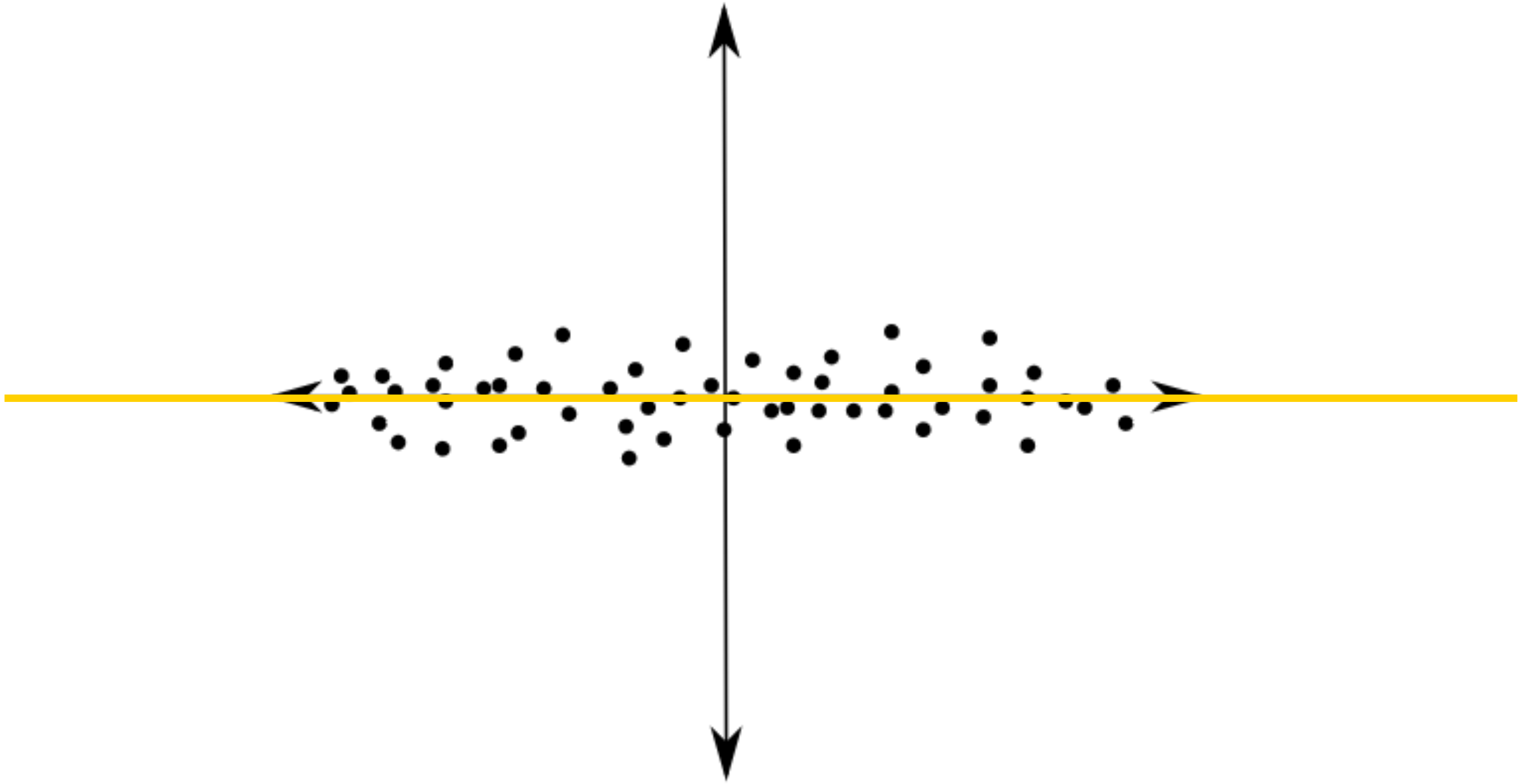
- Imagine that you have some data: m features, n examples.
- Each example $x = (x_1, \dots, x_m)$ – a point in $\mathbb{R}^m$.
- How to visualize this data?

Map into onto a lower-dimensional space!
But preserve as much variance in data as possible.

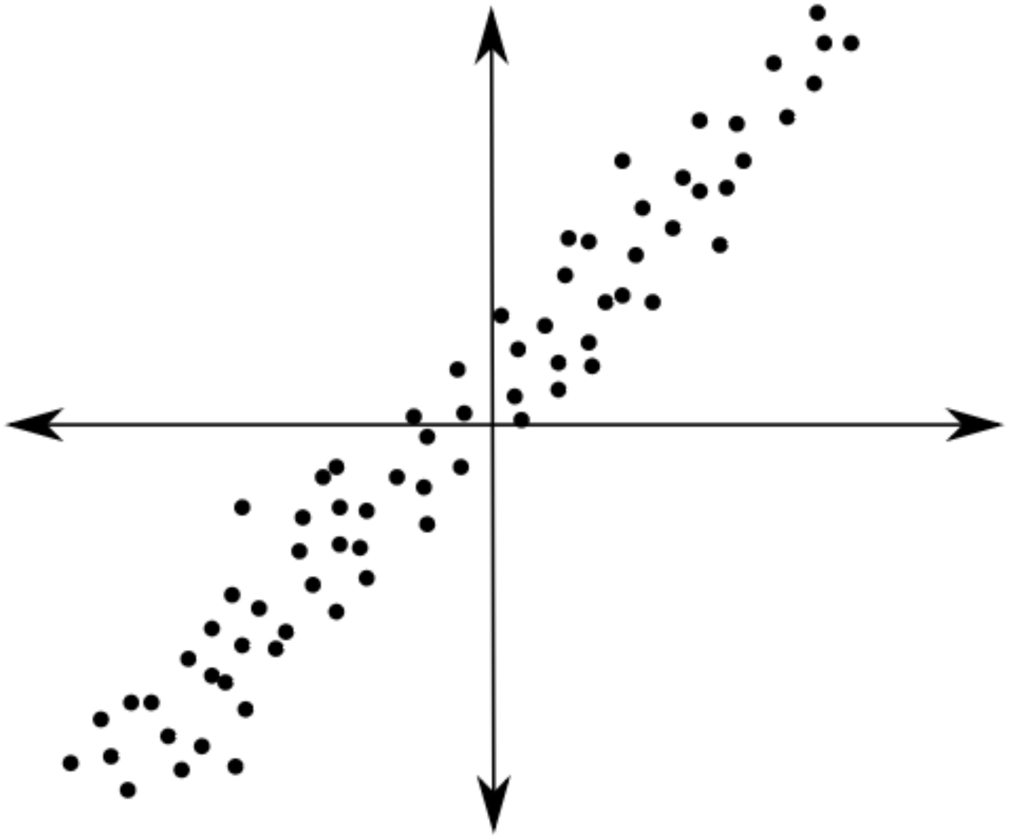
Dimensionality Reduction



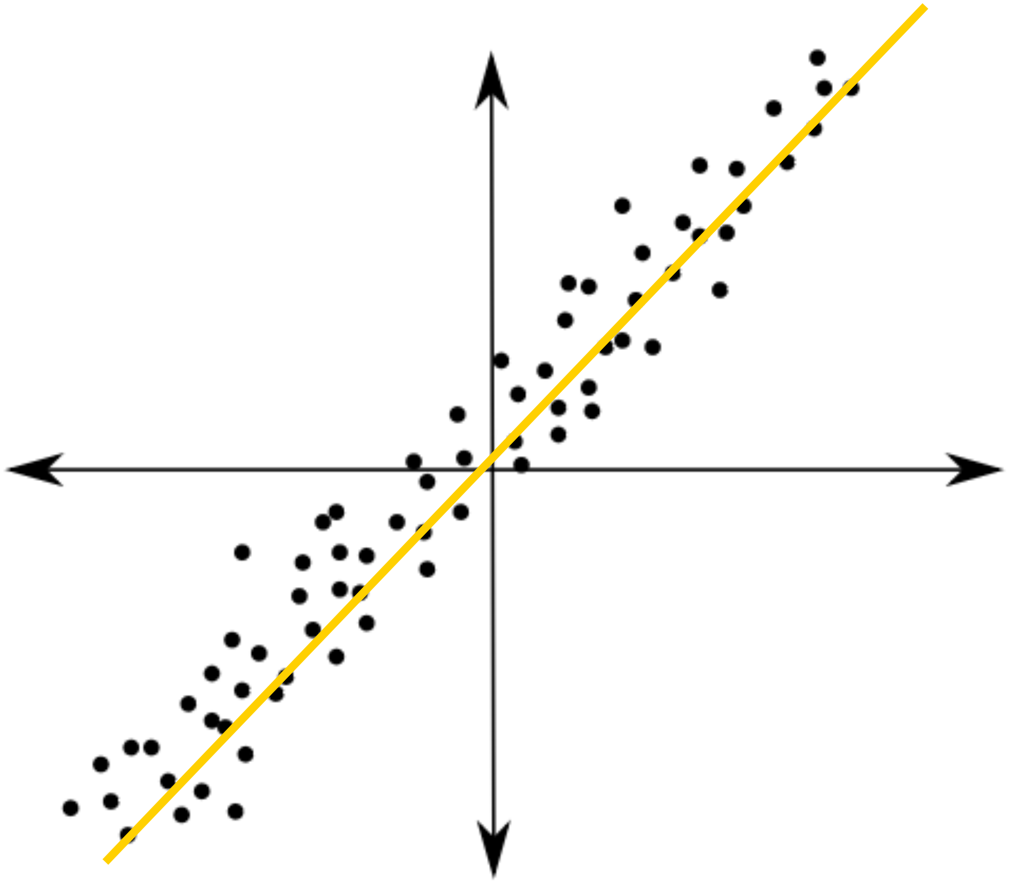
Dimensionality Reduction



Dimensionality Reduction

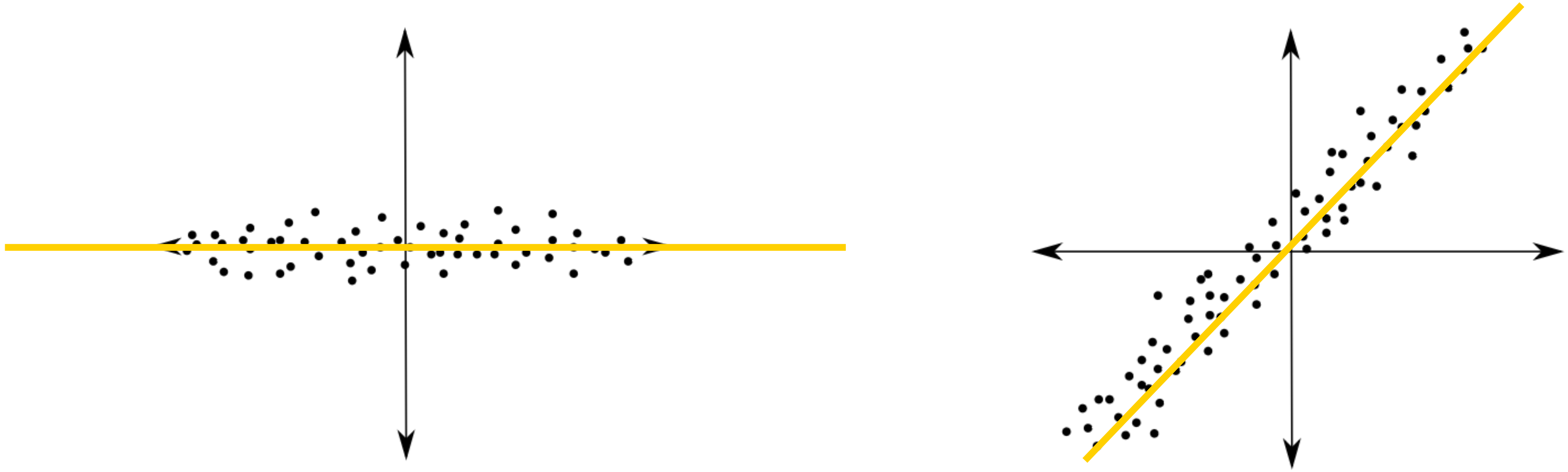


Dimensionality Reduction



Dimensionality Reduction

- How to find this direction?



Let’s review the theory around it

Eigenvalues & Eigenvectors

- Consider an $n{\times}n$ matrix A.
- A – linear transformation of $\mathbb{R}^n$. Every vector gets scaled and rotated:

$$x' = Ax$$

Eigenvalues & Eigenvectors

- Consider an $n \times n$ matrix A.
- A – linear transformation of $\mathbb{R}^n$. Every vector gets scaled and rotated:

$$x' = Ax$$

- Some vectors only get scaled:

$$Av = \lambda v, \qquad \lambda \in \mathbb{R}, \qquad v \neq 0$$

$v \neq 0$ – eigenvector, λ – corresponding eigenvalue.

Eigenvalues & Eigenvectors

- Consider an $n \times n$ matrix A.

- $v \neq 0$ – eigenvector, λ – corresponding eigenvalue:

$$Av = \lambda v$$

How to find λ and v?

$$Av = \lambda v \Leftrightarrow Av - \lambda v = 0 \Leftrightarrow (A - \lambda E)v = 0 \Leftrightarrow$$

Eigenvalues & Eigenvectors

- Consider an $n \times n$ matrix A.
- $v \neq 0$ – eigenvector, λ – corresponding eigenvalue:

$$Av = \lambda v$$

How to find λ and v?

$$Av = \lambda v \Leftrightarrow Av - \lambda v = 0 \Leftrightarrow (A - \lambda E)v = 0 \Leftrightarrow$$
$$\det(A - \lambda E) = 0$$

$\det(A - \lambda E)$ – characteristic polynomial of A.

girafe ai

Eigenvalues & Eigenvectors

- Example: $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$
- Let's find eigenvalues and eigenvectors of A.
- Characteristic polynomial:

$$\det(A - \lambda E) =$$

Eigenvalues & Eigenvectors

- Example: $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$

- Let's find eigenvalues and eigenvectors of A.

- Characteristic polynomial:

$$\det(A - \lambda E) = \begin{vmatrix} -5-\lambda & 2 \\ -7 & 4-\lambda \end{vmatrix} =$$

Eigenvalues & Eigenvectors

- Example: $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$

- Let's find eigenvalues and eigenvectors of A.

- Characteristic polynomial:

$$\det(A - \lambda E) = \begin{vmatrix} -5 - \lambda & 2 \\ -7 & 4 - \lambda \end{vmatrix} = (-5 - \lambda)(4 - \lambda) + 14 = 0 \Leftrightarrow$$

$$\lambda^2 + \lambda - 6 = 0 \Leftrightarrow$$

Eigenvalues & Eigenvectors

- Example: $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$
- Let's find eigenvalues and eigenvectors of A.
- Characteristic polynomial:

$$\det(A - \lambda E) = \begin{vmatrix} -5-\lambda & 2 \\ -7 & 4-\lambda \end{vmatrix} = (-5-\lambda)(4-\lambda) + 14 = 0 \Leftrightarrow$$

$$\lambda^2 + \lambda - 6 = 0 \Leftrightarrow$$

$\boldsymbol{\lambda_1 = 2,\ \lambda_2 = -3}$ – eigenvalues of $\boldsymbol{A}$.

Eigenvalues & Eigenvectors

- Example: $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$

- Let's find eigenvalues and eigenvectors of A.

$\boldsymbol{\lambda_1 = 2,\ \lambda_2 = -3}$ – eigenvalues of $\boldsymbol{A}$.

$$Av_1 = \lambda_1 v_1 \Leftrightarrow$$

Eigenvalues & Eigenvectors

- Example: $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$

- Let's find eigenvalues and eigenvectors of A.

$$\boldsymbol{\lambda_1 = 2,\ \lambda_2 = -3} \quad \text{– eigenvalues of } \boldsymbol{A}.$$

$$Av_1 = \lambda_1 v_1 \Leftrightarrow \begin{bmatrix} -5-2 & 2 \\ -7 & 4-2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0 \ \Leftrightarrow$$

Eigenvalues & Eigenvectors

- Example: $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$

- Let's find eigenvalues and eigenvectors of A.

$$\boldsymbol{\lambda_1 = 2,\ \lambda_2 = -3} \quad \text{– eigenvalues of } \boldsymbol{A}.$$

$$Av_1 = \lambda_1 v_1 \Leftrightarrow \begin{bmatrix} -5-2 & 2 \\ -7 & 4-2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0 \Leftrightarrow v_{11} = \frac{2}{7} v_{21}$$

For example, $v_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$.

Eigenvalues & Eigenvectors

- Example: $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$

- Let's find eigenvalues and eigenvectors of A.

$$\boldsymbol{\lambda_1 = 2,\ \lambda_2 = -3}$$ – eigenvalues of $\boldsymbol{A}$.

$$Av_2 = \lambda_2 v_2 \Leftrightarrow$$

Eigenvalues & Eigenvectors

- Example: $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$

- Let's find eigenvalues and eigenvectors of A.

$$\boldsymbol{\lambda_1 = 2,\ \lambda_2 = -3}$$ – eigenvalues of $\boldsymbol{A}$.

$$Av_2 = \lambda_2 v_2 \Leftrightarrow \begin{bmatrix} -5+3 & 2 \\ -7 & 4+3 \end{bmatrix} \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = 0$$

girafe ai

Eigenvalues & Eigenvectors

- Example: $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$

- Let's find eigenvalues and eigenvectors of A.

$$\boldsymbol{\lambda_1 = 2,\ \lambda_2 = -3} \quad \text{– eigenvalues of } \boldsymbol{A}.$$

$$Av_2 = \lambda_2 v_2 \Leftrightarrow \begin{bmatrix} -5+3 & 2 \\ -7 & 4+3 \end{bmatrix} \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = 0 \Leftrightarrow v_{12} = v_{22}$$

girafe ai

Eigenvalues & Eigenvectors

- Example: $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$

- Let's find eigenvalues and eigenvectors of A.

$\boldsymbol{\lambda_1 = 2,\ \lambda_2 = -3}$ – eigenvalues of $\boldsymbol{A}$.

$$Av_2 = \lambda_2 v_2 \Leftrightarrow \begin{bmatrix} -5+3 & 2 \\ -7 & 4+3 \end{bmatrix} \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = 0 \Leftrightarrow v_{12} = v_{22}$$

For example, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Eigenvalues & Eigenvectors

- Example: $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$
- Eigenvalues and eigenvectors of A:

$$\lambda_1 = 2, \qquad v_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

$$\lambda_2 = -3, \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigenvalues & Eigenvectors

- Example: $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$
- Eigenvalues and eigenvectors of A:

$$\lambda_1 = 2, \qquad v_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

$$\lambda_2 = -3, \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Note that v_1 and v_2 are linearly independent.

Eigenbasis

$$A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$$

- Eigenvalues and eigenvectors: $\lambda_1 = 2,\ v_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \quad \lambda_2 = -3,\ v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$
- v_1 and v_2 are linearly independent. They form eigenbasis.
  What happens to A if we change to basis $\{v_1, v_2\}$?

Eigenbasis

$$A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$$

- Eigenvalues and eigenvectors: $\lambda_1 = 2,\ v_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \quad \lambda_2 = -3,\ v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$
- v_1 and v_2 are linearly independent. They form **eigenbasis**.
  What happens to A if we change to basis $\{v_1, v_2\}$?

$[A]_V$ = {what happens to basis vectors v_1, v_2 after applying A} =

Eigenbasis

$$A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$$

- Eigenvalues and eigenvectors: $\lambda_1 = 2,\ v_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \quad \lambda_2 = -3,\ v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$
- v_1 and v_2 are linearly independent. They form **eigenbasis**.
  What happens to A if we change to basis $\{v_1, v_2\}$?

$[A]_V$ = {what happens to basis vectors v_1, v_2 after applying A} =

$= \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} = \Lambda$ – it becomes diagonal!

Eigendecomposition

$$A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$$

- Eigenvalues and eigenvectors: $\lambda_1 = 2,\ v_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \quad \lambda_2 = -3,\ v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- How do we change to basis $\{v_1, v_2\}$?

Eigendecomposition

$$A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$$

- Eigenvalues and eigenvectors: $\lambda_1 = 2,\ v_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \quad \lambda_2 = -3,\ v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$
- How do we change to basis $\{v_1, v_2\}$?

$V = [v_1 \mid v_2]$ – transition from standard to eigenbasis.

$$A = V[A]_V V^{-1}$$

$A = V\Lambda V^{-1}$ – eigendecomposition.

Eigendecomposition

$$A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$$

- Eigenvalues and eigenvectors: $\lambda_1 = 2,\ v_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \quad \lambda_2 = -3,\ v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$
- How do we change to basis $\{v_1, v_2\}$?

$$A = V\Lambda V^{-1}$$ – eigendecomposition.

Let's check this:

$$\begin{bmatrix} 2 & 1 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 7 & 1 \end{bmatrix}^{-1} =$$

girafe ai

Eigendecomposition

$$A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$$

- Eigenvalues and eigenvectors: $\lambda_1 = 2,\ v_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \quad \lambda_2 = -3,\ v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$
- How do we change to basis $\{v_1, v_2\}$?

$A = V\Lambda V^{-1}$ – eigendecomposition.

Let's check this:

$$\begin{bmatrix} 2 & 1 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 7 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} -0.2 & 0.2 \\ 1.4 & -0.4 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$$

Eigendecomposition

Is it always possible to find an eigenbasis?

No ☹ (see lectures for examples).

But there are good news ☺

The Spectral Theorem

If A is an $n \times n$ symmetric matrix,
then A always n linearly independent eigenvectors $v_1, \ldots, v_n$.

What's more, $v_1, \ldots, v_n$ are mutually orthogonal!
Since we are choosing the scaling, we can make them orthonormal.

The Spectral Theorem

If A is an $n \times n$ symmetric matrix,
then A always n linearly independent eigenvectors $v_1, \ldots, v_n$.

What's more, $v_1, \ldots, v_n$ are mutually orthogonal!
Since we are choosing the scaling, we can make them orthonormal.

So, we can always decompose square symmetric matrices as

$A = V \Lambda V^{-1}$, where

$V -$ orthogonal matrix (columns = eigenvecotrs),

$\Lambda -$ diagonal matrix (diagonal elements = eigenvalues)

Orthogonal Matrices

- A matrix where all columns are mutually orthonormal:

$$A^T A = AA^T = E$$

Orthogonal Matrices

- A matrix where all columns are mutually orthonormal:

$$A^T A = AA^T = E$$

- That means that orthogonal matrices are easy to invert:

$$A^{-1} = A^T.$$

The Spectral Theorem

If A is an $n \times n$ symmetric matrix,
then A always n linearly independent eigenvectors $v_1, \dots, v_n$.

What's more, $v_1, \dots, v_n$ are mutually orthogonal!
Since we are choosing the scaling, we can make them orthonormal.

So, we can always decompose square symmetric matrices as

$\boldsymbol{A = V \Lambda V^{-1}} = \boldsymbol{V \Lambda V^T}$=, where

V – orthogonal matrix (columns = eigenvecotrs),

Λ – diagonal matrix (diagonal elements = eigenvalues)

girafe ai

Back to Dimensionality Reduction

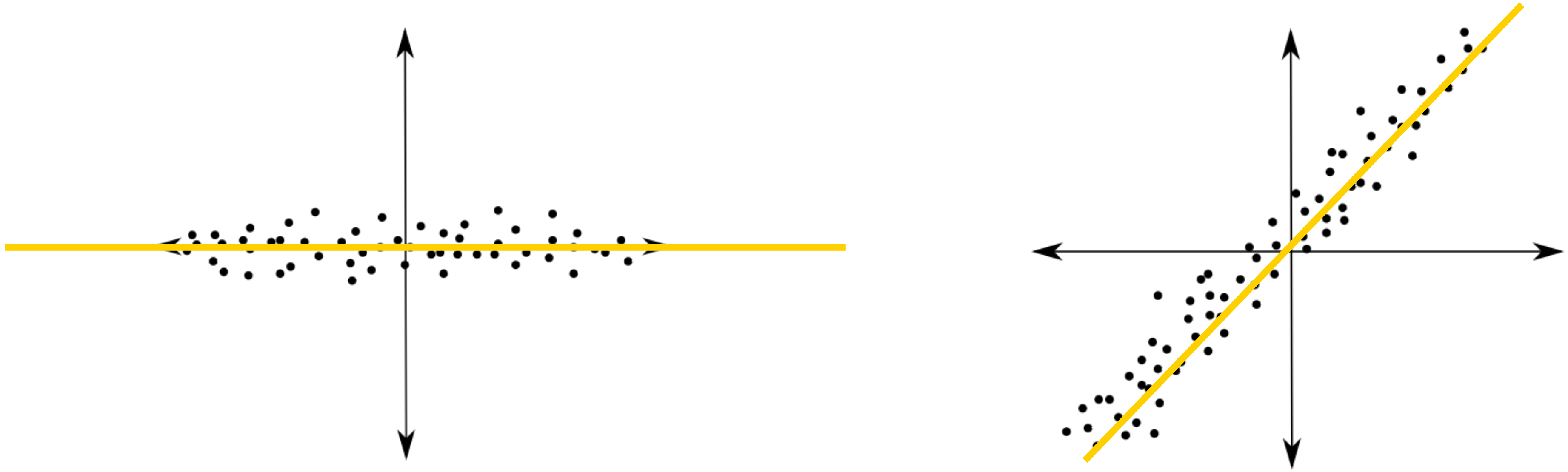
girafe
ai

Dimensionality Reduction

- Imagine that you have some data: m features, n examples.
- Each example $x = (x_1, \dots, x_m)$ – a point in $\mathbb{R}^m$.
- How to visualize this data?
- Map into onto a lower-dimensional space!
  But preserve as much variance in data as possible.

Dimensionality Reduction

- How to find this direction?



PCA

- Let's construct data covariance matrix.

$$S = \frac{1}{n-1} XX^T - m \times m \text{ symmetric matrix.}$$

s_{ij} – covariance between features i and j,

s_{ii} – variance of feature i

PCA

- Let's construct data covariance matrix.

$$S = \frac{1}{n-1} X X^T - m \times m \text{ symmetric matrix.}$$

s_{ij} – covariance between features i and j,

s_{ii} – variance of feature i

- We can apply eigendecomposition to S:

$$S = V \Lambda V^T$$

PCA

$$S = V\Lambda V^{-1} = V\Lambda V^T$$

$$\begin{bmatrix} \boldsymbol{s_{11}} & s_{12} & \cdots & s_{1m} \\ s_{21} & \boldsymbol{s_{22}} & \cdots & s_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ s_{m1} & s_{m2} & \cdots & \boldsymbol{s_{mm}} \end{bmatrix} =$$

$$= \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1m} \\ v_{21} & v_{22} & \cdots & v_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mm} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda_1} & 0 & \cdots & 0 \\ 0 & \boldsymbol{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{\lambda_m} \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1m} \\ v_{21} & v_{22} & \cdots & v_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mm} \end{bmatrix}^T$$

PCA

$$S = V\Lambda V^{-1} = V\Lambda V^{T}$$

$$\begin{bmatrix} \boldsymbol{s_{11}} & s_{12} & \cdots & s_{1m} \\ s_{21} & \boldsymbol{s_{22}} & \cdots & s_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ s_{m1} & s_{m2} & \cdots & \boldsymbol{s_{mm}} \end{bmatrix} =$$

$$= \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1m} \\ v_{21} & v_{22} & \cdots & v_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mm} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda_1} & 0 & \cdots & 0 \\ 0 & \boldsymbol{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{\lambda_m} \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1m} \\ v_{21} & v_{22} & \cdots & v_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mm} \end{bmatrix}^{T}$$

Total variance of the data

$$\mathrm{T} = tr(S) = s_{11} + \cdots + s_{nn} = \lambda_1 + \cdots \lambda_m$$

Projecting the Data

- What happens if we project the data onto the eigenvectors of S? Remember: $\{v_1, .., v_m\}$ is an orthonormal basis!

Projecting the Data

- What happens if we project the data onto the eigenvectors of S? Remember: $\{v_1, .., v_m\}$ is an orthonormal basis!

$$x = x'_1 v_1 + x'_2 v_2 + \cdots + x'_n v_m,$$

Projecting the Data

- What happens if we project the data onto the eigenvectors of S? Remember: $\{v_1, .., v_m\}$ is an orthonormal basis!

$$x = x'_1 v_1 + x'_2 v_2 + \cdots + x'_n v_m, \qquad (x, v_i) = x'_i (v_i, v_i) + 0 = x_i$$

Coordinates in an orthonormal basis = dot products!

Projecting the Data

- What happens if we project the data onto the eigenvectors of S? Remember: $\{v_1, .., v_m\}$ is an orthonormal basis!

$$x = x'_1 v_1 + x'_2 v_2 + \cdots + x'_n v_m, \qquad (x, v_i) = x'_i (v_i, v_i) + 0 = x_i$$

Coordinates in an orthonormal basis = dot products!

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} - \text{one example}, \qquad x_{proj} = V^T x$$

Projecting the Data

- What happens if we project the data onto the eigenvectors of S? Remember: $\{v_1, .., v_m\}$ is an orthonormal basis!

$$x = x'_1 v_1 + x'_2 v_2 + \cdots + x'_n v_m, \qquad (x, v_i) = x'_i (v_i, v_i) + 0 = x_i$$

Coordinates in an orthonormal basis = dot products!

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \text{ – one example,} \qquad x_{proj} = V^T x$$

The whole data: $X_{proj} = V^T X$

girafe ai

Projecting the Data

- How to project the data onto $p < m$ eigenvectors of S?

Projecting the Data

- How to project the data onto $p < m$ eigenvectors of S?

$$V_p = \{v_1 \mid \ldots \mid v_p\}$$

Projecting the Data

- How to project the data onto $p < m$ eigenvectors of S?

$$V_p = \{v_1 \mid \dots \mid v_p\}$$

$$X_p = V_p^T X$$

Projecting the Data

- How to project the data onto $p < m$ eigenvectors of S?

$$V_p = \{v_1 \mid \dots \mid v_p\}$$

$$X_p = V_p^T X$$

- What happens to the covariance matrix?

$$S_{proj} = \frac{1}{n-1} X_p X_p^T = \frac{1}{n-1} V_p^T X X^T V_p$$

girafe ai

Let’s practice!

girafe
ai