Math Refresher for DS

Practical Session 3

girafe
ai

Solving Systems of Linear Equations

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- A – $m \times n$ matrix (= m equations, n variables).

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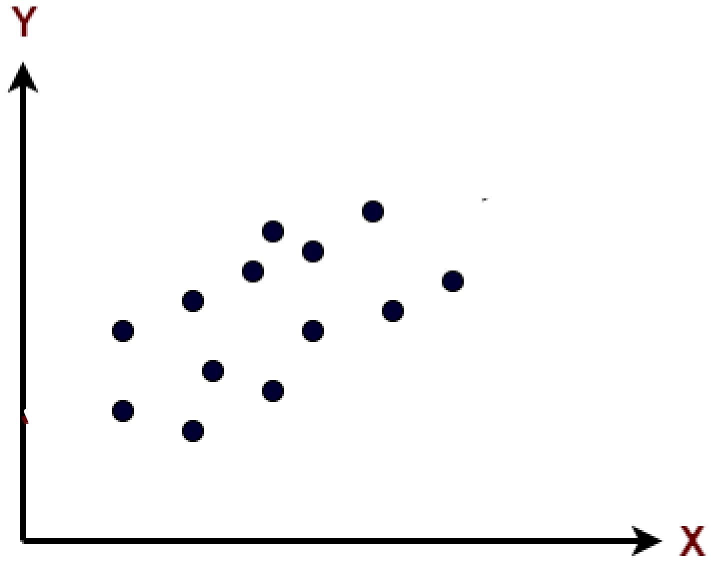
How to find a reasonable approximate solution?

Least Squares

Motivating Example

- Imagine that you have m observations:

$$(x_1, y_1), \qquad (x_2, y_2), \qquad \dots, \qquad (x_m, y_m)$$



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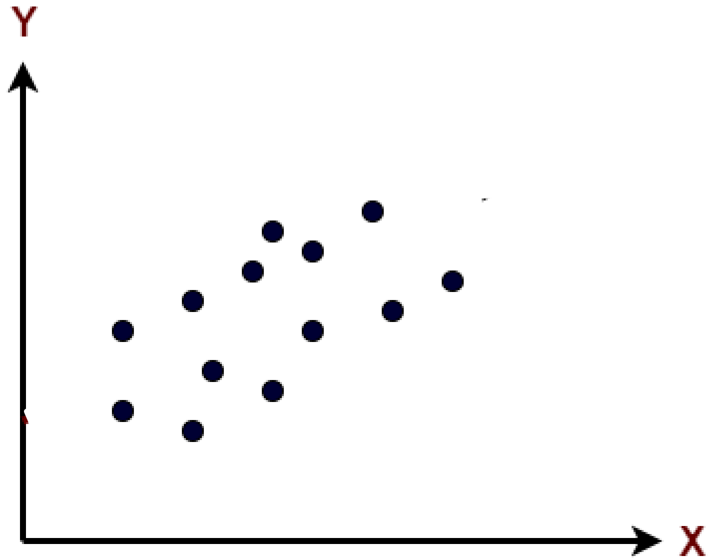

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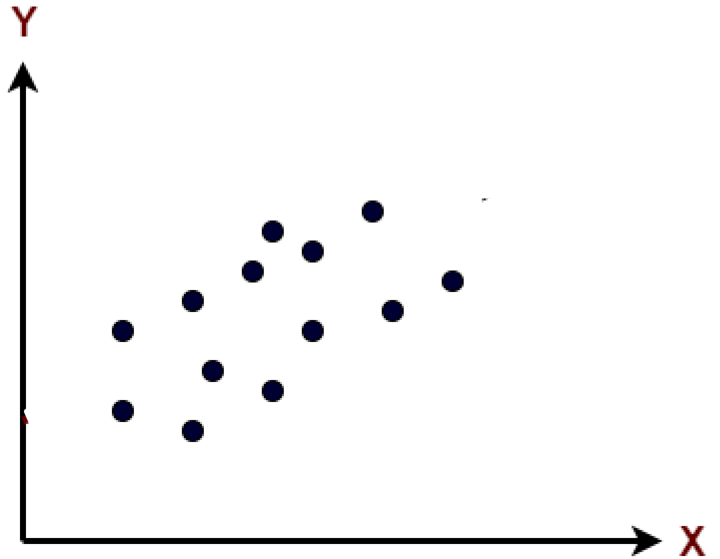

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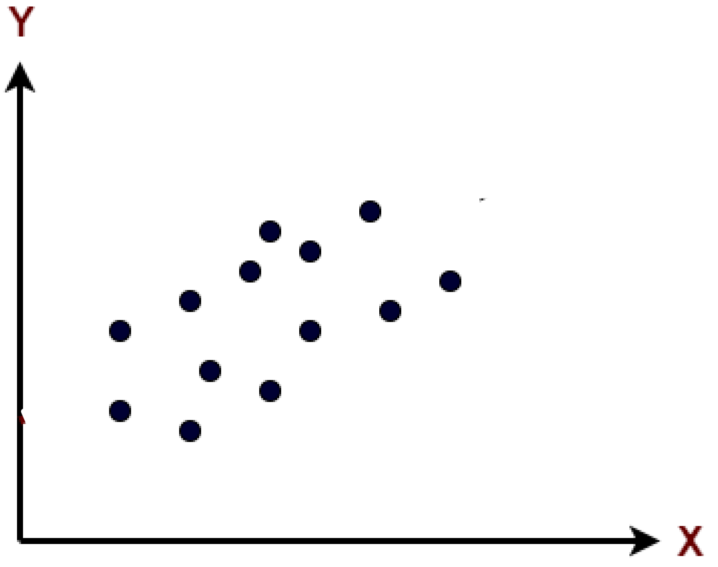

$$\begin{matrix} w_0 + w_1 x_1 = y_1 \\ w_0 + w_1 x_2 = y_n \\ \vdots \\ w_0 + w_1 x_m = y_m \end{matrix} \Leftrightarrow Xw = y, \text{ where } X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & m \end{pmatrix}, w = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

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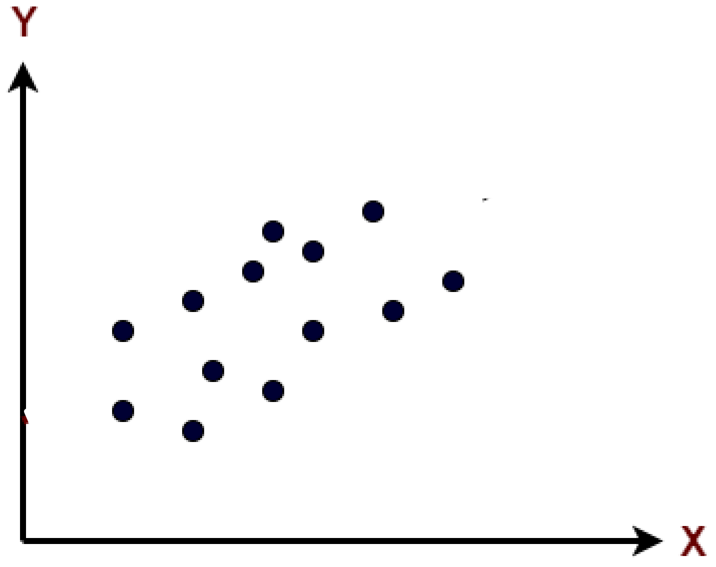

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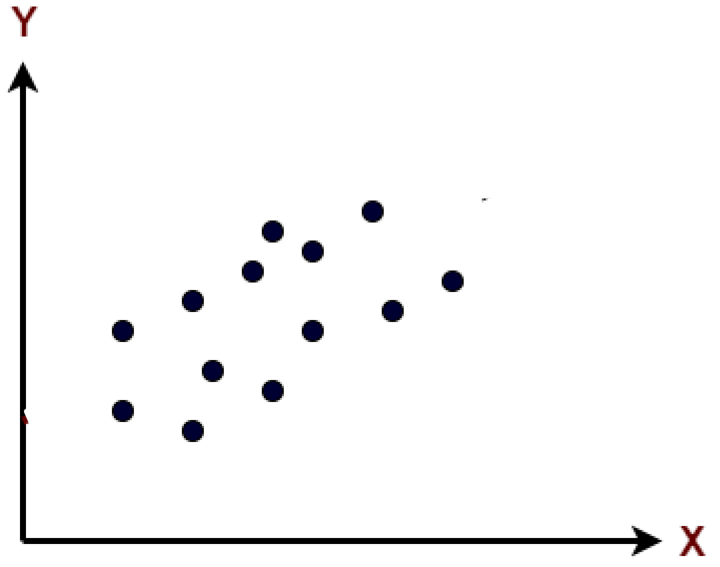

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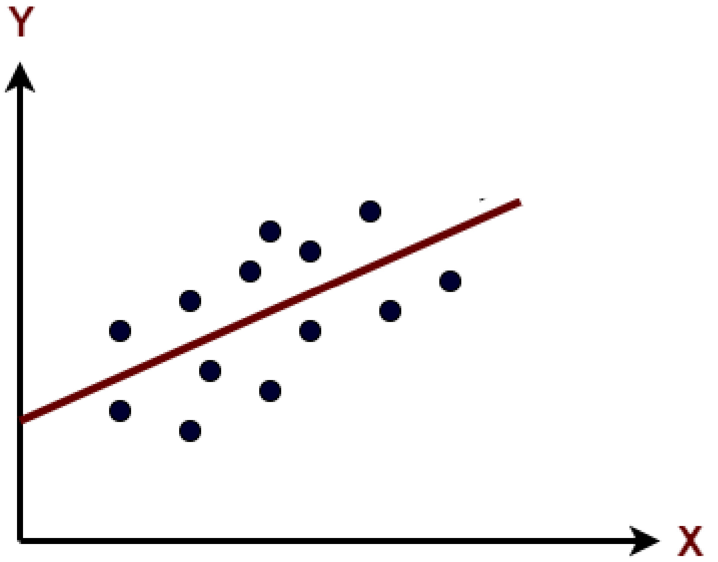

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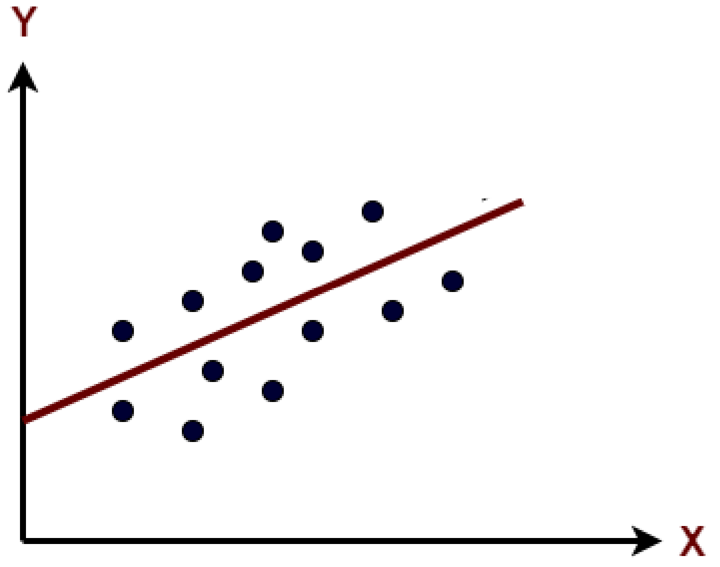

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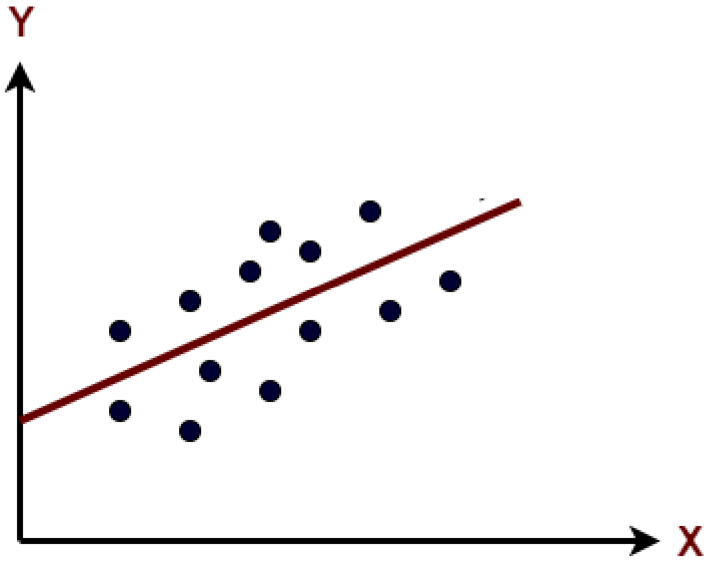

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- Let's look at it from the Linear Algebra perspective.

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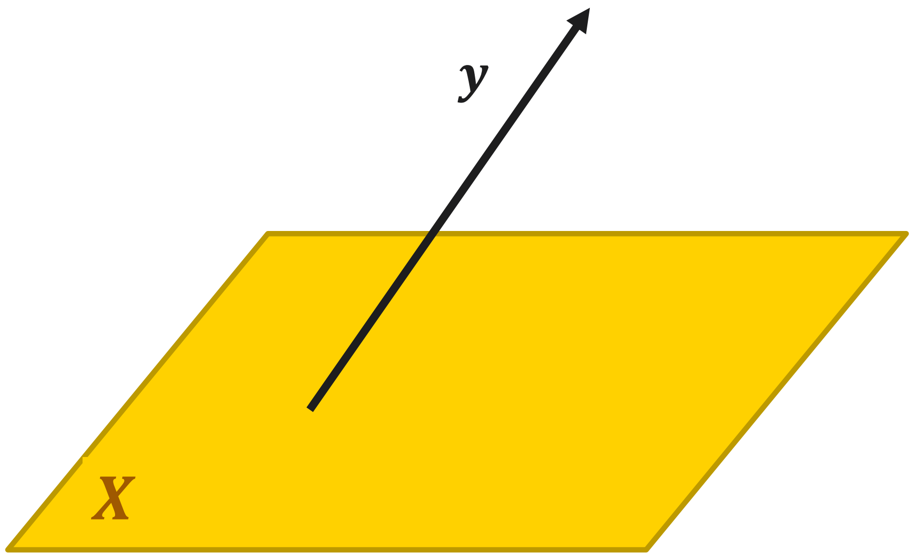
- $Xw = y$ has no solutions $\Leftrightarrow$ y is not in the column space of X.



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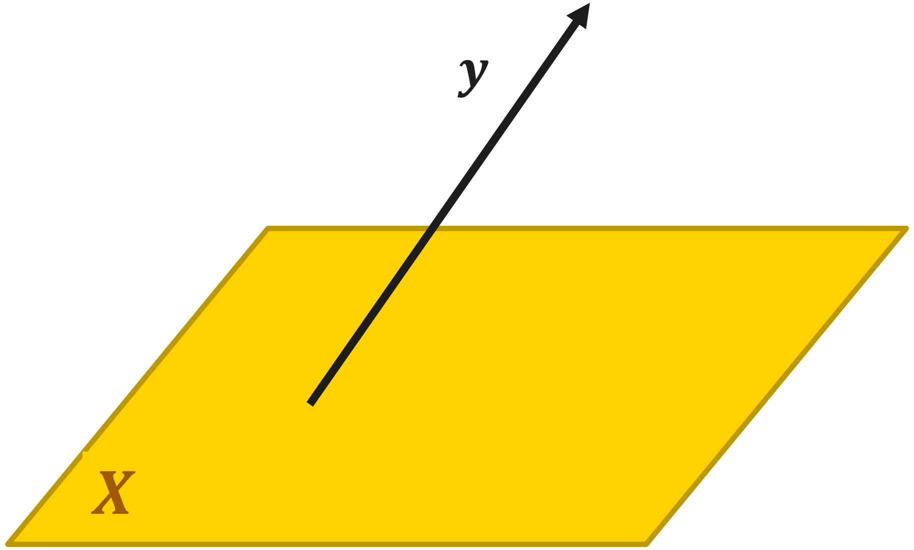

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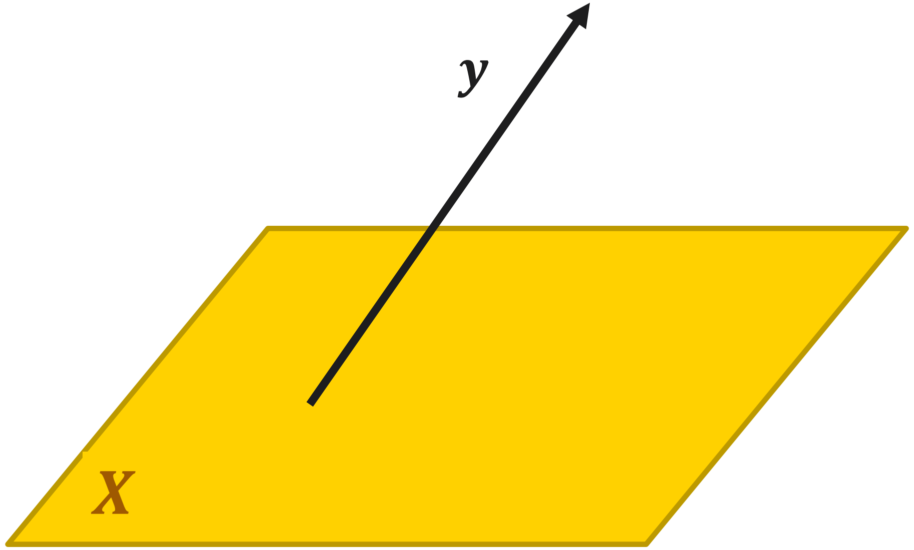

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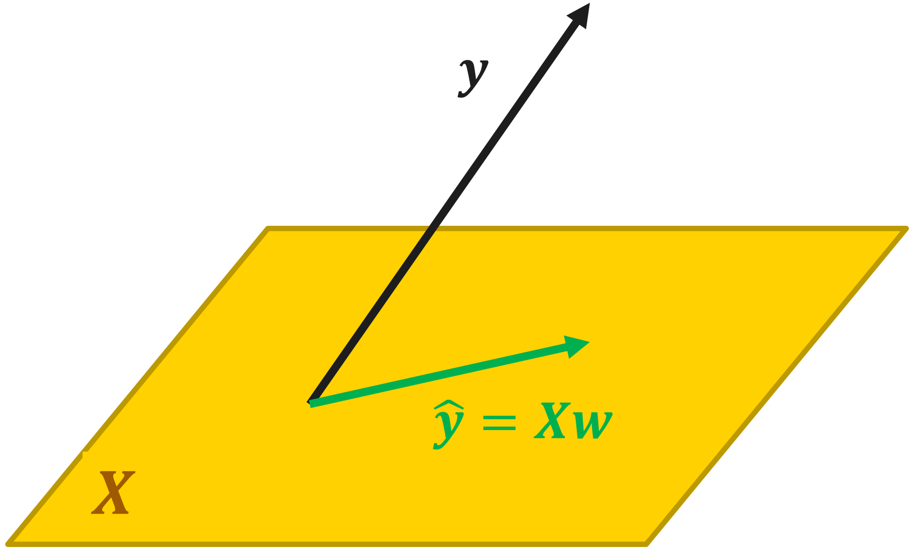

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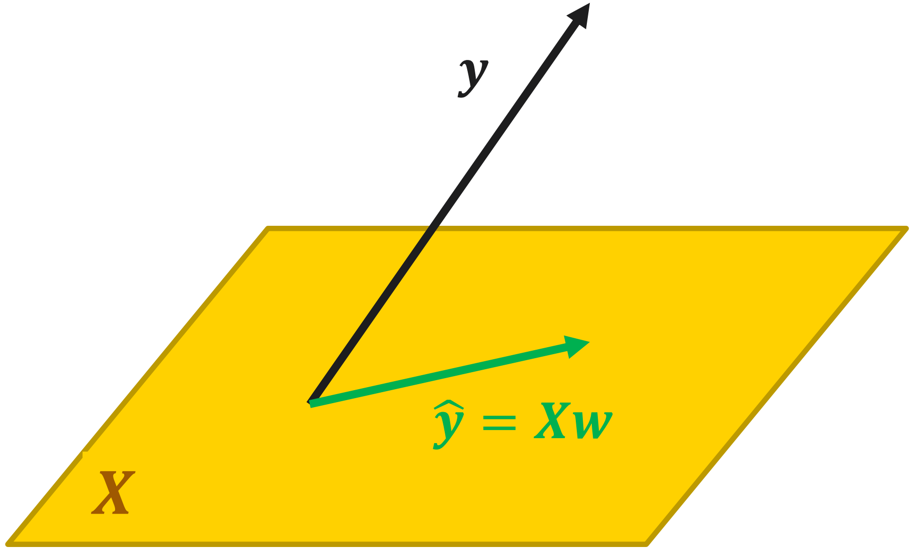

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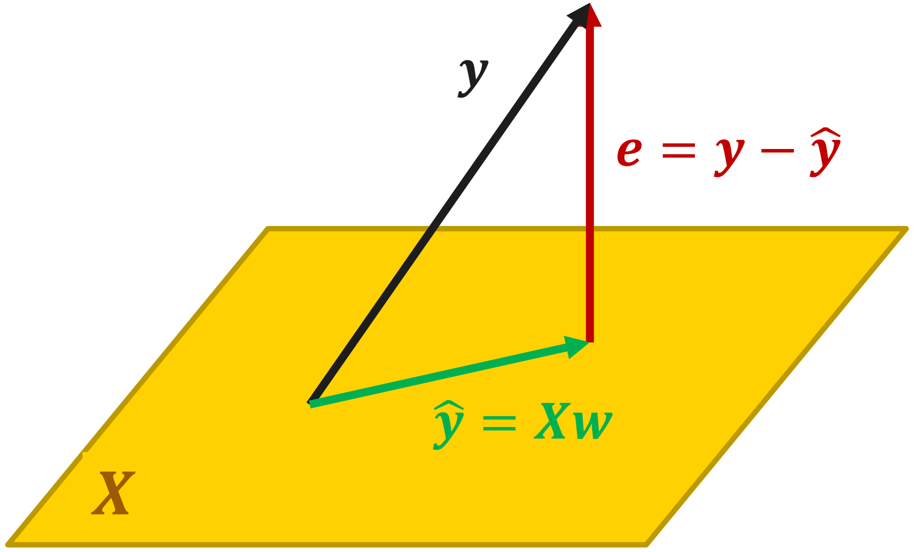

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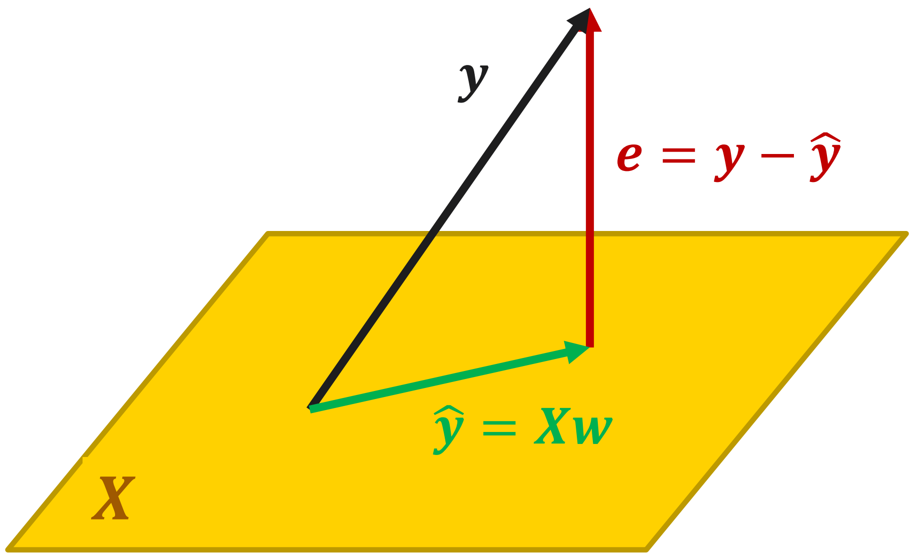

Orthogonal Projections

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- Example:

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- x_W is the closest vector to x in W.

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- Our goal: fit a hyperplane through the data $\left(\mathrm{x}^{i}, y\right)$.
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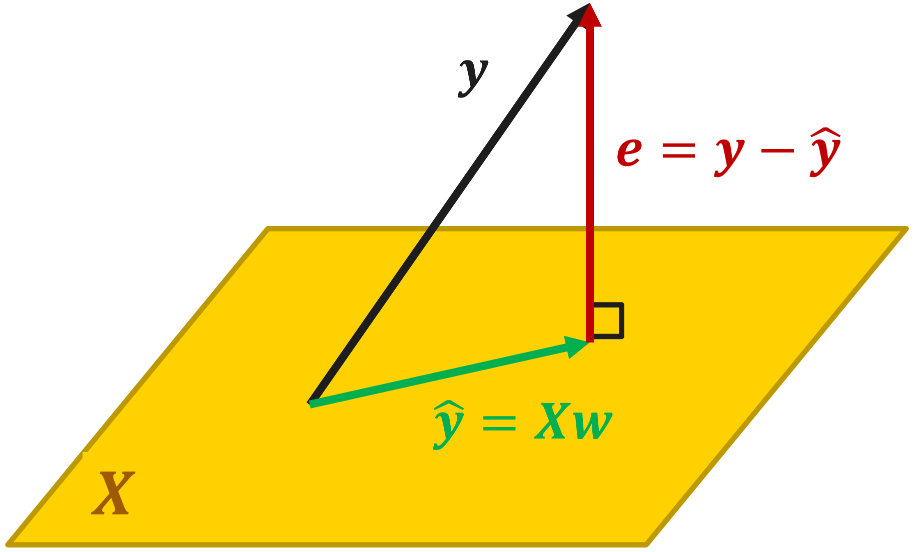

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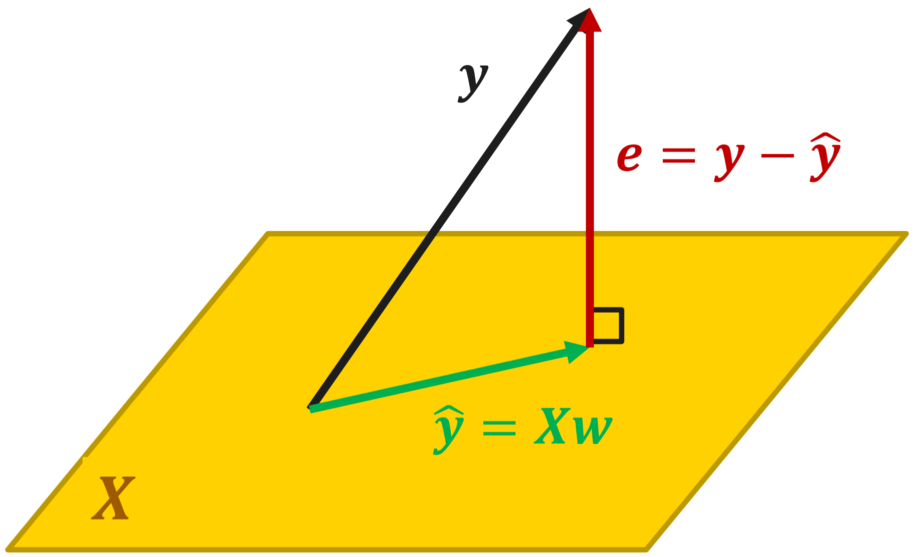

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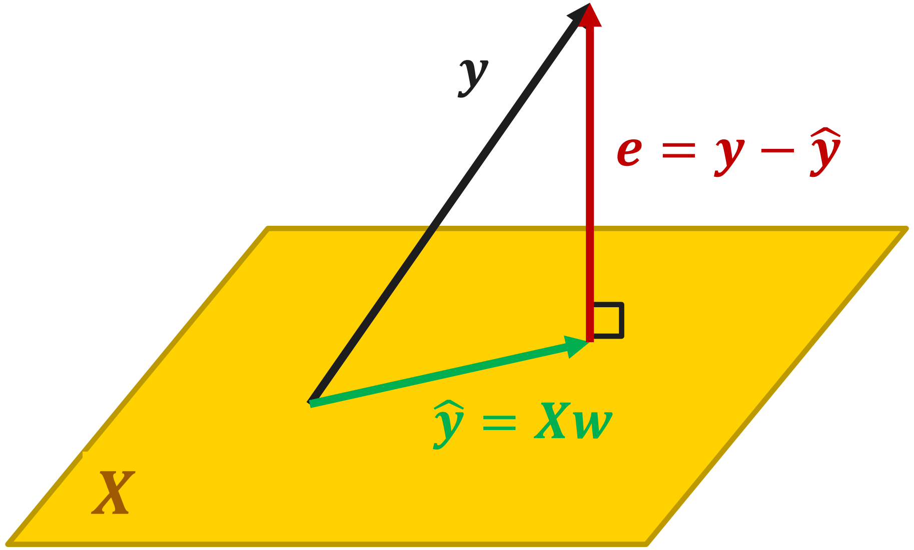

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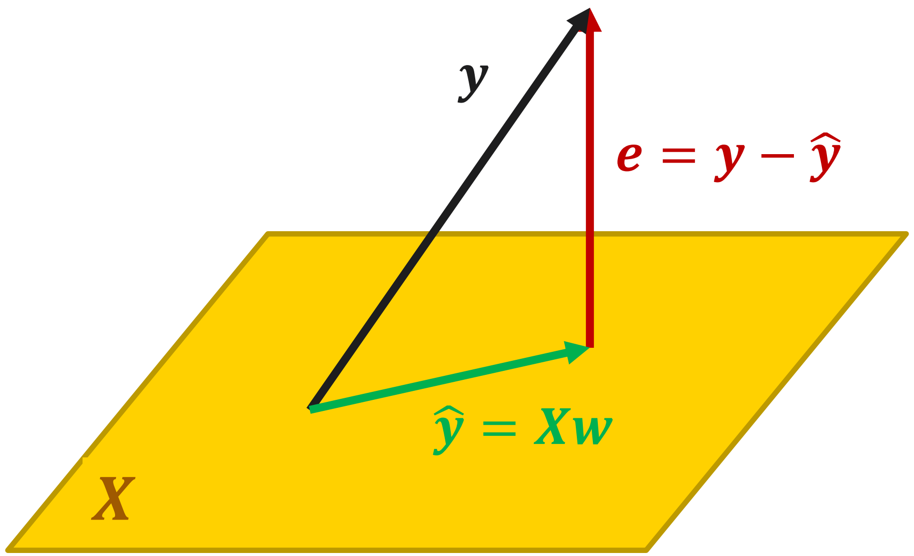
$w^* = ?$ – optimal weights



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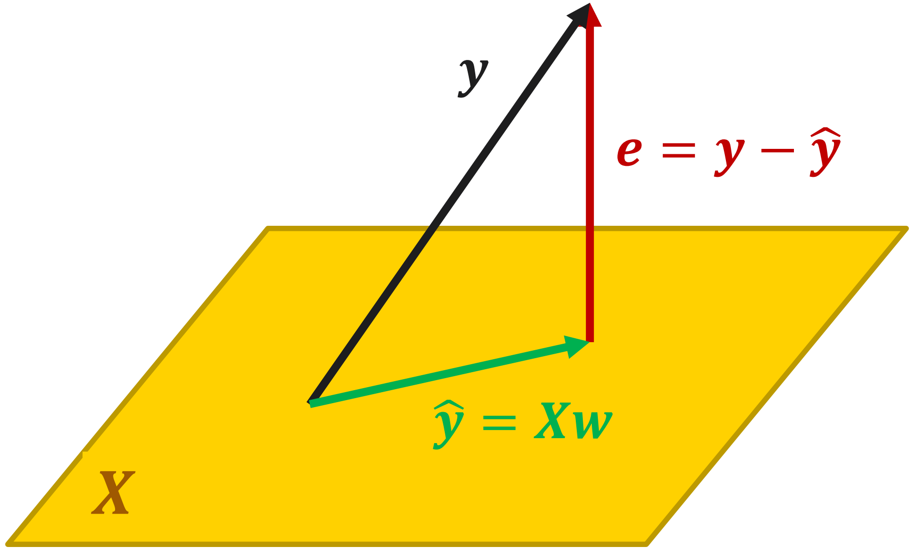

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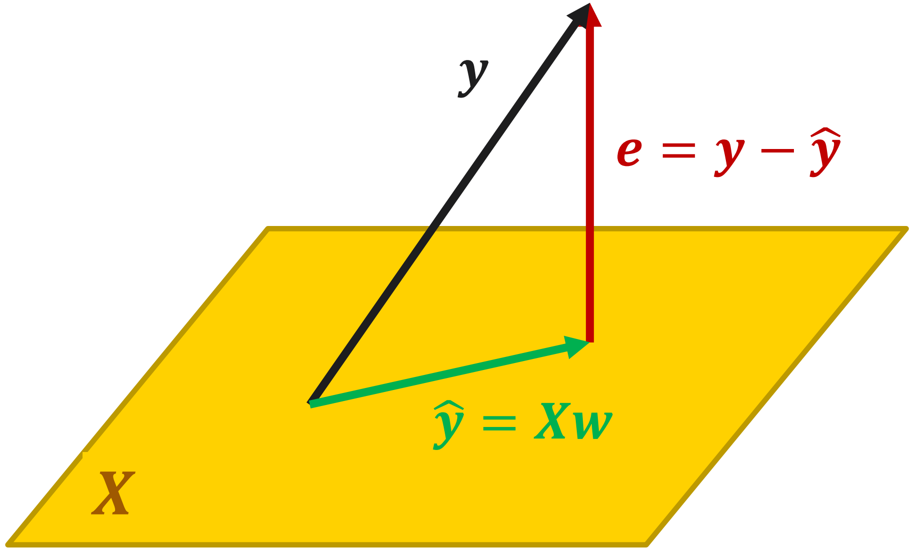

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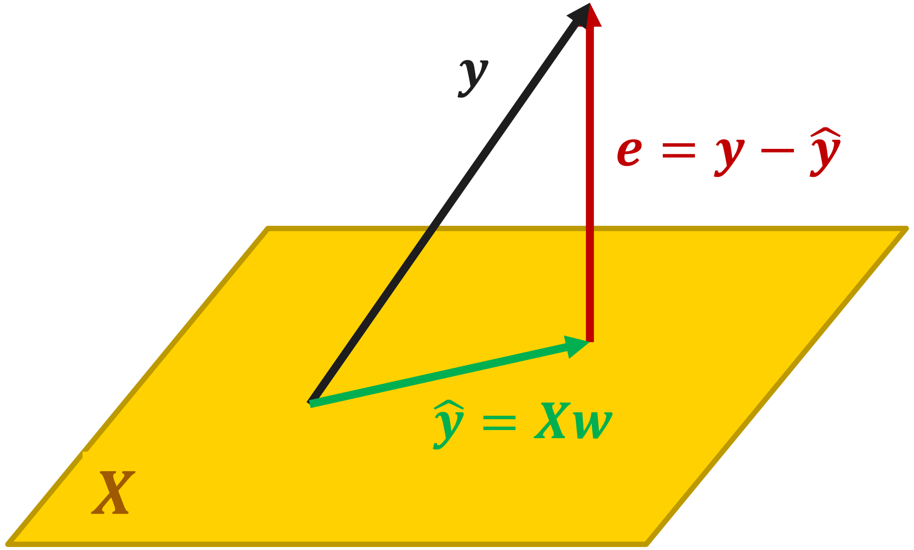

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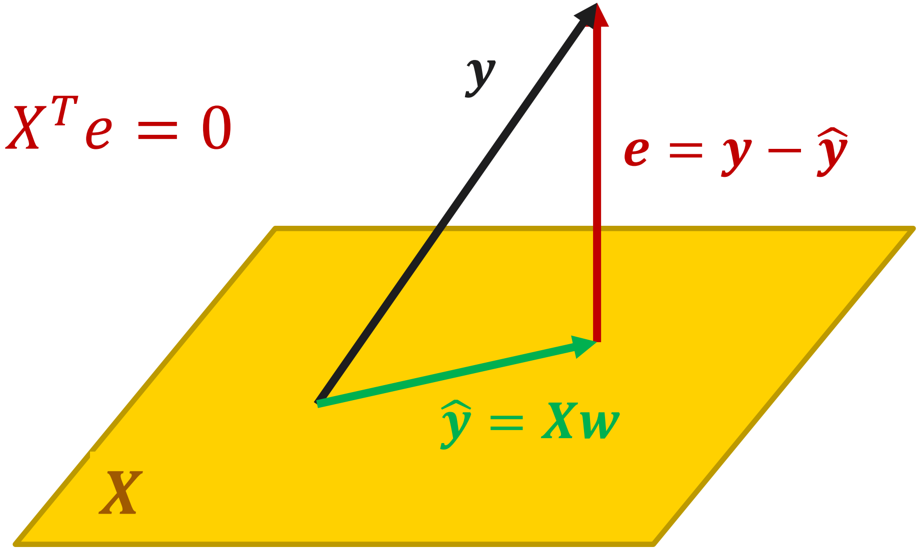

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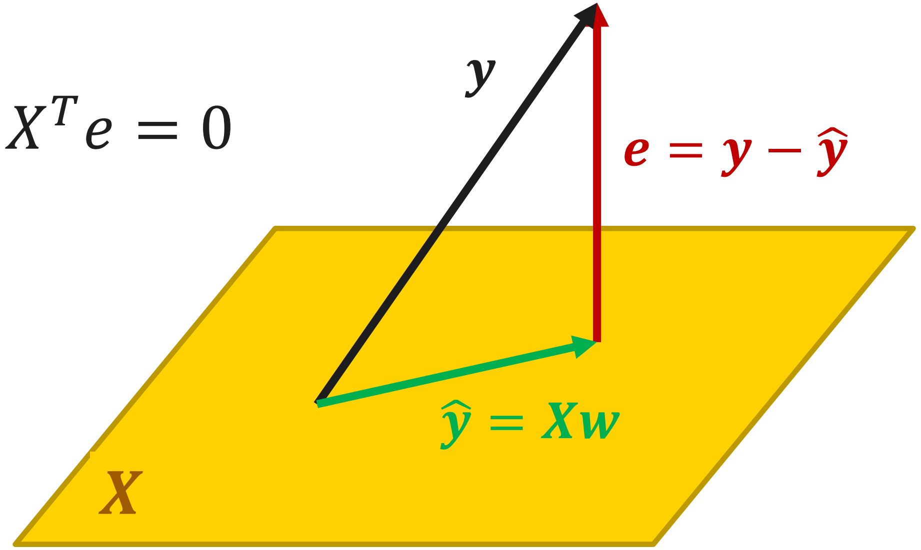

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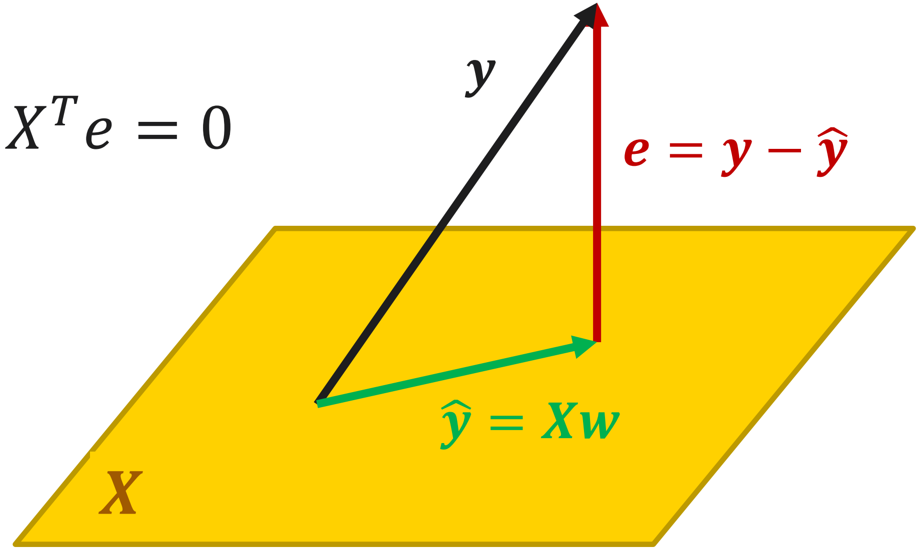

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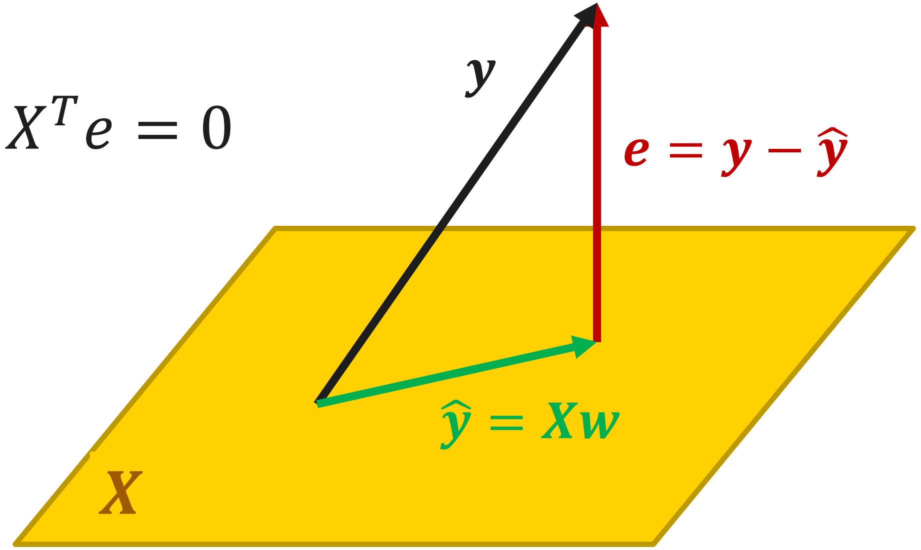

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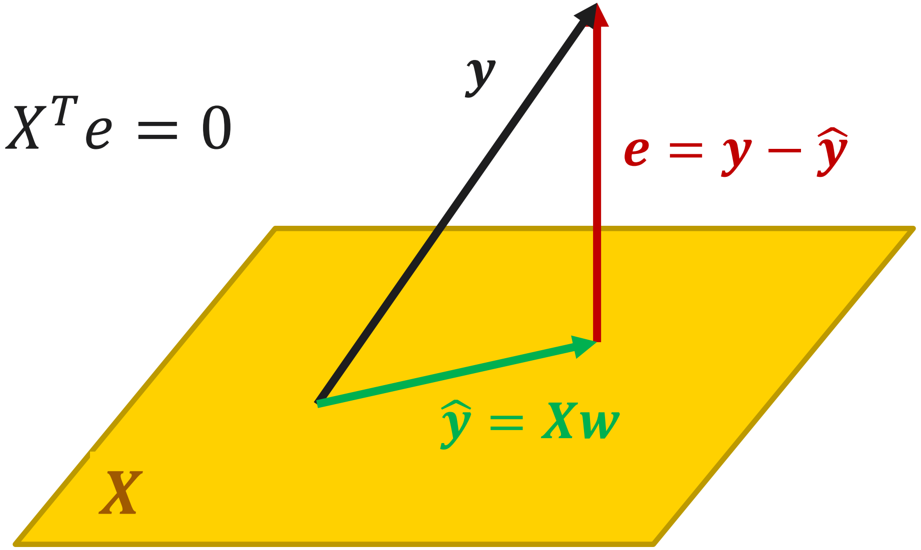

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Method of Least Squares

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$\hat{y}$ - orthogonal projection of y onto $col(X)$

$w^* = ?$ – optimal weights

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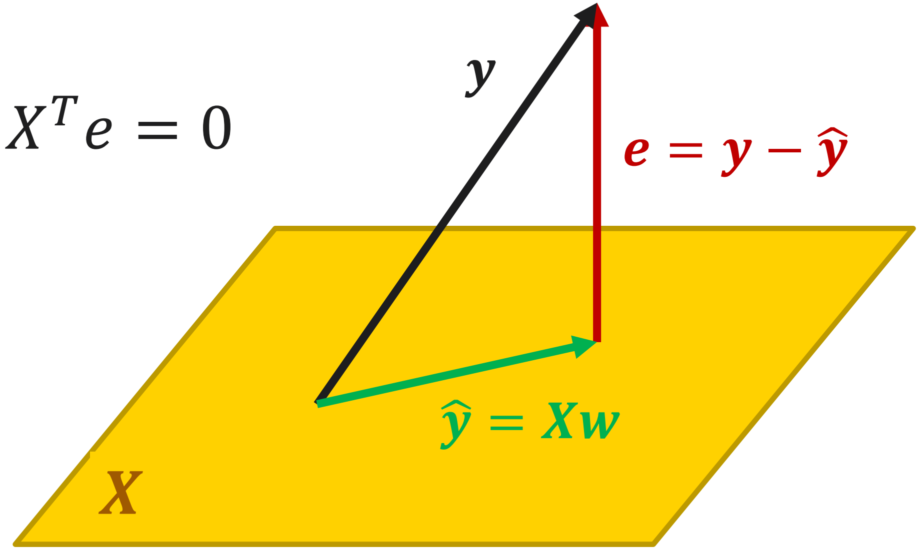

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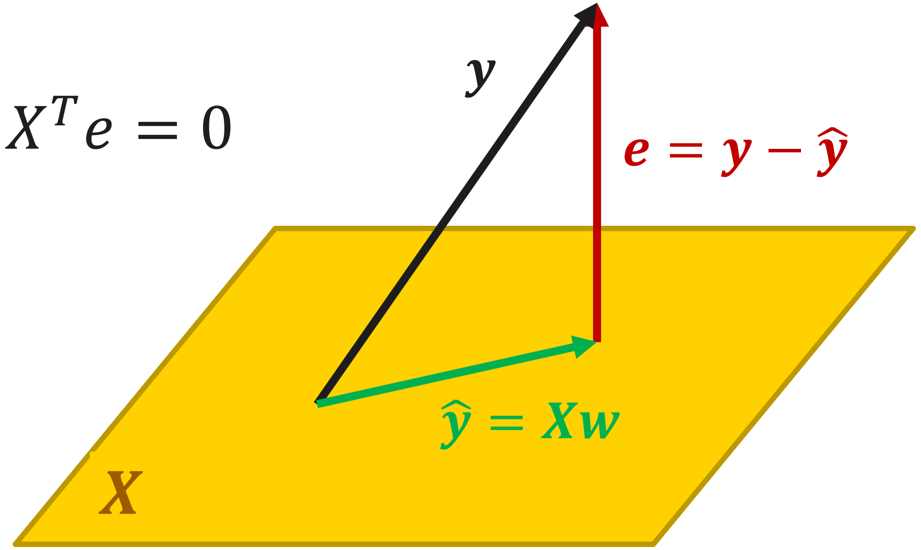

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So, $(X^T X)^{-1}$ exists.

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Toy Example

- Observations (x_i, y_i):

$$(1,1), \qquad (2,3), \qquad (3,2)$$

- With least squares, fit a line $y = w_0 + w_1 x$ through these points.

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$$w = (X^T X)^{-1} X^T y = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$ → regression line $y = 1 + 0.5x$.