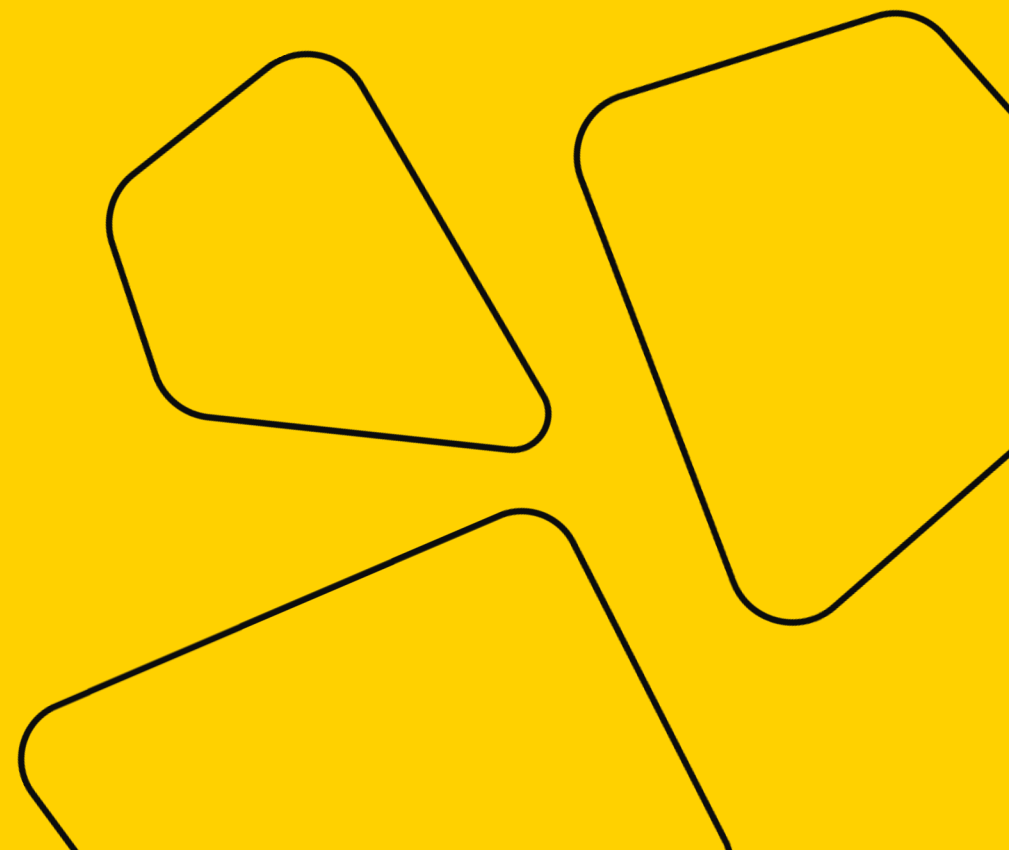




Math Refresher for DS

Lecture 4



Last Time

- Matrices as linear transforms
- More on matrices
 - Rank
 - Determinant
 - Row / Column space
- Solving SLE

Today

- Matrix decompositions
- Eigenvalues & eigenvectors

Matrix Decomposition

- Factorization

$$21 = 3 \times 7$$

Matrix Decomposition

- Factorization

$$21 = 3 \times 7$$

- Matrix factorization: represent a matrix as a product of matrices with specific properties.

LU- Decomposition



LU Decomposition

- A – $n \times n$ matrix.
- Represent A as a product of two matrices:

$$A = LU, \text{ where}$$

$$L = \begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix}$$

Reminder: Gaussian Elimination

$$A = \begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix}$$

Elementary row operations:

$$\begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix} \xrightarrow{(2) + 2 \cdot (1)} \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 3 & -4 & 4 \end{pmatrix} \xrightarrow{(3) + 1 \cdot (1)} \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & -2 & 3 \end{pmatrix} \xrightarrow{(3) - 1 \cdot (2)} \begin{pmatrix} 3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix} = U -$$

an upper triangular matrix.

Key idea: elementary row operations can be represented as matrix operations!

Elementary Matrices

- Elementary row operations can be represented as matrix operations.
- We'll use elementary matrices like

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \textcolor{red}{2} & 0 & 1 \end{pmatrix}$$

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- Let's take a close look at our Gaussian elimination example.

Gaussian Elimination as Matrix Mult.

$$(2)' = (2) + 2 \cdot (1)$$

$$A = \begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix} \sim \begin{pmatrix} -3 & 2 & -1 \\ 6 & -2 & 5 \\ 3 & -4 & 4 \end{pmatrix}$$

$$M_1 = \quad , \quad \begin{pmatrix} -3 & 2 & -1 \\ 6 & -2 & 5 \\ 3 & -4 & 4 \end{pmatrix} = M_1 A$$

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$$M_2 = \quad , \quad \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & -2 & 3 \end{pmatrix} = M_2 M_1 A$$

Gaussian Elimination as Matrix Mult.

$$(2)' = (2) + 2 \cdot (1) \quad (3)' = (3) + (1)$$
$$A = \begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix} \sim \begin{pmatrix} -3 & 2 & -1 \\ 6 & -2 & 5 \\ 3 & -4 & 4 \end{pmatrix} \sim \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & -2 & 3 \end{pmatrix}$$

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Gaussian Elimination as Matrix Mult.

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$(2)' = (2) + 2 \cdot (1)$ $(3)' = (3) + (1)$ $(3)' = (3) - (2)$

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$$M_3 M_2 M_1 A = U \Leftrightarrow A = (M_3 M_2 M_1)^{-1} U = (M_1^{-1} M_2^{-1} M_3^{-1}) U$$

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LU Decomposition: Application

- Solving SLE with different b s:

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$$Ax = b \Leftrightarrow (LU)x = b$$

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$$(1) Ly = b \Leftrightarrow \quad \rightarrow y^*$$

$$(2) Ux = y^* \Leftrightarrow \quad \rightarrow x^* - \text{solution to the original system.}$$

LU Decomposition: Application

- Solving SLE with different b s:

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Let $Ux = y$. Then

$$(1) \quad Ly = b \Leftrightarrow \begin{array}{l} l_{11}y_1 = b_1 \\ l_{21}y_1 + l_{22}y_2 = b_2 \\ \vdots \\ l_{n1}y_1 + l_{n2}y_2 + \cdots + l_{nn}y_n = b_n \end{array} \rightarrow y^*$$

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LU Decomposition: Application

- Solving SLE with different b s:

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$$(2) Ux = y^* \Leftrightarrow \begin{array}{l} u_{11}x_1 + u_{12}x_2 + \cdots + u_{1n}x_n = y_1^* \\ u_{22}x_2 + \cdots + u_{1n}x_n = y_2^* \\ \vdots \\ u_{nn}x_n = y_n^* \end{array} \rightarrow x^* - \text{solution to the original system.}$$

Eigenvalues & Eigenvectors



Eigenvectors and Eigenvalues

- Matrix A = some linear transformation.
- A changes vectors in V :

$$Ax = x'$$

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- For some vector $v \neq 0$ it might happen so that

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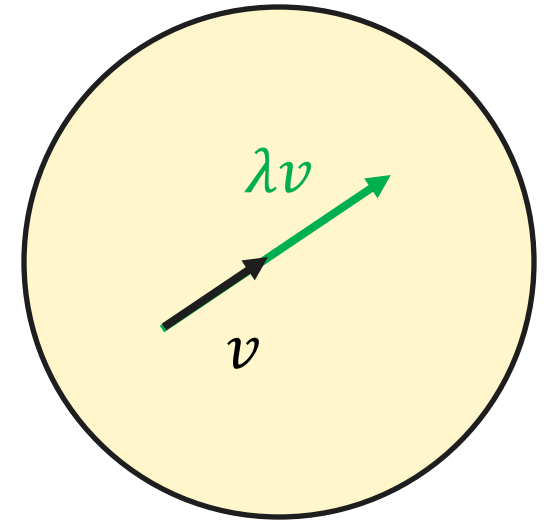
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Eigenvectors and Eigenvalues

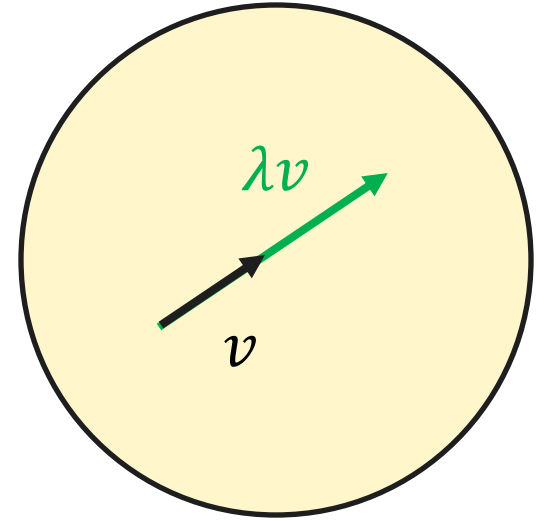
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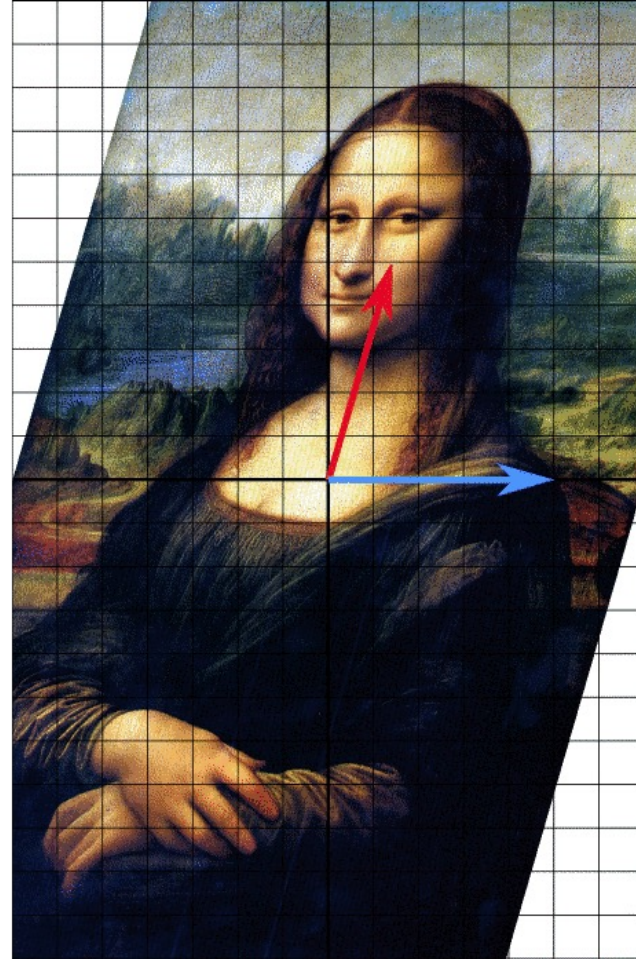
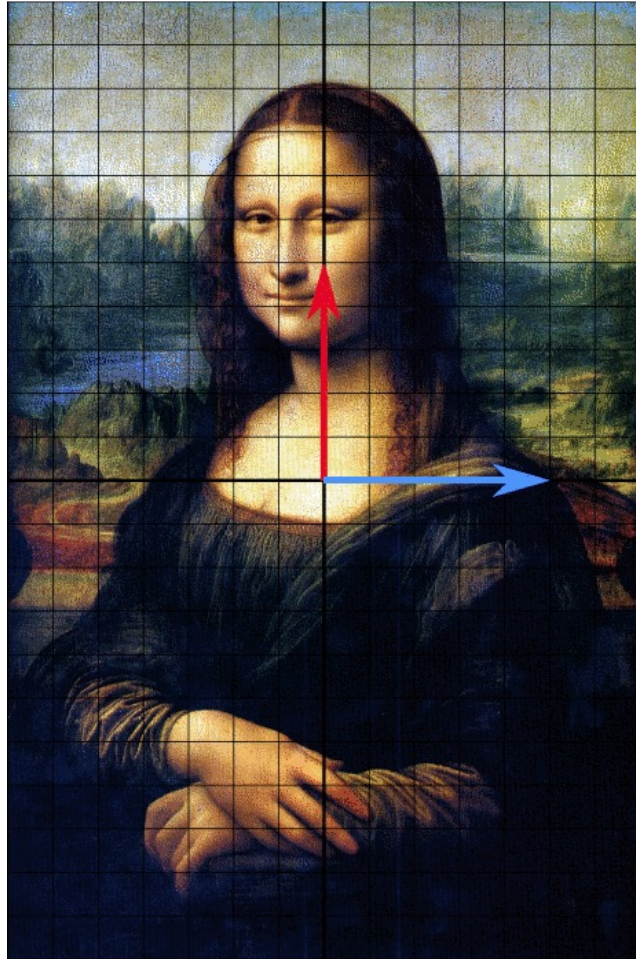
λ – eigenvalue, v – corresponding eigenvector



Eigenvectors and Eigenvalues

eigenvector = a vector that stays on its line after applying A
and only gets stretched by λ .

Eigenvectors and Eigenvalues



Source: [Wikipedia](#)

Eigenvectors and Eigenvalues: Example 1

- Consider rotation in 3D.

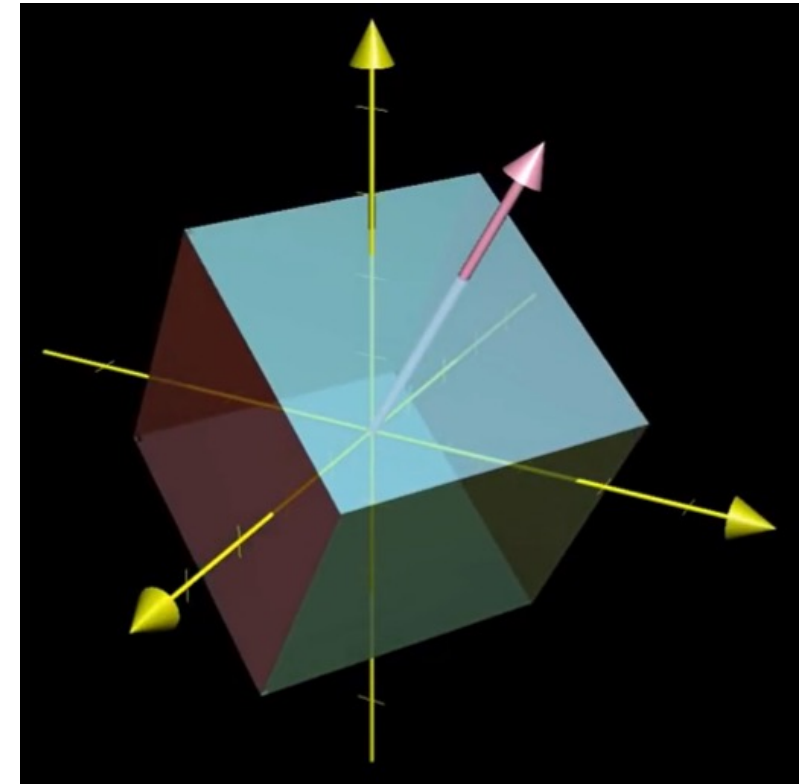


Image source:

<http://andrewmacthoughts.blogspot.com/2019/05/visualizing-linear-algebra-eigenvectors.html>

Eigenvectors and Eigenvalues: Example 1

- Consider rotation in 3D.
- Eigenvector = axis of the rotation.

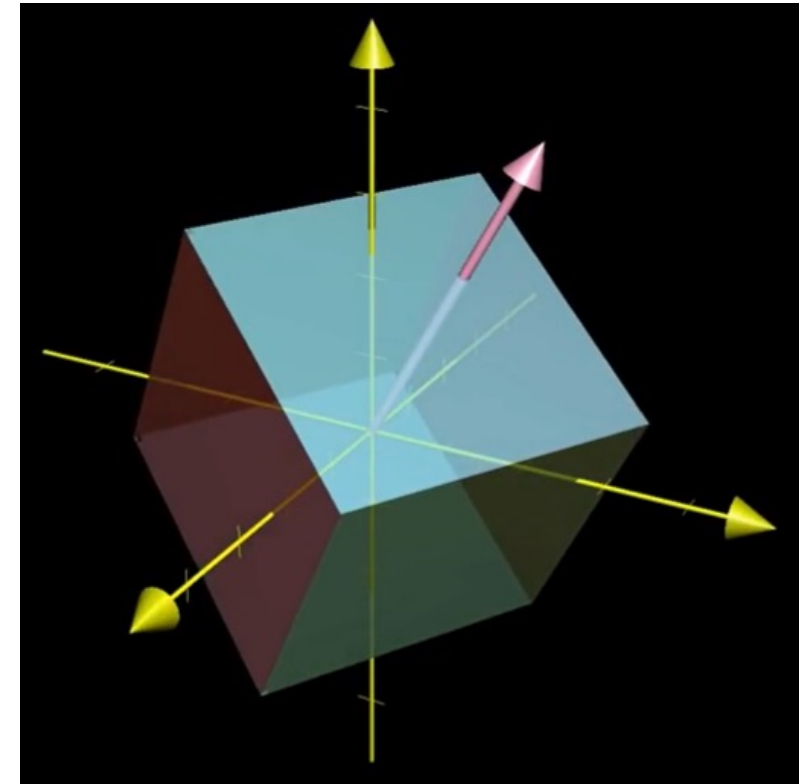


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Eigenvectors and Eigenvalues: Example 1

- Consider rotation in 3D.
- Eigenvector = axis of the rotation.
- Corresponding eigenvalue is 1 (rotation doesn't change lengths)

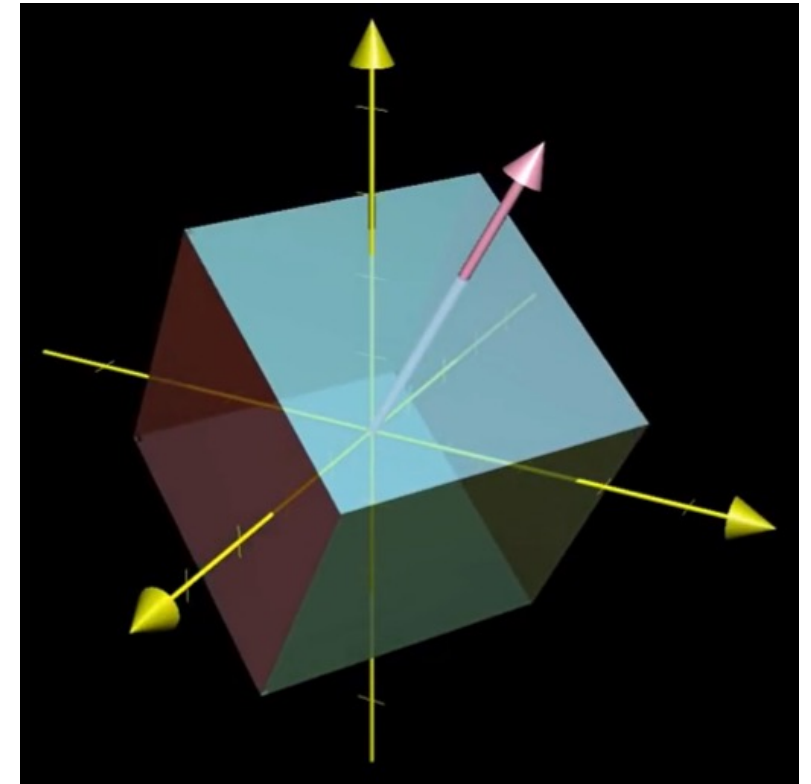


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Eigenvectors and Eigenvalues:

Example 2

- Consider a transformation $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$.

Eigenvectors and Eigenvalues:

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- Basis vector $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with $\lambda_1 = 3$
(see first column of A).

Eigenvectors and Eigenvalues:

Example 2

- Consider a transformation $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$.
- Basis vector $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with $\lambda_1 = 3$
(see first column of A).
- Vector $v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is also an eigenvector! Indeed:

$$Av = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \quad \lambda_2 = 2.$$

There Are Many Eigenvectors

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- If v is an eigenvector, αv as well.
- Example:

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \text{eigenvector with } \lambda = 3.$$

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$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \text{eigenvector with } \lambda = 3.$$

$$e'_1 = \begin{bmatrix} 10 \\ 0 \end{bmatrix} \text{ as well! Indeed: } \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 30 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

Finding Eigenvalues & Eigenvectors

Finding Eigenvalues

- If v is an eigenvector with the corresponding eigenvalue λ , then

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Finding Eigenvalues

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Since $v \neq 0$, this is only possible if and only if

$$\det(A - \lambda E) = 0$$

Finding Eigenvalues

- $v \neq 0$ is an eigenvector with the corresponding eigenvalue $\lambda \Leftrightarrow$

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- Eigenvalues = roots of the characteristic polynomial:

$$\det(A - \lambda E) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0 \Leftrightarrow$$

Polynomial of degree $n = n$ (possibly repeating) roots:

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$\{\lambda_1, \dots, \lambda_k\}$ – spectrum of A .

Finding Eigenvalues: Example 1

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

Characteristic polynomial:

$$\det(A - \lambda E) = \begin{vmatrix} 3 - \lambda & 1 \\ 0 & 2 - \lambda \end{vmatrix} = (3 - \lambda)(2 - \lambda) = 0$$

$$\lambda_1 = 3, \quad \lambda_2 = 2$$

Finding Eigenvectors: Example 1

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$$E_{\lambda_1} = \{v \in V \mid Av = \lambda_1 v\} = \text{span}\{v_1\}$$

Finding Eigenvectors: Example 1

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, \quad \lambda_1 = 3, \quad \lambda_2 = 2$$

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$$\begin{bmatrix} 3-2 & 1 \\ 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 3-2 & 1 \\ 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \mathbf{v}_2 = \begin{bmatrix} \beta \\ -\beta \end{bmatrix}, \text{ e.g. } \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\lambda_1 = 3, \quad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \dim E_{\lambda_1} = 1$$

$$\lambda_2 = 2, \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \dim E_{\lambda_2} = 1$$

Finding Eigenvalues: Example 2

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Characteristic polynomial:

$$\det(A - \lambda E) = \begin{vmatrix} 1 - \lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^2 = 0$$

$$\lambda_{1,2} = \lambda = 1$$

Finding Eigenvalues: Example 2

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \lambda = 1$$

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Finding Eigenvalues: Example 2

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\lambda = 1, \quad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \dim E_\lambda = 2$$

Finding Eigenvalues: Example 3

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Characteristic polynomial:

$$\det(A - \lambda E) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0$$

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Algebraic multiplicity = 2

$$E_\lambda = \{v \in V \mid Av = \lambda v\} = \text{span}\{v_1\}, \quad \dim E_\lambda = 1$$

Geometric multiplicity = 1

λ – degenerate eigenvalue

Useful Properties

- A – $n \times n$ matrix, $\lambda_1, \dots, \lambda_k$ – eigenvalues.
 - **$\det A = \lambda_1 \cdot \dots \cdot \lambda_k$**

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 - $\text{tr } A = a_{11} + a_{22} + \dots + a_{nn} = \lambda_1 + \dots + \lambda_k$

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(1). A is invertible $\Leftrightarrow \lambda_i \neq 0, i = 1, \dots, k$:

Indeed, A is invertible $\Leftrightarrow 0 \neq \det A = \lambda_1 \cdot \dots \cdot \lambda_k$

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Indeed, A is invertible $\Leftrightarrow 0 \neq \det A = \lambda_1 \cdot \dots \cdot \lambda_k$

(2). A^{-1} has eigenvalues $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_k}$.

(Eigenvectors of A and A^{-1} are the same:

$$Av_i = \lambda_i v_i \Leftrightarrow v_i = \lambda_i A^{-1} v_i \Leftrightarrow \frac{1}{\lambda_i} v_i = A^{-1} v_i$$

Eigen- decomposition



Eigenbasis

- A – $n \times n$ matrix.
- Suppose that A has n linearly independent eigenvectors v_1, \dots, v_n .

$\{v_1, \dots, v_n\}$ – eigenbasis.

Eigendecomposition

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$$= \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} = \Lambda - \text{a diagonal matrix.}$$

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$[A]_V = \Lambda$ – a diagonal matrix with $d_{ii} = \lambda_i$

$A = V\Lambda V^{-1}$ – eigendecomposition of A .

Matrix Diagonalization



Diagonalizable Matrix

- A – $n \times n$ matrix
- v_1, \dots, v_n - linearly independent eigenvectors
- $\lambda_1, \dots, \lambda_n$ – corresponding eigenvalues

Eigendecomposition of A : $A = V\Lambda V^{-1}$ –

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\Leftrightarrow

Diagonalization of A : $\Lambda = V^{-1}AV$

Matrix Diagonalization: Example

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$$\Lambda = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Diagonalizable Matrix

- But now all matrices have n linearly independent eigenvectors.
- Example (see beginning of the lecture):

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \lambda_{1,2} = \lambda = 0, \quad v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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So, when is a matrix diagonalizable?

The Spectral Theorem



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 3. v_1, \dots, v_n are orthogonal (we can chose orthonormal).

Orthogonal Matrices

- A – $n \times n$ matrix
- A is orthogonal if its columns are mutually orthonormal:

$$A^T A = A A^T = E$$

Orthogonal Matrices

- Suppose that A is orthogonal.
- Orthogonal vectors are linearly independent $\rightarrow A$ is a full rank matrix. So, A has an inverse!

A is orthogonal \Leftrightarrow

$$A^T A = A A^T = E \Leftrightarrow$$

$$A^{-1} = A^T.$$

The Spectral Theorem

In other words, if A is a real symmetric matrix,
 A is orthogonally diagonalizable:

$$\Lambda = V^{-1}AV = V^TAV$$

where Λ is a diagonal matrix and V is an orthogonal matrix.

Power of a Matrix



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$$\Lambda^n = \begin{bmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_m^n \end{bmatrix}$$

Principle Component Analysis



PCA

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$$\mathbf{x}^j = \begin{bmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{mj} \end{bmatrix} - \text{observations for example } j, \mathbf{X} = [\mathbf{x}^1 \mid \dots \mid \mathbf{x}^n] - m \times n \text{ data matrix.}$$

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 - is there a way to visualize the data?
 - which features are the most significant in describing the full dataset?

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- Key idea: project your $m \times n$ data onto a p -dimensional subspace ($p < m$) in such a way that we preserve as much variance in our data as possible.

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How to find B ?

Turns out we should project on the p eigenvectors of the data covariance matrix that correspond to p largest eigenvalues!

PCA

- (*Probability Theory*) Covariance between two random variables = measure of the joint variability.
- We have a dataset X :

$$X = [\mathbf{x}^1 \mid \dots \mid \mathbf{x}^n], \quad \mathbf{x}^j = \begin{bmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{mj} \end{bmatrix} \text{ -- observations for example } j$$

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Assuming that X is centered, otherwise we should center it first.

- (*Statistics*) Sample covariance matrix: $S = \frac{1}{n-1}XX^T$

s_{ij} , $i \neq j$ – sample covariance between features i and j ,

s_{ii} – sample variance of feature i .

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$$\frac{1}{n-1}XX^T = S = V\Lambda V^{-1} = V\Lambda V^T$$

$V = [v_1 \mid \dots \mid v_m]$ – eigenvectors of S , Λ – diagonal matrix with λ_i .

PCA

- Let's order eigenvalues and eigenvectors so that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$.

$$S = V\Lambda V^{-1} = V\Lambda V^T$$

$$\begin{bmatrix} \mathbf{s_{11}} & s_{12} & \cdots & s_{1m} \\ s_{21} & \mathbf{s_{22}} & \cdots & s_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ s_{n1} & s_{n2} & \cdots & \mathbf{s_{mm}} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1m} \\ v_{21} & v_{22} & \cdots & v_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{mm} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1m} \\ v_{21} & v_{22} & \cdots & v_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{mm} \end{bmatrix}^T$$

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Total variance of the data $T = tr(S) = s_{11} + \dots + s_{nn}$

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Orthogonal eigenvectors v_1, \dots, v_n – principal components of the data

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Orthogonal eigenvectors v_1, \dots, v_n – principal components of the data

Direction of v_i describes λ_i out of the total variance T .

To Sum Up

- Eigenvalues and eigenvectors
- Matrix factorization
 - LU
 - Eigendecomposition
 - Diagonalization