Math Refresher for DS

Practical Session 2

girafe ai

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, $b_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $b_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$ are linearly independent.

• Therefore, $B = \{b_1, b_2, b_3\}$ - basis in \mathbb{R}^3 .



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• How do we go from the standard basis
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$$x_E = [6, 9, 14], \quad x_B = [x_1, x_2, x_3] = ?$$



Change of Coordinates

• There was a small typo in the lecture!



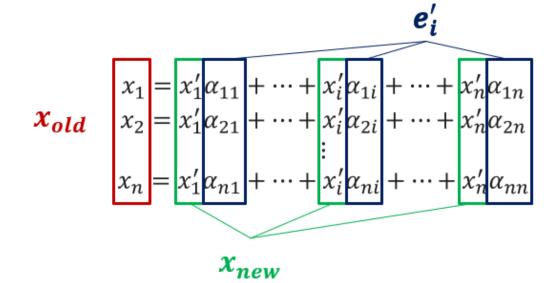
Coordinate Change: Matrix Notation



Result obtained before:

$$e_1, ..., e_n$$
 - old basis $e'_1, ..., e'_n$ - new basis

$$x_{old} = [x_1, ..., x_n], \qquad x_{new} = [x'_1, ..., x'_n]$$



 Transition matrix: columns = coordinates of the new basis in the old one.

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{21} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix}$$

$$x_{old} = A^T x_{new}$$

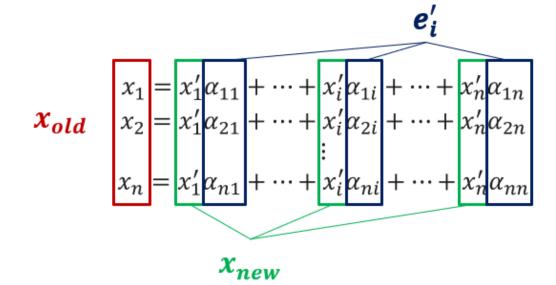
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$$x_{old} = A^{\mathsf{Y}} x_{new}$$

Coordinate Change: Example (again)

- Consider \mathbb{R}^2 with basis $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- New basis: $e'_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $e'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$
- $x_{old} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $x_{new} = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = ?$

$$A = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ -1 \end{bmatrix} = x_{old} = A^T x_{new} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

$$x_{new} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$
.



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 - basis.

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$$x_E = [6, 9, 14]^T$$
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$$A_{E \to B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$



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$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_1 + x_2 + 2x_3 = 9 \Leftrightarrow \\ x_1 + 2x_2 + 3x_3 = 14 \end{cases}$$



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$$\begin{bmatrix} 6 \\ 9 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_{1} \\ x_{2} \\ x_{3} \end{bmatrix} = \begin{bmatrix} x_{1} + x_{2} + x_{3} \\ x_{1} + x_{2} + 2x_{3} \\ x_{1} + 2x_{2} + 3x_{3} \end{bmatrix}$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_1 + x_2 + 2x_3 = 9 \\ x_1 + 2x_2 + 3x_3 = 14 \end{cases} \Leftrightarrow \begin{cases} (2) - (1) \\ x_3 = 3 \\ (2) - (1) \\ (3) = 3 \end{cases} \Leftrightarrow$$



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- Some are more convenient than the other ones.



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- Some are more convenient than the other ones.
- Orthonormal basis = all vectors are pairwise orthogonal $((e_i, e_j) = 0)$ + of unit length $(||e_i|| = 1)$.
- Any basis can be transformed into orthonormal basis!



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Let's construct an orthogonal basis $V = \{v_1, v_2, v_3\}$ from it.

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2. Let's look for v_2 of the form $v_2 \coloneqq b_2 + \alpha v_1$, $\alpha \in \mathbb{R}$.



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$$0 = (v_1, v_2) = (v_1, b_2 + \alpha v_1) = (v_1, b_2) + \alpha(v_1, v_1) \Leftrightarrow$$



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$$v_2 = b_2 - \frac{(v_1, b_2)}{(v_1, v_1)} v_1, \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{1+1+2}{1+1+1} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}.$$



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$$v_3 = b_3 + \alpha_1 v_1 + \alpha_2 v_2 = b_3 - \frac{(v_1, b_3)}{(v_1, v_1)} v_1 - \frac{(v_2, b_3)}{(v_2, v_2)} v_2$$

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$$v_3 = b_3 - \frac{(v_1, b_3)}{(v_1, v_1)} v_1 - \frac{(v_2, b_3)}{(v_2, v_2)} v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{1 + 2 + 3}{1 + 1 + 1} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-\frac{1}{3} - \frac{2}{3} + 2}{\frac{1}{9} + \frac{4}{9}} \cdot \begin{bmatrix} -\frac{1}{3} - \frac{2}{3} + 2 \\ -\frac{1}{3} - \frac{2}{3} + 2 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}.$$



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$$B = \left\{b_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, b_2 = \begin{bmatrix}1\\1\\2\end{bmatrix}, b_3 = \begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$$
 - basis.

Orthogonal basis $V = \{v_1, v_2, v_3\}$ from B:

$$v_1 \coloneqq b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}$$



Gram-Schmidt Process: General Case

- Some basis $B = \{b_1, ..., b_n\}$.
- Constructing orthogonal basis $V = \{v_1, ..., v_n\}, (v_i, v_j) = 0$:

$$v_1 = b_1$$

$$v_2 = b_2 - \frac{(v_1, b_2)}{(v_1, v_1)} v_1$$

$$\vdots$$

$$v_k = b_k - \frac{(v_1, b_k)}{(v_1, v_1)} v_1 - \frac{(v_2, b_k)}{(v_2, v_2)} v_2 - \dots - \frac{(v_{k-1}, b_k)}{(v_{k-1}, v_{k-1})} v_{k-1}$$

• If we additionally normalize v_i , we get orthonormal basis.

