Math Refresher for DS

Practical Session 6

Plan for Today

- Logistics
- Short Quiz
- PCA: final overview
- Intro to SVD

Logistics for Next Weeks

- No class on Wednesday

- Tomorrow:
  - Submission deadline Graded Assignment 2
  - Graded Assignment 3 will be out

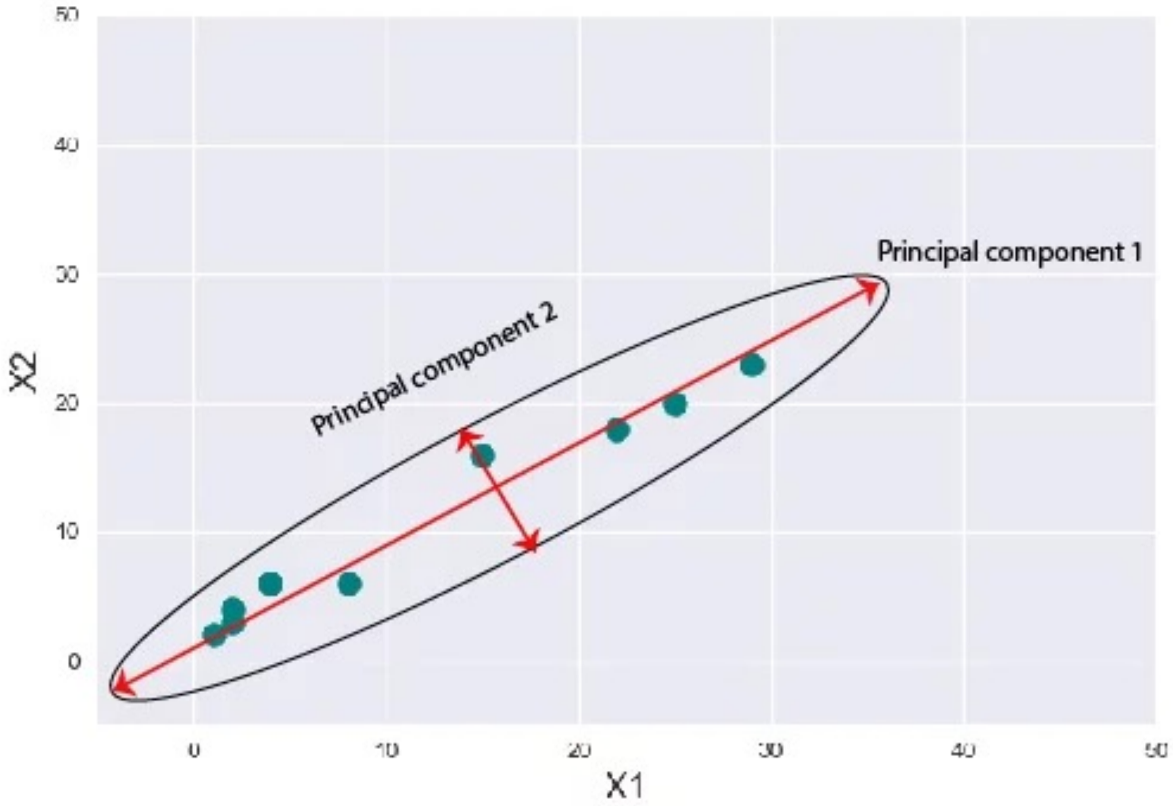
- Next Monday: starting Calculus

- Single final exam (still 2 parts)

Short Quiz

https://forms.gle/ohsrDkAJPCwDTZGaA

PCA STEP-BY-STEP

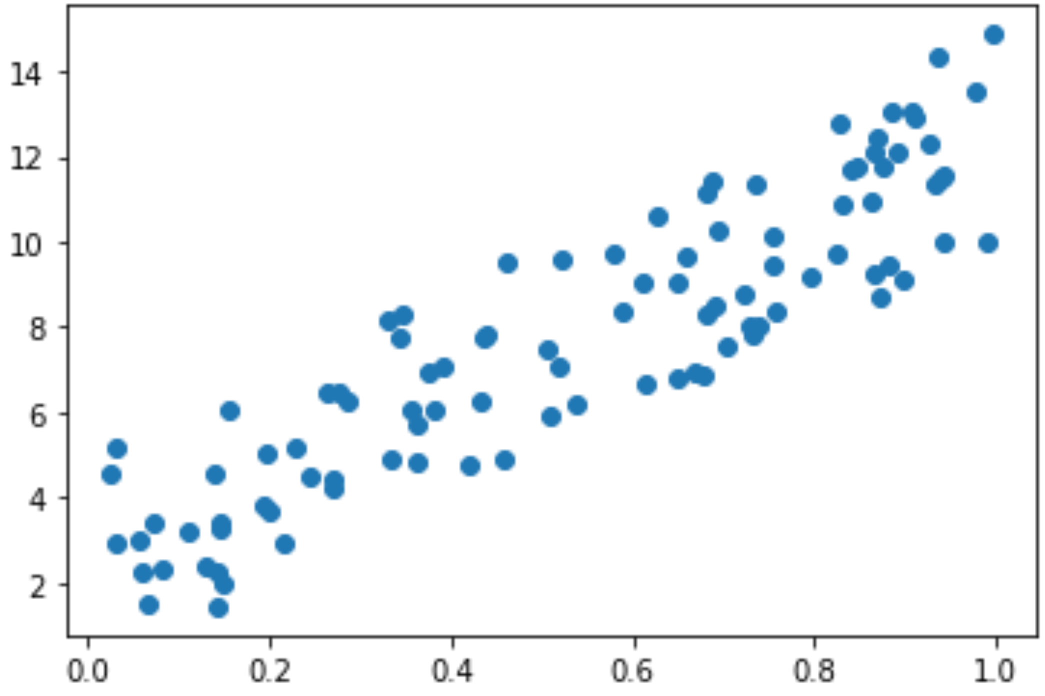


PCA STEP 1: COLLECT THE DATA

- $X - m{\times}n$ matrix.
  m features, n examples.

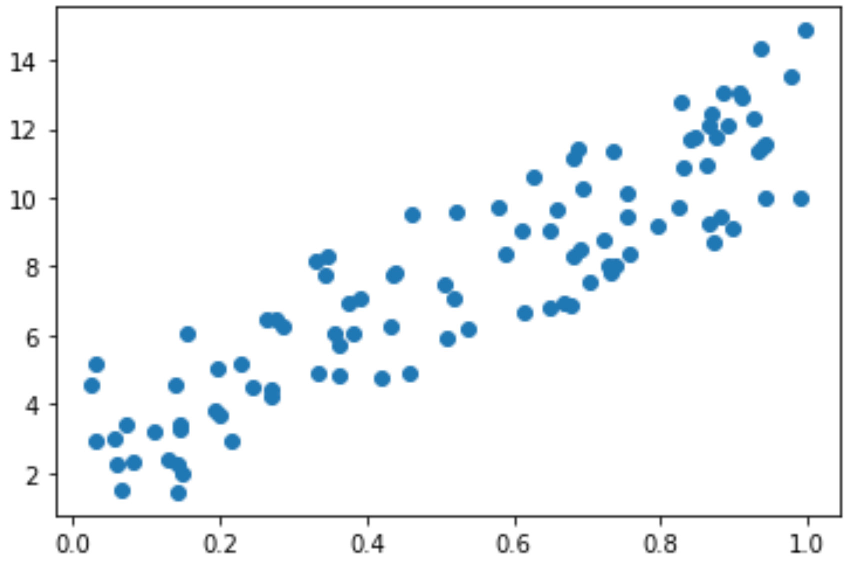
- Columns = examples,
  rows = features.

- Each example = a point
  in an m –dimensional space.



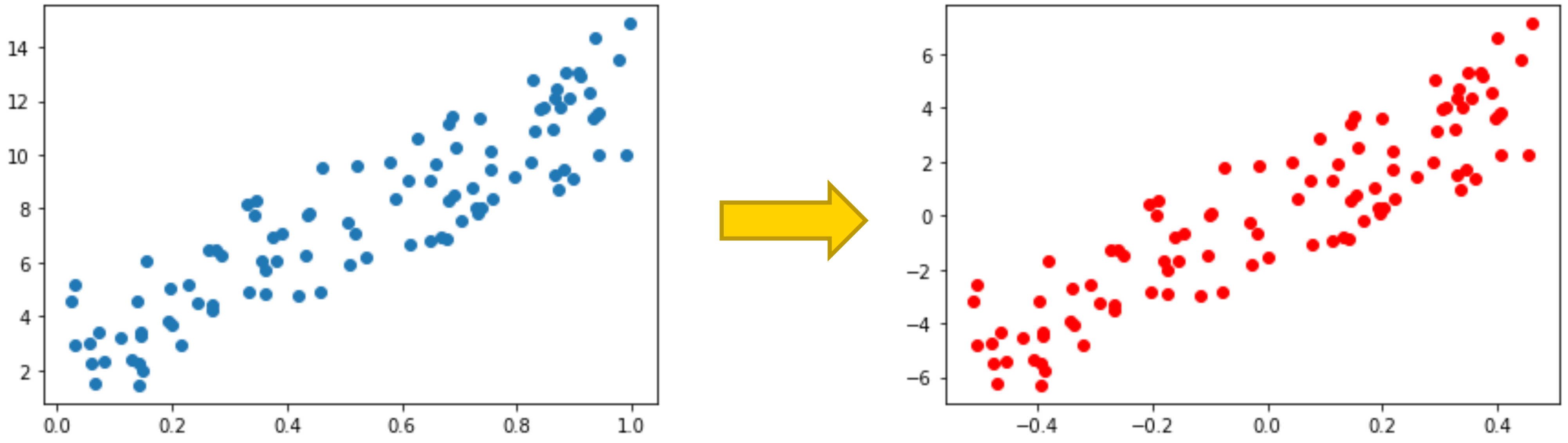
PCA STEP 2: CENTER THE DATA

- (This is needed to construct the covariance matrix)
- From each feature *(row)*, subtract its mean.



PCA STEP 2: CENTER THE DATA

- (This is needed to construct the covariance matrix)
- From each feature *(row)*, subtract its mean.

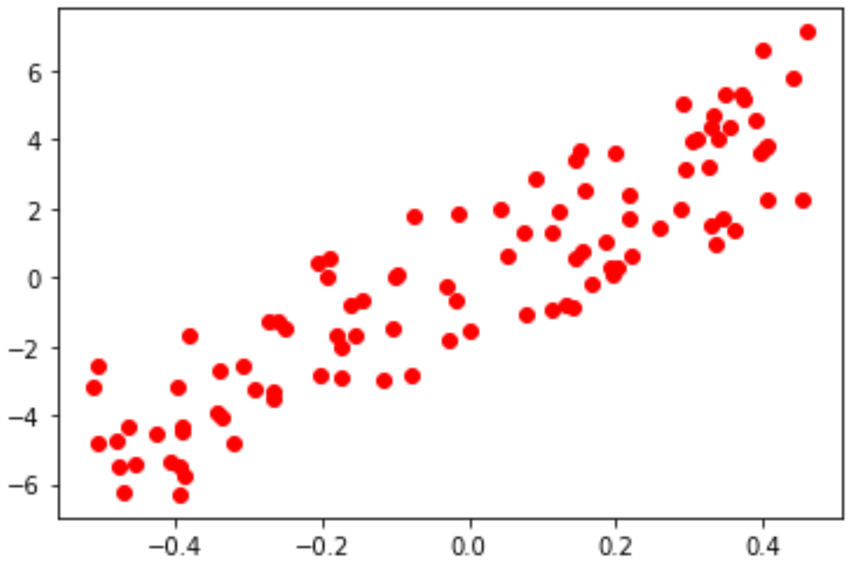


girafe ai

PCA STEP 3: BUILD COVARIANCE MATRIX S

- Sample covariance matrix with centered X:

$$S = \frac{1}{n-1} X X^t - n \times n \text{ matrix.}$$



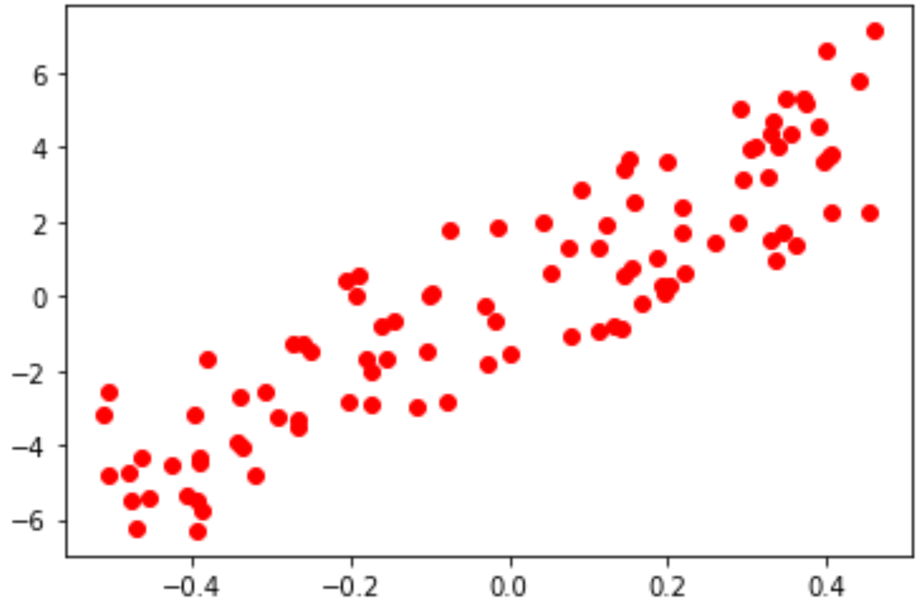

```
(1./(n-1))*np.matmul(X_centered, X_centered.transpose())
```

```
array([[ 0.08830318,  0.88570231],
       [ 0.88570231, 10.93980675]])
```

PCA STEP 4: DECOMPOSE S

- S is a symmetric matrix, so it has an eigendecomposition:

$$S = V \Lambda V^T$$



PCA STEP 4: DECOMPOSE S

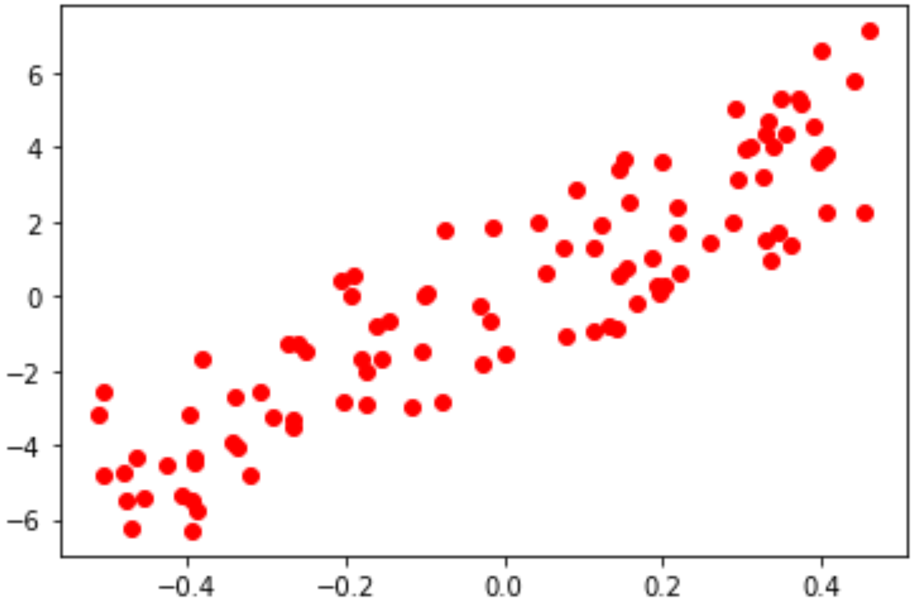
- S is a symmetric matrix, so it has an eigendecomposition:

$$S = V \Lambda V^T$$

- Eigenvalues of S (ordered from large to small):

```
l, V = np.linalg.eig(S)
ind = np.argsort(l)[::-1]
l[ind]
```

```
array([11.0116227 ,  0.01648723])
```



PCA STEP 4: DECOMPOSE S

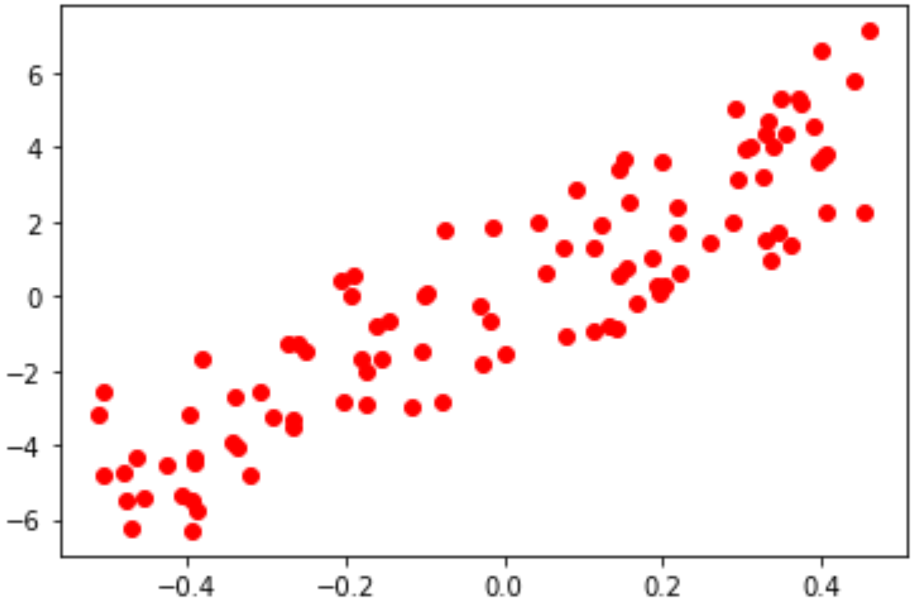
- S is a symmetric matrix, so it has an eigendecomposition:

$$S = V\Lambda V^T$$

- Eigenvalues of S (ordered from large to small):

```
l, V = np.linalg.eig(S)
ind = np.argsort(l)[::-1]
l[ind]
```

```
array([11.0116227 ,  0.01648723])
```



- Corresponding eigenvectors of S
(= principal components of the data):

```
V[:,ind]
```

```
array([[-0.08081839, -0.99672884],
       [-0.99672884,  0.08081839]])
```

girafe ai

PCA STEP 5: PROJECT THE DATA

- Let’s project the data onto the first principal component.
- We know orthogonal projections from the first lecture!

PCA STEP 5: PROJECT THE DATA

- Let's project the data onto the first principal component.
- We know orthogonal projections from the first lecture!

Projecting a single example (= one column):

$$x_{v_1} =$$

PCA STEP 5: PROJECT THE DATA

- Let's project the data onto the first principal component.
- We know orthogonal projections from the first lecture!

Projecting a single example (= one column):

$$x_{v_1} = \frac{(x, v_1)}{(v_1 . v_1)} v_1 =$$

PCA STEP 5: PROJECT THE DATA

- Let's project the data onto the first principal component.
- We know orthogonal projections from the first lecture!

Projecting a single example (= one column):

$$x_{v_1} = \frac{(x, v_1)}{(v_1 . v_1)} v_1 = (x, v_1) \cdot v_1$$

girafe ai

PCA STEP 5: PROJECT THE DATA

- Let's project the data onto the first principal component.
- We know orthogonal projections from the first lecture!

Projecting a single example (= one column):

$$x_{v_1} = \frac{(x, v_1)}{(v_1 . v_1)} v_1 = (x, v_1) \cdot v_1$$

Projecting the whole data matrix (= all columns):

$$X_{v_1} =$$

PCA STEP 5: PROJECT THE DATA

- Let's project the data onto the first principal component.
- We know orthogonal projections from the first lecture!

Projecting a single example (= one column):

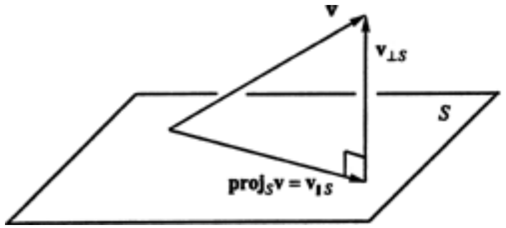
$$x_{v_1} = \frac{(x, v_1)}{(v_1 . v_1)} v_1 = (x, v_1) \cdot v_1$$

Projecting the whole data matrix (= all columns):

$$X_{v_1} = v_1^T X$$

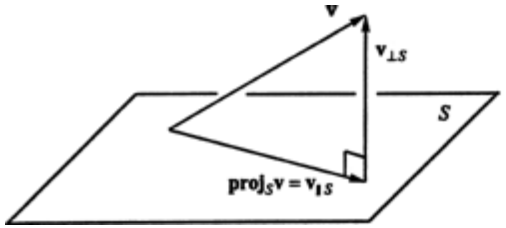
girafe ai

PCA STEP 5: PROJECT THE DATA



- Let's project the data onto p principal components.
- How to project onto a subspace spanned by these p principal components (= its orthonormal basis)?

PCA STEP 5: PROJECT THE DATA

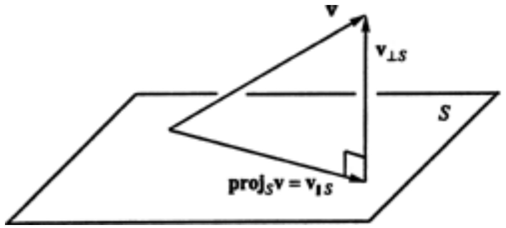


- Let's project the data onto p principal components.

- How to project onto a subspace spanned by these p principal components (= its orthonormal basis)?

Projecting one example (= a single column):

PCA STEP 5: PROJECT THE DATA



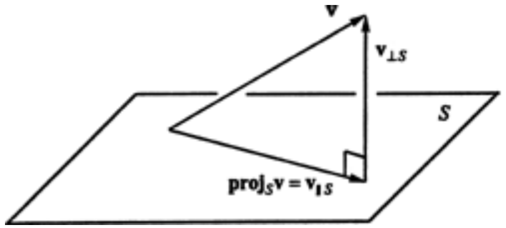
- Let's project the data onto p principal components.
- How to project onto a subspace spanned by these p principal components (= its orthonormal basis)?

Projecting one example (= a single column):

$$x_{proj} = \left[(x, v_1), (x, v_2), \dots, (x, v_p)\right]^T = V_p^T x$$

Projecting the whole data (= every column):

PCA STEP 5: PROJECT THE DATA



- Let's project the data onto p principal components.
- How to project onto a subspace spanned by these p principal components (= its orthonormal basis)?

Projecting one example (= a single column):

$$x_{proj} = \left[(x, v_1), (x, v_2), \dots, (x, v_p)\right]^T = V_p^T x$$

Projecting the whole data (= every column):

$$X_{proj} = V_p^T X, \qquad X_{proj} - p \times n \text{ matrix.}$$

girafe ai

Singular Values and Singular Vectors

Reminder: Eigendecomposition

- Consider an $n \times n$ symmetric matrix A.

- Eigendecomposition of A:

$$A_{n\times n} = V_{n\times n} \Lambda_{n\times n} V_{n\times n}^T$$

Columns of V – eigenvectors of A,
V – orthogonal matrix: $V^T = V^{-1}$.

Λ – diagonal matrix,
diagonal elements $\lambda_1, \dots, \lambda_n$ – eigenvalues of A.

SVD: Motivation

- Eigendecomposition is great ☺
- But it only works for square and symmetric matrices ☹

Singular Value Decomposition:
generalization of eigendecomposition
for *any* rectangular matrix.

girafe ai

Singular Values and Singular Vectors

- For square matrices: eigenvalue + eigenvector:

$$Av = \lambda v$$

girafe ai

Singular Values and Singular Vectors

- For square matrices: eigenvalue + eigenvector:

$$Av = \lambda v$$

- For non-square matrices:

u, v – unit vectors, $\sigma > 0$ – some number such that

$$Av = \sigma u, \qquad A^T u = \sigma v$$

u – left singular vector, v – right singular vector, σ – singular number

Singular Values and Singular Vectors

- For square matrices: eigenvalue + eigenvector:

$$Av = \lambda v$$

- For non-square matrices:

u, v – unit vectors, $\sigma > 0$ – some number such that

$$Av = \sigma u, \qquad A^T u = \sigma v$$

u – left singular vector, v – right singular vector, σ – singular number

What are those vectors and numbers?

girafe ai

SVD

SVD: Main Idea

- Let A be an $m \times n$ matrix.
- *(SVD)*: A can be decomposed as

$$A_{m \times n} = U_{m \times m} \Sigma_{m \times n} (V_{n \times n})^T, \text{ where}$$

$$U = [u_1 | \dots | u_m], \quad V = [v_1 | \dots | v_n] - \text{orthogonal matrices},$$

Σ – “diagonal matrix”

SVD: Main Idea

- Let A be an $m{\times}n$ matrix.
- *(SVD)*: A can be decomposed as

$$A_{m\times n} = U_{m\times m}\Sigma_{m\times n}(V_{n\times n})^T, \text{ where}$$

$$U = [u_1 \,|\ldots|\, u_m],\ \ V = [v_1 \,|\ldots|\, v_n] - \text{orthogonal matrices,}$$

Σ – “diagonal matrix”

$$m \geq n: \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_n \\ & & 0 & \end{bmatrix}$$

girafe ai

SVD: Main Idea

- Let A be an $m{\times}n$ matrix.
- *(SVD)*: A can be decomposed as

$$A_{m\times n} = U_{m\times m}\Sigma_{m\times n}(V_{n\times n})^T, \text{ where}$$

$$U = [u_1 \,|\ldots|\, u_m],\;\; V = [v_1 \,|\ldots|\, v_n] - \text{orthogonal matrices},$$

Σ – “diagonal matrix”

$$m \geq n: \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_n \\ \hline & & 0 & \end{bmatrix}$$

$\boldsymbol{m} - \boldsymbol{n}$ zero rows

girafe ai

SVD: Main Idea

- Let A be an $m\times n$ matrix.
- *(SVD)*: A can be decomposed as

$$A_{m\times n} = U_{m\times m}\Sigma_{m\times n}(V_{n\times n})^T, \text{ where}$$

$$U = [u_1 | \ldots | u_m], \;\; V = [v_1 | \ldots | v_n] \text{ – orthogonal matrices,}$$

Σ – "diagonal matrix"

$$m \geq n: \;\; \Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_n \\ & & \boxed{0} & \end{bmatrix}, \qquad m < n: \;\; \Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & \\ 0 & \sigma_2 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & \sigma_m & \end{bmatrix}$$

$\boldsymbol{m-n}$ zero rows

girafe ai

SVD: Main Idea

- Let A be an $m{\times}n$ matrix.
- *(SVD)*: A can be decomposed as

$$A_{m\times n} = U_{m\times m}\Sigma_{m\times n}(V_{n\times n})^T, \text{ where}$$

$$U = [u_1 \,|\ldots|\, u_m],\ \ V = [v_1 \,|\ldots|\, v_n] - \text{orthogonal matrices,}$$

Σ – “diagonal matrix”

$$m \geq n:\quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_n \\ \hline & & 0 & \end{bmatrix}, \qquad m < n:\quad \Sigma = \left[\begin{array}{cccc|c} \sigma_1 & 0 & \cdots & 0 & \\ 0 & \sigma_2 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & \sigma_m & \end{array}\right]$$

$\boldsymbol{m-n}$ zero rows

$\boldsymbol{n-m}$ zero columns

SVD: Main Idea

- Let A be an $m{\times}n$ matrix.
- *(SVD)*: A can be decomposed as

$$A_{m\times n} = U_{m\times m}\Sigma_{m\times n}(V_{n\times n})^T, \text{ where}$$

$$U = [u_1\,|\ldots|u_m],\;\; V = [v_1\,|\ldots|v_n] \text{ – orthogonal matrices,}$$

$$\Sigma \text{ – "diagonal matrix" with } \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0,\;\; \sigma_{r+1} = \cdots = \sigma_{\max(m,n)} = 0$$

$$m \geq n:\quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_n \\ \hline & \boxed{0} & & \end{bmatrix}, \qquad m < n:\quad \Sigma = \left[\begin{array}{cccc|c} \sigma_1 & 0 & \cdots & 0 & \\ 0 & \sigma_2 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \boxed{0} \\ 0 & 0 & \cdots & \sigma_m & \end{array}\right]$$

$\boldsymbol{m-n}$ zero rows

$\boldsymbol{n-m}$ zero columns

SVD: Main Ingredients

$$A_{m\times n} = U_{m\times m}\Sigma_{m\times n}(V_{n\times n})^T, \text{ where}$$

$$U = [u_1 \,|\ldots|\, u_m] - \text{eigenvectors of } AA^T,$$

$$V = [v_1 \,|\ldots|\, v_n] - \text{eigenvectors of } A^TA,$$

$$\sigma_1^2, \ldots, \sigma_r^2 - \text{corresponding non-zero eigenvalues of of } A^TA \,/\, AA^T.$$

Full SVD

$$A_{m\times n} = U_{m\times m}\Sigma_{m\times n}V^T_{n\times n}$$

U, V – orthogonal matrices

Σ – “diagonal” matrix

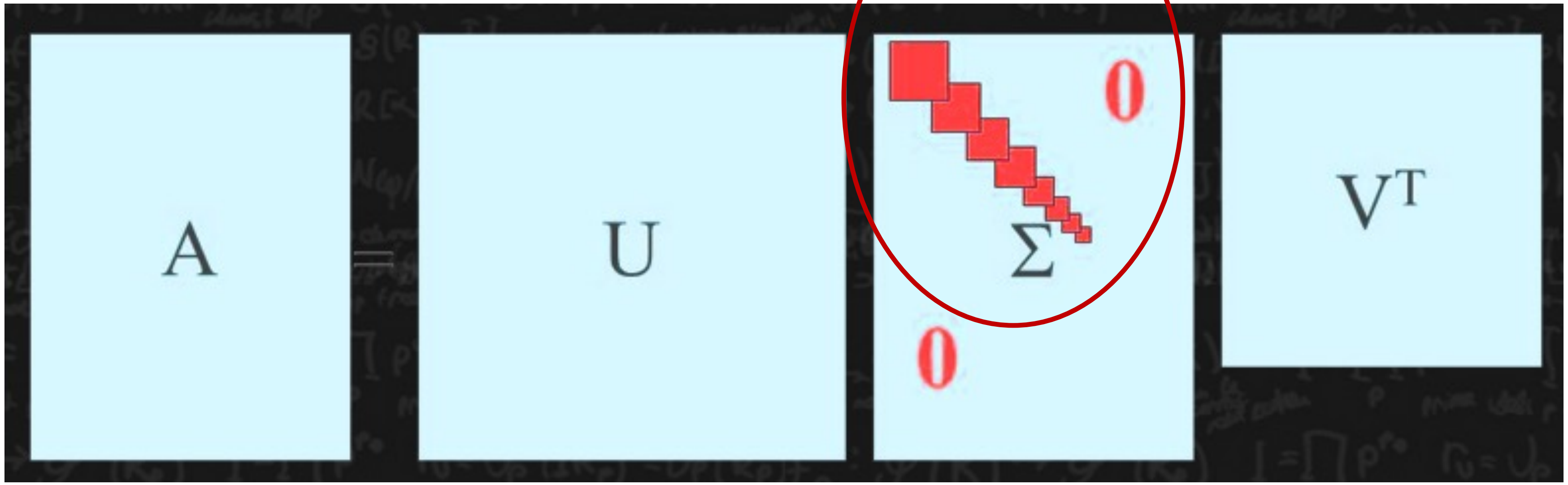
$$m \geq n:\quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_n \\ \hline & & \boxed{0} & \end{bmatrix} \qquad m < n:\quad \Sigma = \left[\begin{array}{cccc|c} \sigma_1 & 0 & \cdots & 0 & \\ 0 & \sigma_2 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \boxed{0} \\ 0 & 0 & \cdots & \sigma_m & \end{array}\right]$$

$\boldsymbol{m-n}$ zero rows

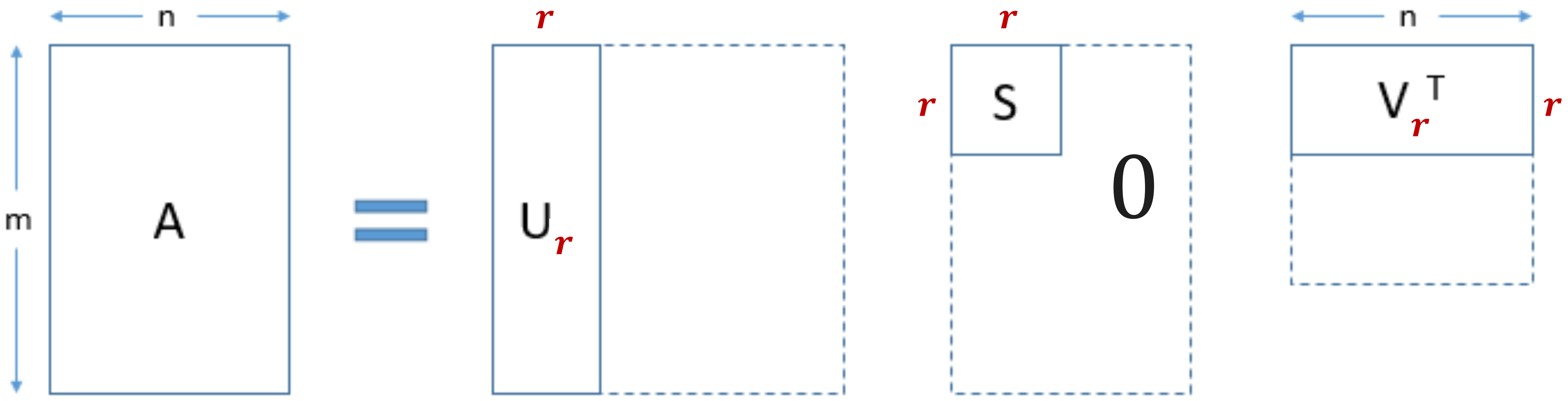
$\boldsymbol{n-m}$ zero columns

Full SVD

Only r non-zero diagonal elements!



Reduced SVD



Reduced SVD

$$A_{m\times n} = U_{m\times \boldsymbol{r}}\Sigma_{\boldsymbol{r}\times \boldsymbol{r}}V^T_{n\times \boldsymbol{r}}$$

$U_{m\times k}, V_{n\times k}$ – orthogonal matrices

$u_1, \dots u_r$ – left singular vectors *= eigenvectors of* AA^T
$v_1, \dots v_r$ – right singular vectors *= eigenvectors of* $A^T A$

$\Sigma_{r\times r}$ – diagonal matrix
$\sigma_1, \dots \sigma_r > 0$ – singular values of A
$\sigma_1^2, \dots \sigma_r^2$ *- non-zero eigenvalues of* AA^T *and* $A^T A$

$$Av_i = \sigma_i u_i$$

girafe ai

Singular Values: Example

- Find singular values of $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$.

$$A_{m\times n} = U_{m\times m}\Sigma_{m\times n}(V_{n\times n})^T, \text{ where}$$

Singular Values: Example

$$m \geq n: \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_n \\ & 0 & & \end{bmatrix}, m < n: \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & \\ 0 & \sigma_2 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & \sigma_m & \end{bmatrix}$$

- Find singular values of $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$.

$A_{m\times n} = U_{m\times m}\boldsymbol{\Sigma}_{m\times n}(V_{n\times n})^T$, where

Singular Values: Example

- Find singular values of $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$.

$$A_{m\times n} = U_{m\times m}\Sigma_{m\times n}(V_{n\times n})^T\text{, where}$$

$A^T A$ is a 3×3 matrix while AA^T is only 2×2 but has the same non-zero eigenvalues.

Singular Values: Example

- Find singular values of $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$.

$$A_{m\times n} = U_{m\times m}\Sigma_{m\times n}(V_{n\times n})^T\text{, where}$$

$A^T A$ is a 3×3 matrix while AA^T is only 2×2 but has the same non-zero eigenvalues.

$$AA^T = \begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix},$$

Singular Values: Example

- Find singular values of $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$.

$$A_{m\times n} = U_{m\times m}\Sigma_{m\times n}(V_{n\times n})^T\text{, where}$$

$A^T A$ is a 3×3 matrix while AA^T is only 2×2 but has the same non-zero eigenvalues.

$$AA^T = \begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix}, \qquad \det(AA^T - \lambda E) = (\lambda - 25)(\lambda - 9) = 0 \Leftrightarrow$$

Singular Values: Example

- Find singular values of $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$.

$$A_{m\times n} = U_{m\times m}\Sigma_{m\times n}(V_{n\times n})^T, \text{ where}$$

$A^T A$ is a 3×3 matrix while AA^T is only 2×2 but has the same non-zero eigenvalues.

$$AA^T = \begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix}, \qquad \det(AA^T - \lambda E) = (\lambda - 25)(\lambda - 9) = 0 \Leftrightarrow$$

$$\sigma_1 = \sqrt{25} = 5, \qquad \sigma_2 = \sqrt{9} = 3$$

Singular Values: Example

- Find singular values of $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$.

$$A_{m\times n} = U_{m\times m}\Sigma_{m\times n}(V_{n\times n})^T, \text{ where}$$

$A^T A$ is a 3×3 matrix while AA^T is only 2×2 but has the same non-zero eigenvalues.

$$AA^T = \begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix}, \qquad \det(AA^T - \lambda E) = (\lambda - 25)(\lambda - 9) = 0 \Leftrightarrow$$

$$\sigma_1 = \sqrt{25} = 5, \qquad \sigma_2 = \sqrt{9} = 3$$

$$\Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

Example

- Let's find SVD and reduced SVD of

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

$$A_{2\times3} = U_{2\times2}\Sigma_{2\times3}(V_{3\times3})^T, \qquad \Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

Columns of V are eigenvectors of $A^T A$.

Eigenvalues of $A^T A$ are 25, 9 and $\mathbf{0}$.

$$A^T A - 25E = \begin{pmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{pmatrix} \sim \dots \to v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}^T$$

girafe ai

Example

- Let's find SVD and reduced SVD of

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

$$A_{2\times 3} = U_{2\times 2}\Sigma_{2\times 3}(V_{3\times 3})^T, \qquad \Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

Columns of V are eigenvectors of $A^T A$.

Eigenvalues of $A^T A$ are 25, 9 and 0.

$$A^T A - 9E = \begin{pmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{pmatrix} \sim \dots \to v_2 = \begin{pmatrix} \frac{1}{3\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} \end{pmatrix}^T$$

Example

- Let's find SVD and reduced SVD of

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

$$A_{2\times 3} = U_{2\times 2}\Sigma_{2\times 3}(V_{3\times 3})^T, \qquad \Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

Columns of V are eigenvectors of $A^T A$.
Eigenvalues of $A^T A$ are 25, 9 and 0.

$$A^T A - 0E = \begin{pmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 10 \end{pmatrix} \sim \dots \to v_3 = \begin{pmatrix} \frac{2}{3} & \frac{-2}{3} & \frac{-1}{3} \end{pmatrix}^T$$

Example

- Let's find SVD and reduced SVD of

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

$$A_{2\times 3} = U_{2\times 2}\Sigma_{2\times 3}(V_{3\times 3})^T,$$

$$\Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}, \qquad \boldsymbol{V} = \begin{pmatrix} \mathbf{1/\sqrt{2}} & \mathbf{1/3\sqrt{2}} & \mathbf{2/3} \\ \mathbf{-1/\sqrt{2}} & \mathbf{-1/3\sqrt{2}} & \mathbf{-2/3} \\ \mathbf{0} & \mathbf{4/3\sqrt{2}} & \mathbf{-1/3} \end{pmatrix}, \qquad U = ?$$

Example

- Let's find SVD and reduced SVD of

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

$$A_{2\times3} = U_{2\times2}\Sigma_{2\times3}(V_{3\times3})^T,$$

$$\Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 1/\sqrt{2} & 1/3\sqrt{2} & 2/3 \\ -1/\sqrt{2} & -1/3\sqrt{2} & -2/3 \\ 0 & 4/3\sqrt{2} & -1/3 \end{pmatrix}, \qquad U = ?$$

Remember: $\boldsymbol{Av_i} = \boldsymbol{\sigma_i u_i}$

Example

- Let's find SVD and reduced SVD of

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

$$A_{2\times 3} = U_{2\times 2}\Sigma_{2\times 3}(V_{3\times 3})^T,$$

$$\Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 1/\sqrt{2} & 1/3\sqrt{2} & 2/3 \\ -1/\sqrt{2} & -1/3\sqrt{2} & -2/3 \\ 0 & 4/3\sqrt{2} & -1/3 \end{pmatrix}, \qquad U = ?$$

$$\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = 5u_1 \Rightarrow u_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

girafe ai

Example

- Let's find SVD and reduced SVD of

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

$$A_{2\times3} = U_{2\times2}\Sigma_{2\times3}(V_{3\times3})^T,$$

$$\Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 1/\sqrt{2} & 1/3\sqrt{2} & 2/3 \\ -1/\sqrt{2} & -1/3\sqrt{2} & -2/3 \\ 0 & 4/3\sqrt{2} & -1/3 \end{pmatrix}, \qquad U = ?$$

$$u_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \qquad \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}\begin{pmatrix} 1/3\sqrt{2} \\ -1/3\sqrt{2} \\ 4/3\sqrt{2} \end{pmatrix} = 3u_1 \Rightarrow u_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

Example

- Let's find SVD and reduced SVD of

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

$$A_{2\times 3} = U_{2\times 2}\Sigma_{2\times 3}(V_{3\times 3})^T,$$

$$\Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1/\sqrt{2} & 1/3\sqrt{2} & 2/3 \\ -1/\sqrt{2} & -1/3\sqrt{2} & -2/3 \\ 0 & 4/3\sqrt{2} & -1/3 \end{pmatrix}, \quad \boldsymbol{U} = \begin{pmatrix} \mathbf{1/\sqrt{2}} & \mathbf{1/\sqrt{2}} \\ \mathbf{1/\sqrt{2}} & \mathbf{-1/\sqrt{2}} \end{pmatrix}$$

Example

- SVD of

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

$$A_{2\times 3} = U_{2\times 2}\Sigma_{2\times 3}(V_{3\times 3})^T,$$

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1/\sqrt{2}} & \mathbf{1/\sqrt{2}} \\ \mathbf{1/\sqrt{2}} & \mathbf{-1/\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{5} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} & \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{1/\sqrt{2}} & \mathbf{1/3\sqrt{2}} & \mathbf{2/3} \\ \mathbf{-1/\sqrt{2}} & \mathbf{-1/3\sqrt{2}} & \mathbf{-2/3} \\ \mathbf{0} & \mathbf{4/3\sqrt{2}} & \mathbf{-1/3} \end{pmatrix}^{\boldsymbol{T}}$$

From SVD to Reduced SVD

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

$$A_{2\times 3} = U_{2\times 2}\Sigma_{2\times 3}(V_{3\times 3})^T,$$

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} & 1/3\sqrt{2} & 2/3 \\ -1/\sqrt{2} & -1/3\sqrt{2} & -2/3 \\ 0 & 4/3\sqrt{2} & -1/3 \end{pmatrix}^T$$

From SVD to Reduced SVD

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

$$A_{2\times3} = U_{2\times2}\Sigma_{2\times3}(V_{3\times3})^T,$$

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} & 1/3\sqrt{2} & 2/3 \\ -1/\sqrt{2} & -1/3\sqrt{2} & -2/3 \\ 0 & 4/3\sqrt{2} & -1/3 \end{pmatrix}^T$$

From SVD to Reduced SVD

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

$$A_{2\times 3} = U_{2\times 2}\Sigma_{2\times 3}(V_{3\times 3})^T,$$

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} & 1/3\sqrt{2} & 2/3 \\ -1/\sqrt{2} & -1/3\sqrt{2} & -2/3 \\ 0 & 4/3\sqrt{2} & -1/3 \end{pmatrix}^T$$

$$A_{2\times 3} = U_{2\times 2}\Sigma_{2\times 2}(V_{3\times 2})^T,$$

From SVD to Reduced SVD

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

$$A_{2\times3} = U_{2\times2}\Sigma_{2\times3}(V_{3\times3})^T,$$

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} & 1/3\sqrt{2} & 2/3 \\ -1/\sqrt{2} & -1/3\sqrt{2} & -2/3 \\ 0 & 4/3\sqrt{2} & -1/3 \end{pmatrix}^T$$

Reduced SVD:

$$A_{2\times3} = U_{2\times2}\Sigma_{2\times\mathbf{2}}(V_{3\times\mathbf{2}})^T,$$

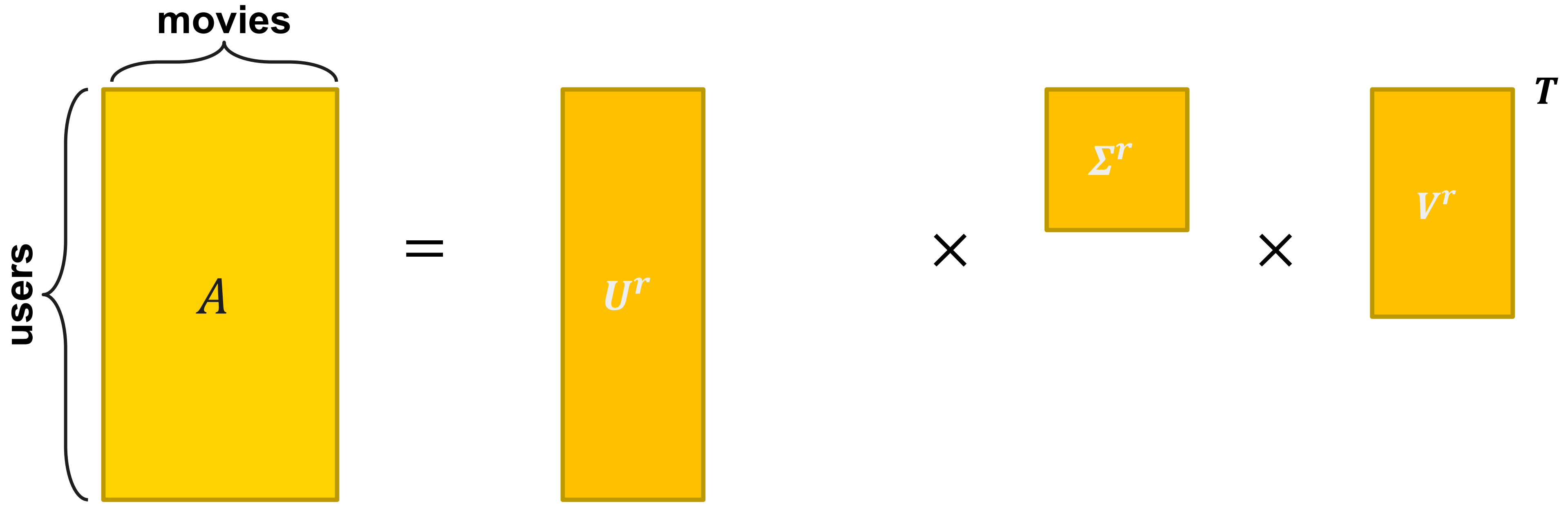
$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} & 1/3\sqrt{2} \\ -1/\sqrt{2} & -1/3\sqrt{2} \\ 0 & 4/3\sqrt{2} \end{pmatrix}^T$$

Reduced SVD: Main Idea

$$A_{m\times n} = U^r_{m\times \mathbf{r}}\Sigma^r_{\mathbf{r}\times \mathbf{r}}(V^r_{n\times \mathbf{r}})^T, \text{ where}$$

$$U^r = [u_1\,|\ldots|u_r],\ \ V^r = [v_1\,|\ldots|v_r] - \text{orthogonal matrices,}$$

$$\Sigma^r - \text{diagonal matrix with } \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0.$$



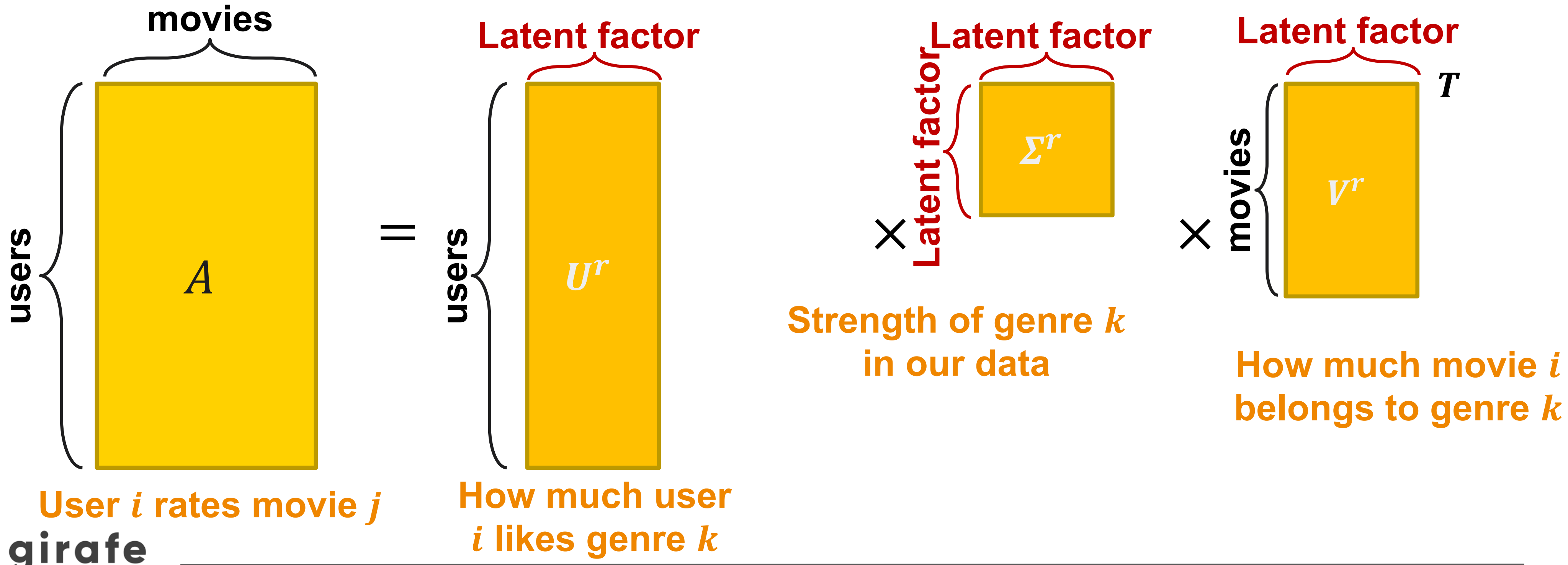
User i rates movie j

Reduced SVD: Main Idea

$$A_{m\times n} = U^r_{m\times \mathbf{r}} \Sigma^r_{\mathbf{r}\times \mathbf{r}} (V^r_{n\times \mathbf{r}})^T, \text{ where}$$

$U^r = [u_1 | \dots | u_r]$, $V^r = [v_1 | \dots | v_r]$ – orthogonal matrices,

Σ^r – diagonal matrix with $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$.

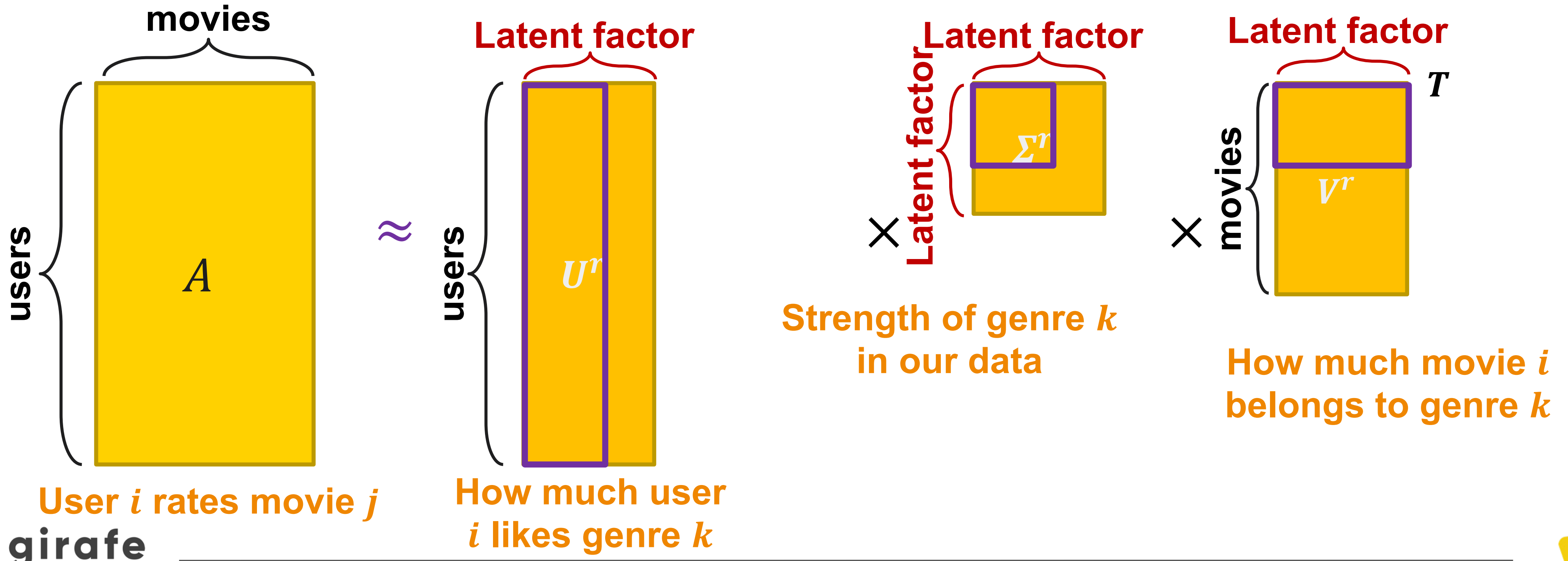


SVD: Dimensionality Reduction

$$A_{m\times n} = U^r_{m\times r}\Sigma^r_{r\times r}(V^r_{n\times r})^T, \text{ where}$$

$$U^r = [u_1 \,|\ldots|\, u_r],\ \ V^r = [v_1 \,|\ldots|\, v_r] - \text{orthogonal matrices},$$

$$\Sigma^r - \text{diagonal matrix with } \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0.$$



girafe ai

Practice!

https://colab.research.google.com/drive/1faLfFmMHZbxtpzgZxxT6MCBuy57CE7Po?usp=sharing