Math Basics for DS

Practical Session 7

Today

- Updated Logistics
- Common mistakes HW 1
- PCA step-by-step
- SVD step-by-step

Logistics

- Graded assignment 2 is OUT
- Deadline extended till

  Sunday, November 7, 23:59 Moscow time.

- No Linear Algebra exam this weekend.
  We’ll have it later, together with Calculus.

Graded Assignment 1

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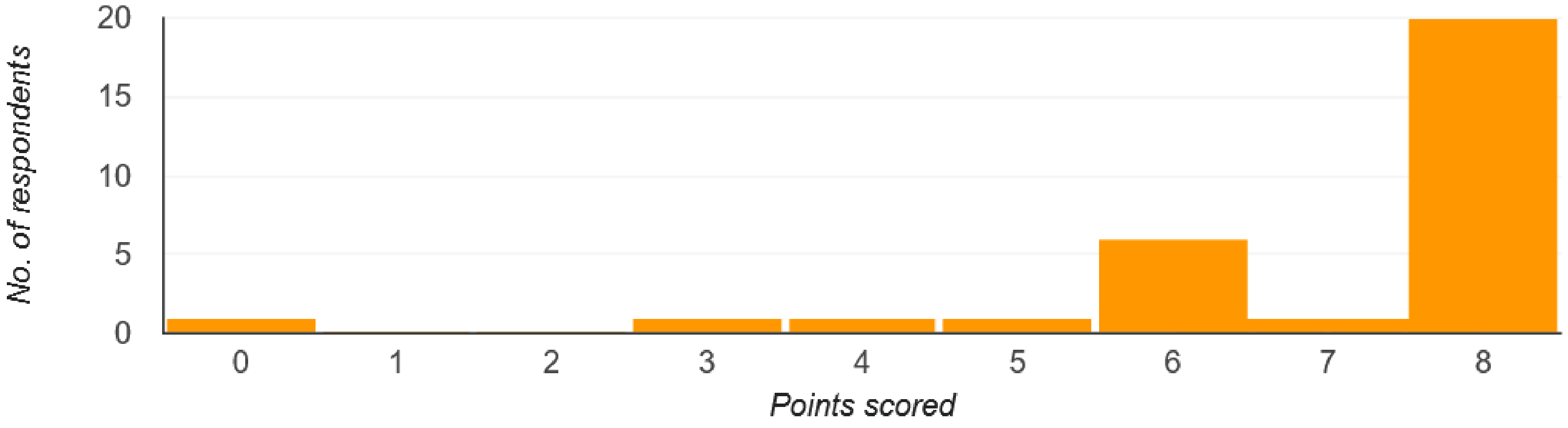
Average	Median	Range
6.94/8 points	8/8 points	0-8 points

Total points distribution



Question 1 - Frequently Missed

- Find $\dim(V)$.
- Select valid bases of V.

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix} v_4 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, V = span\{v1, v2, v3, v4\}.$$

Question 1 - Frequently Missed

- Arrange vectors in a matrix column-wise and convert it to RREF:

$$\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 3 & 3 & 0 \end{pmatrix} \rightarrow$$

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$$\dim(V) = 3, \qquad v_4 = 1v_3 - 1v_2$$

girafe ai

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- What if we arrange vectors row-wise?

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 3 \\ 1 & 2 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & -3 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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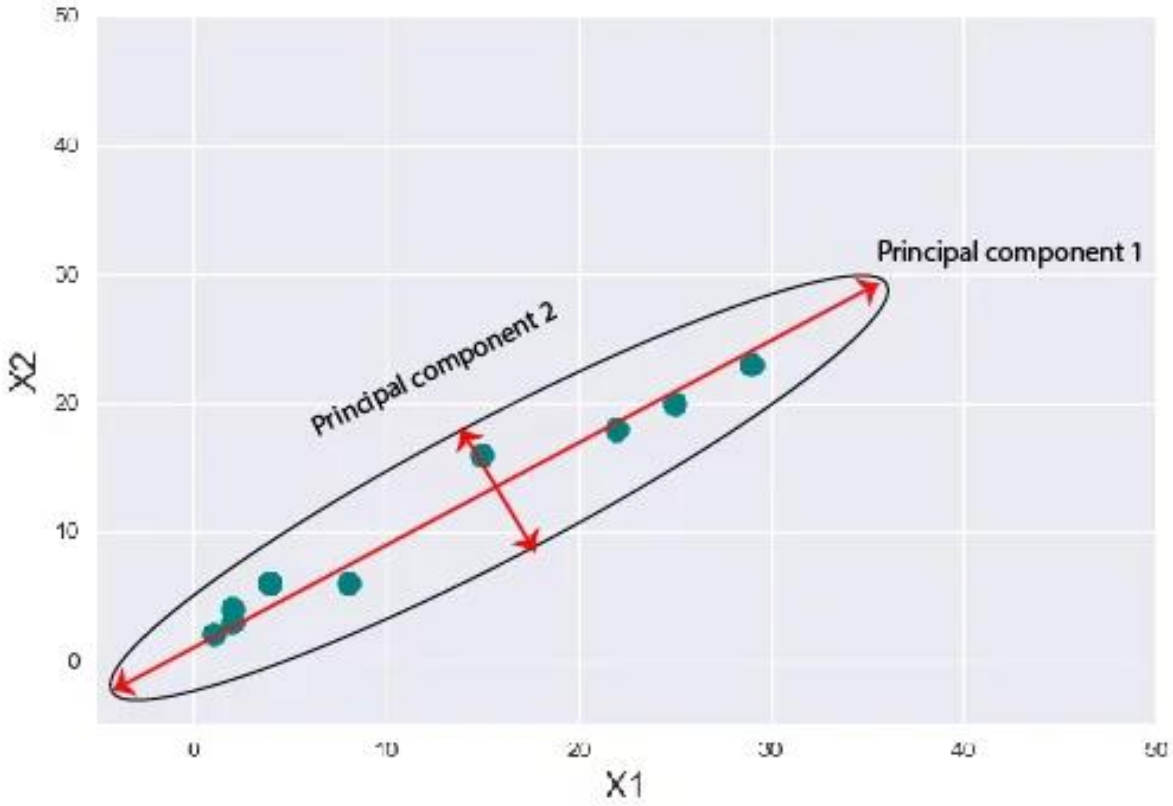
The columns in RREF don't tell anything about the dependencies among rows!

PCA STEP-BY-STEP

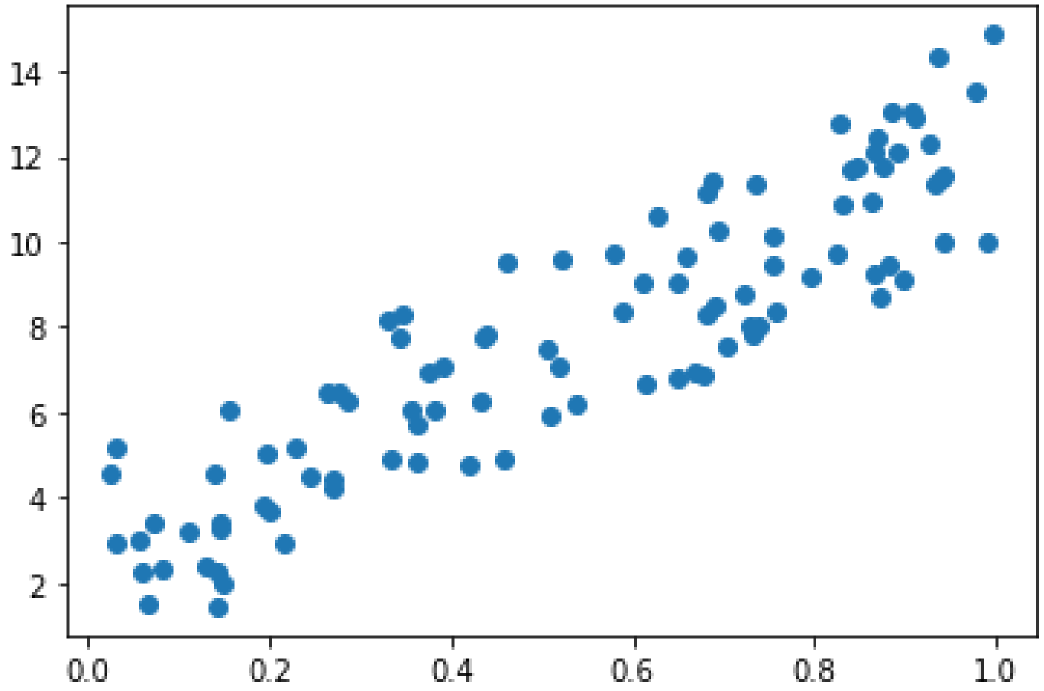


PCA STEP 1: COLLECT THE DATA

- $X - m \times n$ matrix.
  m features, n examples.

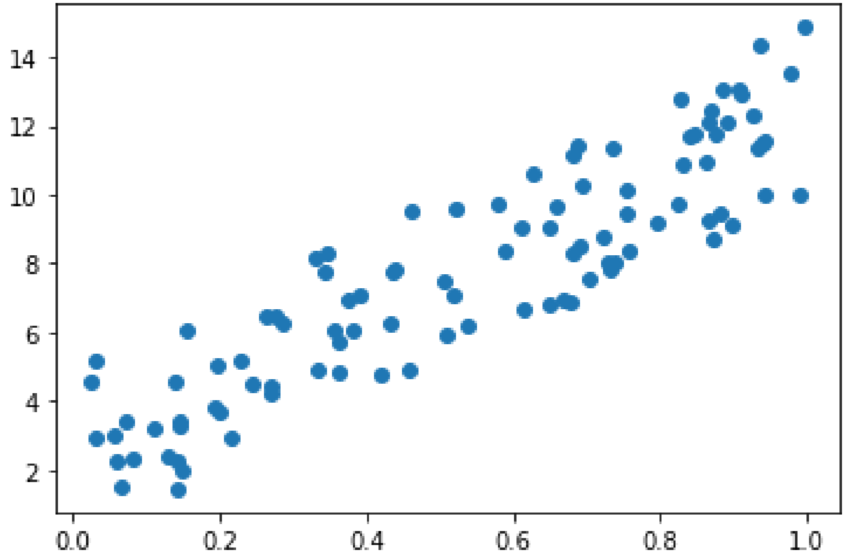
- Columns = examples,
  rows = features.

- Each example = a point
  in an m −dimensional space.



PCA STEP 2: CENTER THE DATA

- (This is needed to construct the covariance matrix)
- From each feature, subtract its mean.



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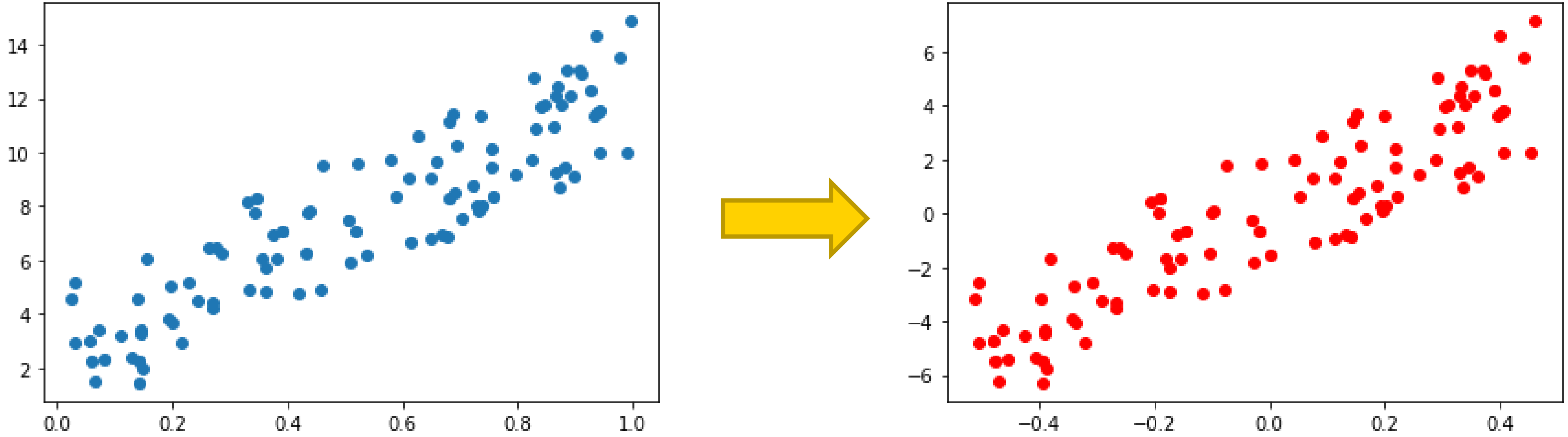
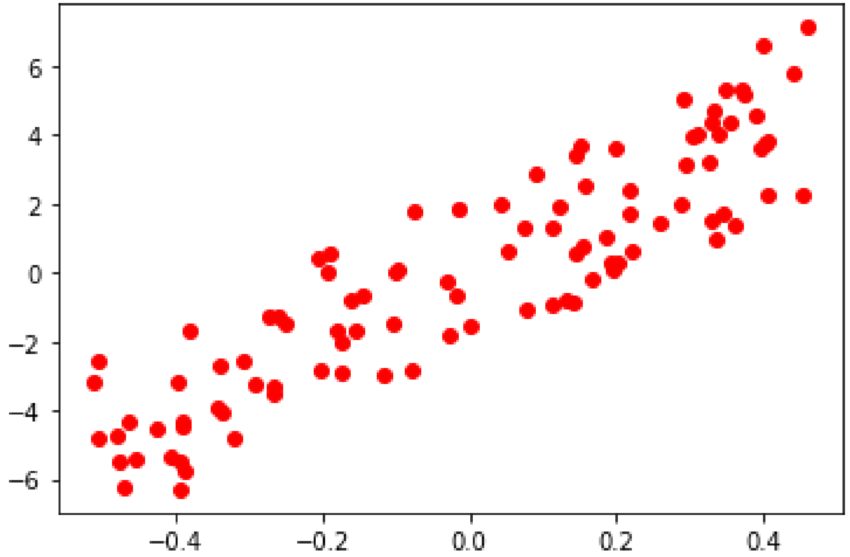


PCA STEP 3: BUILD COVARIANCE MATRIX S

- Sample covariance matrix with centered X:

$$S = \frac{1}{n-1} X X^t - n \times n \text{ matrix.}$$



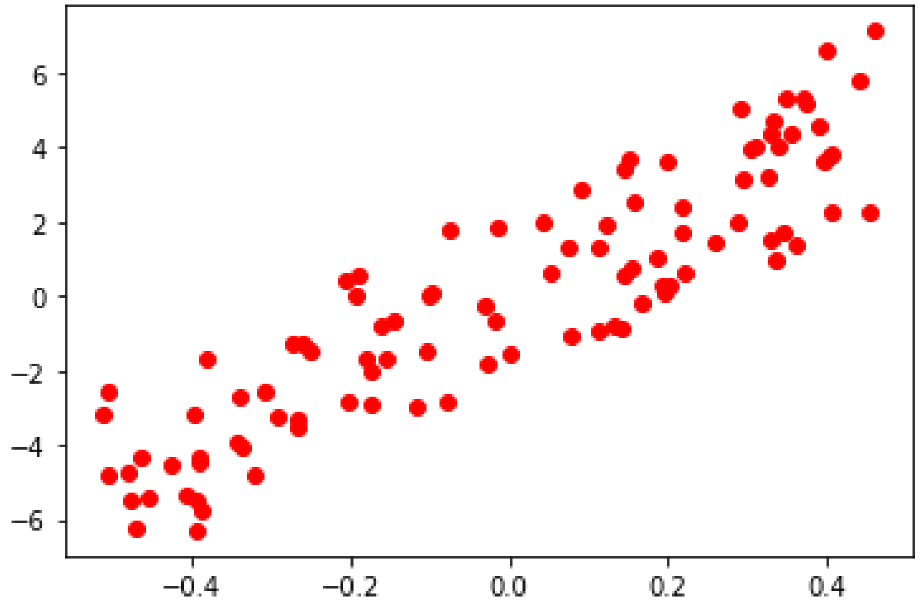

```
(1./(n-1))*np.matmul(X_centered, X_centered.transpose())
```

```
array([[ 0.08830318,  0.88570231],
       [ 0.88570231, 10.93980675]])
```

PCA STEP 4: DECOMPOSE S

- S is a symmetric matrix, so it has an eigendecomposition:

$$S = V\Lambda V^T$$



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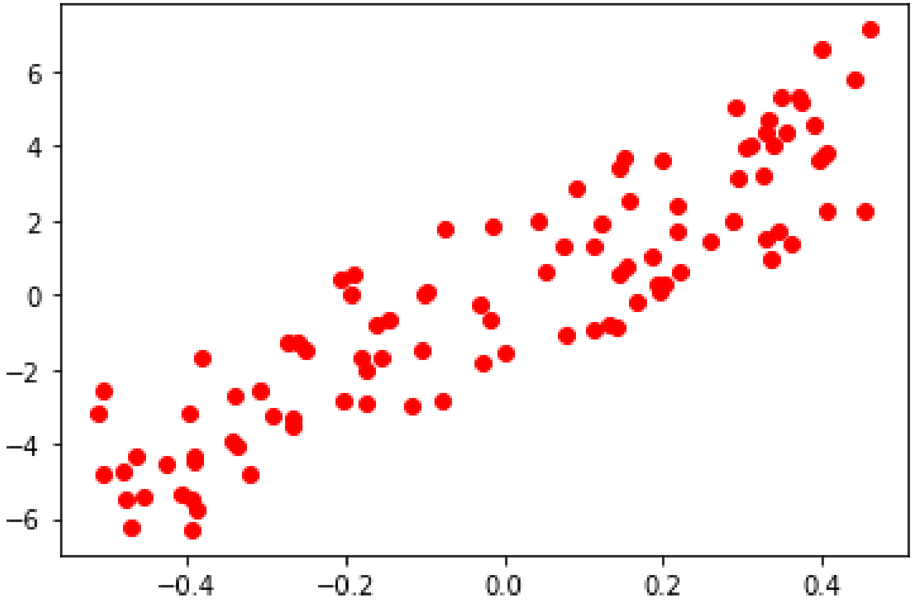
- Eigenvalues of S (ordered from large to small):

```
l, V = np.linalg.eig(S)
ind = np.argsort(l)[::-1]
l[ind]
```

```
array([11.0116227 ,  0.01648723])
```



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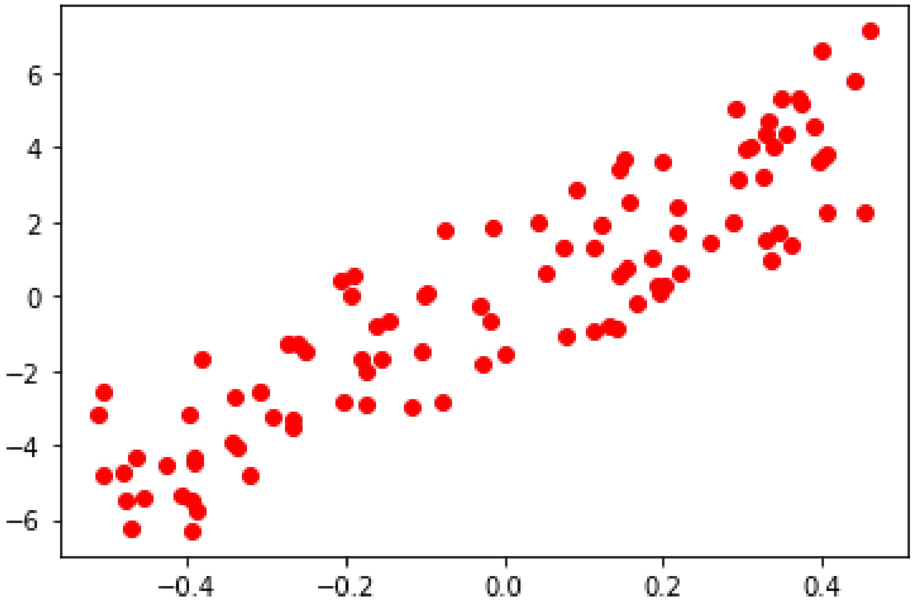
- Corresponding eigenvectors of S (= principal components of the data):

```
V[:,ind]
```

```
array([[-0.08081839, -0.99672884],
       [-0.99672884,  0.08081839]])
```



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- We know orthogonal projections from the first lecture!

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Projecting the whole data matrix (= all columns):

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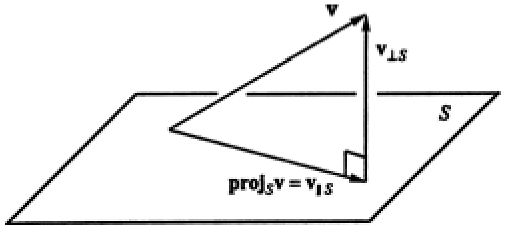

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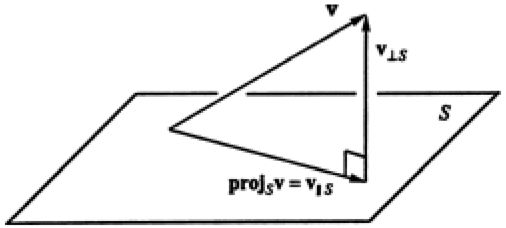

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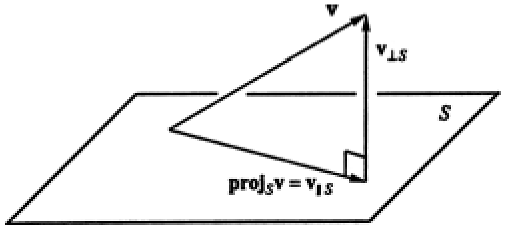

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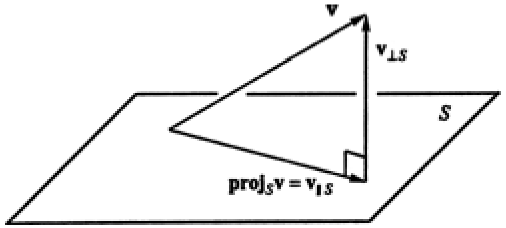

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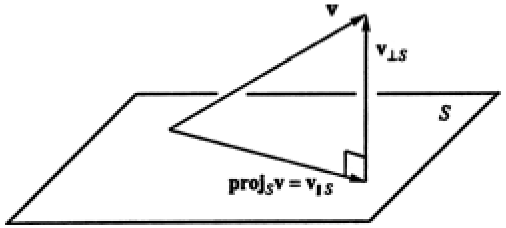

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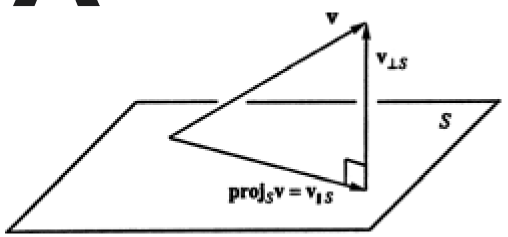

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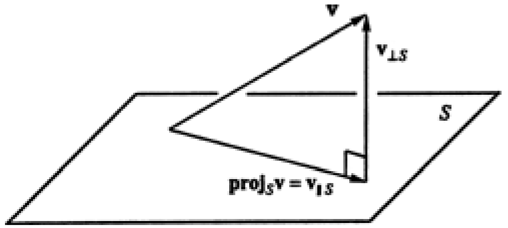

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$$(x, v_i) = 0 + \cdots x_i' \cdot 1 + \cdots + 0 + 0 = x_i'$$

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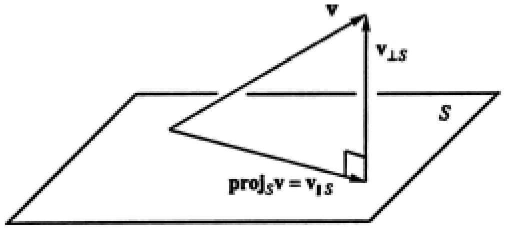

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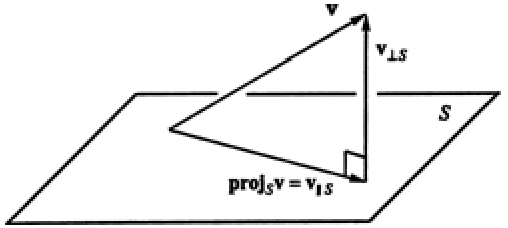

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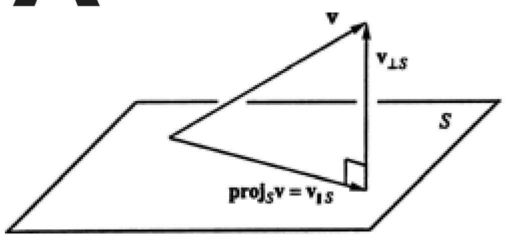

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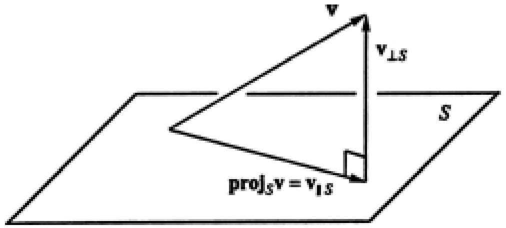

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Projecting the whole data (= every column):

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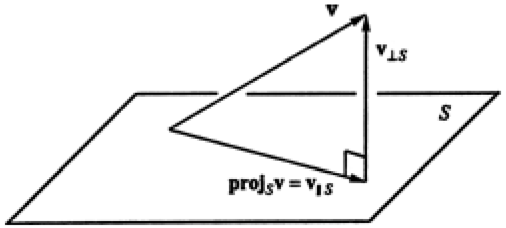

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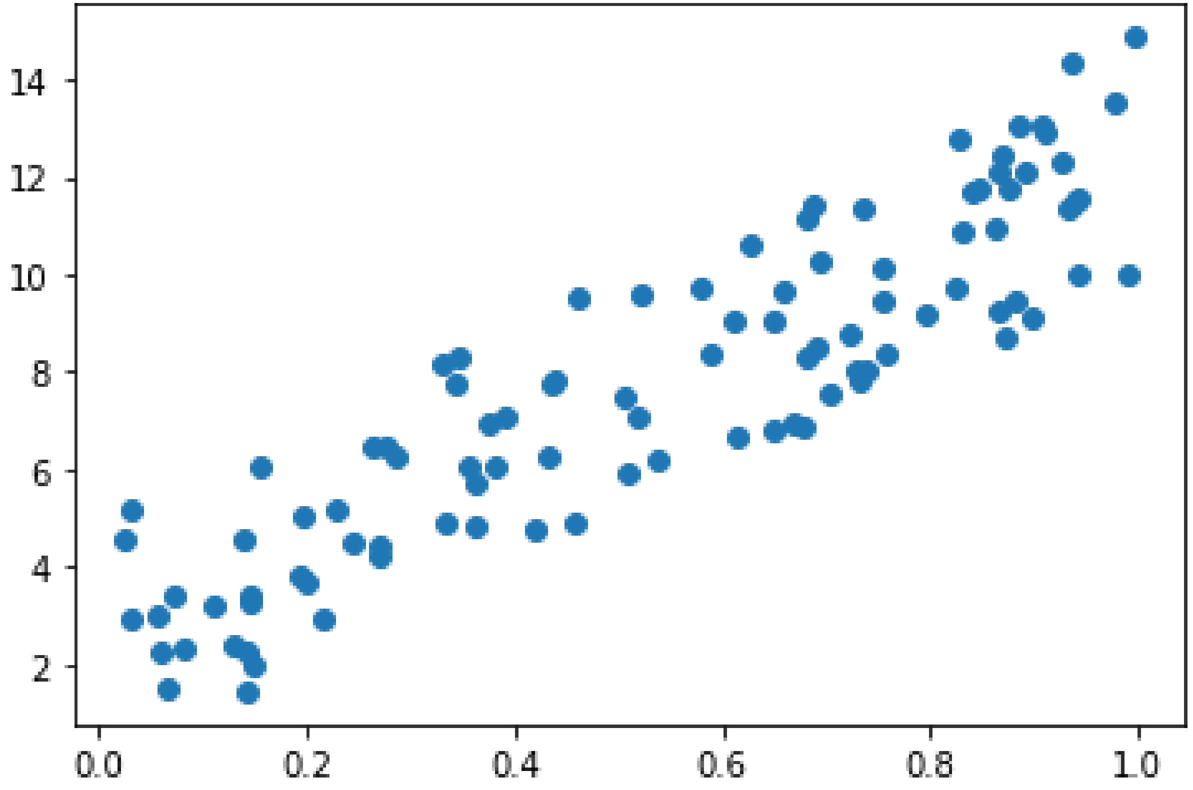
$$X_{proj} = V_p^T X, \qquad X_{proj} - p \times n \text{ matrix.}$$

PCA: RESULT

- What would happen to this data?



PCA: RESULT

Try it out!

https://colab.research.google.com/drive/1FVbnGH1xksvECmbaMVA2WATERCClwsV8?usp=sharing