Math Refresher for DS

Practical Session 8

Today

- SVD step-by-step
- Python practice
- Univariate Calculus basics: quick quiz

Reminder

- Graded Assignment 2 is due Sunday, 23:59 Moscow time.

Reminder: Eigendecomposition

- Consider an $n \times n$ symmetric matrix A.

- Eigendecomposition of A:

$$A_{n\times n} = V_{n\times n}\Lambda_{n\times n}V_{n\times n}^T$$

Columns of V – eigenvectors of A,
V – orthogonal matrix: $V^T = V^{-1}$.

Λ – diagonal matrix,
diagonal elements $\lambda_1, \dots, \lambda_n$ – eigenvalues of A.

girafe ai

SVD: Motivation

- Eigendecomposition is great ☺

- But it only works for square and symmetric matrices ☹

Singular Value Decomposition:
generalization of eigendecomposition
for *any* rectangular matrix.

Full SVD

$$A_{m\times n} = U_{m\times m}\Sigma_{m\times n}V^T_{n\times n}$$

U, V – orthogonal matrices

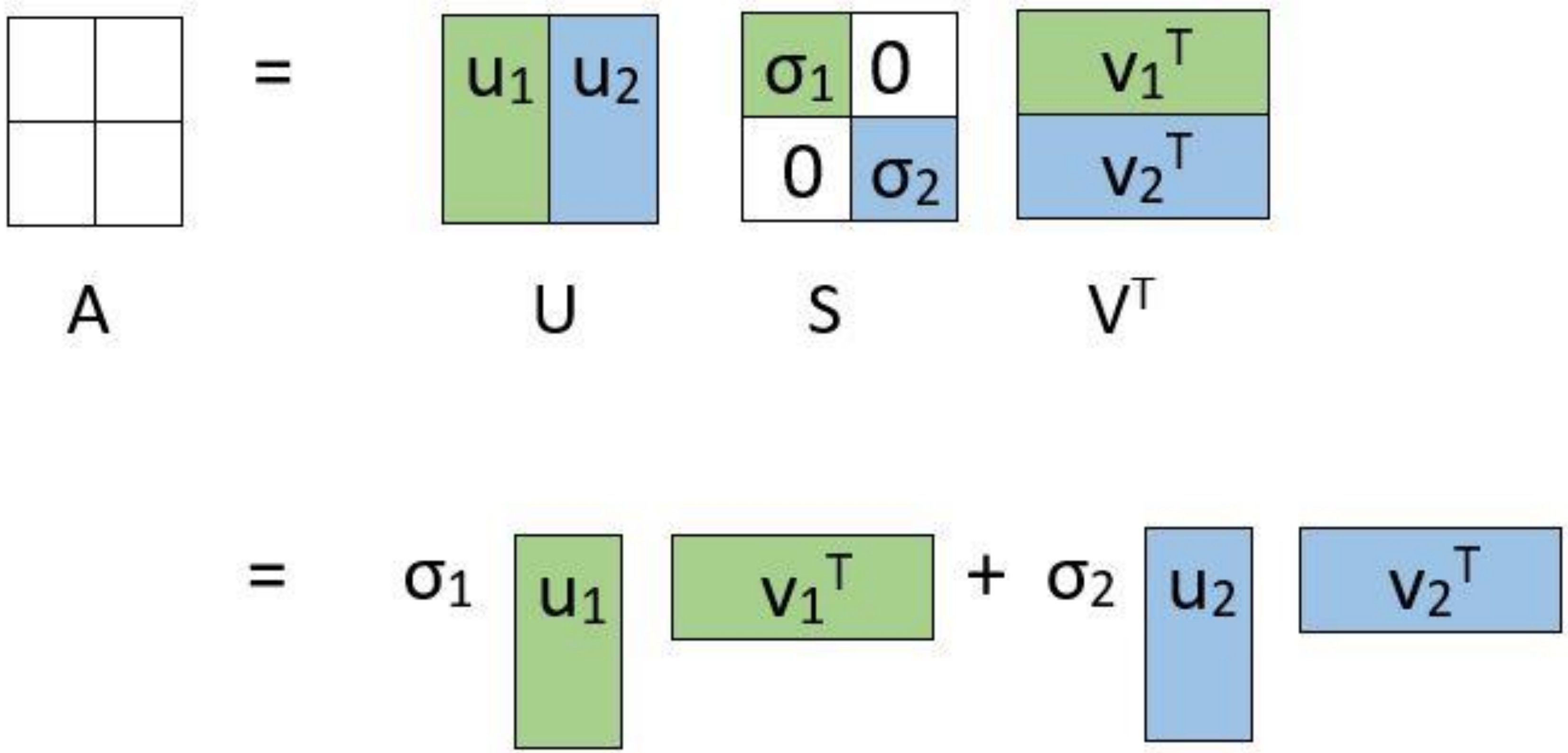
Σ – “diagonal” matrix

$$m \geq n: \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_n \\ \hline & & 0 & \end{bmatrix} \qquad m < n: \quad \Sigma = \left[\begin{array}{cccc|c} \sigma_1 & 0 & \cdots & 0 & \\ 0 & \sigma_2 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & \sigma_m & \end{array}\right]$$

$\boldsymbol{m-n}$ zero rows

$\boldsymbol{n-m}$ zero columns

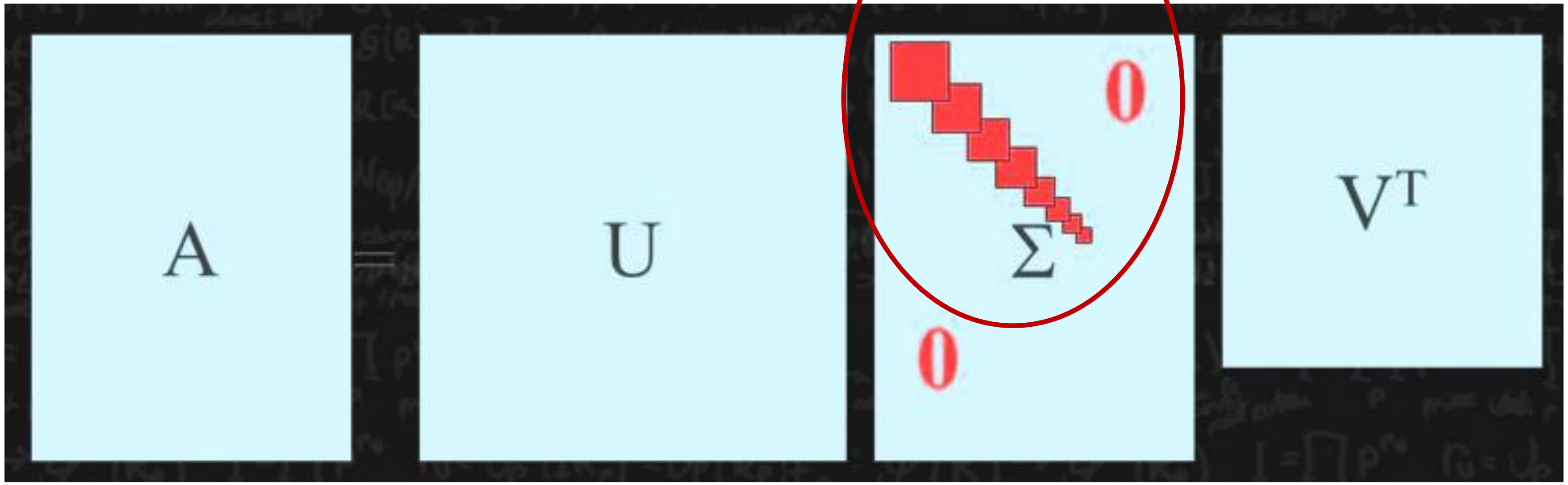
Full SVD



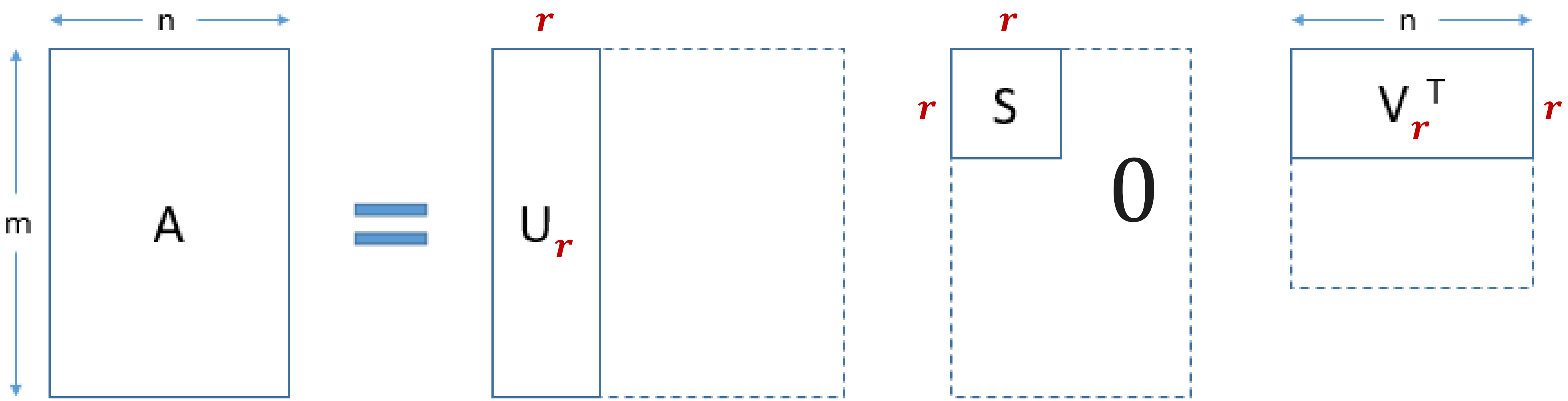
=
u_1 u_2
σ_1 0
0 σ_2
v_1^T
v_2^T
A
U
S
V^T
= σ_1 u_1 v_1^T + σ_2 u_2 v_2^T

Full SVD

Only r non-zero diagonal elements!



Reduced SVD



SVD

$$A_{m\times n} = U_{m\times \boldsymbol{r}}\Sigma_{\boldsymbol{r}\times\boldsymbol{r}}V^T_{n\times\boldsymbol{r}}$$

U, V – orthogonal matrices

Σ – diagonal matrix

r – number of non-zero diagonal elements of Σ

$$\sigma_1, \dots \sigma_r > 0$$

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SVD

$$A_{m\times n} = U_{m\times \boldsymbol{r}}\Sigma_{\boldsymbol{r}\times\boldsymbol{r}}V^T_{n\times \boldsymbol{r}}$$

$U_{m\times k}, V_{n\times k}$ – orthogonal matrices

$u_1, \dots u_r$ – left singular vectors
$v_1, \dots v_r$ – right singular vectors

$\Sigma_{r\times r}$ – diagonal matrix
$\sigma_1, \dots \sigma_r > 0$ – singular values of A

SVD

$$A_{m\times n} = U_{m\times \boldsymbol{r}}\Sigma_{\boldsymbol{r}\times \boldsymbol{r}}V^T_{n\times \boldsymbol{r}}$$

$U_{m\times k}, V_{n\times k}$ – orthogonal matrices

$u_1, \dots u_r$ – left singular vectors *= eigenvectors of AA^T*
$v_1, \dots v_r$ – right singular vectors *= eigenvectors of A^TA*

$\Sigma_{r\times r}$ – diagonal matrix
$\sigma_1, \dots \sigma_r > 0$ – singular values of A
$\sigma_1^2, \dots \sigma_r^2$ *- non-zero eigenvalues of AA^T and A^TA*

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Singular Vectors & Singular Values

- Consider matrix A.

- $u_1, \dots u_r$ – left singular vectors
  $v_1, \dots v_r$ – right singular vectors
  $\sigma_1, \dots \sigma_r > 0$ – singular values

$$Av_i = \sigma_i u_i$$

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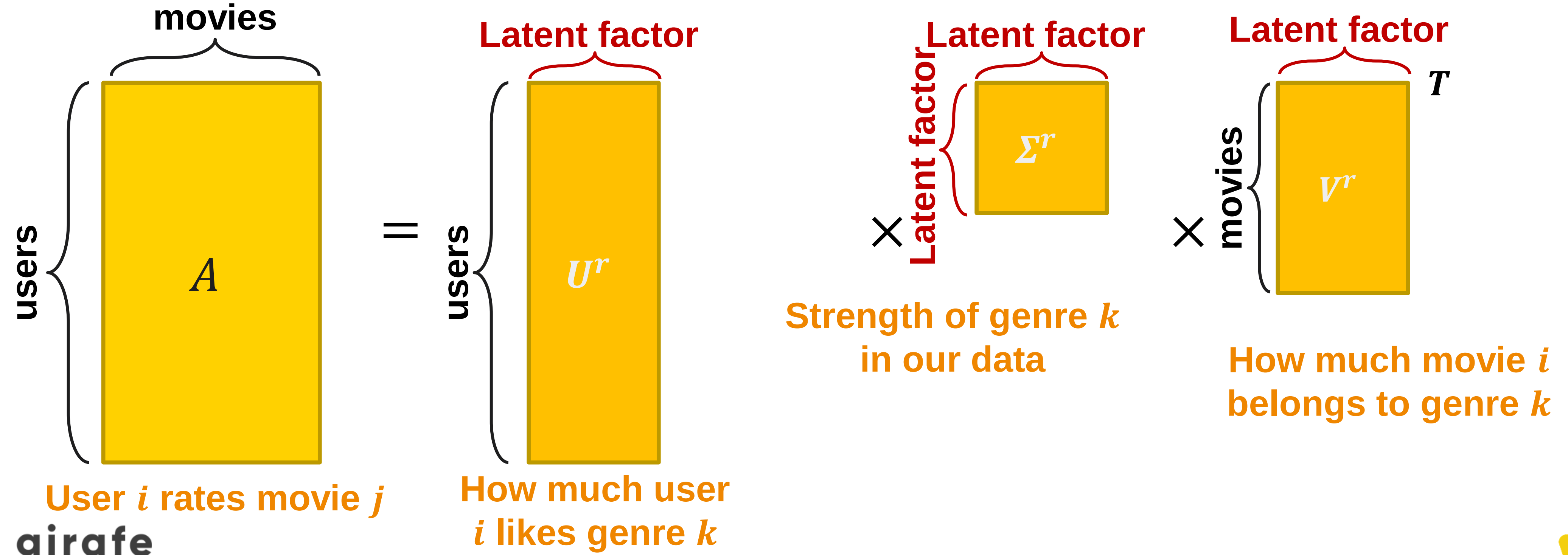
SVD: Interpretation

$$A_{m\times n} = U^r_{m\times \boldsymbol{r}} \Sigma^r_{\boldsymbol{r}\times \boldsymbol{r}} (V^r_{n\times \boldsymbol{r}})^T, \text{ where}$$

$$U^r = [u_1 | \ldots | u_r], \;\; V^r = [v_1 | \ldots | v_r] - \text{orthogonal matrices},$$

$$\Sigma^r - \text{diagonal matrix with } \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0.$$

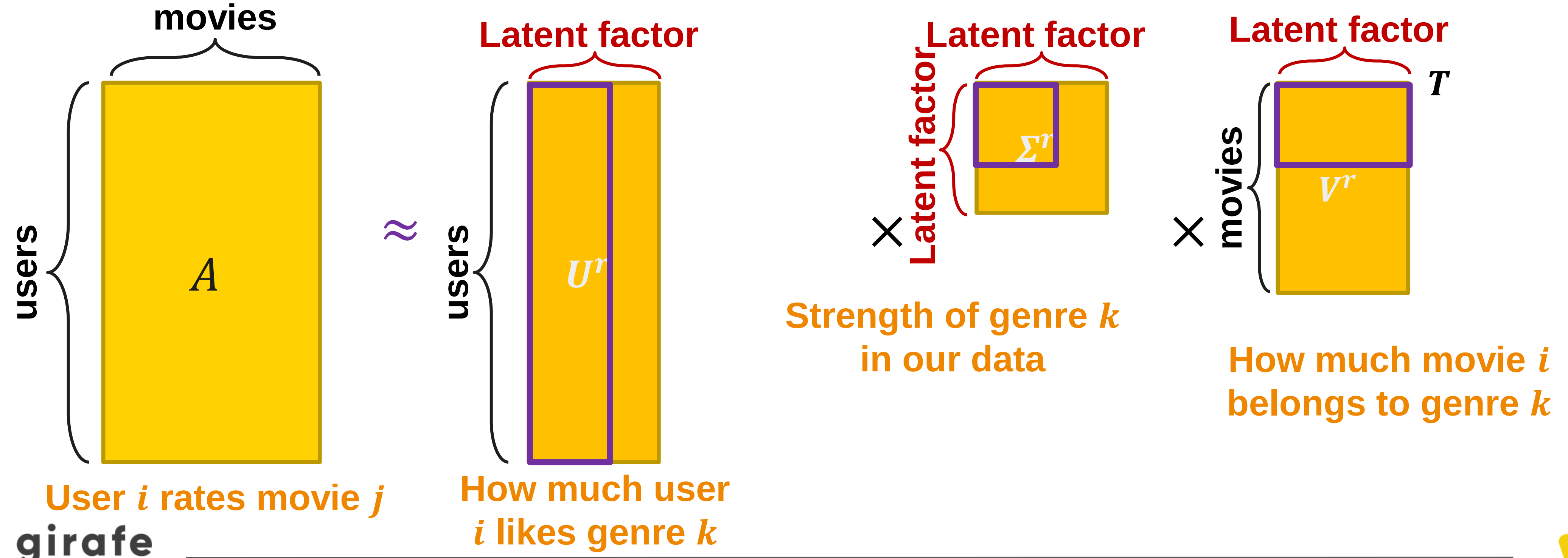


SVD: Dimensionality Reduction

$$A_{m\times n} = U^r_{m\times \mathbf{r}} \Sigma^r_{\mathbf{r}\times \mathbf{r}} (V^r_{n\times \mathbf{r}})^T, \text{ where}$$

$$U^r = [u_1 | \dots | u_r], \;\; V^r = [v_1 | \dots | v_r] - \text{orthogonal matrices},$$

$$\Sigma^r - \text{diagonal matrix with } \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0.$$



Practice!

https://colab.research.google.com/drive/1Xch-StnOTkWkhto2c8jGzqp58CfiakdD?usp=sharing