Graded Assignment 3

Math Basics for Machine Learning

Fall 2022

Instructions

This is the second graded assignment for the Math Basics for Machine Learning course. It contains three tasks. The instructions, as well as links to supplementary material, are given in the task descriptions below.

You should submit the **detailed solutions** to the tasks, as well as your code by filling in the corresponding Google form.

You can earn 10 points for this assignment. This score will contribute to your final score for this course.

You must submit your answers by **Monday, October 31, 23:59 Anywhere on Earth**. *Late submissions will* ***not*** *be accepted.*

It is the idea that you complete this assignment individually. Do not collaborate or copy answers of somebody else.

Have fun!

1 Matrix diagonalization *(1 point)*

In this task, you can use Python for computations if you wish. If you do, don't forget to attach your code to your submission.

Diagonalize the matrix below or explain why it's not possible.

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$$

If the matrix is diagonalizable, your final answer should state how to transform A into a diagonal matrix Λ.

2 Modelling population flows *(6 points)*

To answer the questions in this task, you will likely need to write some Python code. Don't forget to save and attach it to your submission.

Each year, some people from the suburbs move to the nearby city, while some fraction of the city population decides to move to the suburbs. How will the population in the city and in the suburban areas evolve over the years?

In this task, you will answer this question using a simple Linear Algebra model.

Transition matrix

Official statistics shows that every year, 3% of the city dwellers move to the suburbs. At the same time, 5% of suburbanites move to the city.

We can collect this information into a single matrix T, where the first column and the first row correspond to the city, while the second column and the second row are related to the suburbs:

$$T = \begin{bmatrix} 0.97 & 0.05 \\ 0.03 & 0.95 \end{bmatrix}$$

You can read this matrix as follows:

- *first column:* 0.97 of the city population stays in the city, while the remaining 0.03 moves to the suburbs;
- *second column:* 0.05 of the suburban population moves to the city, while the remaining 0.95 stays in the suburbs.

Suppose that in some year 0, there were x_0^{city} people living in the city and x_0^{sub} people living in the suburbs.

Given the migration statistics above, we can estimate the urban and the suburban populations for the following year as follows:

$$x_1 = Tx_0,\ x_0 = \left[x_0^{city}, x_0^{sub}\right]^T$$

Similarly, we can estimate population distributions in k years from the start:

$$\begin{aligned} cx_2 = Tx_1 = TTx_0 = T^2x_0 \\ x_3 = T^3x_0 \\ ... \\ x_k = T^kx_0. \end{aligned}$$

2.1 Population distribution next year *(0.5 points)*

Imagine that in 2022, only 25% of the population lives in the city, while 75% reside in the suburbs. Thus,

$$x_{2022} = [0.25,\ 0.75]$$

.
How will the population distribution look like in 2023?

2.2 Dynamics over the years *(2 points)*

How will the population distribution be changing over the next 100 years (assuming that the transition matrix stays the same)?

(1.5 point) Make a plot to visualize urban and suburban populations for every year from 2022 up to 2122 according to our model. Explain what you see.

(0.5 points) What will the distribution between city dwellers and suburbanites be in 2122?

2.3 A different starting point *(1 point)*

Imagine now that in 2022, 90% of the population already lives in the city (as opposed to only 25% in the example above). So, our starting distribution becomes as follows:

$$x'_{2022} = [0.9,\ 0.1]$$

.
(0.5 points) Visualize the change in the population distribution over the next 100 year again, using the new starting distribution this time, and explain what you see.

(0.5 points) What will the distribution x'_{2122} between city dwellers and suburbanites be in the year 2122 in this case? How does it compare to the distribution x_{2122} that you have gotten previously?

2.4 Eigenvalues eigenvectors *(2.5 points)*

(0.5 points) Compute eigenvalues and eigenvectors of the transition matrix T.

(2 points) Note that one of the eigenvalues of T is equal to 1. How is this related to the development of the city and suburban populations you have observed in the previous task? Explain.

3 SVD for image compression (*3 points*)

This is a programming task dedicated to SVD. You can find the assignment in the Google Colab notebook.

First, make your own copy of the notebook (*File* → *Save a copy in Drive*) or download the notebook to your machine if you prefer to work locally (*File* → *Download*).

Then, implement your solutions to the tasks formulated in the notebook. You can add **code cells** to write some code and **text cells** in case you want to include additional explanations to your answers in plain English.

Finally, save your notebook as a .pdf file and attach it to the submission form. **Make sure that all the cells are executed and all relevant outputs are being printed out**.