Math Refresher for DS

Practical Session 5

Today

- Graded assignment 1
- Least Squares (continued)
- More on coordinates change

Graded Assignment 1 is OUT

- Google-form, link in chat.
- Submit answers and detailed solutions.

- Submission deadline: Tuesday, October 26, 19:00 Moscow time.
- Late submissions won’t be accepted.

Reminder: We Have a Course Repo!

- https://github.com/girafe-ai/math-basics-for-ai
    - Slides
    - Links to colab-notebooks
    - Links to lectures / practical session recordings
    - Additional material

Where we stopped last time...

Method of Least Squares

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  $Xw = \hat{y}$ has a solution w^*
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  and

  $\hat{y}$ is as close to y as possible
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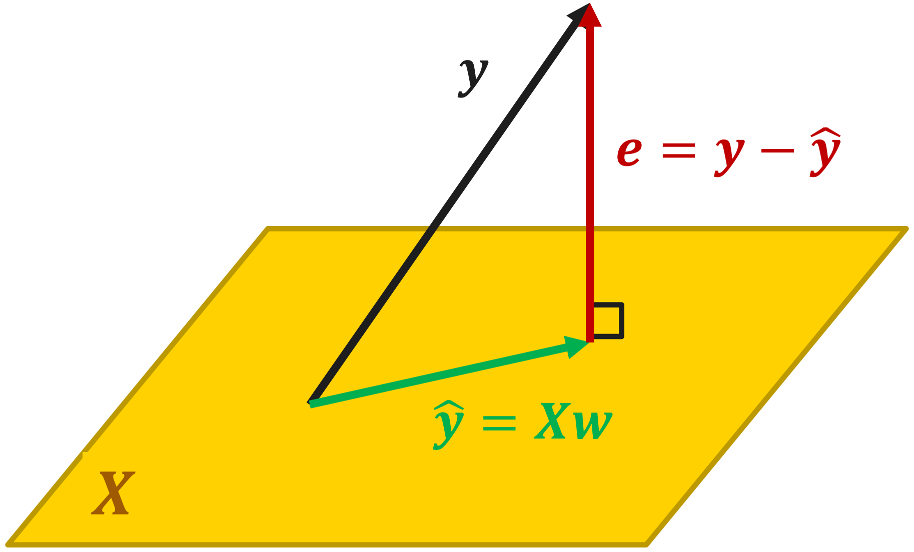

- What do $\hat{y}$ and $\boldsymbol{e}$ look like?

Orthogonal Decomposition

- Consider a vector space V and a subspace W.
- $x \in V$ and $x \notin W$.

- x can be decomposed into a sum of two vectors:

$$x = x_W + x_{W^\perp}, \qquad x_W \in W, x_{W^\perp} \in W^\perp$$

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- x_W is the closest vector to x in W.

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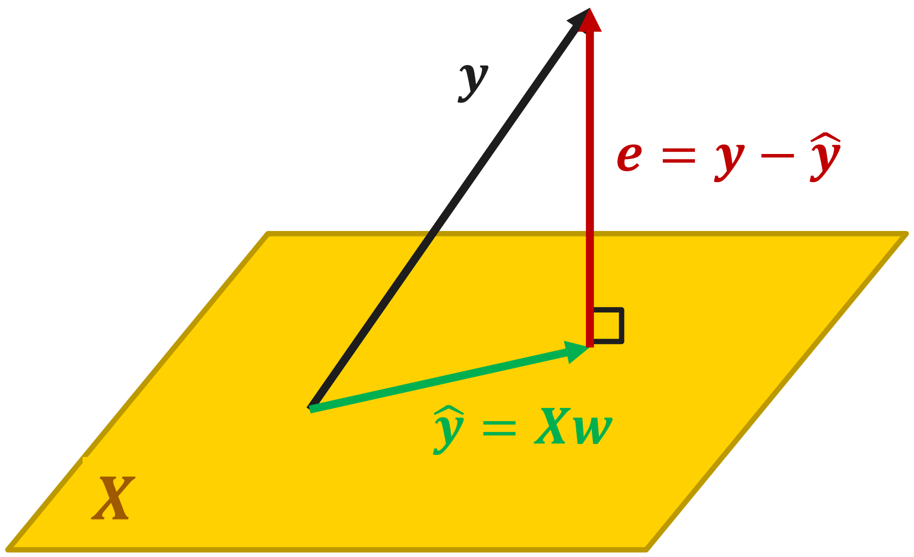

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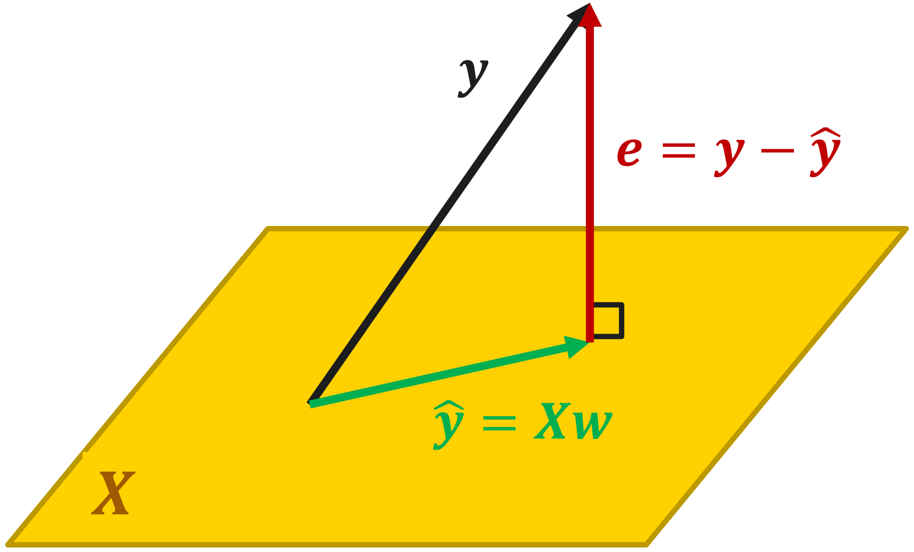

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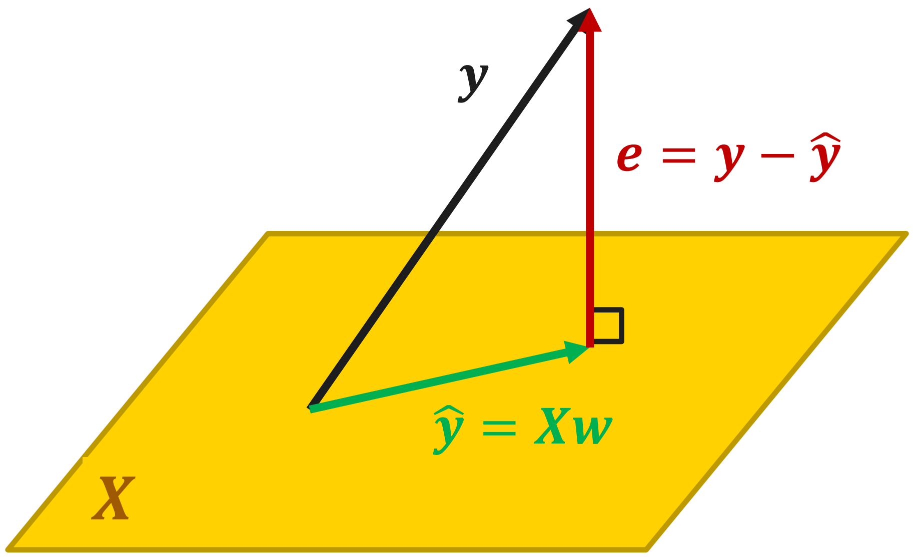

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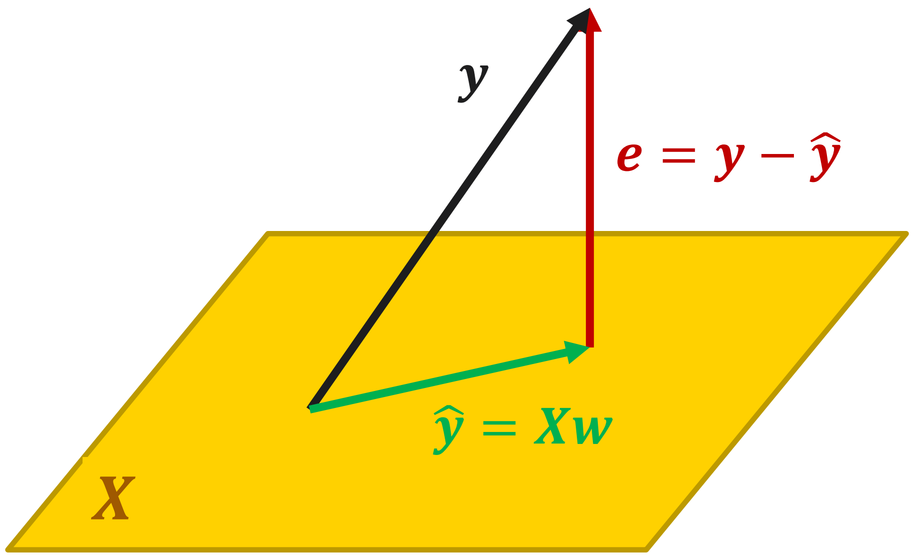
$$Xw^* = \hat{y} = y - e$$

$\hat{y}$ - orthogonal projection of y onto $col(X)$
$w^* = ?$ – optimal weights



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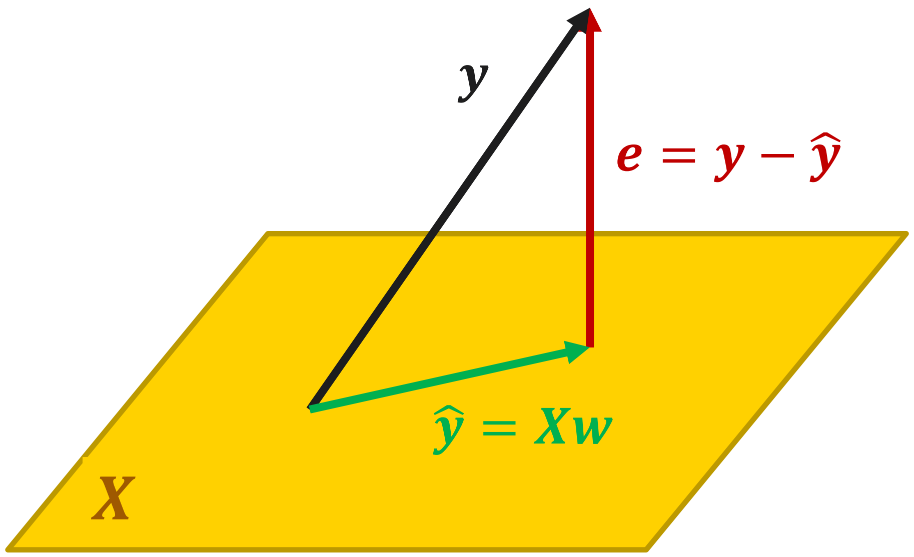

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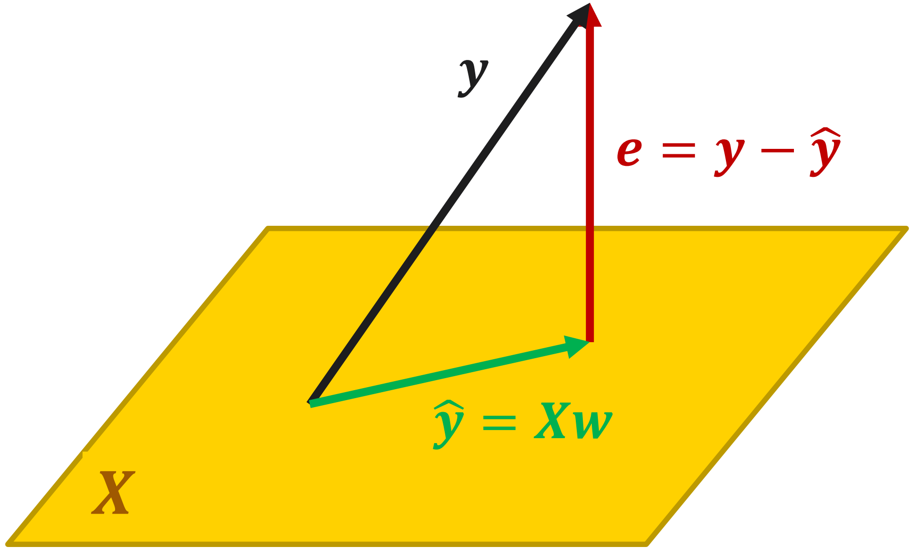

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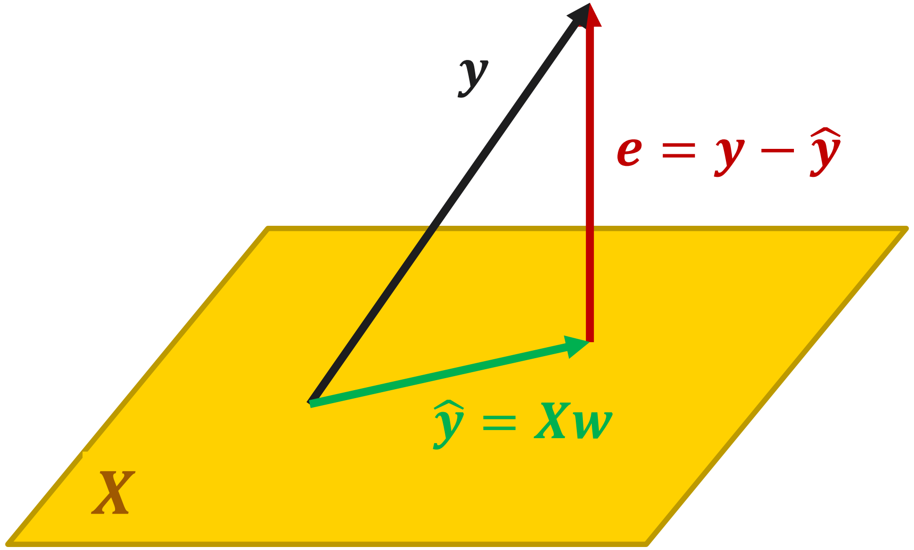

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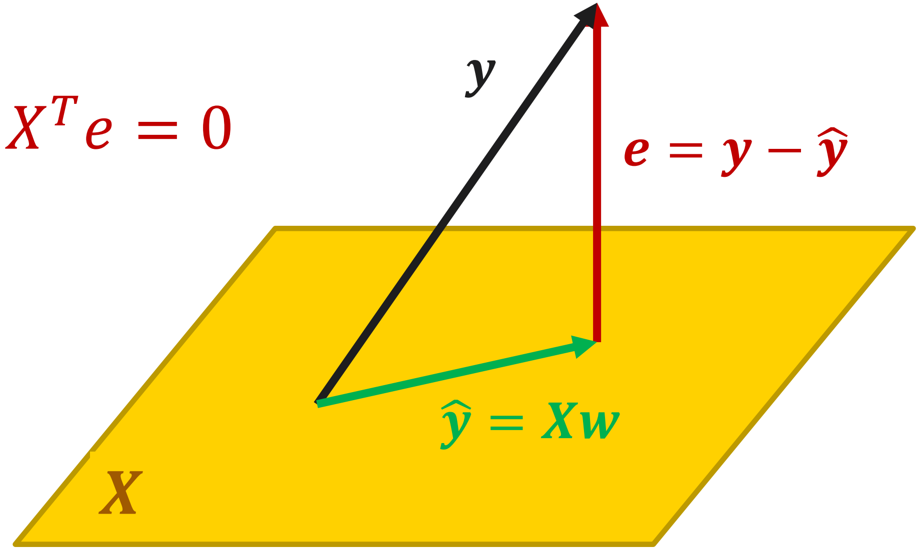

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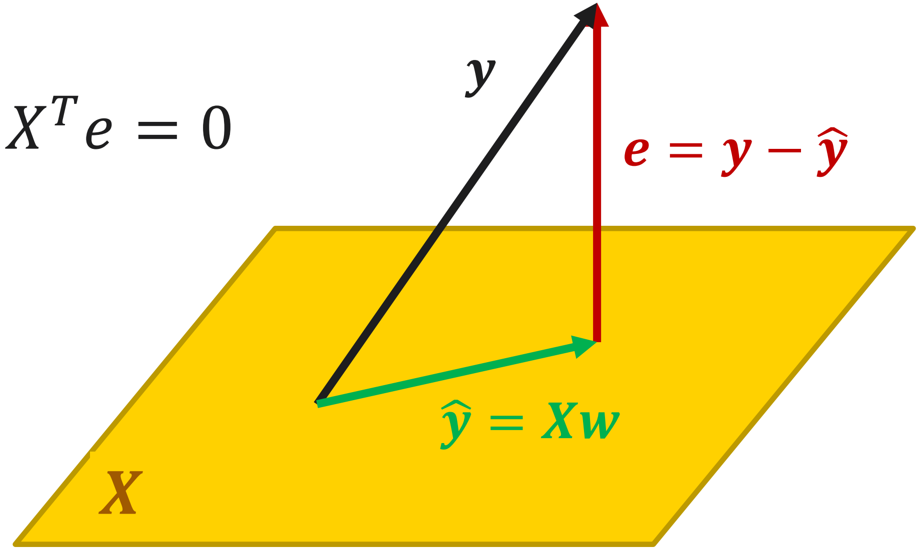

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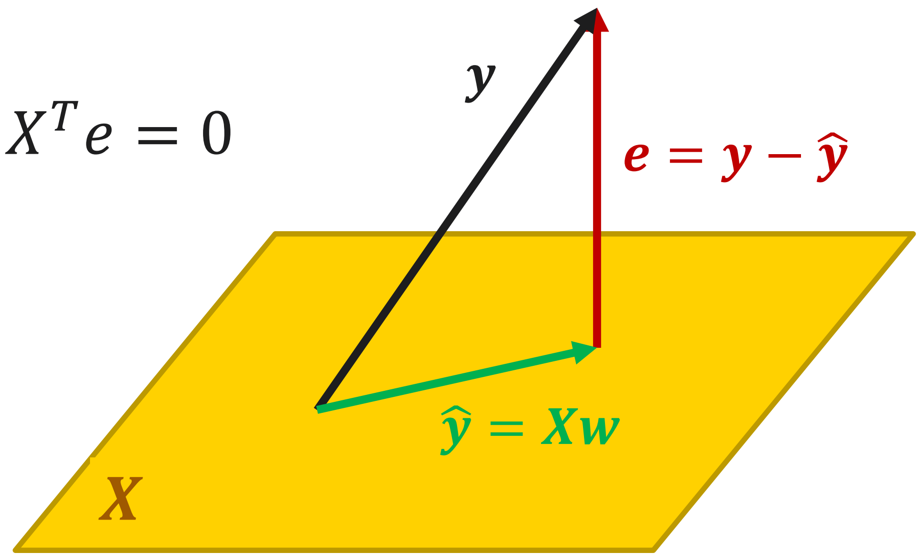

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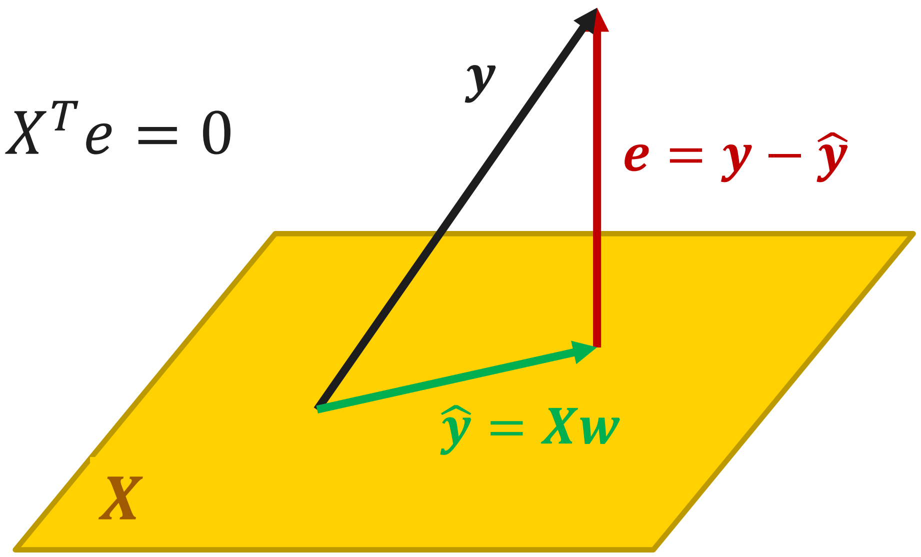

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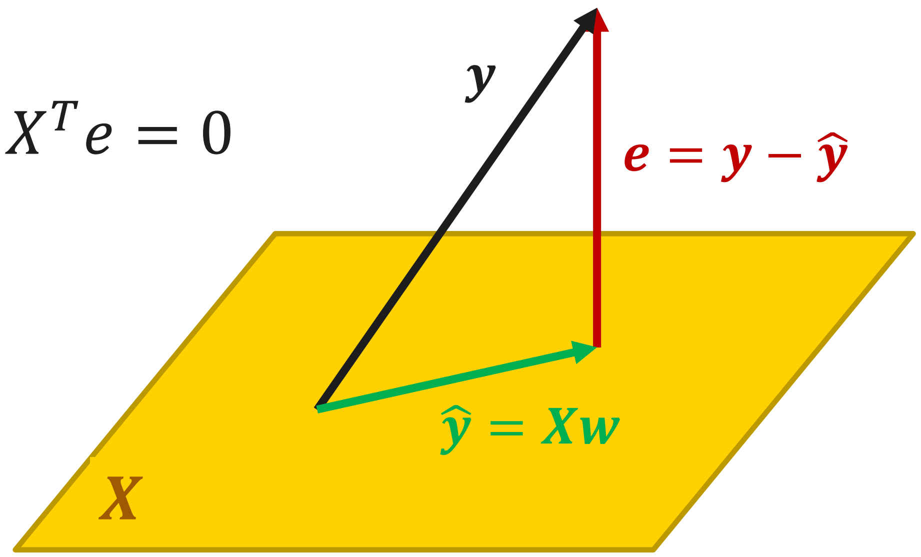

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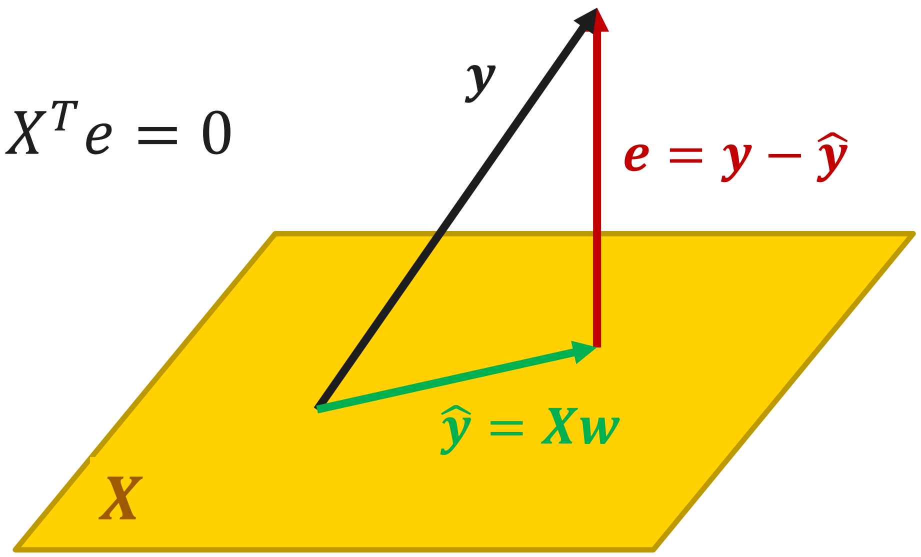

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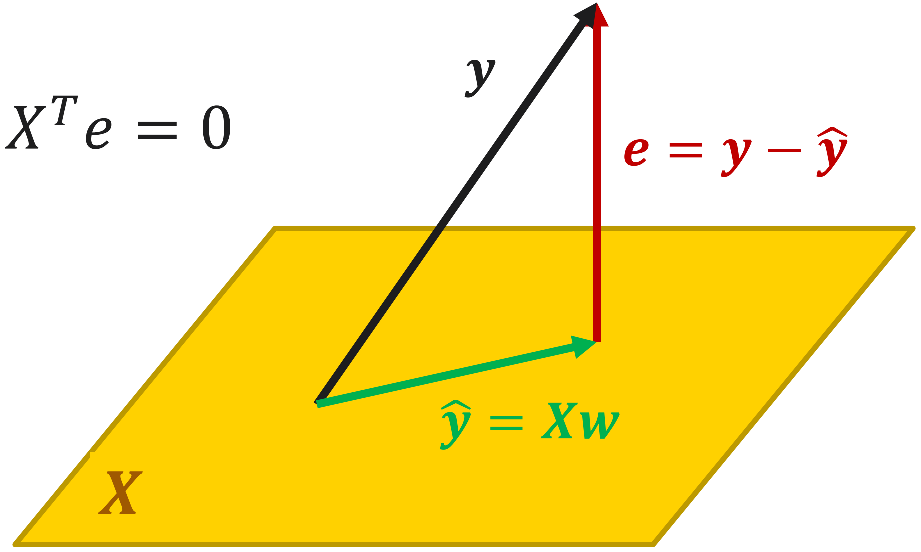

$$\hat{\boldsymbol{y}} = Xw^* = \underbrace{\boldsymbol{X}\left(\boldsymbol{X}^T \boldsymbol{X}\right)^{-1} \boldsymbol{X}^T}_{\text{projection matrix}} \boldsymbol{y}$$

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So, $(X^T X)^{-1}$ exists.

Toy Example

- Observations (x_i, y_i):

$$(1,1), \qquad (2,3), \qquad (3,2)$$

- With least squares, fit a line $y = w_0 + w_1 x$ through these points.

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$$w = (X^T X)^{-1} X^T y = \begin{bmatrix}1\\0.5\end{bmatrix} \rightarrow \text{regression line } y = 1 + 0.5x.$$

Let’s practice more!

https://colab.research.google.com/drive/16UqY0p5h5324atAIQ3kWilF7AZTa5mbX?usp=sharing

Coordinates Change

Change of Basis for Vectors

- V – a vector space.
- $B = \{b_1, \dots, b_n\}$ – current basis, $S = \{s_1, \dots, s_n\}$ – new basis.

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$$x_B = Mx_S,\ M = M_{B \to S} = \big[[s_1]_B \mid \dots \mid [s_n]_B\big] \text{ – transition matrix.}$$

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- We also established that

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- But vectors aren't the only things with coordinates...

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Change of Basis for Linear Transforms

- Consider a linear transform A.
- It’s defined by its matrix: columns = what happens to basis vectors.

- Example: rotation

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

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Change of Basis for Linear Transforms

- Consider a linear transform A.
- It's defined by its matrix: columns = what happens to basis vectors.

- Example: rotation

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- $S = \left\{ s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ – another basis.

How would A look like in this basis?

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Change of Basis for Linear Transforms

- A – linear transform;
- $B = \{b_1, \dots, b_n\}$ – current basis, $S = \{s_1, \dots, s_n\}$ – new basis;
- $M = M_{B \to S}$ – transition matrix.

- x – some vector.

Change of Basis for Linear Transforms

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$$[x']_B = [A]_B \cdot [x]_B$$

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$$[x']_s = \quad [\boldsymbol{A}]_{\boldsymbol{S}} \quad \cdot [x]_S$$

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$$[x']_B = [A]_B \cdot [x]_B$$

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$$[x']_B = [A]_B \cdot [x]_B$$

$$[x']_S = M^{-1}[A]_B M \cdot [x]_S$$

$$[\boldsymbol{A}]_{\boldsymbol{S}} =$$

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- x – some vector.

$$[x']_B = [A]_B \cdot [x]_B$$

$$[x']_S = M^{-1}[A]_B M \cdot [x]_S$$

$$\boldsymbol{[A]_S = M^{-1}[A]_B M}$$

Change of Basis for Linear Transforms

- Back to our example:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad S\left\{s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} -$$ new basis.

$$[A]_S = M^{-1}AM =$$

Change of Basis for Linear Transforms

- Back to our example:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad S\left\{s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} -$$ new basis.

$$[A]_S = M^{-1}AM = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} =$$

Change of Basis for Linear Transforms

- Back to our example:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad S\left\{s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} - \text{ new basis.}$$

$$[A]_S = M^{-1}AM = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} =$$

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$$= \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} =$$

Change of Basis for Linear Transforms

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$$[A]_S = M^{-1}AM = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix}$$

Change of Basis for Linear Transforms

$$[A]_E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \; [A]_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix}, \; S = \left\{ s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$ – new basis;

$$x_S = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad x_S' = [A]_S \cdot x_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$x_E = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \qquad x_E' = [A]_E \cdot x_E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

Change of Basis for Linear Transforms

$[A]_E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $[A]_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix}$, $S = \left\{ s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ – new basis;

$$x_S = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad x'_S = [A]_S \cdot x_S =$$

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Change of Basis for Linear Transforms

$[A]_E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $[A]_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix}$, $S = \left\{ s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ – new basis;

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Change of Basis for Linear Transforms

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Change of Basis for Linear Transforms

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$$x_E =$$

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Change of Basis for Linear Transforms

$$[A]_E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ [A]_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix},\ S = \left\{ s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$ – new basis;

$$x_S = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad x'_S = [A]_S \cdot x_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$x_E = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \qquad x'_E =$$

Change of Basis for Linear Transforms

- Another example:

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, \qquad S = \left\{ s_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} -$$ new basis

$$[A]_S = M^{-1} A M =$$

Change of Basis for Linear Transforms

- Another example:

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, \qquad S = \left\{ s_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} -$$ new basis

$$[A]_S = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} =$$

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We get a diagonal matrix, it’s easier to work with it!