Partial Derivatives

- Derivative for univariate functions:

$$y = f(x), \qquad \frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Partial Derivatives

- Derivative for univariate functions:

$$y = f(x), \qquad \frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

- **Partial derivative** for multivariate functions:

$$y = f(x_1, \dots, x_n), \qquad \frac{\partial y}{\partial x_i} = f'_{x_i} = \lim_{\Delta x \to 0} \frac{f(x_1, \dots, x_i + \Delta x_i, \dots, x_n) - f(x_1, \dots, x_i, \dots x_n)}{\Delta x_i}$$

Compute derivative with respect to x_i
regarding all other variables as constants.

Partial Derivatives - Example

- Let $f(x, y) = x^3 + x^2y^3 - 2y^2$.
- Find $f_x(2,1)$ and $f_y(2,1)$.

girafe ai

Partial Derivatives - Example

- Let $f(x, y) = x^3 + x^2y^3 - 2y^2$.
- Find $f_x(2,1)$ and $f_y(2,1)$.

$$f_x(x, y) = \frac{\partial f}{\partial x} =$$

$$f_y(x, y) = \frac{\partial f}{\partial y} =$$

girafe ai

Partial Derivatives - Example

- Let $f(x,y) = x^3 + x^2y^3 - 2y^2$.
- Find $f_x(2,1)$ and $f_y(2,1)$.

$$f_x(x,y) = \frac{\partial f}{\partial x} = 3x^2 + 2xy^3$$

$$f_y(x,y) = \frac{\partial f}{\partial y} =$$

Partial Derivatives - Example

- Let $f(x, y) = x^3 + x^2y^3 - 2y^2$.
- Find $f_x(2,1)$ and $f_y(2,1)$.

$$f_x(x, y) = \frac{\partial f}{\partial x} = 3x^2 + 2xy^3 \Big|_{(2,1)} =$$

$$f_y(x, y) = \frac{\partial f}{\partial y} =$$

girafe ai

Partial Derivatives - Example

- Let $f(x, y) = x^3 + x^2y^3 - 2y^2$.
- Find $f_x(2,1)$ and $f_y(2,1)$.

$$f_x(x, y) = \frac{\partial f}{\partial x} = 3x^2 + 2xy^3 \Big|_{(2,1)} = 12 + 4 = 16$$

$$f_y(x, y) = \frac{\partial f}{\partial y} =$$

girafe ai

Partial Derivatives - Example

- Let $f(x, y) = x^3 + x^2y^3 - 2y^2$.
- Find $f_x(2,1)$ and $f_y(2,1)$.

$$f_x(x, y) = \frac{\partial f}{\partial x} = 3x^2 + 2xy^3 \Big|_{(2,1)} = 12 + 4 = 16$$

$$f_y(x, y) = \frac{\partial f}{\partial y} = 3x^2y^2 - 4y$$

Partial Derivatives - Example

- Let $f(x, y) = x^3 + x^2y^3 - 2y^2$.
- Find $f_x(2,1)$ and $f_y(2,1)$.

$$f_x(x, y) = \frac{\partial f}{\partial x} = 3x^2 + 2xy^3 \Big|_{(2,1)} = 12 + 4 = 16$$

$$f_y(x, y) = \frac{\partial f}{\partial y} = 3x^2y^2 - 4y \Big|_{(2,1)} =$$

Partial Derivatives - Example

- Let $f(x,y) = x^3 + x^2y^3 - 2y^2$.
- Find $f_x(2,1)$ and $f_y(2,1)$.

$$f_x(x,y) = \frac{\partial f}{\partial x} = 3x^2 + 2xy^3 \Big|_{(2,1)} = 12 + 4 = 16$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = 3x^2y^2 - 4y \Big|_{(2,1)} = 12 - 4 = 8$$

Higher Derivatives

- Consider $f(x, y)$.

- $f'_x(x, y),\ f'_y(x, y)$ – also functions of two variables.
  We can compute their partial derivatives!

Higher Derivatives

- Consider $f(x, y)$.
- $f'_x(x, y),\ f'_y(x, y)$ – also functions of two variables.
  We can compute their partial derivatives!

$$(f'_x)'_x = f''_{xx}(x, y) = \frac{\partial^2 f}{\partial x^2}, \qquad (f'_x)'_y = f''_{xy}(x, y) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\left(f'_y\right)'_y = f''_{yy}(x, y) = \frac{\partial^2 f}{\partial y^2}, \qquad \left(f'_y\right)'_x = f''_{yx}(x, y) = \frac{\partial^2 f}{\partial x \partial y}$$

girafe ai

Higher Derivatives - Example

- Consider $f(x,y) = x^3 + x^2y^3 - 2y^2$.

$$f_x(x,y) = \frac{\partial f}{\partial x} = 3x^2 + 2xy^3, \qquad f_y(x,y) = \frac{\partial f}{\partial y} = 3x^2y^2 - 4y$$

$$f''_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = \qquad , \qquad f''_{xy}(x,y) = \frac{\partial^2 f}{\partial y \partial x} =$$

$$f''_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = \qquad f''_{yx}(x,y) = \frac{\partial^2 f}{\partial x \partial y} =$$

Higher Derivatives - Example

- Consider $f(x,y) = x^3 + x^2y^3 - 2y^2$.

$$f_x(x,y) = \frac{\partial f}{\partial x} = 3x^2 + 2xy^3, \qquad f_y(x,y) = \frac{\partial f}{\partial y} = 3x^2y^2 - 4y$$

$$f''_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = 6x + 2y^3, \qquad f''_{xy}(x,y) = \frac{\partial^2 f}{\partial y \partial x} =$$

$$f''_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = \qquad f''_{yx}(x,y) = \frac{\partial^2 f}{\partial x \partial y} =$$

Higher Derivatives - Example

- Consider $f(x,y) = x^3 + x^2y^3 - 2y^2$.

$$f_x(x,y) = \frac{\partial f}{\partial x} = 3x^2 + 2xy^3, \qquad f_y(x,y) = \frac{\partial f}{\partial y} = 3x^2y^2 - 4y$$

$$f''_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = 6x + 2y^3, \qquad f''_{xy}(x,y) = \frac{\partial^2 f}{\partial y \partial x} = 6xy^2$$

$$f''_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = \qquad f''_{yx}(x,y) = \frac{\partial^2 f}{\partial x \partial y} =$$

Higher Derivatives - Example

- Consider $f(x,y) = x^3 + x^2y^3 - 2y^2$.

$$f_x(x,y) = \frac{\partial f}{\partial x} = 3x^2 + 2xy^3, \qquad f_y(x,y) = \frac{\partial f}{\partial y} = 3x^2y^2 - 4y$$

$$f''_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = 6x + 2y^3, \qquad f''_{xy}(x,y) = \frac{\partial^2 f}{\partial y \partial x} = 6xy^2$$

$$f''_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = 6x^2y - 4, \qquad f''_{yx}(x,y) = \frac{\partial^2 f}{\partial x \partial y} =$$

Higher Derivatives - Example

- Consider $f(x, y) = x^3 + x^2y^3 - 2y^2$.

$$f_x(x, y) = \frac{\partial f}{\partial x} = 3x^2 + 2xy^3, \qquad f_y(x, y) = \frac{\partial f}{\partial y} = 3x^2y^2 - 4y$$

$$f''_{xx}(x, y) = \frac{\partial^2 f}{\partial x^2} = 6x + 2y^3, \qquad f''_{xy}(x, y) = \frac{\partial^2 f}{\partial y \partial x} = 6xy^2$$

$$f''_{yy}(x, y) = \frac{\partial^2 f}{\partial y^2} = 6x^2y - 4, \qquad f''_{yx}(x, y) = \frac{\partial^2 f}{\partial x \partial y} = 6xy^2$$

Hessian

- A matrix of second derivatives.
- $f(x_1, \dots, x_n)$

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 x_n} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n x_1} & \frac{\partial^2 f}{\partial x_n x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

Hessian - Example

- $f(x,y) = x^3 + x^2y^3 - 2y^2$

$$f''_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = 6x + 2y^3, \qquad f''_{xy}(x,y) = \frac{\partial^2 f}{\partial y \partial x} = 6xy^2$$

$$f''_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = 6x^2y - 4, \qquad f''_{yx}(x,y) = \frac{\partial^2 f}{\partial x \partial y} = 6xy^2$$

$$H = \left(\qquad \right)$$

girafe ai

Hessian - Example

- $f(x, y) = x^3 + x^2y^3 - 2y^2$

$$f''_{xx}(x, y) = \frac{\partial^2 f}{\partial x^2} = 6x + 2y^3, \qquad f''_{xy}(x, y) = \frac{\partial^2 f}{\partial y \partial x} = 6xy^2$$

$$f''_{yy}(x, y) = \frac{\partial^2 f}{\partial y^2} = 6x^2y - 4, \qquad f''_{yx}(x, y) = \frac{\partial^2 f}{\partial x \partial y} = 6xy^2$$

$$H = \begin{pmatrix} 6x + 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y - 4 \end{pmatrix}$$

Extrema

- Univariate case: a stationary point x_0 is a local minimum (maximum) if

$$f''(x_0) > 0 \ (f''(x_0) < 0).$$

Extrema

- Univariate case: a stationary point x_0 is a local minimum (maximum) if

$$f''(x_0) > 0 \; (f''(x_0) < 0).$$

- Multivariate case:

x_0 is a stationary point: $\nabla f(x_0) = 0$

$H -$ Hessian.

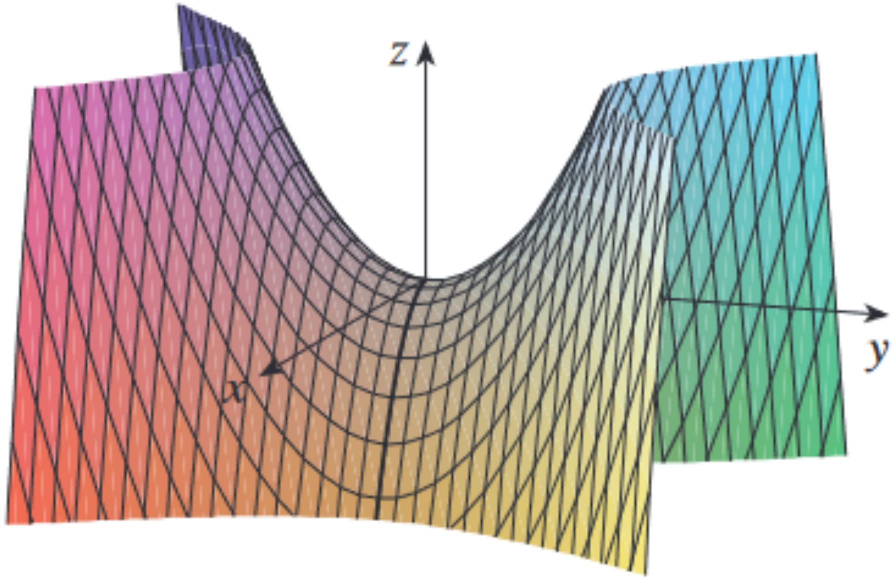
Then, if H is a positive-definite matrix, then x_0 is a local minimum.

If H is a negative-definite matrix, then x_0 is a local maximum.

If $\det H = 0$, we need to check manually.

Otherwise, x_0 is a saddle point.

Saddle Points



Positive vs Negative Definite Matrices

- A matrix A is called positive-definite if

$$x^T A x > 0 \quad \forall x \neq 0$$

- A matrix A is called negative-definite if

$$x^T A x < 0 \quad \forall x \neq 0$$

Positive vs Negative Definite Matrices

- How to check if a matrix is positive (negative) definite?

Positive vs Negative Definite Matrices

- How to check if a matrix is positive (negative) definite?

  Check principal minors D_k!

  For positive definite:

  $D_1 > 0, \quad D_2 > 0, \quad \dots, D_n > 0$

  For negative definite:

  $D_1 < 0, \quad D_2 > 0, \quad D_3 < 0, \dots$

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 x_1} & \cdots & \frac{\partial^2 f}{\partial x_n x_1} \\ \frac{\partial^2 f}{\partial x_1 x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n x_1} & \frac{\partial^2 f}{\partial x_n x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

Extrema – Example

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

Extrema – Example

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

$$\nabla f = \left(f'_x, f'_y\right) =$$

Extrema – Example

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

$$\nabla f = \left(f_x', f_y'\right) = (4x^3 - 4y, \qquad 4y^3 - 4x)$$

Extrema – Example

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

$$\nabla f = \left(f_x', f_y'\right) = (4x^3 - 4y, \qquad 4y^3 - 4x)$$

Stationary points: $\nabla f = 0 \iff$

Extrema – Example

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

$$\nabla f = \left(f_x', f_y'\right) = (4x^3 - 4y, \qquad 4y^3 - 4x)$$

Stationary points: $\nabla f = 0 \iff$

$$\begin{matrix} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{matrix} \iff$$

Extrema – Example

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

$$\nabla f = \left(f'_x, f'_y\right) = (4x^3 - 4y, \qquad 4y^3 - 4x)$$

Stationary points: $\nabla f = 0 \iff$

$$\begin{matrix} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{matrix} \iff \begin{matrix} y = x^3 \\ x^9 - x = 0 \end{matrix} \iff$$

Extrema – Example

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

$$\nabla f = \left(f'_x, f'_y\right) = (4x^3 - 4y, \qquad 4y^3 - 4x)$$

Stationary points: $\nabla f = 0 \quad \Leftrightarrow$

$$\begin{matrix} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} y = x^3 \\ x^9 - x = 0 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} y = x^3 \\ x(x^8 - 1) = 0 \end{matrix}$$

girafe ai

Extrema – Example

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

$$\nabla f = \left(f_x', f_y'\right) = (4x^3 - 4y, \qquad 4y^3 - 4x)$$

Stationary points: $\nabla f = 0 \quad \Leftrightarrow$

$$\begin{matrix} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} y = x^3 \\ x^9 - x = 0 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} y = x^3 \\ x(x^8 - 1) = 0 \end{matrix}$$

Stationary points: $(0,0)$, $(-1,-1)$, $(1,1)$

girafe ai

Extrema – Example

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

$$\nabla f = \left(f'_x, f'_y\right) = (4x^3 - 4y, \qquad 4y^3 - 4x)$$

Stationary points: $(0, 0)$, $(-1, -1)$, $(1, 1)$

$$H = \left(\qquad\qquad \right)$$

girafe ai

Extrema – Example

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

$$\nabla f = \left(f_x', f_y'\right) = (4x^3 - 4y, \qquad 4y^3 - 4x)$$

Stationary points: $(0, 0)$, $(-1, -1)$, $(1, 1)$

$$H = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

Extrema – Example

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

$$\nabla f = \left(f'_x, f'_y\right) = (4x^3 - 4y, \qquad 4y^3 - 4x)$$

Stationary points: $(0,0),\ (-1,-1),\ (1,1)$

$$H = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

$\det H = 144x^2y^2 - 16|_{(0;0)} < 0,\ (0,0)$ – saddle point.

Extrema – Example

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

$$\nabla f = \left(f'_x, f'_y\right) = (4x^3 - 4y, \qquad 4y^3 - 4x)$$

Stationary points: $(0,0),\ (-1,-1),\ (1,1)$

$$H = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

$$\det H = 144x^2y^2 - 16 \qquad\qquad , \qquad \det H_1 = 12x^2$$

girafe ai

Extrema – Example

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

$$\nabla f = \left(f'_x, f'_y\right) = (4x^3 - 4y, \qquad 4y^3 - 4x)$$

Stationary points: $(0,0)$, $(-1,-1)$, $(1,1)$

$$H = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

$$\det H = 144x^2y^2 - 16\Big|_{(1;1)} > 0, \qquad \det H_1 = 12x^2\Big|_{(1,1)} > 0$$

girafe ai

Extrema – Example

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

$$\nabla f = \left(f'_x, f'_y\right) = (4x^3 - 4y, \qquad 4y^3 - 4x)$$

Stationary points: $(0, 0)$, $(-1, -1)$, $(1, 1)$

$$H = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

$$\det H = 144x^2y^2 - 16\,\Big|_{(1;1)} > 0, \qquad \det H_1 = 12x^2\,\Big|_{(1,1)} > 0$$

$(1, 1)$ – local minimum.

girafe ai

Extrema – Example

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

$$\nabla f = \left(f_x', f_y'\right) = (4x^3 - 4y, \qquad 4y^3 - 4x)$$

Stationary points: $(0,0),\ (-1,-1),\ (1,1)$

$$H = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

$$\det H = 144x^2y^2 - 16 \qquad , \qquad \det H_1 = 12x^2$$

girafe ai

Extrema – Example

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

$$\nabla f = \left(f_x', f_y'\right) = (4x^3 - 4y, \qquad 4y^3 - 4x)$$

Stationary points: $(0,0)$, $(-1,-1)$, $(1,1)$

$$H = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

$$\det H = 144x^2y^2 - 16\Big|_{(-1;-1)} > 0, \qquad \det H_1 = 12x^2\Big|_{(-1,-1)} > 0$$

$(-1,-1)$ – local minimum.

girafe ai

Extrema – Example

$$f(x, y) = x^4 + y^4 - 4xy + 1$$