Math Refresher for DS

Practical Session 3

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Linear Subspace – Quiz

- Is the following a linear subspace?
    1. All vectors in $\mathbb{R}^n$ with integer coordinates?

Linear Subspace – Quiz

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       No, this set is not closed on scalar multiplication.

Linear Subspace – Quiz

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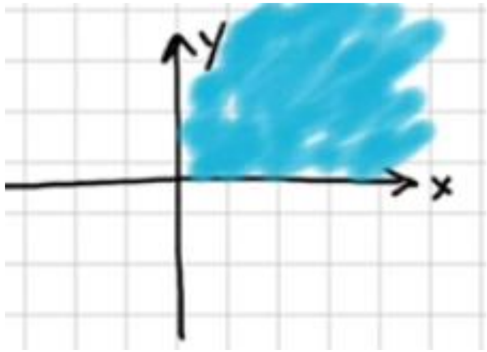
  2. All vectors in $\mathbb{R}^n$ with positive coordinates?



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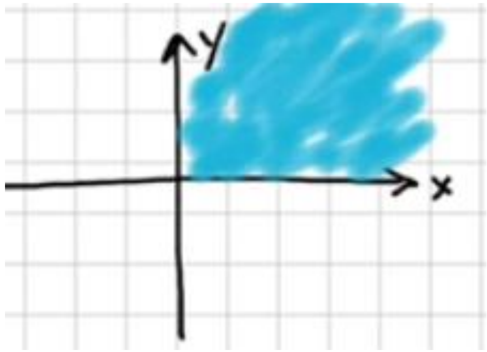

1. All vectors in $\mathbb{R}^n$ with integer coordinates?
   No, this set is not closed on scalar multiplication.

2. All vectors in $\mathbb{R}^n$ with positive coordinates?
   No, this set is not closed on vector addition and scalar multiplication.

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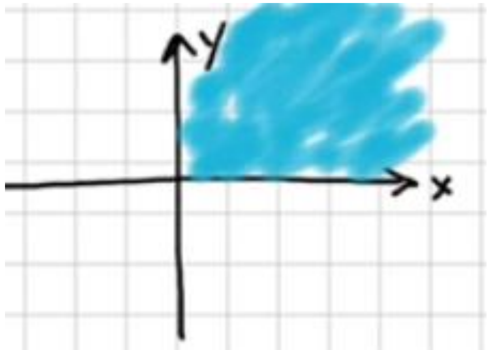

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3. All vectors in $\mathbb{R}^n$ with first coordinate equal to a given number c?

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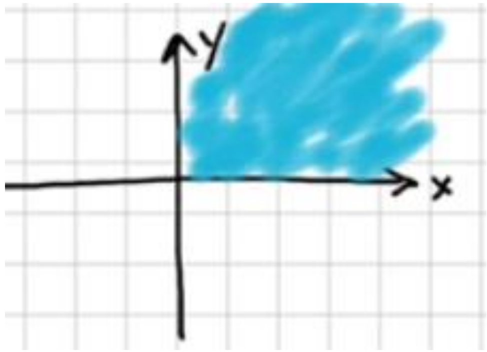

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Linear Subspace – Quiz

- Consider $U = \left\{ x = \begin{bmatrix} 2r + q \\ 3r \\ r - q \end{bmatrix} \forall\, r, q \in \mathbb{R} \right\}$. Is U a linear subspace of $\mathbb{R}^3$?

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- Note that $x = \begin{bmatrix} 2r+q \\ 3r \\ r-q \end{bmatrix} \forall\, r, q \in \mathbb{R} \;\Leftrightarrow$

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- Note that $x = \begin{bmatrix} 2r+q \\ 3r \\ r-q \end{bmatrix} \forall\, r, q \in \mathbb{R} \Leftrightarrow x = r\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + q\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \forall\, r, q \in \mathbb{R}$.

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- Note that $x = \begin{bmatrix} 2r+q \\ 3r \\ r-q \end{bmatrix} \forall\, r, q \in \mathbb{R} \Leftrightarrow x = r\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + q\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \forall\, r, q \in \mathbb{R}$.

- In other words, $U = span\left(\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$.

Linear Subspace – Quiz

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- Consider $U = span\left(\begin{bmatrix}2\\3\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right)$. Is U a linear subspace of $\mathbb{R}^3$?
  - $0 \in U$
  - $\forall x, y \in U \; (x + y) \in U$
  - $\forall x \in U \;\; \lambda x \in U \; \forall \lambda \in \mathbb{R}$

Linear Subspace – Quiz

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  - $0 \in U$
  - $\forall x, y \in U \ (x + y) \in U$
  - $\forall x \in U \ \ \lambda x \in U \ \forall \lambda \in \mathbb{R}$

$\Longrightarrow$ Yes, U is a subspace of $\mathbb{R}^3$!

Linear Independence - Example

- Consider vectors

$$b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ b_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},\ b_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

- Are they linearly independent?

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- Are they linearly independent?

- We need to check if it's possible to find $\lambda_1, \lambda_2, \lambda_3$ with at least one $\lambda_i \neq 0$ such that

$$\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 = 0$$

Linear Independence - Example

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$$\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 = 0 \Leftrightarrow$$

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \lambda_1 + \lambda_2 + 2\lambda_3 = 0 \\ \lambda_1 + 2\lambda_2 + 3\lambda_3 = 0 \end{cases} \Leftrightarrow$$

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vectors b_1, b_2, b_3 are linearly independent.

Change of Coordinates - Example

- $b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ b_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \ b_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$ are linearly independent.

- Therefore, $B = \{b_1, b_2, b_3\}$ - basis in $\mathbb{R}^3$.

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- Therefore, $B = \{b_1, b_2, b_3\}$ - basis in $\mathbb{R}^3$.
- How do we go from the standard basis $E = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ to B?

Change of Coordinates - Example

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- How do we go from the standard basis $E = \left\{e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ to B?
    - $x_E = [6, 9, 14\ ],\quad x_B = [x_1, x_2, x_3] =?$

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Change of Coordinates

- There was a small typo in the lecture!

Coordinate Change: Matrix Notation

- Result obtained before:

  $e_1, \dots, e_n$ - old basis
  $e'_1, \dots, e'_n$ - new basis

  $x_{old} = [x_1, \dots, x_n], \qquad x_{new} = [x'_1, \dots, x'_n]$

$$\boldsymbol{e'_i}$$

$$\boldsymbol{x_{old}} \quad \begin{matrix} x_1 = x'_1\alpha_{11} + \cdots + x'_i\alpha_{1i} + \cdots + x'_n\alpha_{1n} \\ x_2 = x'_1\alpha_{21} + \cdots + x'_i\alpha_{2i} + \cdots + x'_n\alpha_{2n} \\ \vdots \\ x_n = x'_1\alpha_{n1} + \cdots + x'_i\alpha_{ni} + \cdots + x'_n\alpha_{nn} \end{matrix}$$

$$\boldsymbol{x_{new}}$$

- Transition matrix: columns = coordinates of the new basis in the old one.

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{21} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix}$$

$$x_{old} = A^T x_{new}$$

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  e'_i

  x_{old}

$$\begin{aligned} x_1 &= x'_1\alpha_{11} + \cdots + x'_i\alpha_{1i} + \cdots + x'_n\alpha_{1n} \\ x_2 &= x'_1\alpha_{21} + \cdots + x'_i\alpha_{2i} + \cdots + x'_n\alpha_{2n} \\ &\vdots \\ x_n &= x'_1\alpha_{n1} + \cdots + x'_i\alpha_{ni} + \cdots + x'_n\alpha_{nn} \end{aligned}$$

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$$x_{old} = A^{\cancel{T}} x_{new}$$

Coordinate Change: Example (again)

- Consider $\mathbb{R}^2$ with basis $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- New basis: $e'_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ e'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$
- $x_{old} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},\ x_{new} = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = ?$

$$A = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ -1 \end{bmatrix} = x_{old} = A^{\cancel{T}} x_{new} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}$$

$$x_{new} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

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Change of Coordinates - Example

- $B = \left\{ b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ b_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},\ b_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ - basis.
- $x_E = [6, 9, 14\ ]^T, \quad x_B = [x_1, x_2, x_3]^T = ?$

$$x_E = A_{E \to B} \cdot x_B$$

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$A_{E\to B}$ – transition matrix (columns = coordinates of b_1, b_2, b_3 in E).

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$$A_{E \to B} =$$

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- $x_E = [6, 9, 14\,]^T, \quad x_B = [x_1, x_2, x_3]^T = ?$

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$A_{E \to B}$ – transition matrix (columns = coordinates of b_1, b_2, b_3 in E).

$$A_{E \to B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Change of Coordinates - Example

- $B = \left\{ b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ b_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},\ b_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ - basis.
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$$x_E = A_{E \to B} \cdot x_B$$

$$\begin{bmatrix} 6 \\ 9 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

Change of Coordinates - Example

- $B = \left\{ b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ b_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},\ b_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ - basis.

- $x_E = [6, 9, 14\,]^T, \quad x_B = [x_1, x_2, x_3]^T = ?$

$$x_E = A_{E \to B} \cdot x_B$$

$$\begin{bmatrix} 6 \\ 9 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 + 2x_3 \\ x_1 + 2x_2 + 3x_3 \end{bmatrix}$$

Change of Coordinates - Example

- $B = \left\{ b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ b_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},\ b_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ - basis.
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$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_1 + x_2 + 2x_3 = 9 \\ x_1 + 2x_2 + 3x_3 = 14 \end{cases} \Leftrightarrow$$

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$$\left\{ \begin{array}{r} x_1 + x_2 + x_3 = 6 \\ x_1 + x_2 + 2x_3 = 9 \\ x_1 + 2x_2 + 3x_3 = 14 \end{array} \right. \overset{(2)-(1)}{\Leftrightarrow} \left\{ \begin{array}{c} \\ x_3 = 3 \\ \\ \end{array} \right. \Leftrightarrow$$

Change of Coordinates - Example

- $B = \left\{ b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ b_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},\ b_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ - basis.
- $x_E = [6, 9, 14\,]^T, \quad x_B = [x_1, x_2, x_3]^T = ?$

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$$\begin{bmatrix} 6 \\ 9 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 + 2x_3 \\ x_1 + 2x_2 + 3x_3 \end{bmatrix}$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_1 + x_2 + 2x_3 = 9 \\ x_1 + 2x_2 + 3x_3 = 14 \end{cases} \Leftrightarrow \begin{cases} x_1 + x_2 = 3 \\ x_3 = 3 \\ x_1 + 2x_2 = 5 \end{cases} \Leftrightarrow$$

Change of Coordinates - Example

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- $x_E = [6, 9, 14\,]^T, \quad x_B = [x_1, x_2, x_3]^T = ?$

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$$\begin{bmatrix} 6 \\ 9 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 + 2x_3 \\ x_1 + 2x_2 + 3x_3 \end{bmatrix}$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_1 + x_2 + 2x_3 = 9 \\ x_1 + 2x_2 + 3x_3 = 14 \end{cases} \Leftrightarrow \begin{cases} x_1 + x_2 = 3 \\ x_3 = 3 \\ x_1 + 2x_2 = 5 \end{cases} \overset{\mathbf{(3)-(1)}}{\Leftrightarrow} \begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases} \Leftrightarrow \qquad .$$

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$$x_E = A_{E \to B} \cdot x_B$$

$$\begin{bmatrix} 6 \\ 9 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 + 2x_3 \\ x_1 + 2x_2 + 3x_3 \end{bmatrix}$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_1 + x_2 + 2x_3 = 9 \\ x_1 + 2x_2 + 3x_3 = 14 \end{cases} \Leftrightarrow \begin{cases} x_1 + x_2 = 3 \\ x_3 = 3 \\ x_1 + 2x_2 = 5 \end{cases} \Leftrightarrow \begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases} \Leftrightarrow \ x_B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Change of Coordinates – Another Example

- $S = \left\{ s_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, s_3 = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix} \right\}$ – basis.
- $B = \left\{ b_1 = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}, b_2 = \begin{bmatrix} 5 \\ 14 \\ 13 \end{bmatrix}, b_3 = \begin{bmatrix} 1 \\ 9 \\ 2 \end{bmatrix} \right\}$ – also basis.
- What is the transition matrix $A_{S \to B} = ?$

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- What is the transition matrix $A_{S \to B} =?$

- $A_{S \to B}$: columns = coordinates of b_1, b_2, b_3 in S.

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- $A_{S \to B}$: columns = coordinates of b_1, b_2, b_3 in S.
- Now, b_1, b_2, b_3 are in standard basis E. How do we change to S?

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Change of Coordinates – Another Example

- $S = \left\{ s_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, s_3 = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix} \right\}, \quad B = \left\{ b_1 = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}, b_2 = \begin{bmatrix} 5 \\ 14 \\ 13 \end{bmatrix}, b_3 = \begin{bmatrix} 1 \\ 9 \\ 2 \end{bmatrix} \right\}$

- What is the transition matrix $A_{S \to B} =?$
    - $A_{S \to B}$: columns = coordinates of b_1, b_2, b_3 in S.

$$[b_i]_E = A_{E \to S} \cdot [b_i]_S, \qquad A_{E \to S} =$$

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Change of Coordinates – Another Example

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$$\begin{bmatrix} \mathbf{3} \\ \mathbf{5} \\ \mathbf{8} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 1 & 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$$

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Change of Coordinates – Another Example

- $S = \left\{ s_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, s_3 = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix} \right\}$, $B = \left\{ b_1 = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}, b_2 = \begin{bmatrix} 5 \\ 14 \\ 13 \end{bmatrix}, b_3 = \begin{bmatrix} 1 \\ 9 \\ 2 \end{bmatrix} \right\}$.

- Transition matrix $A_{S \to B}$: columns = coordinates of b_1, b_2, b_3 in S.

$$A_{S \to B} = \begin{bmatrix} -27 & ? & ? \\ 9 & ? & ? \\ 4 & ? & ? \end{bmatrix}$$

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Change of Coordinates – Another Example

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$$[b_i]_E = A_{E\to S} \cdot [b_i]_S, \qquad A_{E\to S} = \begin{bmatrix}1 & 2 & 3\\2 & 3 & 8\\1 & 3 & 2\end{bmatrix}$$

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Change of Coordinates – Another Example

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$$\begin{bmatrix} \mathbf{5} \\ \mathbf{14} \\ \mathbf{13} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 1 & 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y + 3z \\ 2x + 3y + 8z \\ x + 3y + 2z \end{bmatrix} \Leftrightarrow [b_2]_S = \begin{bmatrix} -71 \\ 20 \\ 12 \end{bmatrix}$$

Change of Coordinates – Another Example

- $S = \left\{ s_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, s_3 = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix} \right\}$, $B = \left\{ b_1 = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}, b_2 = \begin{bmatrix} 5 \\ 14 \\ 13 \end{bmatrix}, b_3 = \begin{bmatrix} 1 \\ 9 \\ 2 \end{bmatrix} \right\}$.
- Transition matrix $A_{S \to B}$: columns = coordinates of b_1, b_2, b_3 in S.

$$A_{S \to B} = \begin{bmatrix} -27 & -71 & ? \\ 9 & 20 & ? \\ 4 & 12 & ? \end{bmatrix}$$

Change of Coordinates – Another Example

- $S = \left\{ s_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, s_3 = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix} \right\}, \quad B = \left\{ b_1 = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}, b_2 = \begin{bmatrix} 5 \\ 14 \\ 13 \end{bmatrix}, \boldsymbol{b_3} = \begin{bmatrix} \mathbf{1} \\ \mathbf{9} \\ \mathbf{2} \end{bmatrix} \right\}$

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$$[b_2]_S = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{1} \\ \mathbf{9} \\ \mathbf{2} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 1 & 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$[b_3]_S =$$

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Change of Coordinates – Another Example

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$$[b_i]_E = A_{E\to S} \cdot [b_i]_S, \qquad A_{E\to S} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 1 & 3 & 2 \end{bmatrix}$$

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$$\begin{bmatrix} \mathbf{1} \\ \mathbf{9} \\ \mathbf{2} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 1 & 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y + 3z \\ 2x + 3y + 8z \\ x + 3y + 2z \end{bmatrix} \Leftrightarrow [b_3]_S = \begin{bmatrix} -41 \\ 9 \\ 8 \end{bmatrix}$$

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Change of Coordinates – Another Example

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- Transition matrix $A_{S \to B}$: columns = coordinates of b_1, b_2, b_3 in S.

$$A_{S \to B} = \begin{bmatrix} -27 & -71 & -41 \\ 9 & 20 & 9 \\ 4 & 12 & 8 \end{bmatrix}$$

Let's practice!

Orthogonal Basis

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- There is more than one basis in a vector space.

- Some are more convenient than the other ones.

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Orthogonal Basis

- There is more than one basis in a vector space.
- Some are more convenient than the other ones.
- Orthonormal basis = all vectors are pairwise orthogonal ($\left(e_i, e_j\right) = 0$) + of unit length ($\|e_i\| = 1$).
- Any basis can be transformed into orthonormal basis!

Gram-Schmidt Process

- $B = \left\{ b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ b_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},\ b_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ - basis.

  Let's construct an orthogonal basis $V = \{v_1, v_2, v_3\}$ from it.

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1. $v_1 := b_1$
2. Let's look for v_2 of the form $v_2 := b_2 + \alpha v_1,\ \alpha \in \mathbb{R}$.

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   How to chose α? v_1 and v_2 must be orthogonal!

$$0 = (v_1, v_2) = (v_1, b_2 + \alpha v_1) = (v_1, b_2) + \alpha (v_1, v_1) \Leftrightarrow$$

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Gram-Schmidt Process

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$$v_2 = b_2 - \frac{(v_1, b_2)}{(v_1, v_1)} b_2, \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{1+1+2}{1+1+1} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}.$$

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Gram-Schmidt Process

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$$v_3 = b_3 + \alpha_1 v_1 + \alpha_2 v_2$$

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$$0 = (v_1, v_3) =$$

$$0 = (v_2, v_3)$$

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Gram-Schmidt Process

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$$0 = (v_1, v_3) = (v_1, b_3) + \alpha_1 (v_1, v_1) + \alpha_2 (v_1, v_2) = (v_1, b_3) + \alpha_1 (v_1, v_1)$$

$$0 = (v_2, v_3)$$

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Gram-Schmidt Process

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$$0 = (v_2, v_3)$$

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  Let's construct an orthogonal basis $V = \{v_1, v_2, v_3\}$ from it.

$$v_1 \coloneqq b_1, \qquad v_2 = \begin{bmatrix}-1/3\\-1/3\\2/3\end{bmatrix}$$

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$$v_3 = b_3 + \alpha_1 v_1 + \alpha_2 v_2 = b_3 - \frac{(v_1, b_3)}{(v_1, v_1)} v_1 - \frac{(v_2, b_3)}{(v_2, v_2)} v_2$$

$$0 = (v_1, v_3) = (v_1, b_3) + \alpha_1 (v_1, v_1) + \alpha_2 (v_1, v_2) = (v_1, b_3) + \alpha_1 (v_1, v_1) \Leftrightarrow \alpha_1 = -\frac{(v_1, b_3)}{(v_1, v_1)}$$

$$0 = (v_2, v_3) = (v_2, b_3) + \alpha_1 (v_2, v_1) + \alpha_2 (2, v_2) = (v_1, b_3) + \alpha_1 (v_1, v_1) \Leftrightarrow \alpha_2 = -\frac{(v_2, b_3)}{(v_2, v_2)}$$

Gram-Schmidt Process

- $B = \left\{ b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ b_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},\ b_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ - basis.

  Let's construct an orthogonal basis $V = \{v_1, v_2, v_3\}$ from it.

$$v_1 \coloneqq b_1, \qquad v_2 = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$$v_3 = b_3 - \frac{(v_1, b_3)}{(v_1, v_1)} v_1 - \frac{(v_2, b_3)}{(v_2, v_2)} v_2 =$$

Gram-Schmidt Process

- $B = \left\{ b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ b_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},\ b_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ - basis.

  Let's construct an orthogonal basis $V = \{v_1, v_2, v_3\}$ from it.

$$v_1 := b_1, \qquad v_2 = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$$v_3 = b_3 - \frac{(v_1, b_3)}{(v_1, v_1)} v_1 - \frac{(v_2, b_3)}{(v_2, v_2)} v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{1+2+3}{1+1+1} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-\frac{1}{3} - \frac{2}{3} + 2}{\frac{1}{9} + \frac{1}{9} + \frac{4}{9}} \cdot \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

Gram-Schmidt Process

- $B = \left\{ b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ b_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},\ b_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ - basis.

  Let's construct an orthogonal basis $V = \{v_1, v_2, v_3\}$ from it.

$$v_1 := b_1, \qquad v_2 = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

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$$v_3 = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}.$$

Gram-Schmidt Process

- $B = \left\{ b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ b_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},\ b_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ - basis.

  Orthogonal basis $V = \{v_1, v_2, v_3\}$ from B:

$$v_1 \coloneqq b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

Gram-Schmidt Process: General Case

- Some basis $B = \{b_1, \dots, b_n\}$.
- Constructing orthogonal basis $V = \{v_1, \dots, v_n\}$, $\left(v_i, v_j\right) = 0$:

$$v_1 = b_1$$

$$v_2 = b_2 - \frac{(v_1, b_2)}{(v_1, v_1)} v_1$$

$$\vdots$$

$$v_k = b_k - \frac{(v_1, b_k)}{(v_1, v_1)} v_1 - \frac{(v_2, b_k)}{(v_2, v_2)} v_2 - \dots - \frac{(v_{k-1}, b_k)}{(v_{k-1}, v_{k-1})} v_{k-1}$$

- If we additionally normalize v_i , we get orthonormal basis.