Math Refresher for DS

Practical Session 10

Today: Integrals

- Indefinite integrals

$$\int f(x)dx$$

- Definite integrals

$$\int_a^b f(x)dx$$

- Improper integrals

$$\int_{-\infty}^{+\infty} f(x)dx$$

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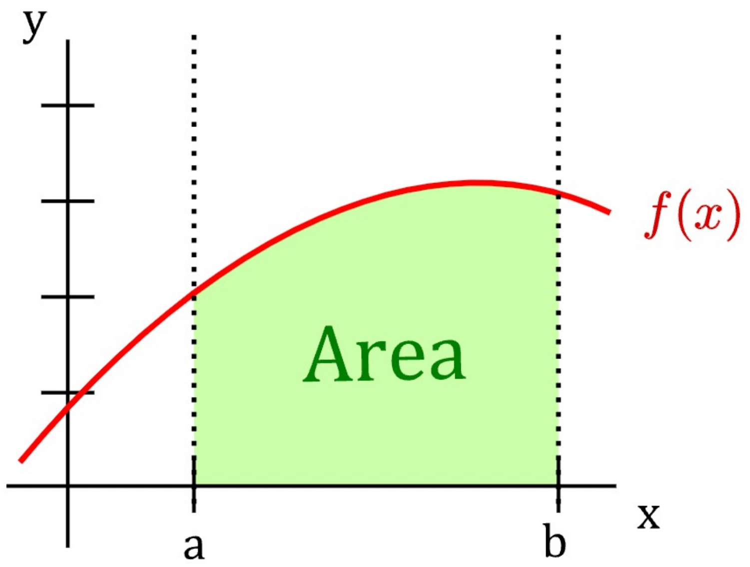
Defnite integrals

- Fundamental Theorem of Calculus: if f is a continuous function on the closed interval $[a;\ b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is an antiderivative of f.



Last time

$$\int_{1}^{2} x dx =$$

Last time

$$\int_1^2 x dx = \frac{1}{2} x^2 \Big|_1^2 =$$

Last time

$$\int_1^2 x dx = \frac{1}{2}x^2\Big|_1^2 = \frac{1}{2}\cdot(4-1) =$$

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Last time

$$\int_1^2 x dx = \frac{1}{2} x^2 \Big|_1^2 = \frac{1}{2} \cdot (4 - 1) = 1.5$$

Last time

$$\int_{-1}^{1} \frac{1}{x^2}\, dx =$$

Last time

$$\int_{-1}^{1} \frac{1}{x^2} dx =$$

$\frac{1}{x^2}$ isn't defined at $\mathbf{0}$

Not a definite integral!

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Definition

- An integral is called *improper* if
  - one or both **limits of integration** are **infinity**:

$$\int_{1}^{+\infty} \frac{1}{x^2}\,dx$$

  - it has a **discontinuous integrand**:

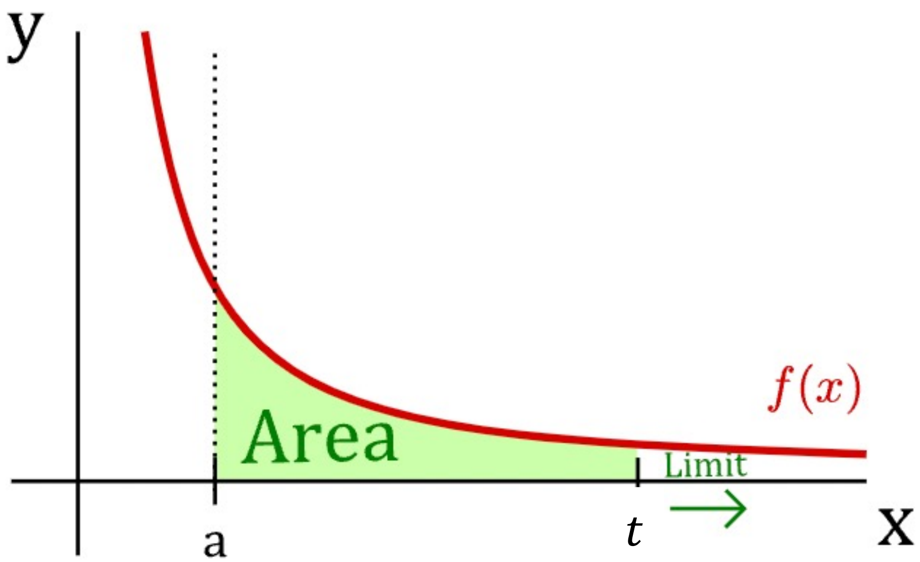
$$\int_{-1}^{1} \frac{1}{x^2}\,dx$$

Infinite interval

Definition

- If $f(x)$ is continuous on $[a; +\infty)$, then

$$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} \int_a^t f(x)dx$$



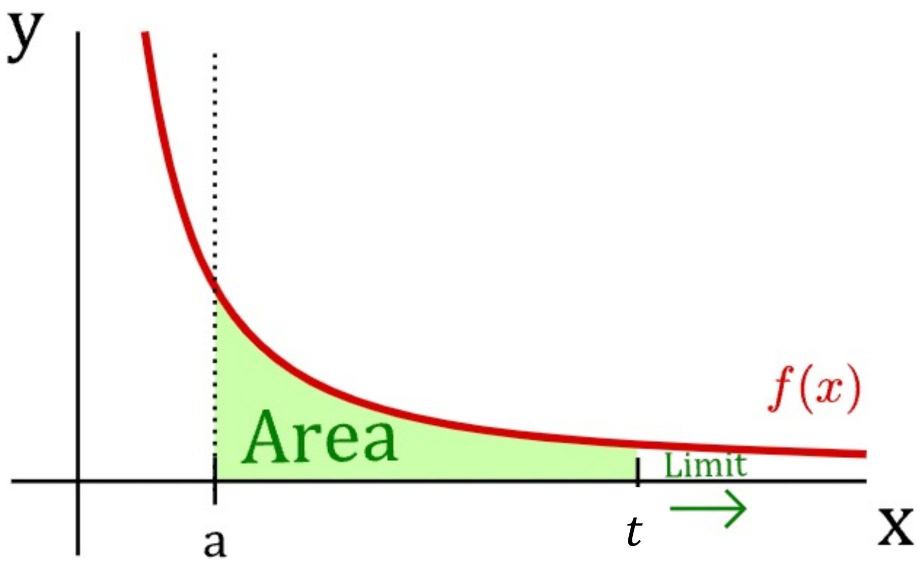
Definition

- If $f(x)$ is continuous on $[a;\ +\infty)$, then

$$\int_{a}^{+\infty} f(x)dx = \lim_{t\to+\infty} \int_{a}^{t} f(x)dx$$

- If $f(x)$ is continuous on $(-\infty; b]$, then

$$\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_{t}^{b} f(x)dx$$



Example

$$\int_{1}^{+\infty} \frac{1}{x^2} dx =$$

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$$= \lim_{t\to+\infty} \left(-\frac{1}{x}\bigg|_1^t \right) =$$

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Example

$$\int_1^{+\infty} \frac{1}{x^2} dx = \lim_{t \to +\infty} \int_1^t \frac{1}{x^2} dx =$$

$$= \lim_{t \to +\infty} \left(-\frac{1}{x} \Big|_1^t \right) = \lim_{t \to +\infty} \left(-\frac{1}{t} \right) + 1 =$$

Example

$$\int_{1}^{+\infty} \frac{1}{x^2}\,dx = \lim_{t\to+\infty} \int_{1}^{t} \frac{1}{x^2}\,dx =$$

$$= \lim_{t\to+\infty} \left(-\frac{1}{x} \Big|_{1}^{t} \right) = \lim_{t\to+\infty} \left(-\frac{1}{t} \right) + 1 =$$

$$= 0 + 1 = 1$$

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Divergent integrals

- We call integrals **convergent** if associated limits exist, and **divergent** otherwise.
- Example:

$$\int_{1}^{+\infty} \frac{1}{x} dx =$$

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Divergent integrals

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- Example:

$$\int_1^{+\infty} \frac{1}{x}\, dx = \lim_{t \to +\infty} \int_1^{t} \frac{1}{x}\, dx =$$

$$= \lim_{t \to +\infty} \left(\log x \Big|_1^t \right) =$$

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$$= \lim_{t \to +\infty} \log t + 0$$

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Divergent integrals

- We call integrals **convergent** if associated limits exist, and **divergent** otherwise.
- Example: the following integral in divergent

$$\int_1^{+\infty} \frac{1}{x}\, dx = \lim_{t\to+\infty} \int_1^t \frac{1}{x}\, dx =$$

$$= \lim_{t\to+\infty} \left(\log x \Big|_1^t \right) =$$

$$= \lim_{t\to+\infty} \log t + 0 \to +\infty$$

One more example

- For which p is the following integral convergent ($a > 0$)?

$$\int_a^{+\infty} \frac{1}{x^p}\,dx =$$

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- For which p is the following integral convergent ($a > 0$)?

$$\int_a^{+\infty} \frac{1}{x^p} dx =$$

$$= -\frac{1}{p-1} \cdot \lim_{t \to +\infty} \frac{1}{x^{p-1}} \Big|_a^t =$$

One more example

- For which p is the following integral convergent ($a > 0$)?

$$\int_a^{+\infty} \frac{1}{x^p} dx =$$

$$= -\frac{1}{p-1} \cdot \lim_{t \to +\infty} \frac{1}{x^{p-1}}\bigg|_a^t =$$

$$= -\frac{1}{p-1} \cdot \lim_{t \to +\infty} \left(\frac{1}{t^{p-1}} - \frac{1}{a^{p-1}} \right) =$$

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One more example

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$$= -\frac{1}{p-1} \cdot \lim_{t\to+\infty} \left(\frac{1}{t^{p-1}} - \frac{1}{a^{p-1}}\right) =$$

$$= \frac{1}{p-1} \cdot \frac{1}{a^{p-1}}$$

when $p - 1 > 0 \Leftrightarrow p > 1.$

One more example

- For which p is the following integral convergent $(a > 0)$?

$$\int_a^{+\infty} \frac{1}{x^p} dx =$$

$$= -\frac{1}{p-1} \cdot \lim_{t \to +\infty} \frac{1}{x^{p-1}} \bigg|_a^t =$$

$$= -\frac{1}{p-1} \cdot \lim_{t \to +\infty} \left(\frac{1}{t^{p-1}} - \frac{1}{a^{p-1}} \right) =$$

$$= \frac{1}{p-1} \cdot \frac{1}{a^{p-1}}$$

when $p - 1 > 0 \Leftrightarrow p > 1.$

If $p \leq 1$, the limit doesn't exist.

Two infinite limits

- If both $\int_{-\infty}^{a} f(x)dx$ and $\int_{a}^{+\infty} f(x)dx$ are convergent, then the improper integral of f over $(-\infty;\ +\infty)$ is

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{+\infty} f(x)dx$$

Example 2

- Is the following integral convergent or divergent?

$$\int_{-\infty}^{+\infty} xe^{-x^2}dx =$$

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$$\int_{-\infty}^{+\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{+\infty} xe^{-x^2}dx =$$

Example 2

- Is the following integral convergent or divergent?

$$\int_{-\infty}^{+\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{+\infty} xe^{-x^2}dx = \begin{Bmatrix} y = x^2 \\ dy = 2xdx \end{Bmatrix} =$$

Example 2

- Is the following integral convergent or divergent?

$$\int_{-\infty}^{+\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{+\infty} xe^{-x^2}dx = \begin{Bmatrix} y = x^2 \\ dy = 2xdx \end{Bmatrix} =$$

$$= \frac{1}{2}\int_{-\infty}^{0} e^{-y}dy + \frac{1}{2}\int_{0}^{+\infty} e^{-y}dy =$$

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Example 2

- Is the following integral convergent or divergent?

$$\int_{-\infty}^{+\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{+\infty} xe^{-x^2}dx = \begin{Bmatrix} y = x^2 \\ dy = 2xdx \end{Bmatrix} =$$

$$= \frac{1}{2}\int_{-\infty}^{0} e^{-y}dy + \frac{1}{2}\int_{0}^{+\infty} e^{-y}dy =$$

$$= -\frac{1}{2}e^{y}\Big|_{-\infty}^{0} - \frac{1}{2}e^{y}\Big|_{0}^{+\infty} =$$

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$$= \frac{1}{2}\int_{-\infty}^{0} e^{-y}dy + \frac{1}{2}\int_{0}^{+\infty} e^{-y}dy =$$

$$= -\frac{1}{2}e^{y}\Big|_{-\infty}^{0} - \frac{1}{2}e^{y}\Big|_{0}^{+\infty} =$$

$$= -\frac{1}{2}e^{0} + 0 - 0 + \frac{1}{2}e^{0} =$$

Example 2

- Is the following integral convergent or divergent?

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$$= \frac{1}{2}\int_{-\infty}^{0} e^{-y}dy + \frac{1}{2}\int_{0}^{+\infty} e^{-y}dy =$$

$$= -\frac{1}{2}e^{y}\Big|_{-\infty}^{0} - \frac{1}{2}e^{y}\Big|_{0}^{+\infty} =$$

$$= -\frac{1}{2}e^{0} + 0 - 0 + \frac{1}{2}e^{0} = -\frac{1}{2} + \frac{1}{2} = 0.$$

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Comparison test

- There are many techniques to check if an integral is convergent or not.
- *Example:* comparison test

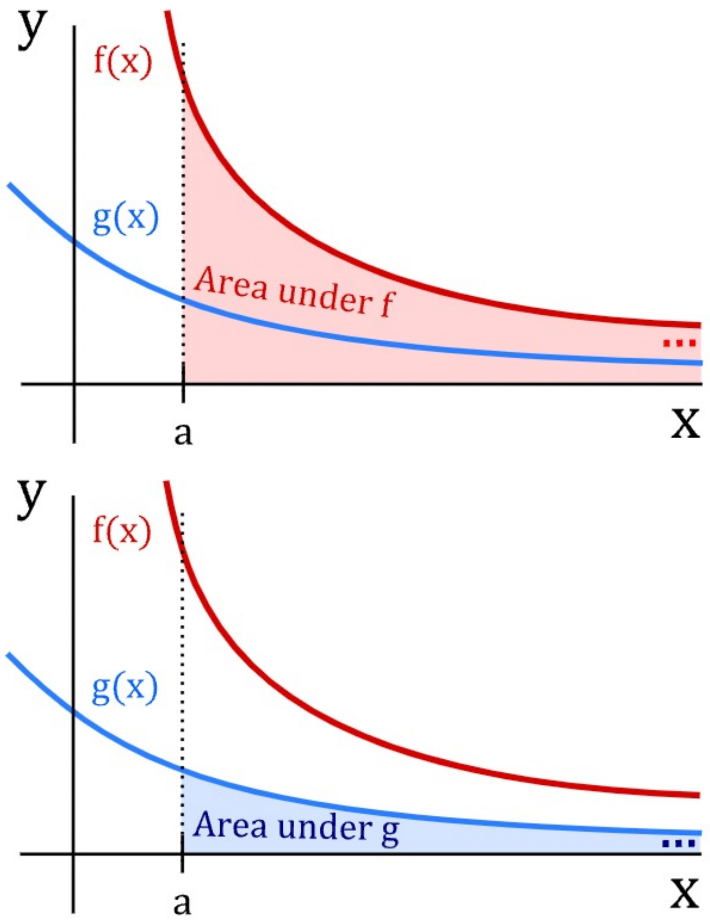
Suppose that $f(x) \geq g(x) \geq 0$ for $x \geq a$. Then

- if $\int_a^{+\infty} f(x)dx$ converges,

  $\int_a^{+\infty} g(x)dx$ also converges

- if $\int_a^{+\infty} f(x)dx$ diverges,

  $\int_a^{+\infty} g(x)dx$ also diverges



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Comparison test - Example

- Check if the following integral converges:

$$\int_{2}^{+\infty} \frac{\cos^2 x}{x^2} dx$$

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Let's use the comparison test:

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$$0 \leq \frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}$$

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Let's use the comparison test:

$$0 \le \frac{\cos^2 x}{x^2} \le \frac{1}{x^2}$$

$\int_{2}^{+\infty} \frac{1}{x^2} dx$ converges →

Comparison test - Example

- Check if the following integral converges:

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Let's use the comparison test:

$$0 \leq \frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}$$

$\int_{2}^{+\infty} \frac{1}{x^2} dx$ converges →

$\int_{2}^{+\infty} \frac{\cos^2 x}{x^2} dx$ also converges!

Discontinuous integrand

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Definition - 1

- If $f(x)$ is continuous on $(a;b]$, then the improper integral of f over $[a;b]$ is

$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$$

- If $f(x)$ is continuous on $[a;b)$, then the improper integral of f over $[a;b]$ is

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$$

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Example - 1

$$\int_0^1 \frac{1}{x^2}\,dx =$$

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$$\int_0^1 \frac{1}{x^2}\,dx =$$

$$= \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}\,dx =$$

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$$= \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}\,dx =$$

$$= \lim_{t\to 0^+} \left(-\frac{1}{x}\right)\Bigg|_t^1 =$$

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Example - 1

$$\int_0^1 \frac{1}{x^2}\,dx =$$

$$= \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}\,dx =$$

$$= \lim_{t\to 0^+} \left(-\frac{1}{x}\right)\Bigg|_t^1 =$$

$$= -1 + \lim_{t\to 0^+} \frac{1}{t}$$

Example - 1

$$\int_0^1 \frac{1}{x^2}\,dx =$$

$$= \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}\,dx =$$

$$= \lim_{t\to 0^+} \left(-\frac{1}{x}\right)\Bigg|_t^1 =$$

$$= -1 + \lim_{t\to 0^+} \frac{1}{t} \to \infty$$

Definition - 2

- If $f(x)$ has a discontinuity at $x = c \in [a; b]$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example - 2

$$\int_{-1}^{1} \frac{1}{x^2}\, dx =$$

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Example - 2

$$\int_{-1}^{1} \frac{1}{x^2} dx =$$

$$= \int_{-1}^{0} \frac{1}{x^2} dx + \int_{0}^{1} \frac{1}{x^2} dx$$

Example - 2

$$\int_{-1}^{1} \frac{1}{x^2}\, dx =$$

$$= \int_{-1}^{0} \frac{1}{x^2}\, dx + \int_{0}^{1} \frac{1}{x^2}\, dx$$

Example - 3

$$\int_0^{+\infty} \frac{1}{x^2} dx =$$

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Example - 3

$$\int_0^{+\infty} \frac{1}{x^2} dx =$$

$$= \int_0^1 \frac{1}{x^2} dx + \int_1^{+\infty} \frac{1}{x^2} dx =$$

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Example - 3

$$\int_0^{+\infty} \frac{1}{x^2}\,dx =$$

$$= \int_0^1 \frac{1}{x^2}\,dx + \int_1^{+\infty} \frac{1}{x^2}\,dx =$$

$$= \int_0^1 \frac{1}{x^2}\,dx + 1$$

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Example - 3

$$\int_0^{+\infty} \frac{1}{x^2} dx =$$

$$= \int_0^1 \frac{1}{x^2} dx + \int_1^{+\infty} \frac{1}{x^2} dx =$$

$$= \int_0^1 \frac{1}{x^2} dx + 1$$

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Example - 3

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$$= \int_0^1 \frac{1}{x^2} dx + 1$$

$$\int_0^1 \frac{1}{x^2} dx = \lim_{t \to 0^+} \left(-\frac{1}{x} \bigg|_t^1 \right) =$$

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Example - 3

$$\int_0^{+\infty} \frac{1}{x^2} dx =$$

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$$= \int_0^1 \frac{1}{x^2} dx + 1$$

$$\int_0^1 \frac{1}{x^2} dx = \lim_{t \to 0^+} \left(-\frac{1}{x} \Big|_t^1 \right) =$$

$$= \lim_{t \to 0^+} \left(-1 + \frac{1}{t} \right)$$

Example - 3

$$\int_0^{+\infty} \frac{1}{x^2} dx =$$

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$$\int_0^1 \frac{1}{x^2} dx = \lim_{t \to 0^+} \left(-\frac{1}{x} \Big|_t^1 \right) =$$

$$= \lim_{t \to 0^+} \left(-1 + \frac{1}{t} \right) \to \infty$$

Example - 3

$$\int_0^{+\infty} \frac{1}{x^2} dx =$$

$$= \int_0^1 \frac{1}{x^2} dx + \int_1^{+\infty} \frac{1}{x^2} dx =$$

$$= \int_0^1 \frac{1}{x^2} dx + 1 \to \infty$$

$$\int_0^1 \frac{1}{x^2} dx = \lim_{t \to 0^+} \left(-\frac{1}{x} \bigg|_t^1 \right) =$$

$$= \lim_{t \to 0^+} \left(-1 + \frac{1}{t} \right) \to \infty$$

Let's practice