Final Exam

Math Basics for Machine Learning

Fall 2022

Instructions

This is the final exam for the Math Basics for Data Science course.

This exam is in two parts:

- PART 1 consists of a quiz similar to the short quizes from our classes. You will need to answer a number of questions, but you do **not** need to provide detailed solutions.
- PART 2 consists of six free-response questions for which you will need to attach detailed solutions, similar to the graded assignments.

You can get 30 points for each of the parts, resulting in 60 points in total for the exam.

You can submit the exam by filling in the corresponding Google form. **You can submit your answers just once.** After you have submitted the form, you should receive a confirmation email. If you have submitted your solutions but did not receive any confirmation, contact me.

You must submit your answers by Monday, December 19, 23:59 Anywhere on Earth. *Late submissions will not be accepted.*

It is the idea that you complete the exam individually. Do not collaborate, share your solutions with anyone or copy answers of somebody else.

Feel free to contact me if you have any questions.

Good luck!

Part 1

The first part of the exam is a quiz. You will need to answer a number of questions, but you do **not** need to provide detailed solutions.

The questions for this part can be found in the submission Google form.

Part 2

The second part of the exam contains six free-response questions for which you will need to attach detailed solutions. You can find problem formulations below.

Always show how you arrive to the solution and provide reasonably detailed explanations. Answers stated without any comments won't be accepted.

It is preferred if you type your solutions using Latex. If you decide to take a picture of hand-written notes, make sure that the picture is not blurred and that the solutions are readable.

For some tasks, it might be convenient to use Python rather than perform the computations by hand. If you do so, attach your code as well.

You can get 30 points for this part.

Problem 1 *(3 points)*

Consider a set of vectors $B = \{b_1 = (0, 2),\ b_2 = (-1, 1)\}$. Note that B forms a basis in $\mathbb{R}^2$.

1. *(1 point)* Find the transition matrix $A_{E \to B}$ from the canonical basis $E = \{e_1 = (1, 0),\ e_2 = (0, 1)\}$ to B.
2. *(1 point)* Find the transition matrix $A_{B \to E}$ from B to the canonical basis E.
3. *(1 point)* Suppose that in the canonical basis E, vector x has coordinates $x_E = (2, 3)$. Find the coordinates of x in the basis B.

Problem 2 *(5 points)*

Consider the following three vectors:

$$u = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix},\ v = \begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix},\ w = \begin{bmatrix} 10 \\ 7 \\ h + 5 \end{bmatrix}$$

where h is some real number.

Find all possible values of h for which $\{u, v, w\}$ form a basis for $\mathbb{R}^3$.

Problem 3 *(4 points)*

Consider the following linearly independent vectors:

$$u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ w = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

Apply the Gram-Schmidt process to obtain an *orthonormal* basis for $\mathbb{R}^3$.

Problem 4 *(4 points)*

Consider the following matrix:

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

Can it be diagonalized? If so, determine its diagonal form and a basis in which A is diagonal. If not, explain why.

Problem 5 *(8 points)*

Let $a_1, ..., a_n$ be some real numbers. Consider the following function:

$$f(x) = \prod_{i=1}^{n} [x^{a_i} e^{-x}],\ x > 0$$

Find the value of $x > 0$ that maximizes $f(x)$.

Problem 6 *(6 points)*

Find and classify all the critical points of the following function:

$$f(xy) = 8x - x\sqrt{y-1} + x^3 + 0.5y - 12x^2$$

Note: if the second derivative test is indecisive, you don't need to investigate the correspondent point any further.