Math Refresher for DS

Practical Session 10

Today: Integrals

- Indefinite integrals

$$\int f(x)dx$$

- Definite integrals

$$\int_a^b f(x)dx$$

- Improper integrals

$$\int_{-\infty}^{+\infty} f(x)dx$$

Indefinite Integral

Antiderivatives

- Which function $F(x)$ should we differentiate to get

$$f(x) = x^4 + 3x - 9$$

Antiderivatives

- Which function $F(x)$ should we differentiate to get

$$f(x) = x^4 + 3x - 9$$

$$F(x) = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + C, \qquad C \in \mathbb{R}$$

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Antiderivatives

Given a function, $f(x)$, an **anti-derivative** of $f(x)$ is any function $F(x)$ such that

$$F'(x) = f(x)$$

If $F(x)$ is any anti-derivative of $f(x)$ then the most general anti-derivative of $f(x)$ is called an **indefinite integral** and denoted,

$$\int f(x)\, dx = F(x) + c, \qquad c \text{ is any constant}$$

In this definition the $\int$ is called the **integral symbol**, $f(x)$ is called the **integrand**, x is called the **integration variable** and the "c" is called the **constant of integration**.

Indefinite Integral

$$\int f(x)\,dx$$

Indefinite Integral: Example

$$\int x^n \, dx = \qquad\qquad\qquad\qquad , \qquad n \neq -1$$

$$\int \frac{1}{x} dx =$$

$$\int \sin x \, dx =$$

$$\int e^x dx =$$

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Indefinite Integral: Example

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \qquad n \neq -1$$

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Indefinite Integral: Example

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \qquad n \neq -1$$

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$$\int e^x dx = e^x + C$$

Indefinite Integral: Example

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$$f(0) = 15 = -2\cos 0 + 7e^0 = 5 \quad \rightarrow \quad C = 10$$

$$f(x) = x^4 - 9x\ - 2\cos x + 7e^x + 10$$

Substitution Rule

- Compute the following integral:

$$\int 18x^2 \sqrt[4]{6x^3 + 5}\, dx =$$

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$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du, \quad \text{where, } u = g(x)$$

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$$= \int \sqrt[4]{u}\, du = \frac{4}{5} u^{5/4}$$

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- Compute the following integral:

$$\int 18x^2 \sqrt[4]{6x^3+5}\, dx = \{u = 6x^3+5, \qquad du = 18x^2 dx\} =$$

$$= \int \sqrt[4]{u}\, du = \frac{4}{5}u^{5/4} = \frac{4}{5}\sqrt[4]{(6x^3+5)^5} + C.$$

- Substitution rule:

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du, \quad \text{where, } u = g(x)$$

Substitution Rule - Example

$$\int 3(8y-1)e^{4y^2-y}\,dy =$$

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Substitution Rule - Example

$$\int 3(8y-1)e^{4y^2-y}dy = \int 3e^{4y^2-y}d(4y^2-y) =$$

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$$\int 3(8y-1)e^{4y^2-y}dy = \int 3e^{4y^2-y}d(4y^2-y) =$$

$$= 3e^{4y^2-y} + C$$

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Integration by Parts

- Consider the following integrals:

$$\int e^x dx =$$

$$\int x e^{x^2} dx =$$

$$\int x e^{6x} dx =$$

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Integration by Parts

- Consider the following integrals:

$$\int e^x dx = e^x + C$$

$$\int xe^{x^2} dx =$$

$$\int xe^{6x} dx =$$

Integration by Parts

- Consider the following integrals:

$$\int e^x dx = e^x + C$$

$$\int xe^{x^2} dx = \frac{1}{2}\int e^{x^2} dx^2 =$$

$$\int xe^{6x} dx =$$

Integration by Parts

- Consider the following integrals:

$$\int e^x dx = e^x + C$$

$$\int xe^{x^2} dx = \frac{1}{2}\int e^{x^2} dx^2 = \frac{1}{2}e^{x^2} + C$$

$$\int xe^{6x} dx =$$

Integration by Parts

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$$\int xe^{x^2} dx = \frac{1}{2}\int e^{x^2} dx^2 = \frac{1}{2} e^{x^2} + C$$

$$\int xe^{6x} dx = \cdots ?$$

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- Chain rule:

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

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$$u \cdot v = \int (u \cdot v)'\, dx = \int u' \cdot v\, dx + \int u \cdot v'\, dx = \int v\, du + \int u\, dv$$

Integration by Parts

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$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

$$u \cdot v = \int (u \cdot v)'\, dx = \int u' \cdot v\, dx + \int u \cdot v'\, dx = \int v\, du + \int u\, dv$$

$$\int u\, dv = u \cdot v - \int v\, du$$

Integration by Parts

- Consider the following integrals:

$$\int e^x dx = e^x + C$$

$$\int x e^{x^2} dx = \frac{1}{2} \int e^{x^2} dx^2 = \frac{1}{2} e^{x^2} + C$$

$$\int x e^{6x} dx = \frac{1}{6} \int x d e^{6x} = \frac{1}{6} x e^{6x} - \frac{1}{6} \int e^{6x} dx = \frac{1}{6} x e^{6x} - \frac{1}{36} e^{6x} + C$$

Integration by Parts

- Consider the following integrals:

$$\int e^x dx = e^x + C$$

$$\int xe^{x^2} dx = \frac{1}{2}\int e^{x^2} dx^2 = \frac{1}{2}e^{x^2} + C$$

$$\int xe^{6x} dx =$$

Integration by Parts

- Consider the following integrals:

$$\int e^x dx = e^x + C$$

$$\int xe^{x^2} dx = \frac{1}{2}\int e^{x^2} dx^2 = \frac{1}{2}e^{x^2} + C$$

$$\int xe^{6x} dx = \frac{1}{6}\int x de^{6x} =$$

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Integration by Parts

- Consider the following integrals:

$$\int e^x dx = e^x + C$$

$$\int xe^{x^2} dx = \frac{1}{2}\int e^{x^2} dx^2 = \frac{1}{2}e^{x^2} + C$$

$$\int xe^{6x} dx = \frac{1}{6}\int x de^{6x} = \frac{1}{6}xe^{6x} - \frac{1}{6}\int e^{6x} dx =$$

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Integration by Parts - – Example 2

$$\int x^2 \sin 10x \, dx =$$

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$$\int x^2 \sin 10x \, dx = -0.1 \int x^2 d \cos 10x =$$

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$$\int x^2 \sin 10x \, dx = -0.1 \int x^2 d \cos 10x =$$

$$-0.1x^2 \cos 10x + 0.1 \int \cos 10x \, dx^2 =$$

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Integration by Parts - – Example 2

$$\int x^2 \sin 10x \, dx = -0.1 \int x^2 d \cos 10x =$$

$$-0.1x^2 \cos 10x + 0.1 \int \cos 10x \, dx^2 = -0.1x^2 \cos 10x + 0.2 \int x \cos 10x \, dx$$

Integration by Parts - – Example 2

$$\int x^2 \sin 10x\, dx = -0.1 \int x^2 d \cos 10x =$$

$$-0.1x^2 \cos 10x + 0.1 \int \cos 10x\, dx^2 = -0.1x^2 \cos 10x + 0.2 \int x \cos 10x\, dx$$

$$= -0.1x^2 \cos 10x + 0.02 \sin 10x + 0.002 \cos 10x + C$$

Integration by Parts - – Example 2

$$\int x^2 \sin 10x\, dx = -0.1 \int x^2 d \cos 10x =$$

$$-0.1x^2 \cos 10x + 0.1 \int \cos 10x\, dx^2 = -0.1x^2 \cos 10x + 0.2 \int x \cos 10x\, dx$$

$$= -0.1x^2 \cos 10x + 0.02 \sin 10x + 0.002 \cos 10x + C$$

$$\int x \cos 10x\, dx =$$

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$$\int x^2 \sin 10x\, dx = -0.1 \int x^2 d \cos 10x =$$

$$-0.1x^2 \cos 10x + 0.1 \int \cos 10x\, dx^2 = -0.1x^2 \cos 10x + 0.2 \int x \cos 10x\, dx$$

$$= -0.1x^2 \cos 10x + 0.02 \sin 10x + 0.002 \cos 10x + C$$

$$\int x \cos 10x\, dx = 0.1 \int xd \sin 10x = 0.1 \sin 10x - 0.1 \int \sin 10x\, dx$$

$$=$$

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Integration by Parts - – Example 2

$$\int x^2 \sin 10x \, dx = -0.1 \int x^2 d \cos 10x =$$

$$-0.1x^2 \cos 10x + 0.1 \int \cos 10x \, dx^2 = -0.1x^2 \cos 10x + 0.2 \int x \cos 10x \, dx$$

$$= -0.1x^2 \cos 10x + 0.02 \sin 10x + 0.002 \cos 10x + C$$

$$\int x \cos 10x \, dx = 0.1 \int x d \sin 10x = 0.1 \sin 10x - 0.1 \int \sin 10x \, dx$$

$$= 0.1 \sin 10x + 0.01 \cos 10x + C.$$

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Integration by Parts – Example 3

$$\int \ln x \, dx =$$

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$$\int \ln x \, dx =$$

$$= x \ln x - \int x d \ln x =$$

Integration by Parts – Example 3

$$\int \ln x \, dx =$$

$$= x \ln x - \int x d \ln x = x \ln x - \int x \cdot \frac{1}{x} dx =$$

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Integration by Parts – Example 3

$$\int \ln x \, dx =$$

$$= x \ln x - \int x d \ln x = x \ln x - \int x \cdot \frac{1}{x} dx =$$

$$= x \ln x - x + C$$

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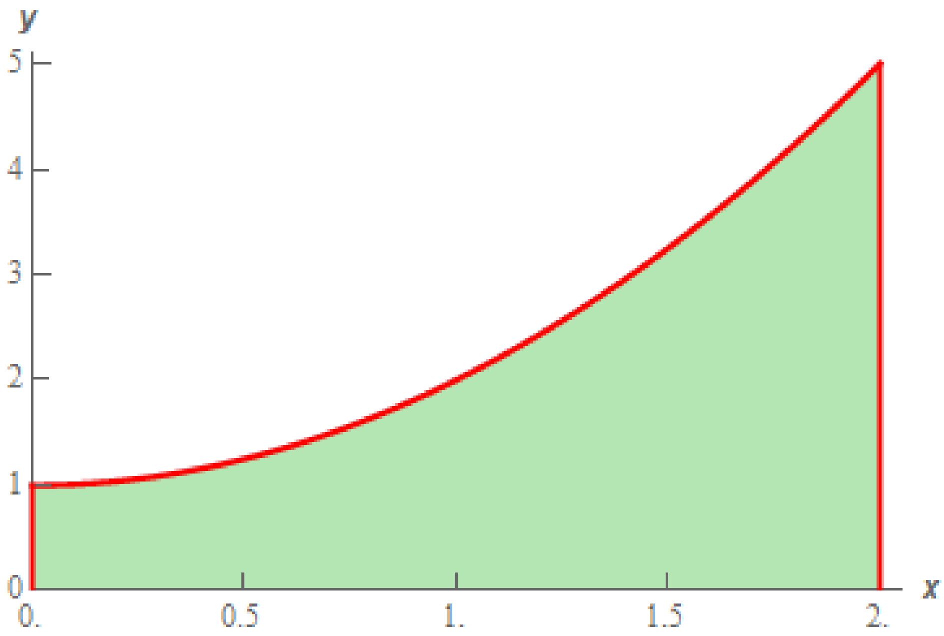
Definite Integral

Definite Integral



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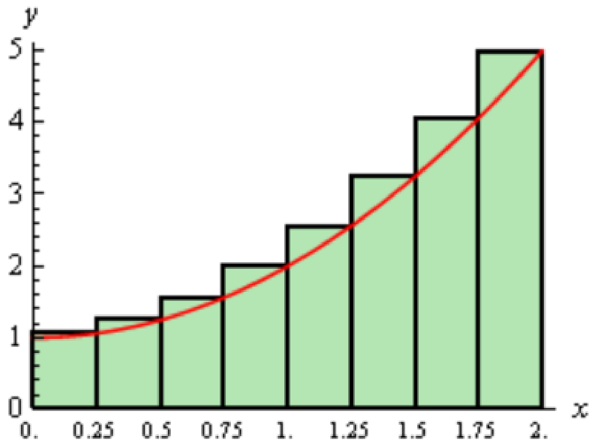
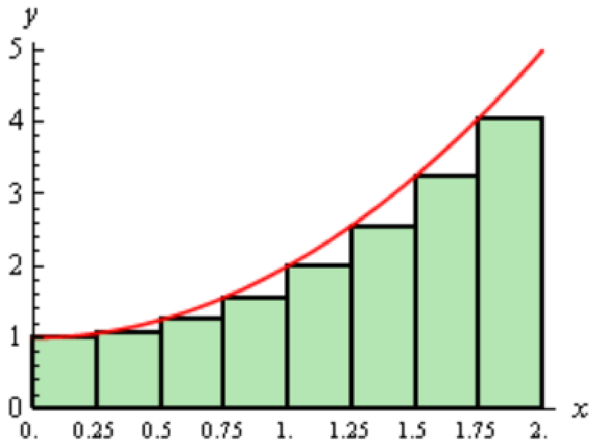
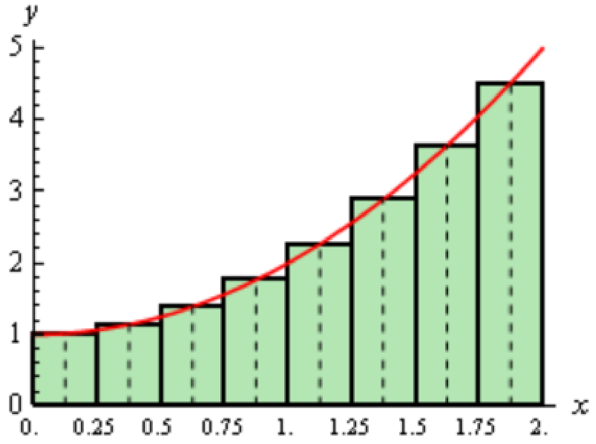
Definite Integral







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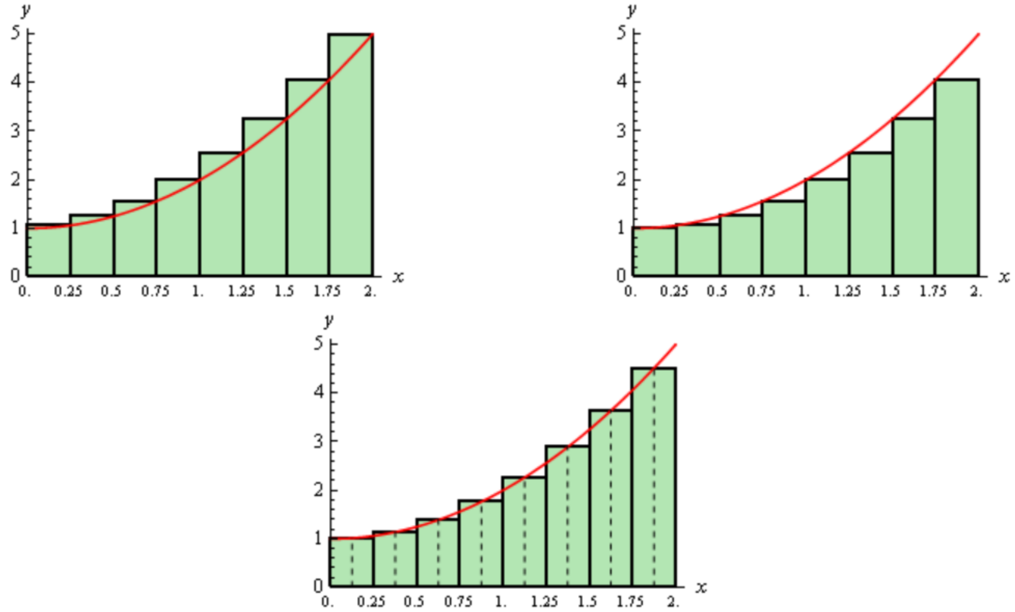
Definite Integral

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$



Definite Integral

- The fundamental theorem of Calculus:

$$\int_a^b f(x)dx = F(b) - F(a)$$

- Example:

$$\int_0^1 x^2 dx = \frac{1}{3}x^3 \Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

Definite Integral: Properties

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. We can interchange the limits on any definite integral, all that we need to do is tack a minus sign onto the integral when we do.

2. $\int_a^a f(x)\,dx = 0$. If the upper and lower limits are the same then there is no work to do, the integral is zero.

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any number. So, as with limits, derivatives, and indefinite integrals we can factor out a constant.

4. $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$. We can break up definite integrals across a sum or difference.

5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where c is any number. This property is more important than we might realize at first. One of the main uses of this property is to tell us how we can integrate a function over the adjacent intervals, $[a, c]$ and $[c, b]$. Note however that c doesn't need to be between a and b.

Definite Integral – Example 2

$$\int_0^1 2e^{-2x}dx = -e^{-2x}\Big|_0^1 = -e^{-2} + 1 = 1 - \frac{1}{e^2}$$

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