Math Basics for DS

Lecture 6

girafe
ai

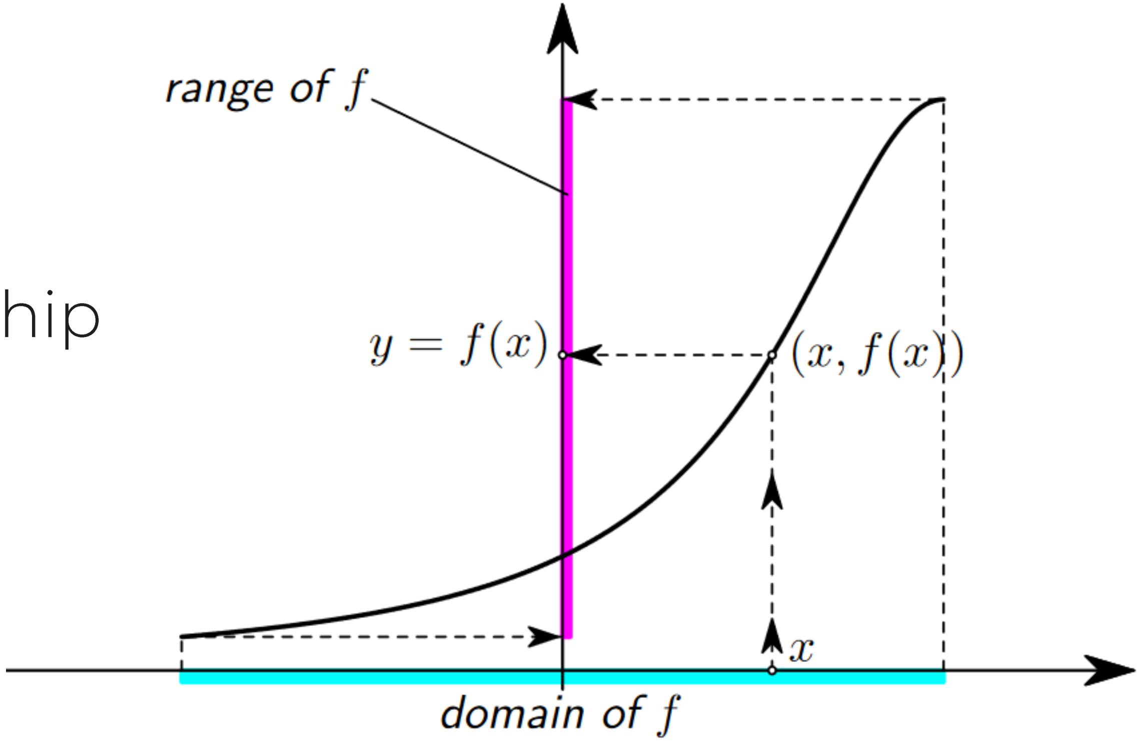
Today

- Univariate functions
- Derivatives

Functions

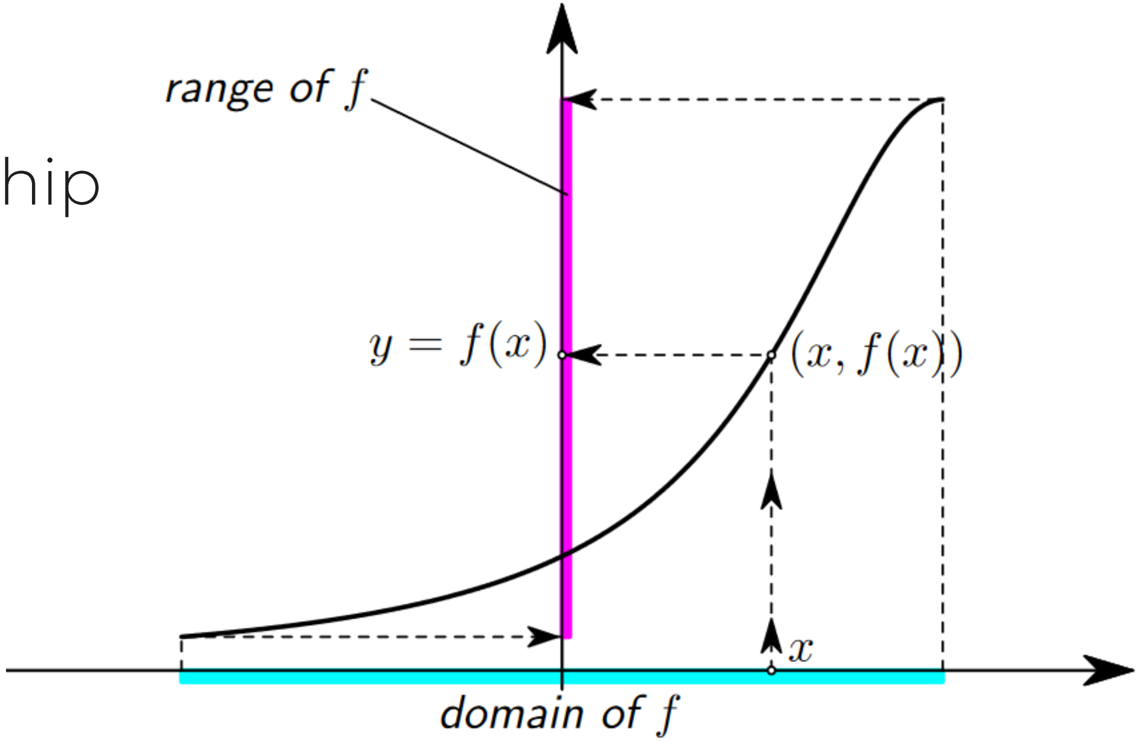
What is a Function

- Function describes the relationship between x and y.



What is a Function

- **Function** describes the relationship between x and y.
- **Domain**: the set of numbers for which a function is defined.

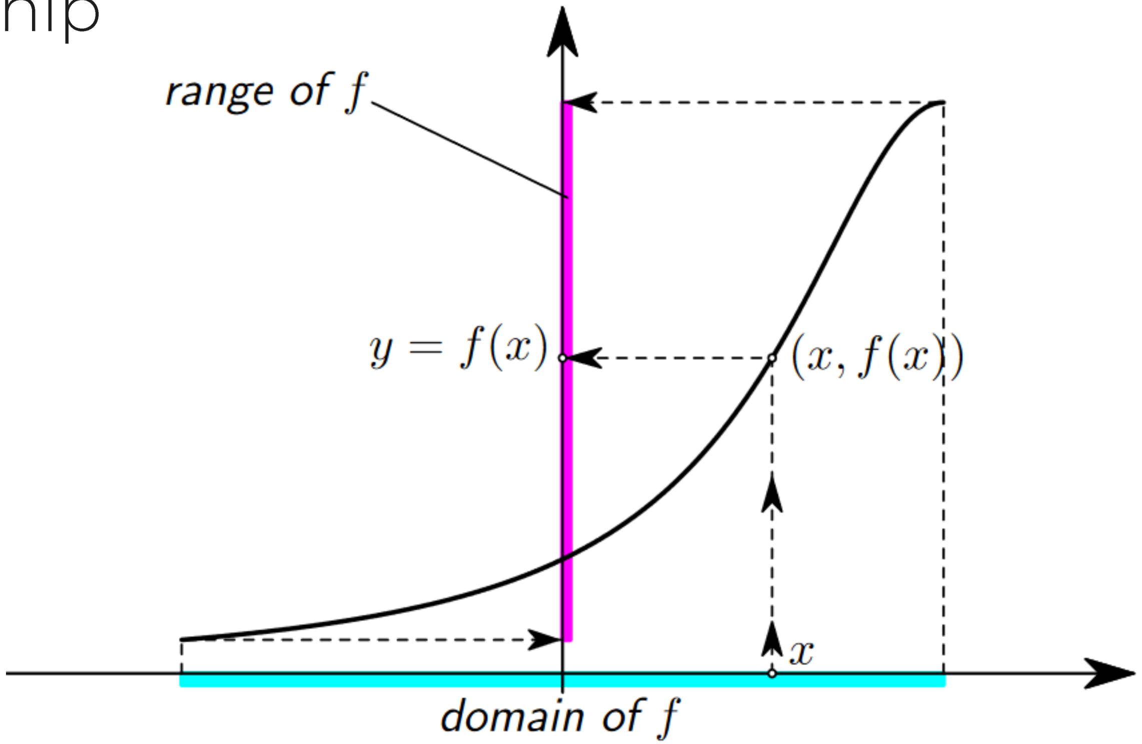


What is a Function

- Function describes the relationship between x and y.

- Domain: the set of numbers for which a function is defined.

- Range: the set of all possible numbers $f(x)$ as x runs over its domain.



Some Univariate Functions

- A linear function:

$$f(x) = 2x + 1, \qquad f: \mathbb{R} \to \mathbb{R}$$

Some Univariate Functions

- A linear function:

intercept

$$f(x) = 2x + 1 \qquad f: \mathbb{R} \to \mathbb{R}$$

slope

Some Univariate Functions

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$$f(x) = 2x + 1, \qquad f: \mathbb{R} \rightarrow \mathbb{R}$$

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$$f(x) = x^2 - 2x + 1, \qquad f: \mathbb{R} \rightarrow \mathbb{R}^+$$

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$$f(x) = 10^x, \qquad f: \mathbb{R} \to \mathbb{R}^+$$

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- A trigonometric function:

$$f(x) = \sin x\,, \qquad f: \mathbb{R} \to [0,1]$$

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Limit of a Function

Limit

$$\lim_{x \to a} f(x) = L$$

- "The limit of $f(x)$ as x approaches a is L".

- Informally: for x close to a, $f(x)$ is close to L.
  The closer x gets to a, the closer $f(x)$ gets to L.

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- Formally:

$$\forall \varepsilon > 0 \; \exists \, \delta > 0 \text{ such that } |x - a| < \delta \rightarrow |f(x) - L| < \varepsilon.$$

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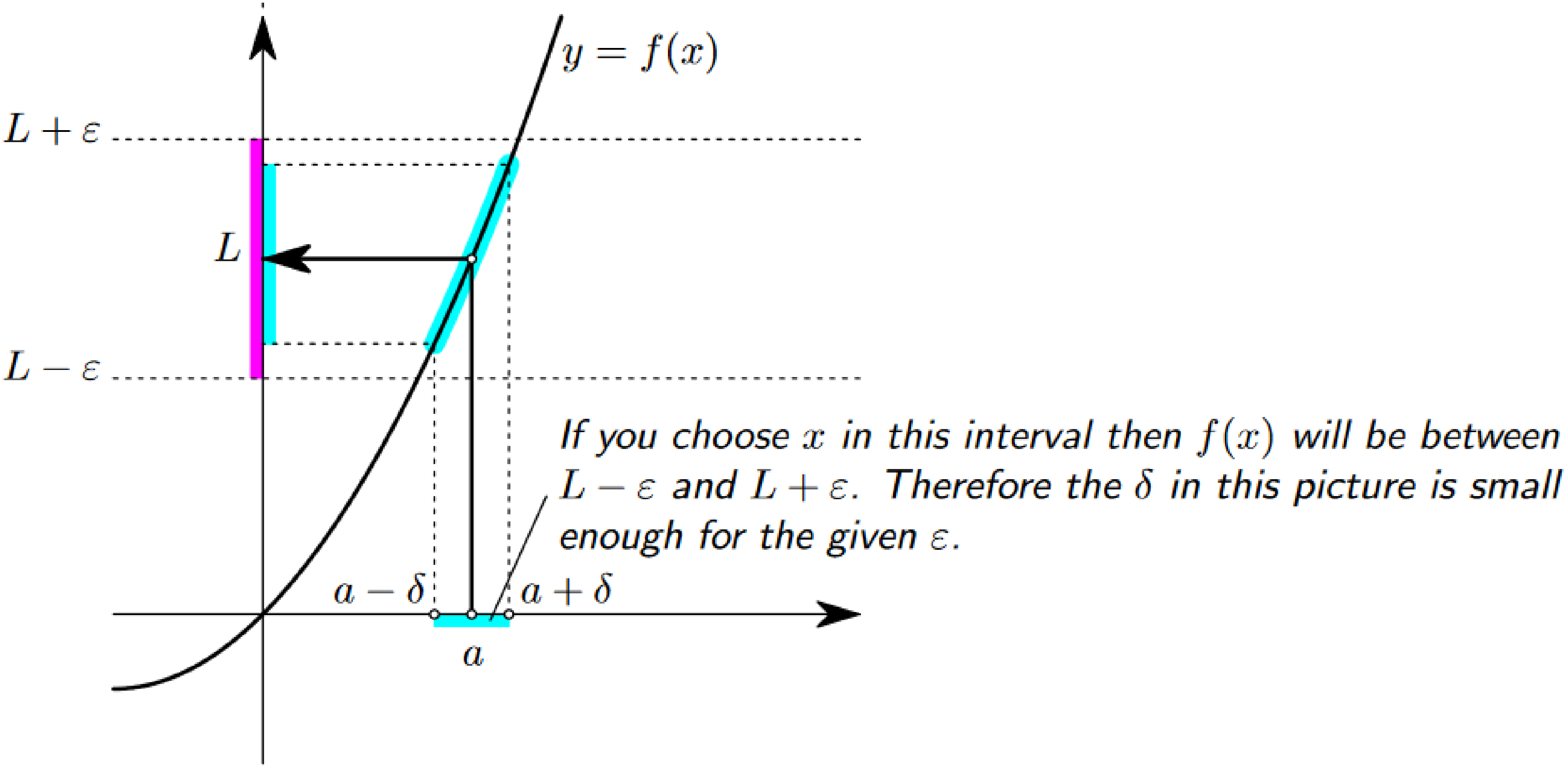
Limit



$y = f(x)$
$L + \varepsilon$
L
$L - \varepsilon$
$a - \delta$
$a + \delta$
a
If you choose x in this interval then $f(x)$ will be between $L - \varepsilon$ and $L + \varepsilon$. Therefore the δ in this picture is small enough for the given ε.

Limit - Examples

$$\lim_{x \to +\infty} \frac{1}{x} = 0$$

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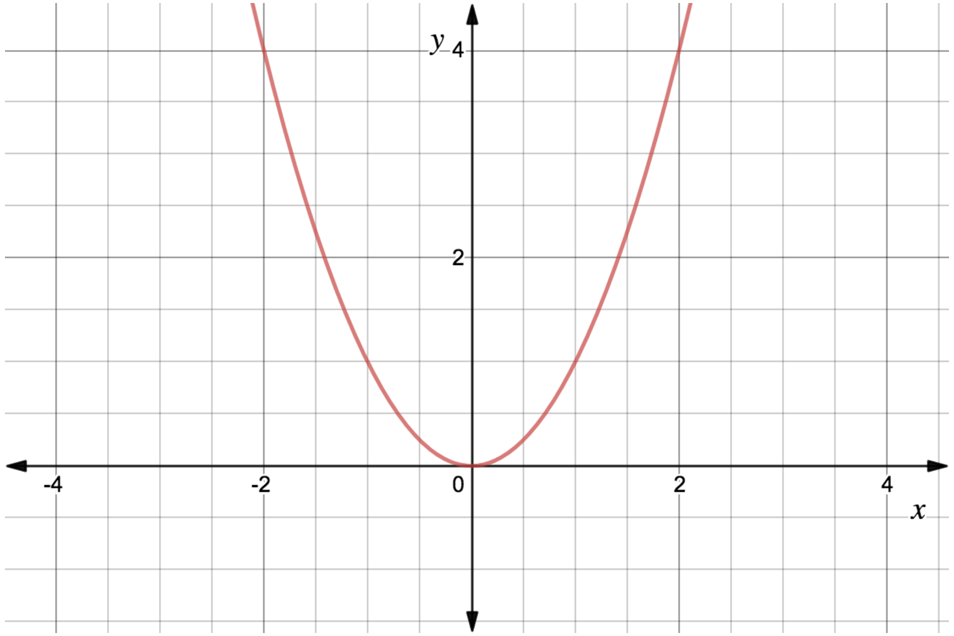
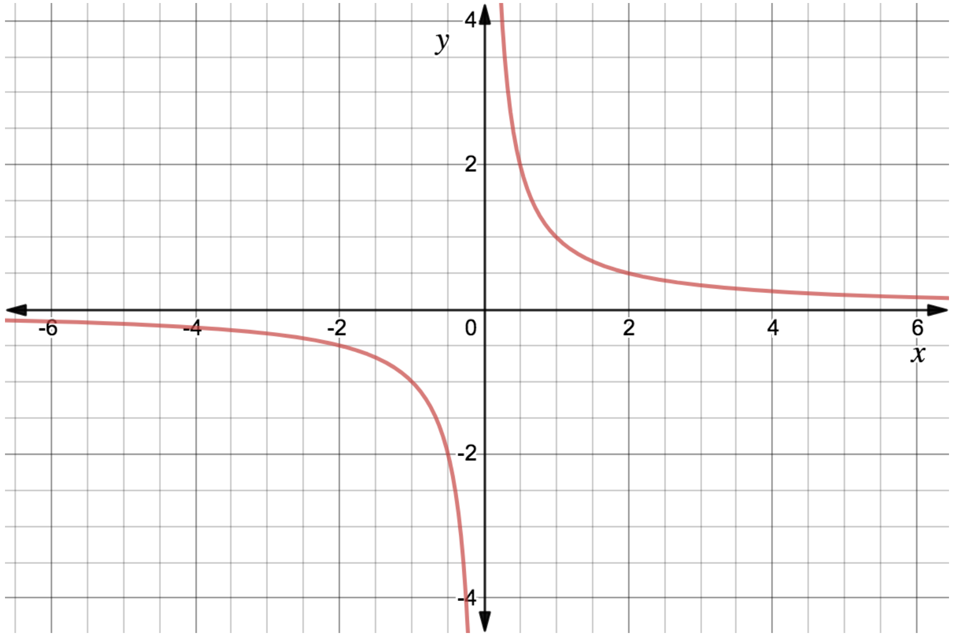
$$\lim_{x \to 2} \frac{x^2 - 2x}{x^2 - 4} = \lim_{x \to 2} \frac{x(x-2)}{(x-2)(x+2)} = \lim_{x \to 2} \frac{x}{x+2} = 0.5$$

Properties of Functions

Continuity Informally

- Very basic definition: a **continuous function** is one that can be drawn in one continuous stroke.





Source: https://www.expii.com/t/precalculus-concepts-continuous-vs-discontinuous-4767

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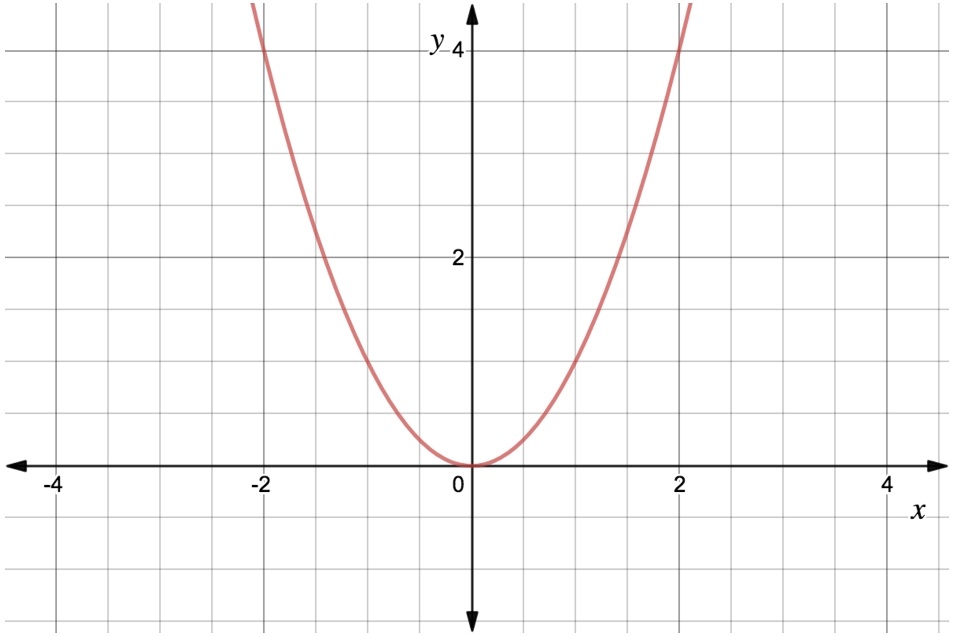

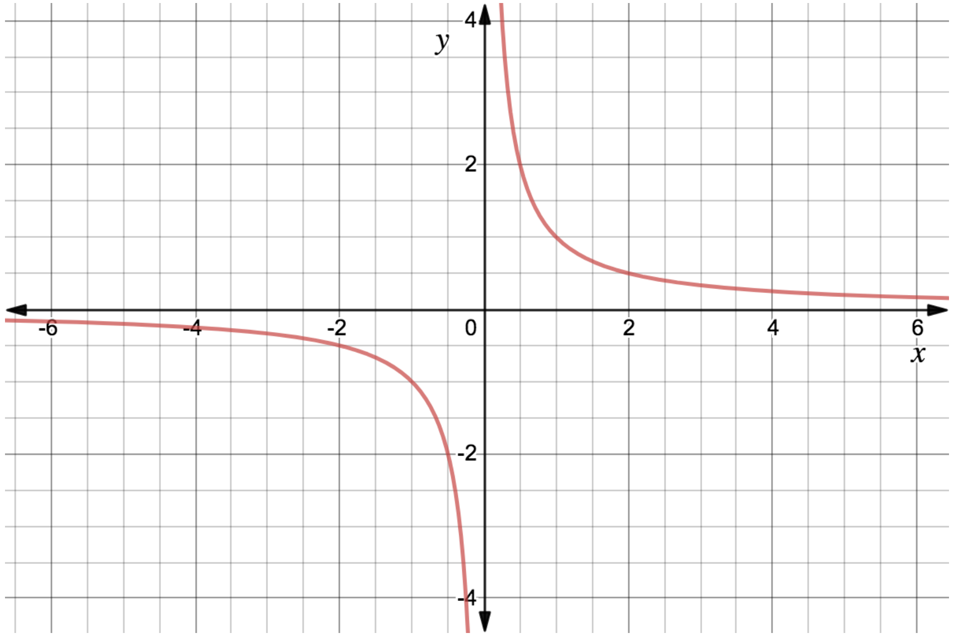

- **Intermediate value property:** if a continuous function takes on two values, it must also take on all values in between.

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Continuity Formally

- A function $f(x)$ is continuous if for every x_0 in its domain

$$\lim_{x \to x_0} f(x) = f(x_0)$$

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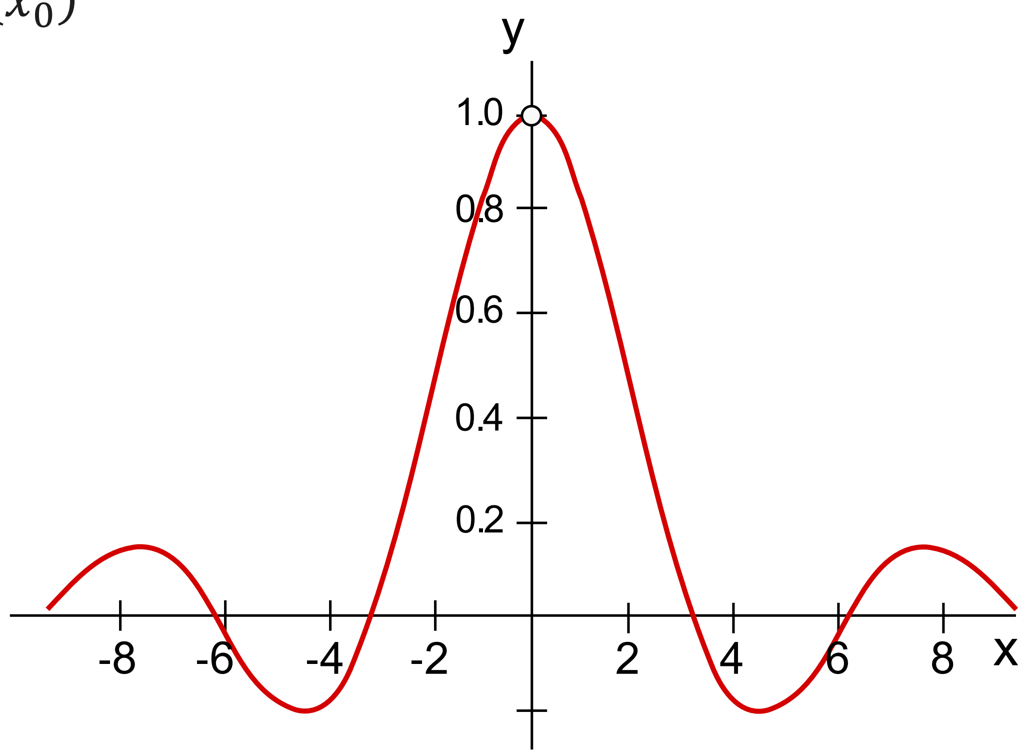
- <u>Example:</u>

$$f(x) = \frac{\sin x}{x}$$

Not defined at $x_0 = 0$, but $\lim_{x\to 0} \frac{\sin x}{x} = 1$.



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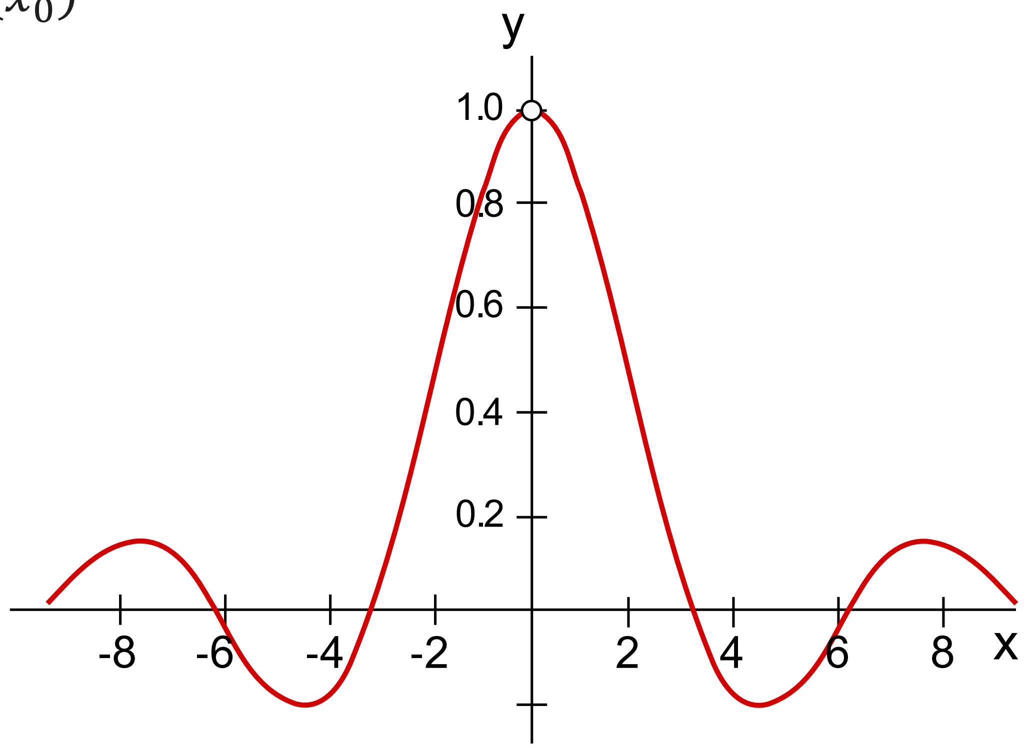
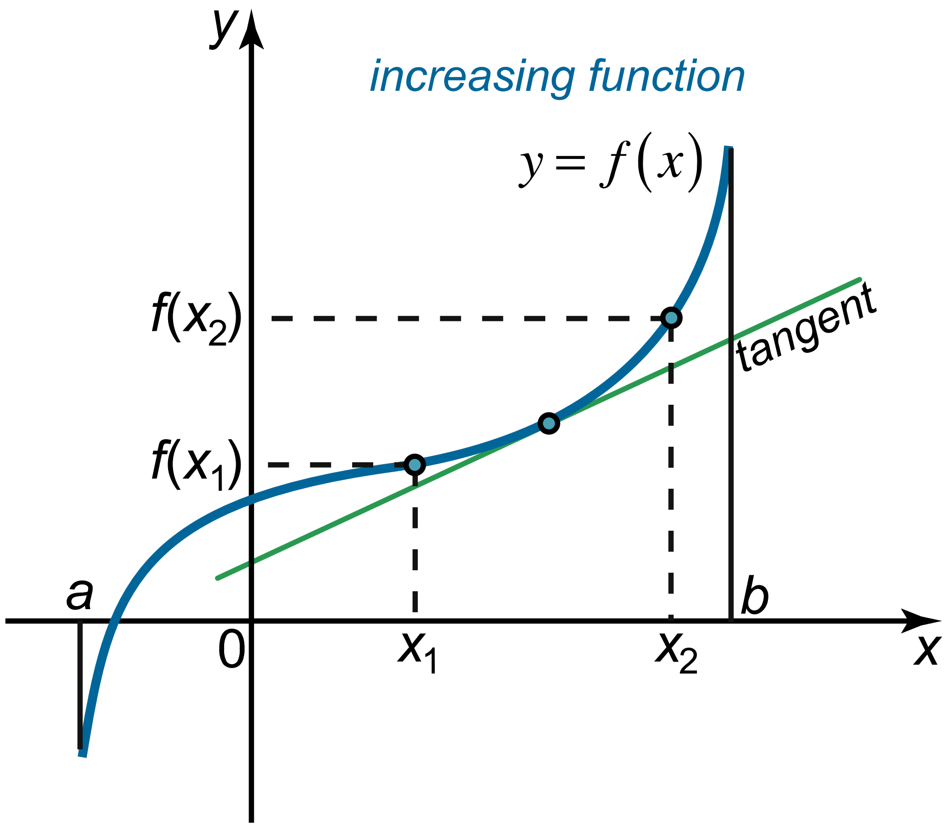
Not defined at $x_0 = 0$, but $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

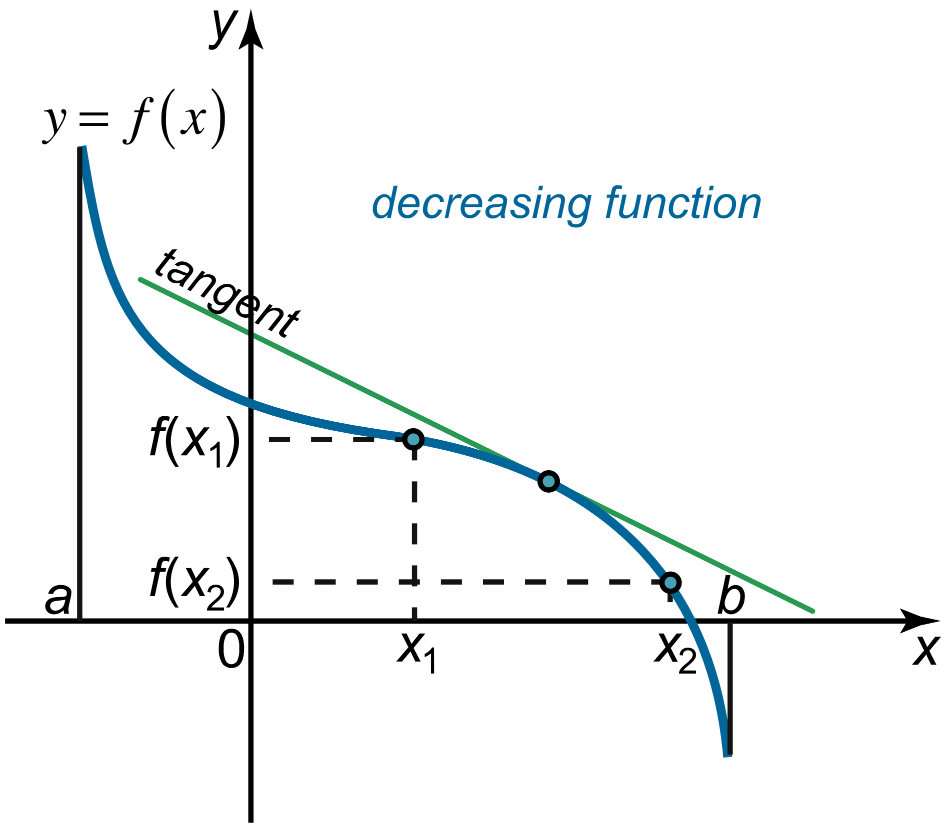
$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$ is a continuous function!



Increasing / Decreasing



$f(x_2) > f(x_1) \quad \forall x_2 > x_1 \in [a; b]$

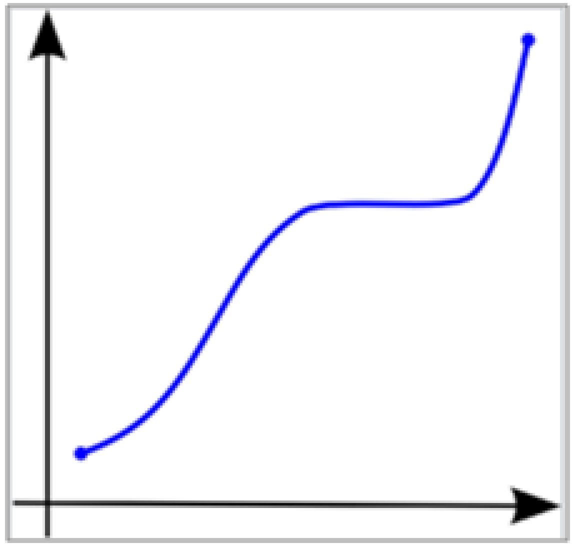


$f(x_2) < f(x_1) \quad \forall x_2 > x_1 \in [a; b]$

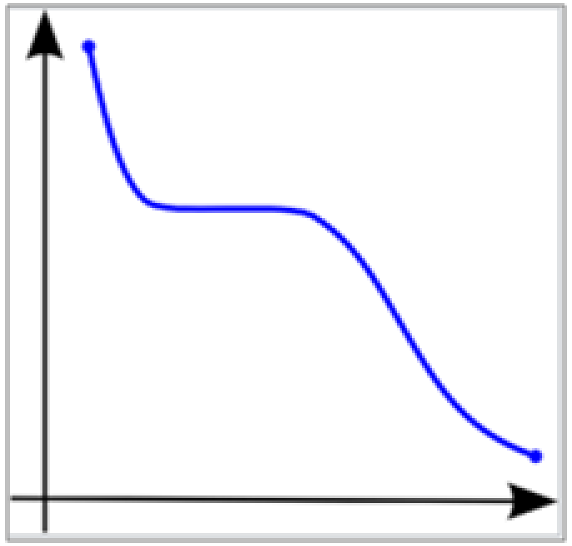
Source: https://math24.net/increasing-decreasing-functions.html

Monotonicity

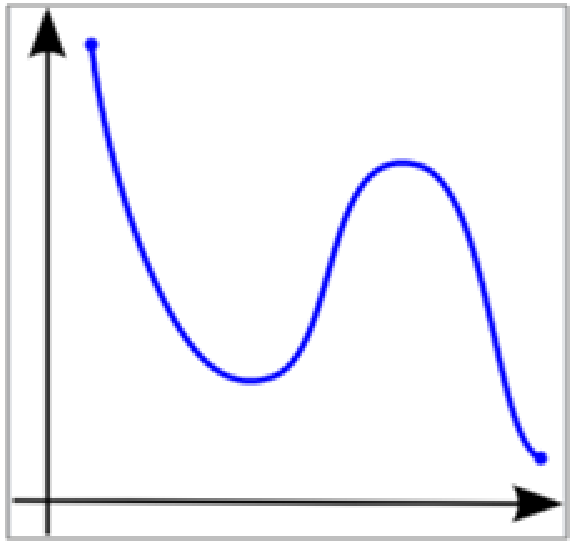
- A **monotonic function** = a (non-) increasing / decreasing function over the whole domain.



A monotonically non-decreasing function.



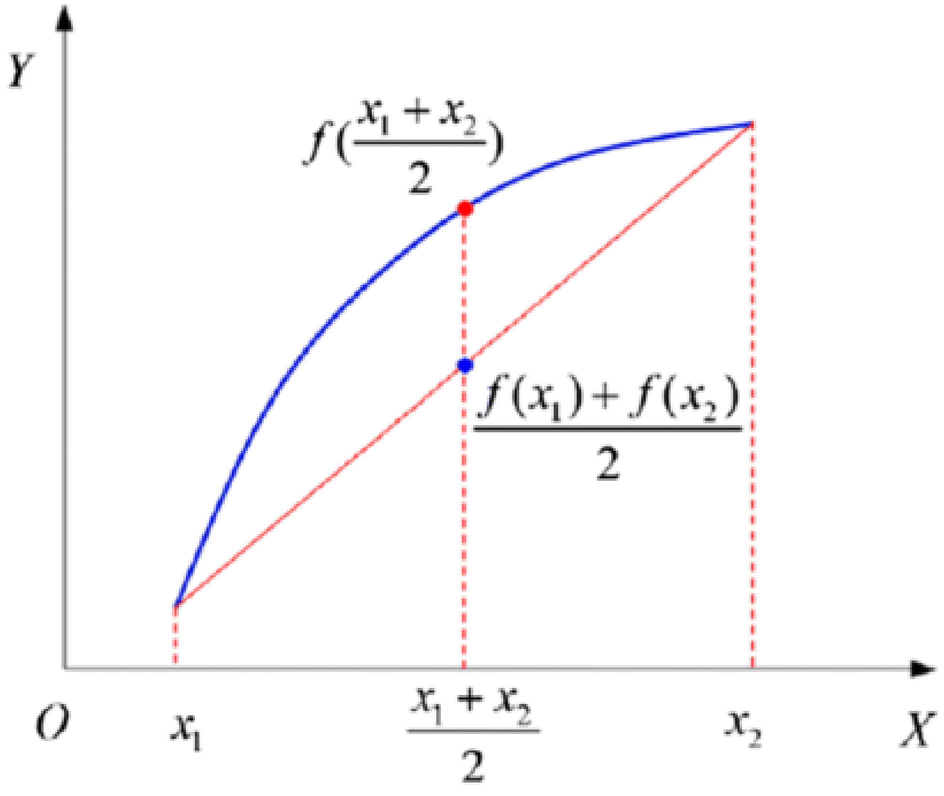
A monotonically non-increasing function.



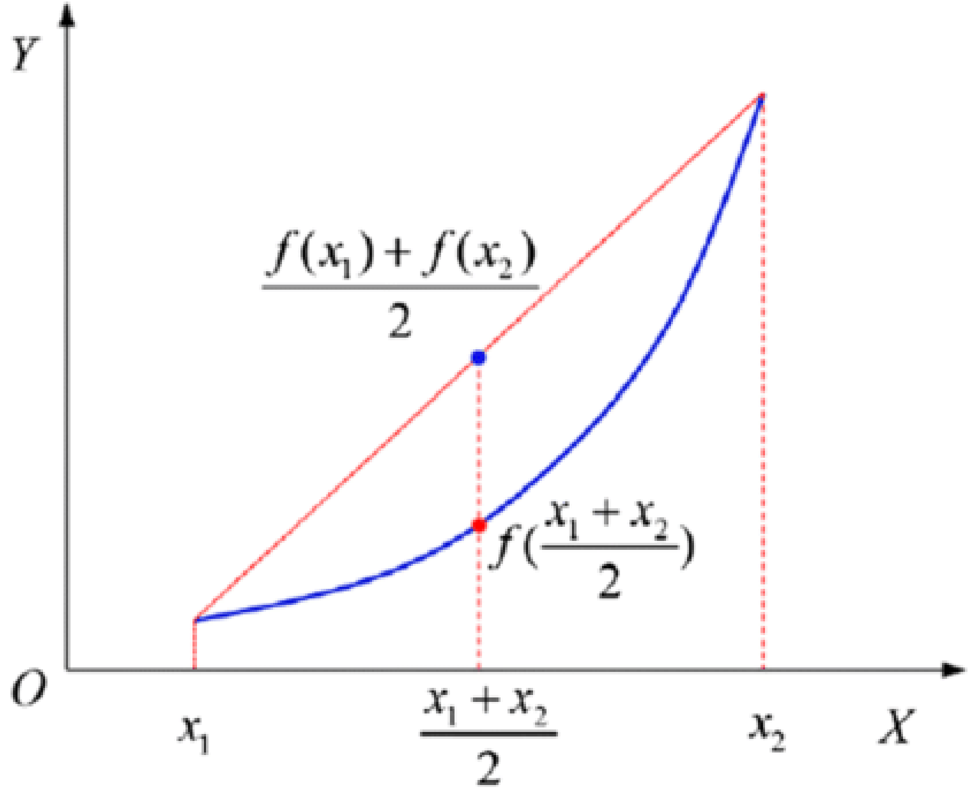
A non-monotonic function.

Source: https://en.wikipedia.org/wiki/Monotonic_function

Convexity



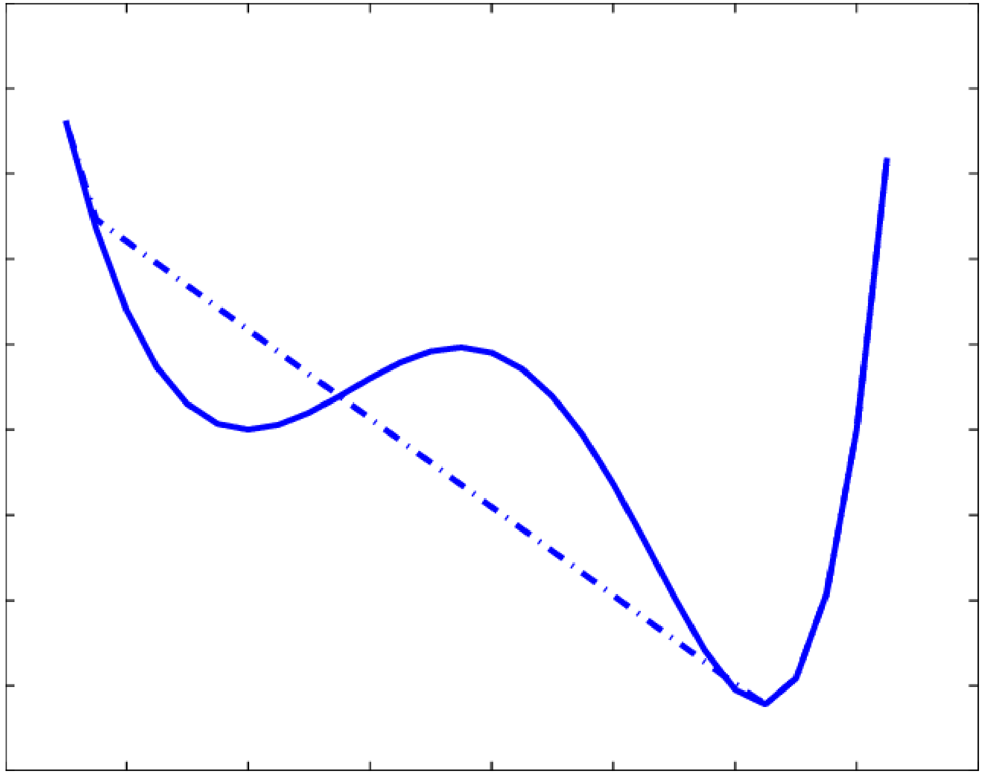
Concave function



Convex function

Source: https://www.researchgate.net/figure/Judgment-of-concavity-and-convexity-a-Convex-curve-b-Concave-curve_fig3_339939083

Convexity



A non-convex curve

Derivatives

Derivative

- A way to measure change:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

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$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

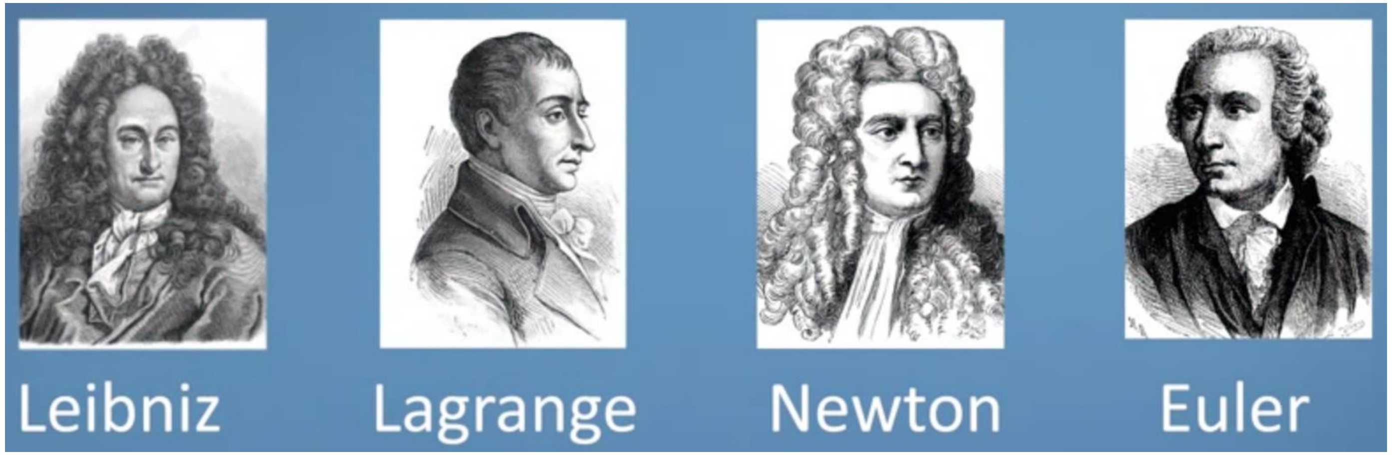
- Derivative of the function f at the point x tells us how much the function f changes as the input x changes by a small amount Δx:

$$f(x + \Delta x) \approx f(x) + \Delta x \cdot f'(x)$$

Derivatives - Example

$$\left(\frac{1}{x}\right)' = \lim_{\Delta x \to 0} \frac{\frac{1}{x + \Delta x} - \frac{1}{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x \cdot x(x + \Delta x)} = \lim_{\Delta x \to 0} \frac{-1}{x^2 + x\Delta x} = -\frac{1}{x^2}.$$

Derivatives – Other Notation



$$f'(x) = f'_x(x) = \frac{d}{dx}f(x) = \frac{\partial}{\partial x}f(x)$$

Derivatives

$$(c)' = 0 \quad (c = \text{const}), \qquad (x^\alpha)' = \alpha x^{\alpha-1},$$

$$(e^x)' = e^x, \qquad (a^x)' = a^x \ln a,$$

$$(\ln x)' = \frac{1}{x}, \qquad (\log_a x)' = \frac{1}{x \ln a},$$

$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x,$$

$$(\operatorname{tg} x)' = \frac{1}{\cos^2 x}, \qquad (\operatorname{ctg} x)' = -\frac{1}{\sin^2 x},$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \qquad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}},$$

$$(\operatorname{arctg} x)' = \frac{1}{1+x^2}, \qquad (\operatorname{arcctg} x)' = -\frac{1}{1+x^2}.$$

Sum Rule

$$[u(x) + v(x)]' = u'(x) + v'(x)$$

- Example:

$$(x^2 + x^3)' = 2x + 3x^2$$

Product Rule

$$[u(x) \cdot v(x)]' = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

- Example:

$$(xe^x)' = 1 \cdot e^x + x \cdot e^x$$

$$\left(\frac{1-x}{x}\right)' = (1-x) \cdot \frac{1}{x} = -\frac{1}{x} - \frac{1-x}{x^2}$$

Chain Rule

- Tells us how to compute the derivative of the composition of functions:

$$f\big(g(x)\big)' = f'\big(g(x)\big) \cdot g'(x)$$

- Other notation:

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

Chain Rule - Example

$$\left(\frac{1}{1-x}\right)' = -\frac{1}{(1-x)^2} \cdot (1-x)' = \frac{1}{(1-x)^2}$$

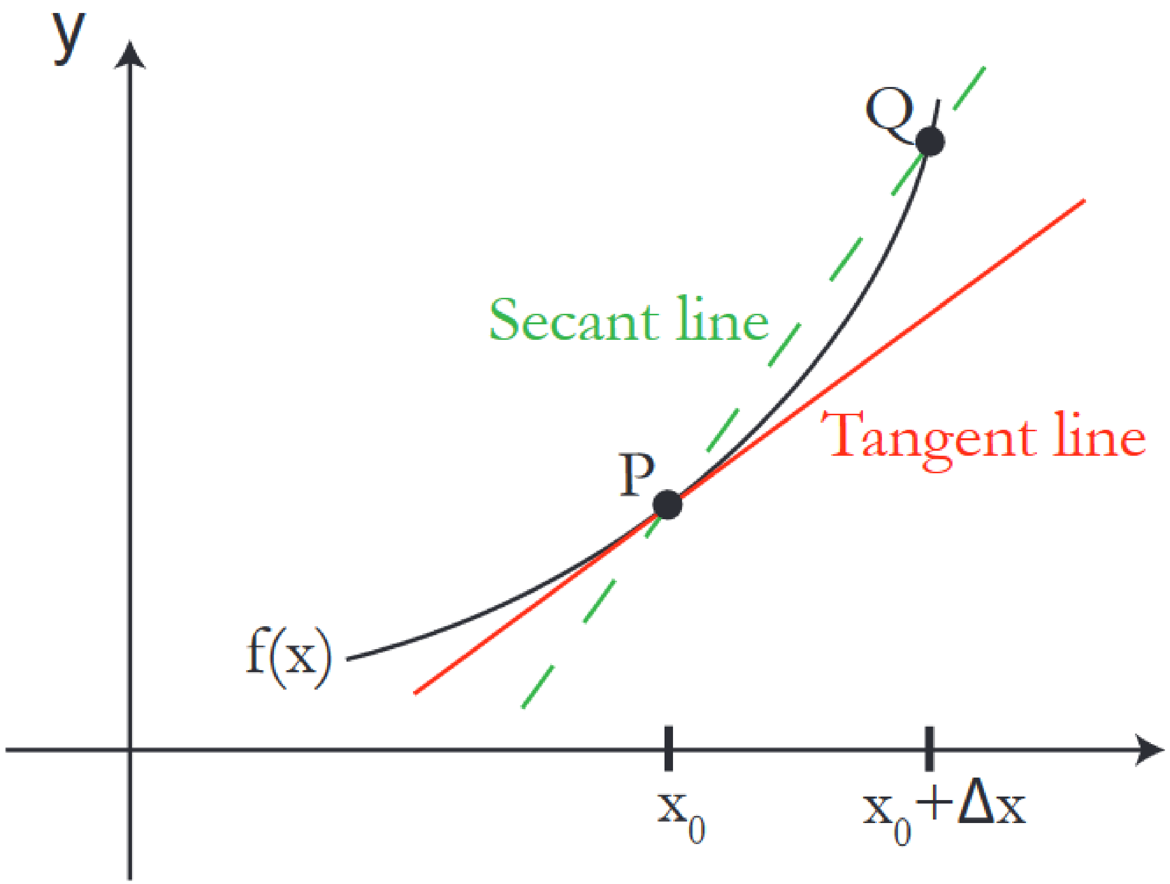
$$\left(e^{x^2}\right)' = e^{x^2} \cdot (x^2)' = e^{x^2} \cdot 2x$$

Quotient Rule

$$\frac{u(x)}{v(x)} = \left[u(x) \cdot \frac{1}{v(x)} = u'(x) \cdot \frac{1}{v(x)} - u(x) \cdot \frac{1}{\left(v(x)\right)^2} \cdot v'(x)\right] =$$

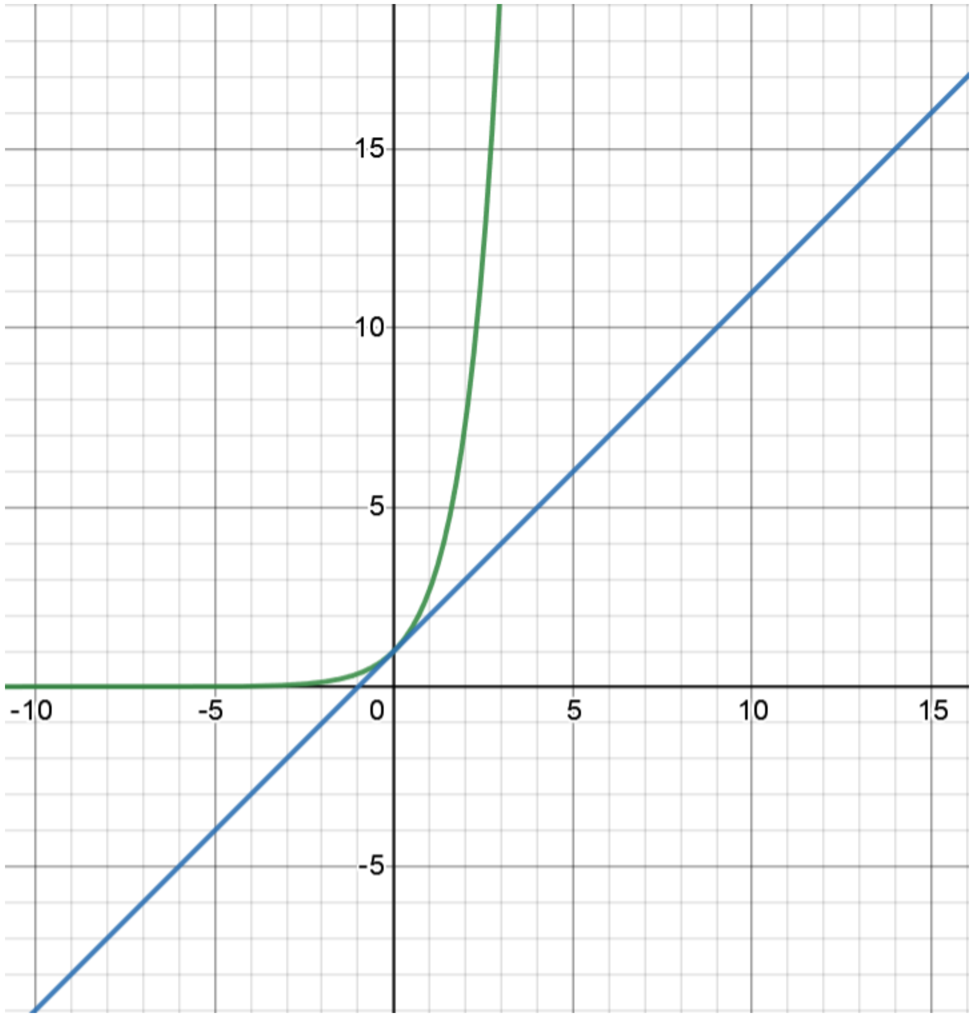
$$= \frac{u'(x)v(x) - u(x)v'(x)}{\left(v(x)\right)^2}$$

Geometric Meaning of a Derivative



Tangent Line - Example

- Find a tangent line to $y = e^x$ at $x_0 = 0$.



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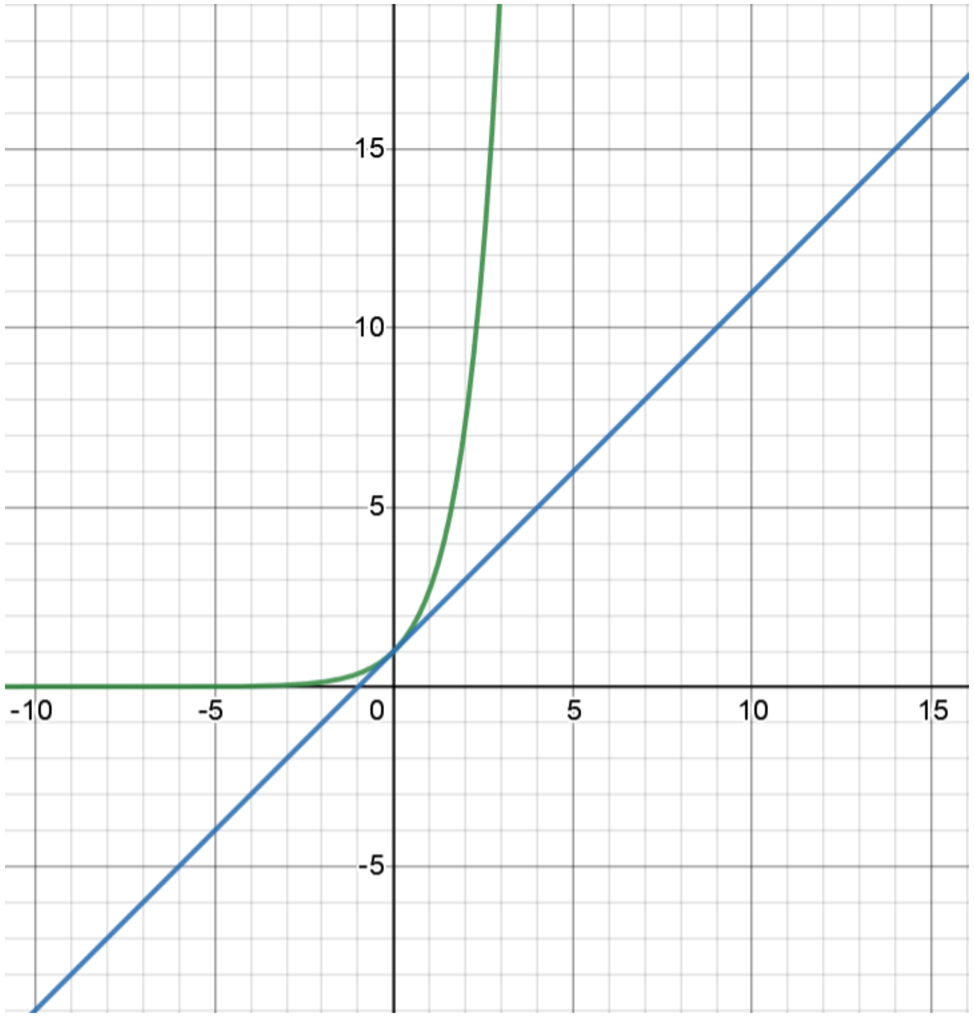
Tangent line: $y = kx + b$



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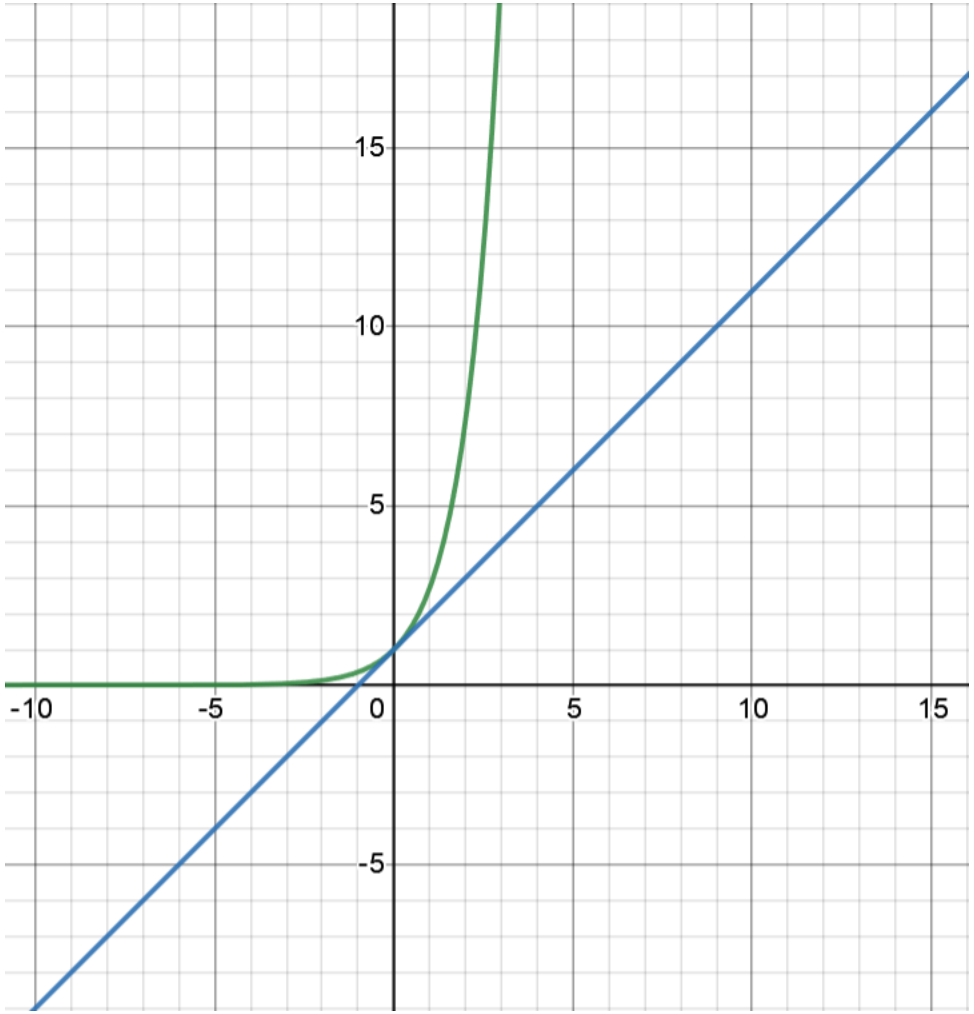

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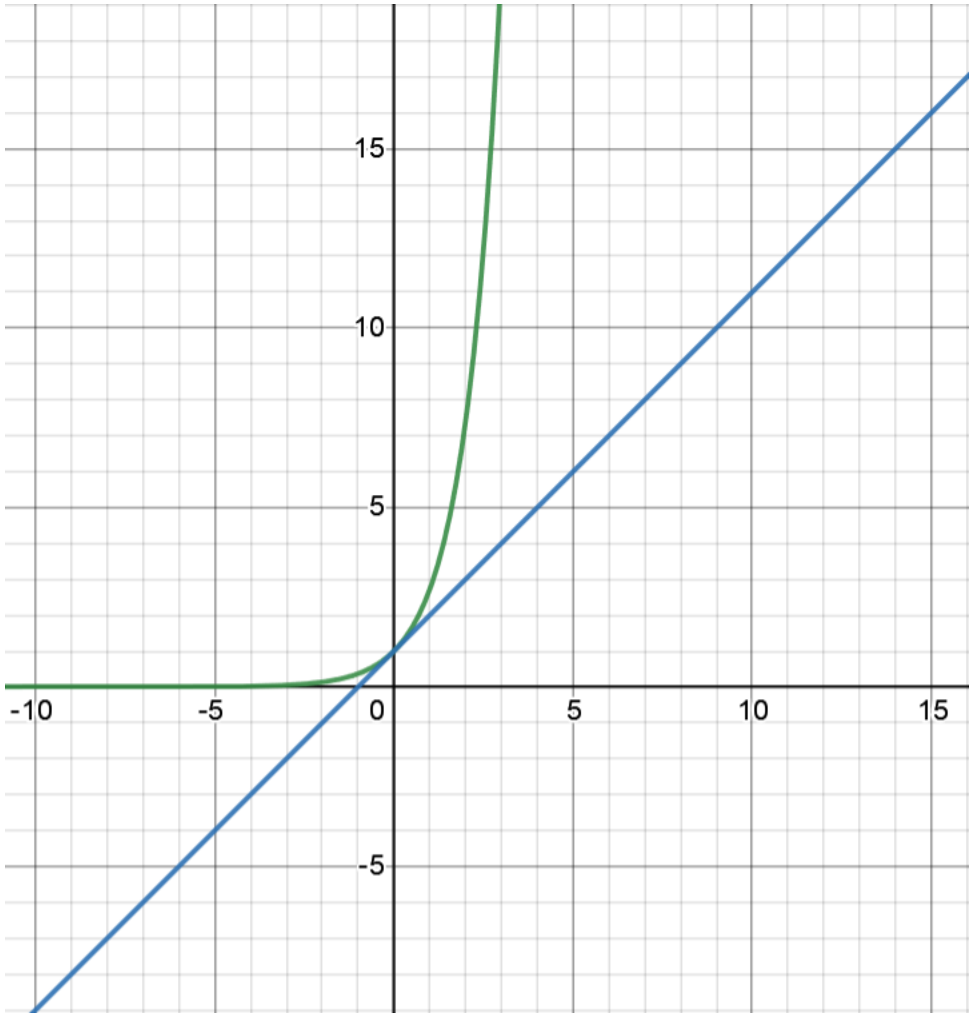
Tangent line touches the graph at $x_0 = 0$:



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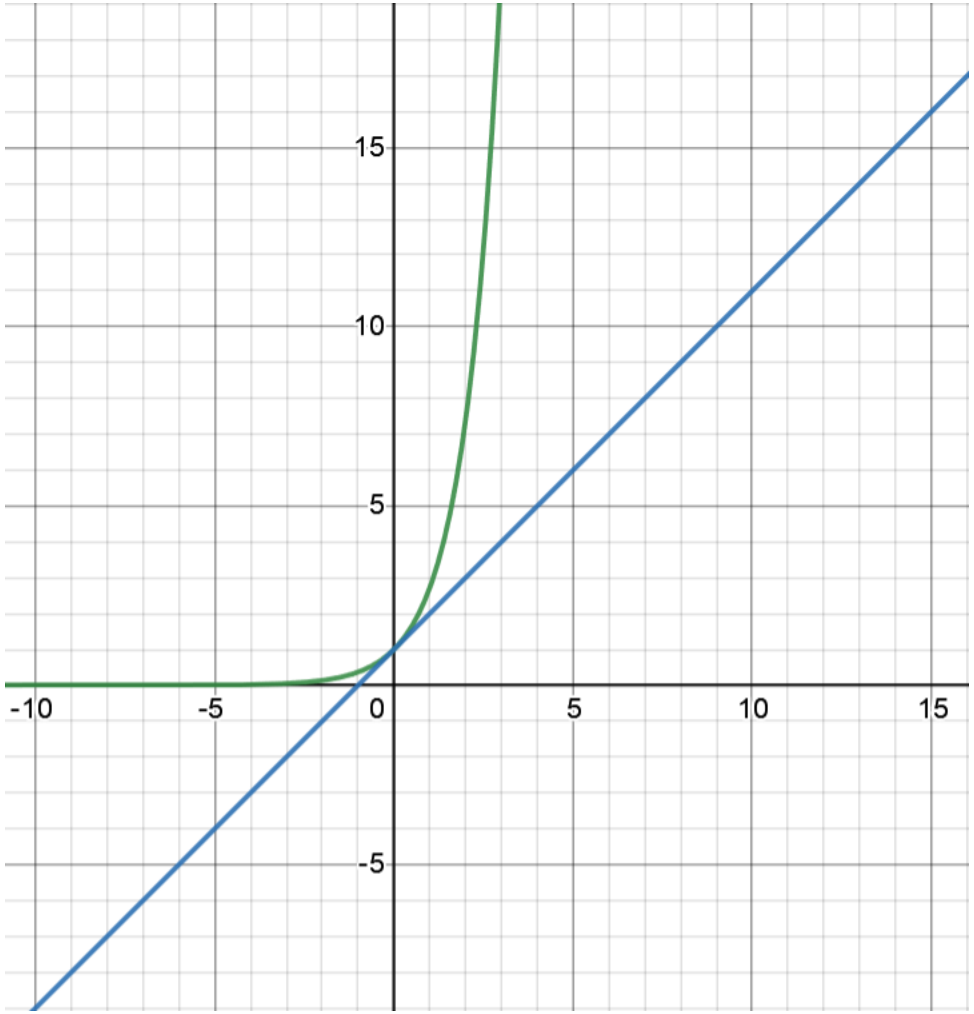
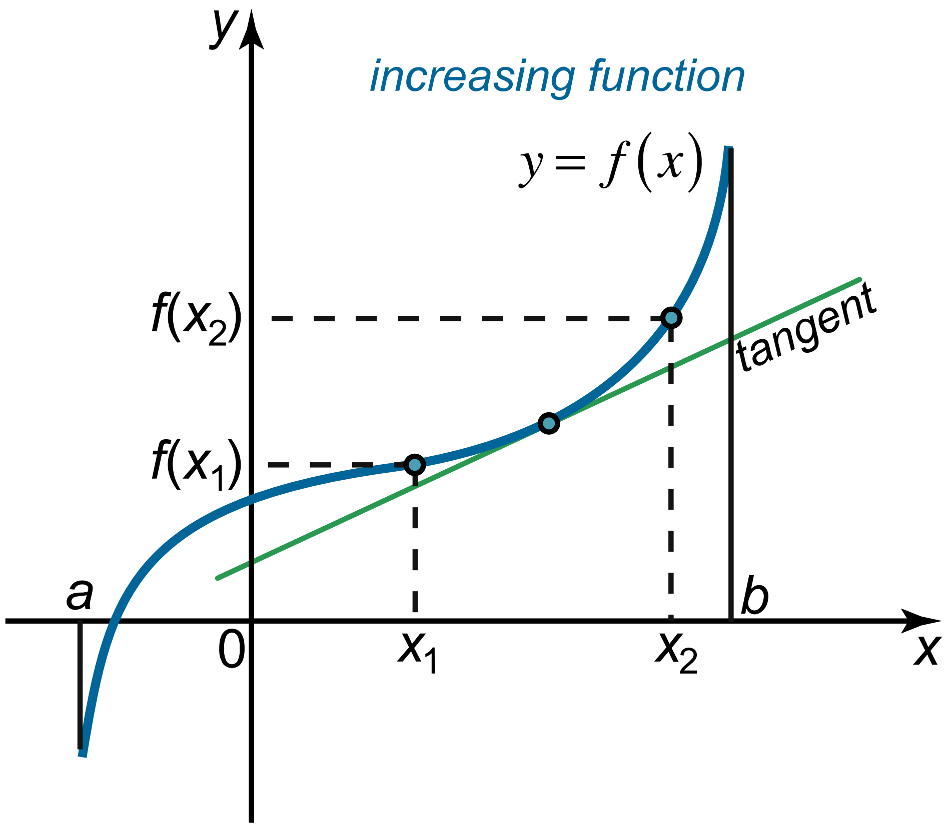
Tangent line touches the graph at $x_0 = 0$:

$$1 \cdot 0 + b = e^0 = 1, \qquad b = 1$$

Tangent line: $y = x + 1$



Increasing / Decreasing



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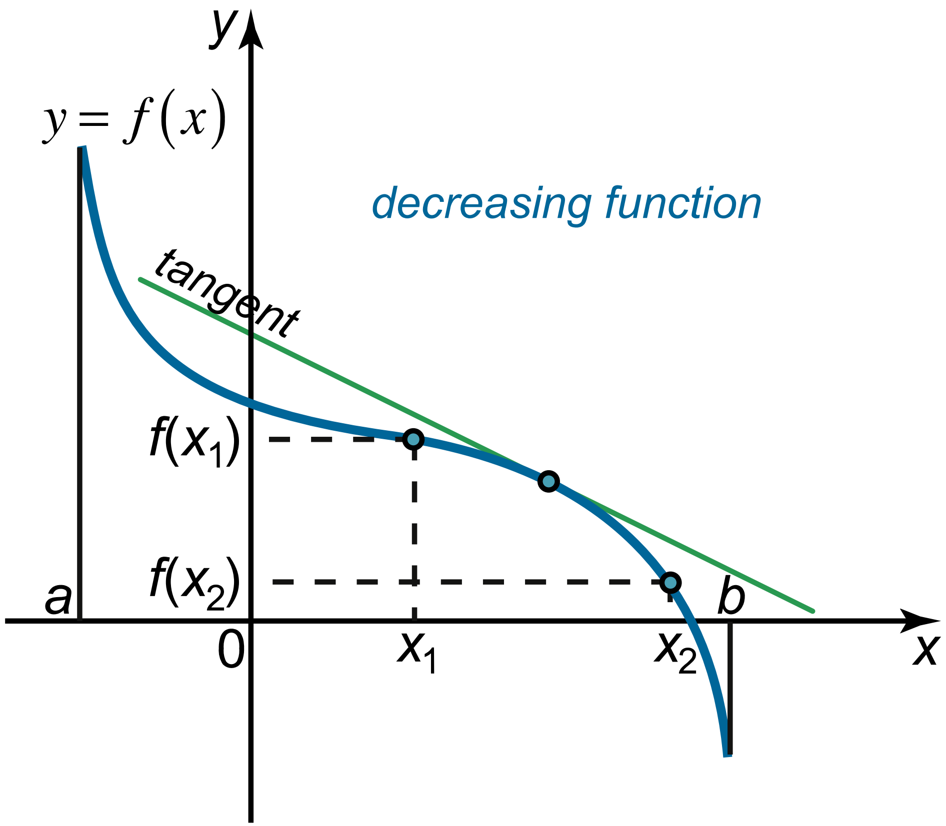

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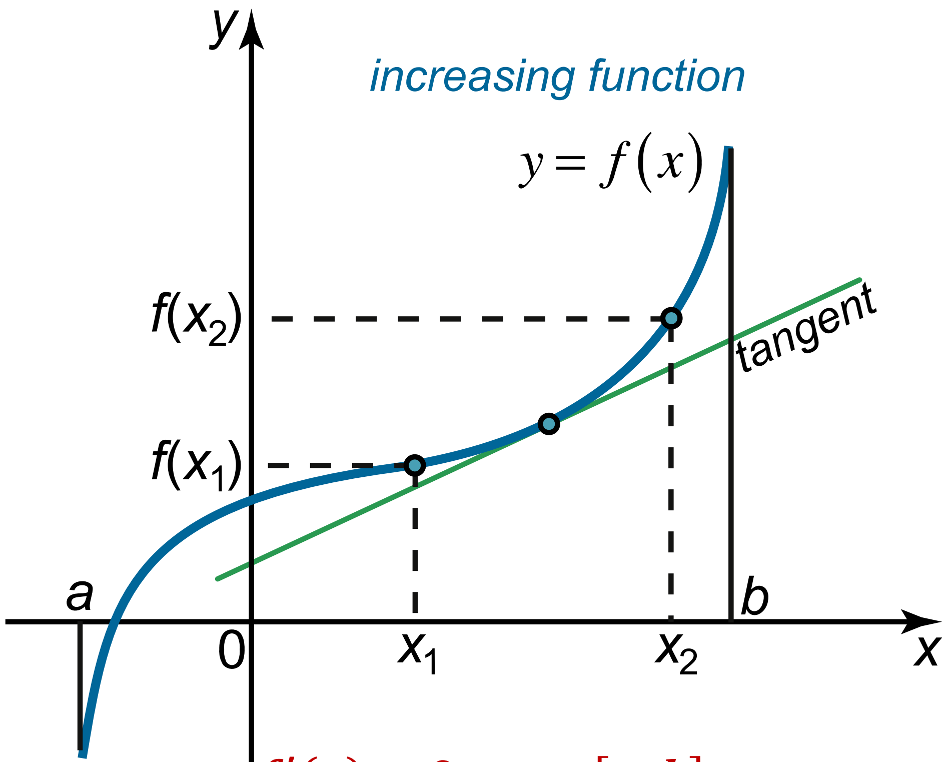

$$f'(x) > 0 \ \ \forall x \in [a; b]$$

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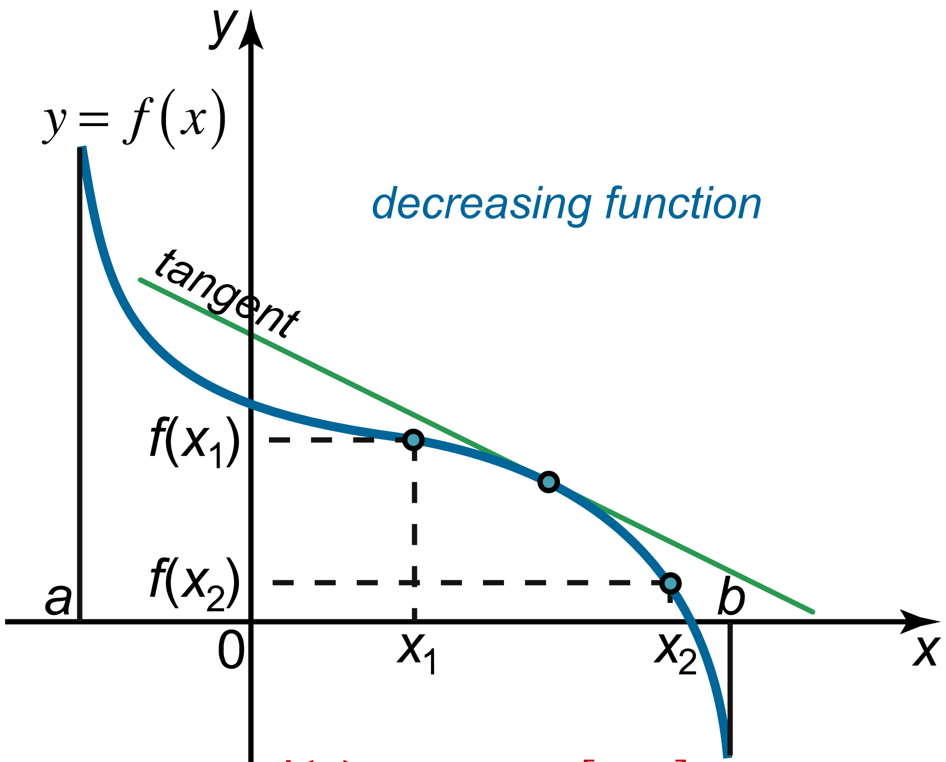

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Exploring a Function with Its Derivative

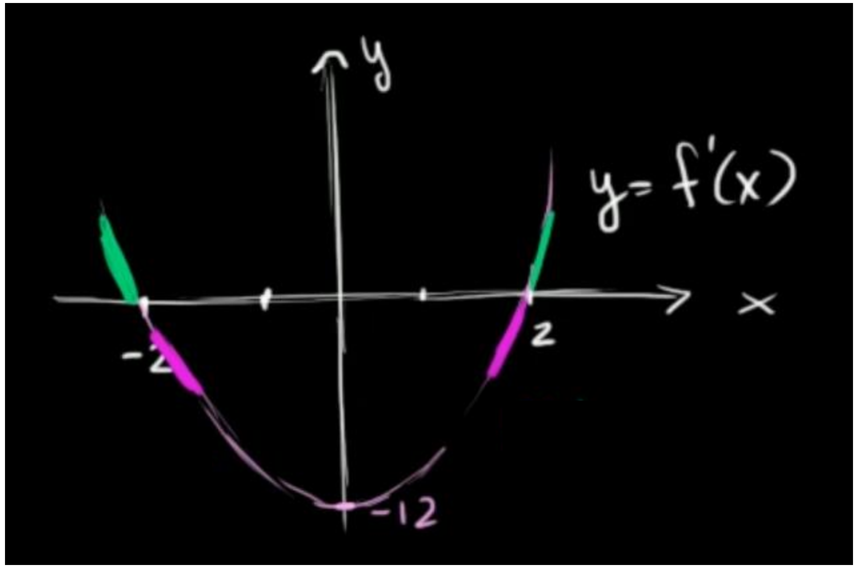
- Consider a function $f(x) = x^3 - 12x + 2$.

Source: https://www.khanacademy.org/math/ap-calculus-ab/ab-diff-analytical-applications-new/ab-5-8/v/analyzing-a-function-with-its-derivative

Exploring a Function with Its Derivative

- Consider a function $f(x) = x^3 - 12x + 2$.

Derivative: $f'(x) = 3x^2 - 12 = 0$



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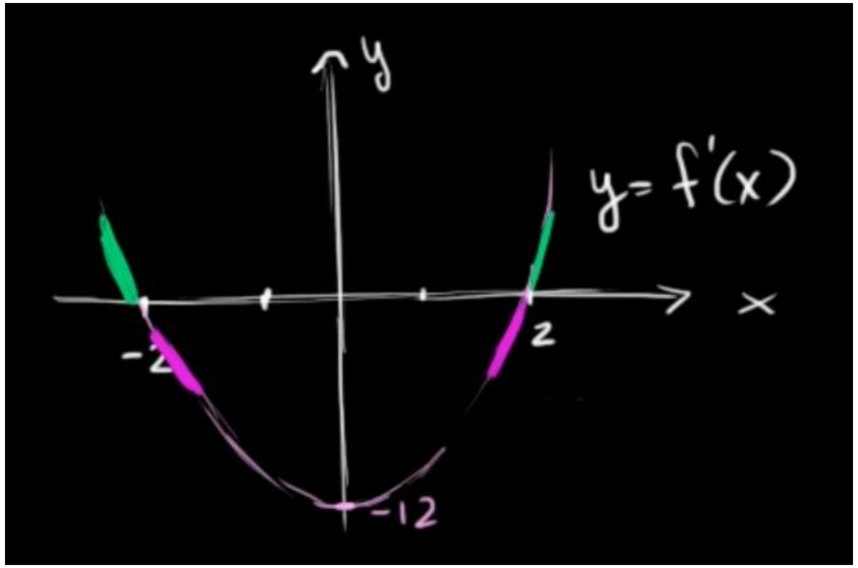
$f'(x) \Leftrightarrow x = \pm 2$



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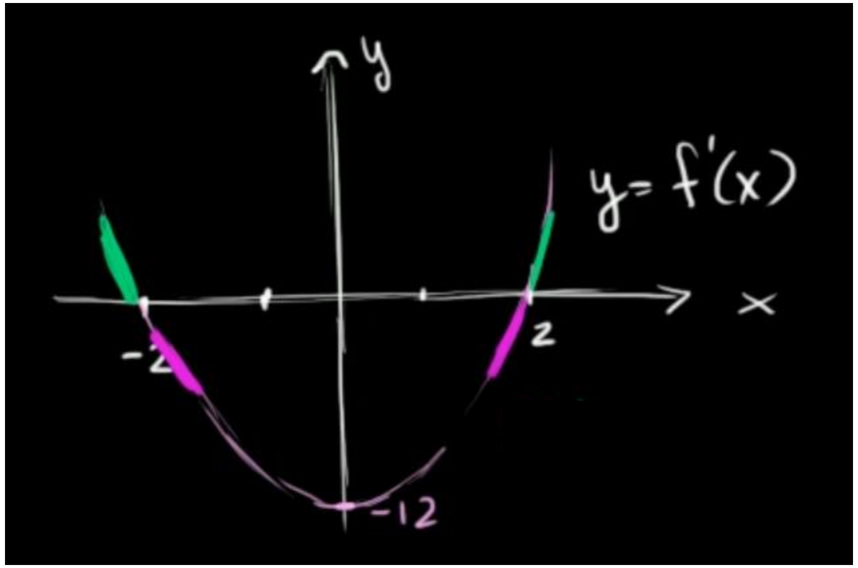

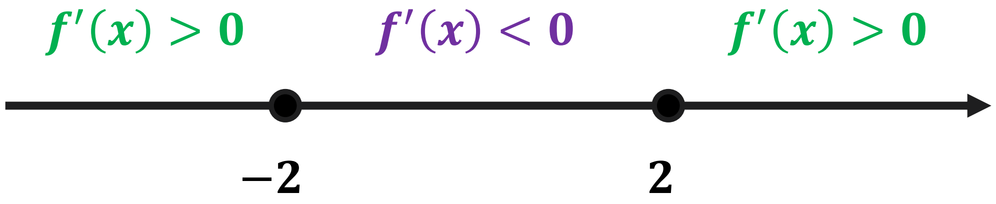

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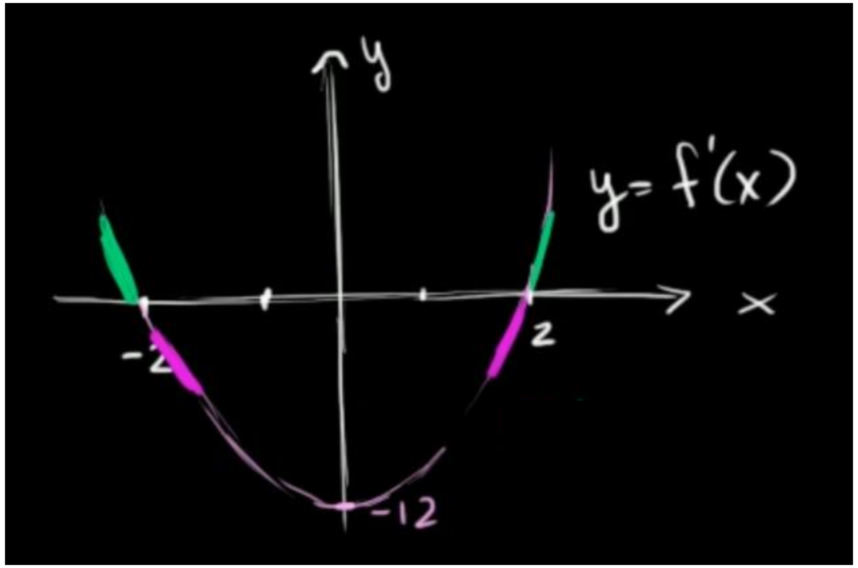

$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
$-\infty$ to -2	-2 to 2	2 to ∞
$f(x)$ increasing	$f(x)$ decreasing	$f(x)$ increasing

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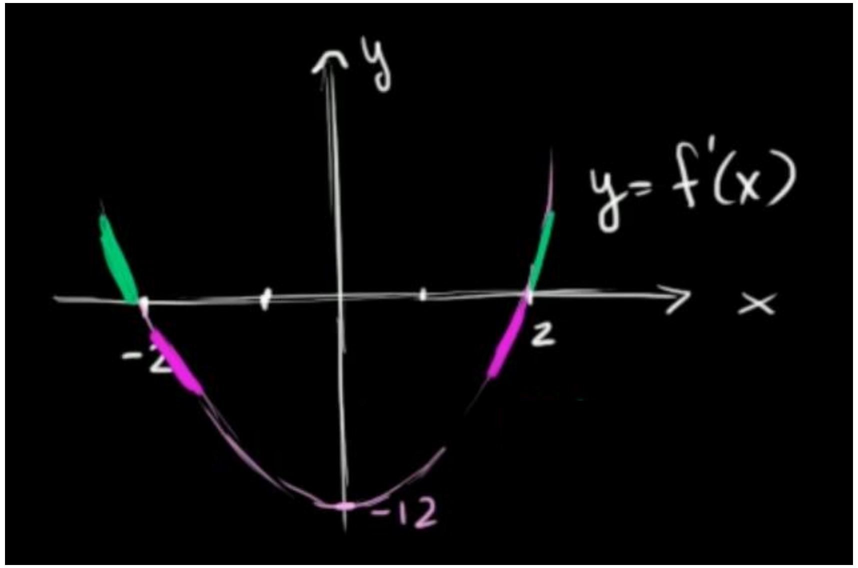

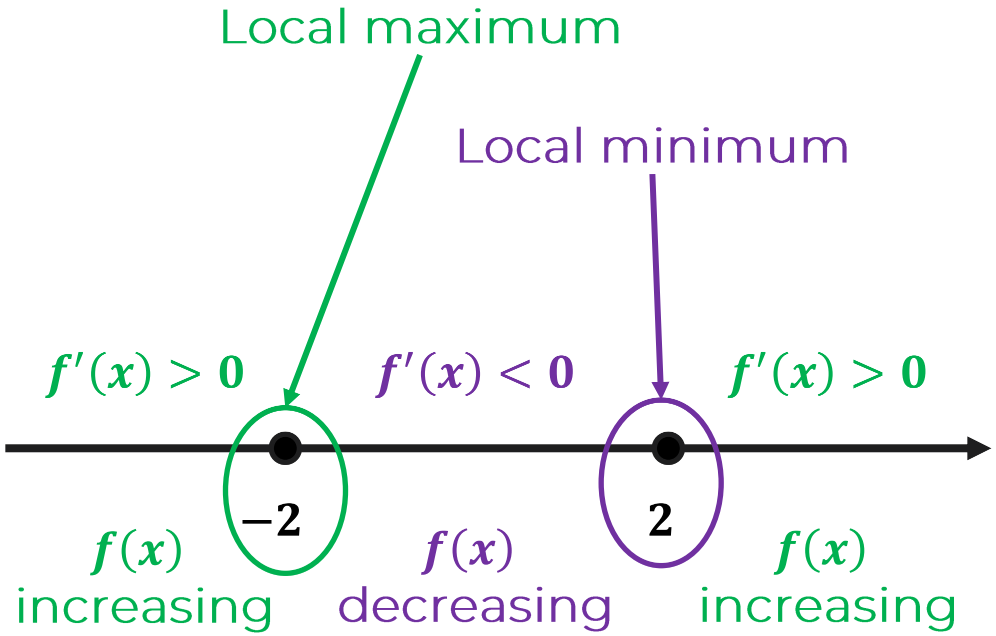

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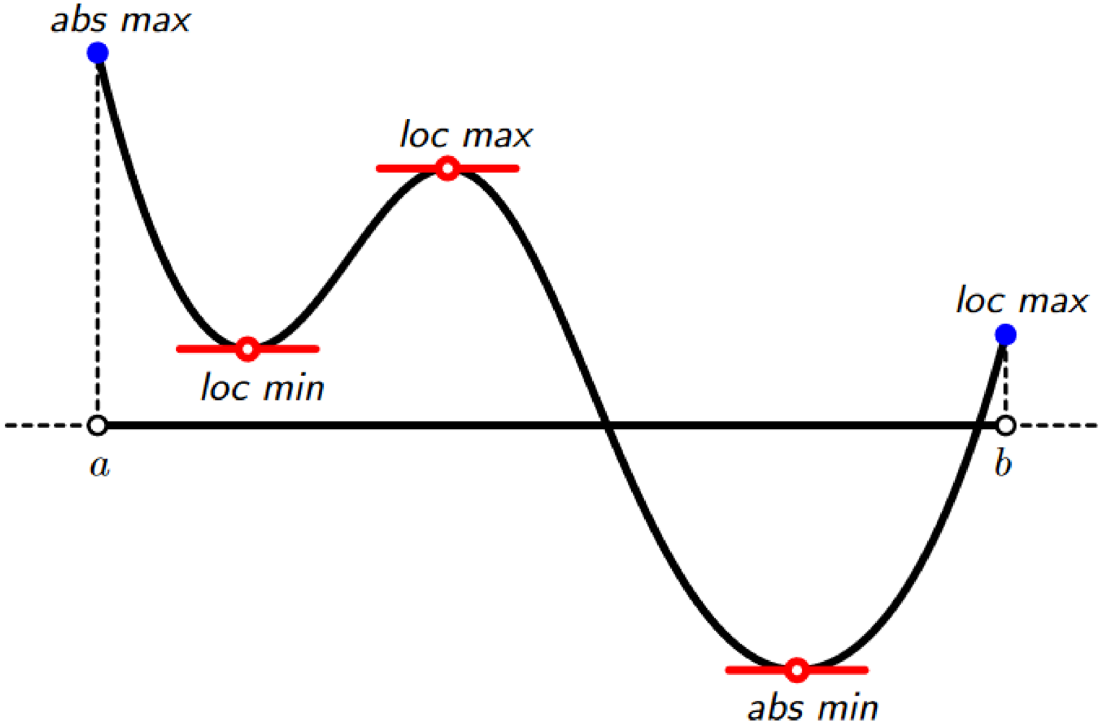
Extrema

Extrema of a Function

- $f(x)$ reaches its local minima *(maxima)* at x_0 if $f(x_0)$ is the smallest *(highest)* value of $f(x)$ around x_0 .



- $f(x)$ reaches its global minima *(maxima)* at x_0 if $f(x_0)$ is the smallest *(highest)* value of $f(x)$ on the interval of interest.

Critical Point

- A stationary point of $f(x)$ is a point x_0 such that $f'(x_0) = 0$

Source:http://www.conejousd.org/Portals/49/Departments/Math/Sansing/Math%20AnalyCP/Notes%203-6%20Critical%20Points%20and%20Extrema.pdf

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  - $f(x_0) = 0$ (x_0 is a stationary point) or
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- Critical points: those points on a graph at which a line drawn tangent to the curve is horizontal or vertical.

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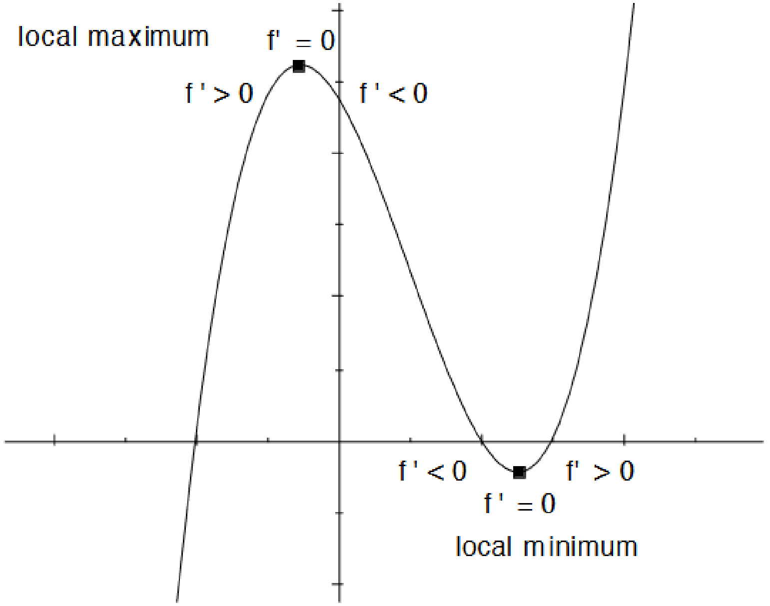
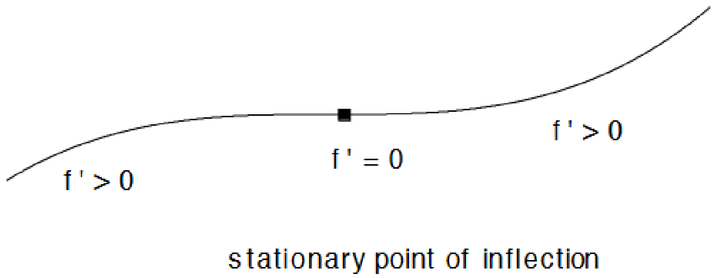
First Derivative Test

- Let x_0 be a critical point of $f(x)$.
- If $f'(x) < 0$ for $x < x_0$ and $f'(x) > 0$ for $x > x_0$ then x_0 is a point of a local minimum.





- If $f'(x) > 0$ for $x < x_0$ and $f'(x) < 0$ for $x > x_0$ then x_0 is a point of a local maximum.

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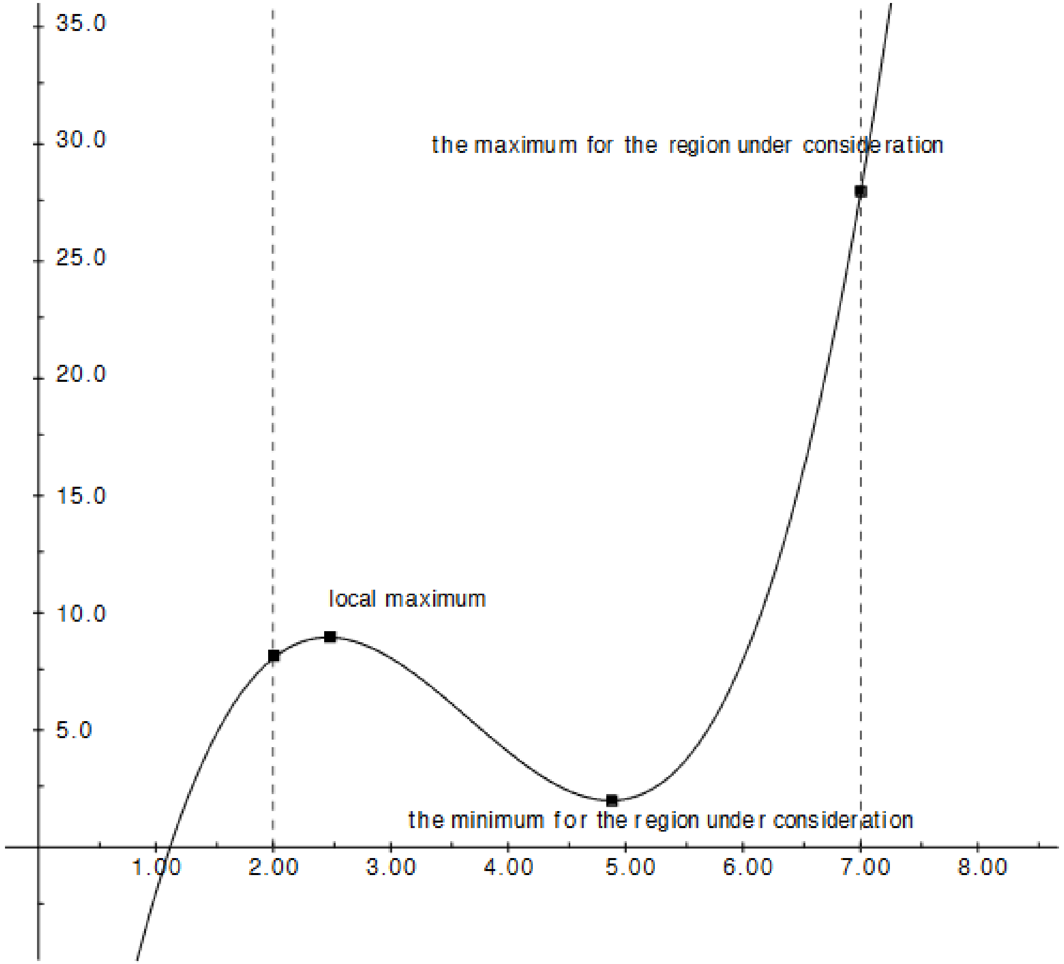
Source: https://www.sydney.edu.au/content/dam/students/documents/mathematics-learning-centre/introduction-to-differential-calculus.pdf

Don't Forget the Endpoints!



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Algorithm for Finding Global Extrema

- Suppose you need to find global maxima (minima) of $f(x)$ on $[a; b]$.
- Here is s recipe:
  1. Find all critical points of $f(x)$ on $[a; b]$;
  2. Determine which of them are the local maxima (minima);
  3. Compute $f(x)$ at the endpoints: $f(a)$ and $f(b)$.
  4. Pick the point from (2) – (3) corresponding to the largest (smallest) function value.

Finding Extrema - Example

- Find the global minimum of $f(x) = x^2 e^x$ on $[-4, 1]$.

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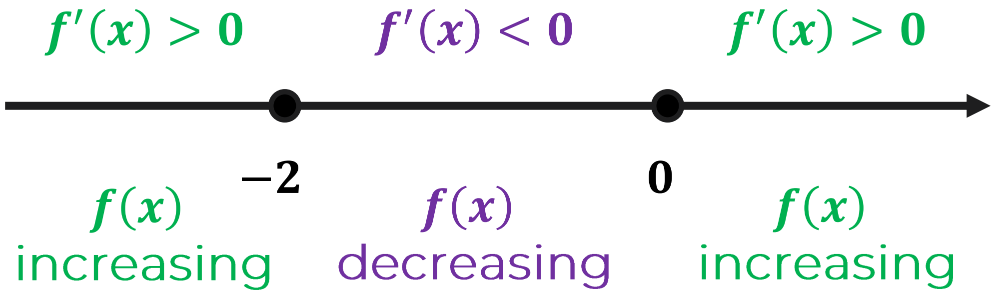

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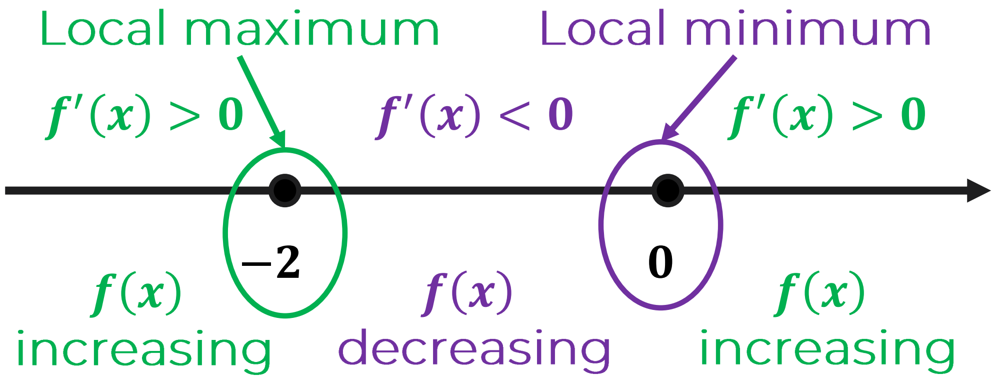

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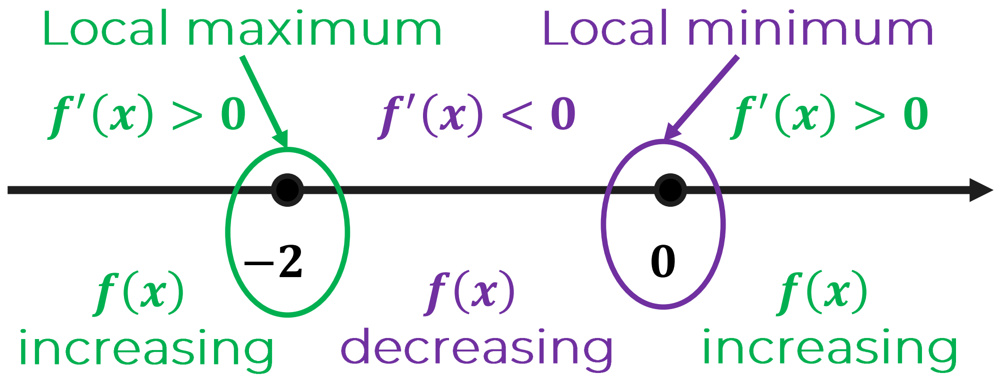

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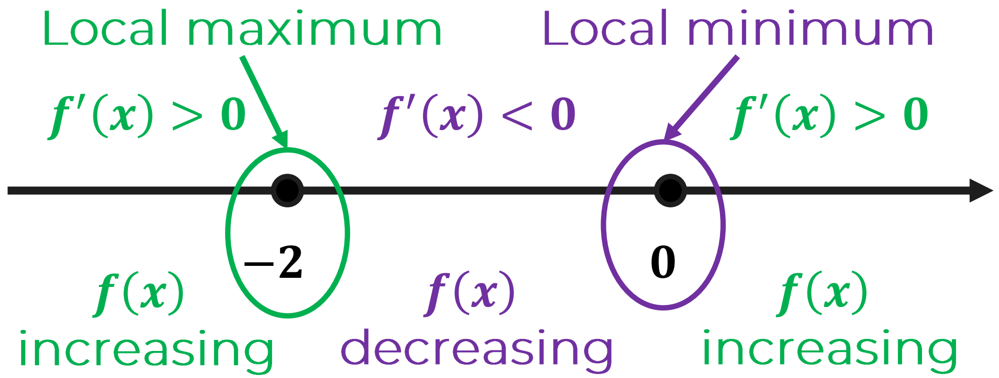

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Higher Derivatives

Higher Derivatives

- Derivatives of the derivatives:

$$f''(x) = \left(f'(x)\right)', \qquad f'''(x) = \left(f''(x)\right)', \qquad \dots$$

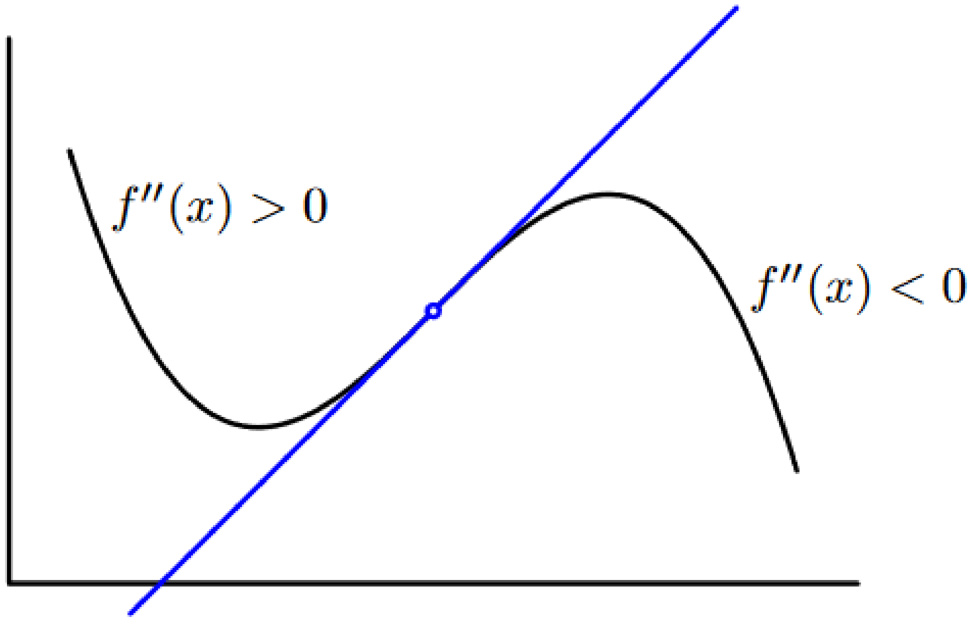
- Pretty straightforward!

- Example:

$$(3x^3 + 2x^2 + x)'' = (9x^2 + 4x + 1)' = 18x + 4$$

Second Derivative and Convexity

- A function is convex on some interval $[a; b]$ if and only if $\boldsymbol{f''(x) > 0}$ for all $x \in [a; b]$.



Source: https://people.math.wisc.edu/~angenent/Free-Lecture-Notes/free221.pdf

Second Derivative Test

- Consider a differentiable function $f(x)$.

- Let x_0 be its stationary point: $f'(x_0) = 0$.

- If $\boldsymbol{f''(x_0) < 0}$ then $f(x)$ has a local maximum at $\boldsymbol{x_0}$, and if $\boldsymbol{f''(x_0) > 0}$ then $f(x)$ has a local minimum at $\boldsymbol{x_0}$.

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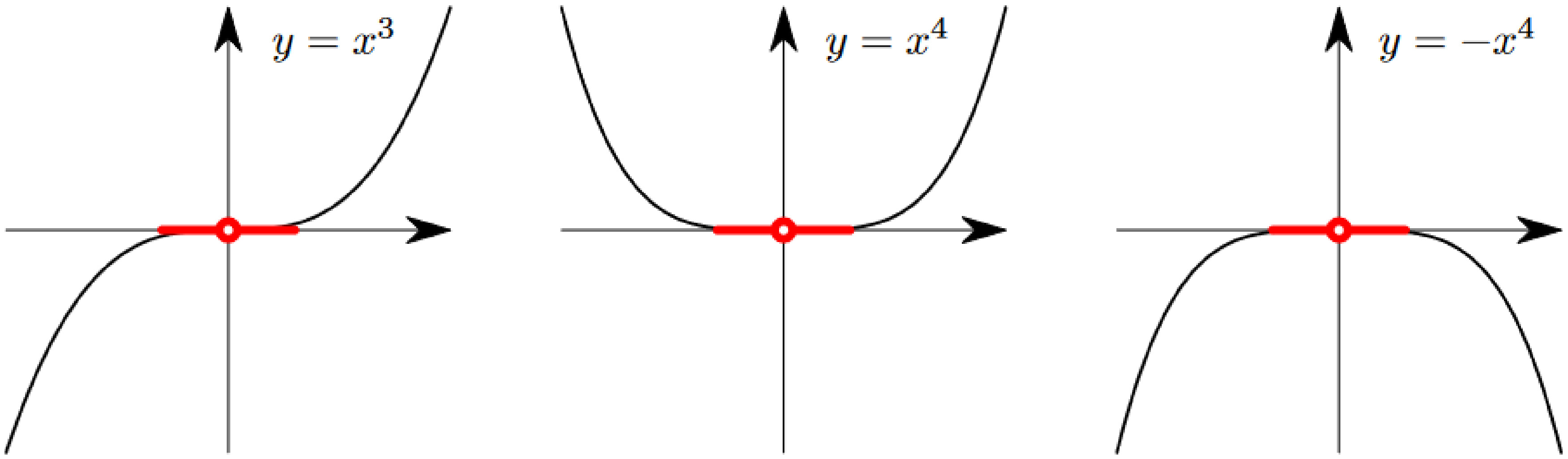

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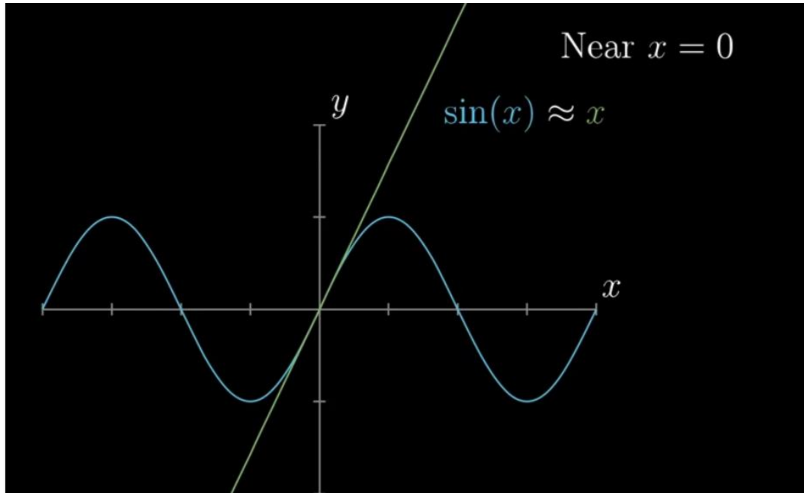
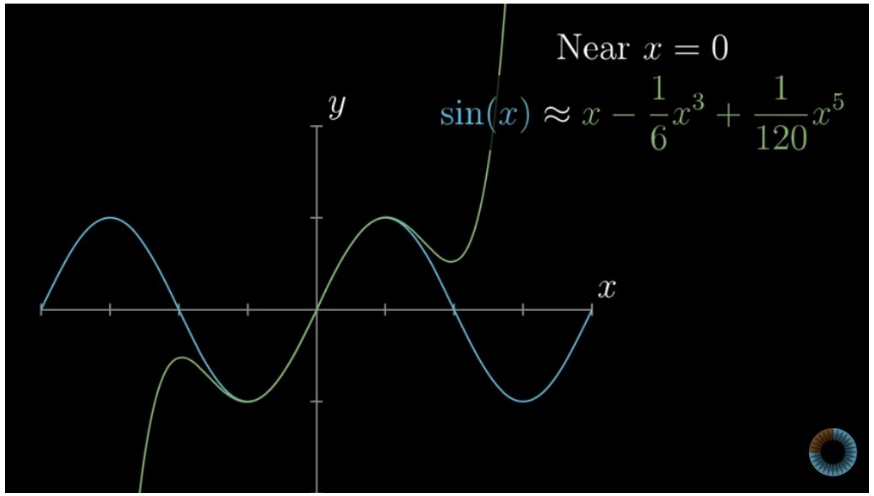
Taylor Series

Taylor Series

- Key idea: take a non-polynomial function and approximate it with a polynomial near some input.

- What for? Polynomial functions are easier!





Taylor Series

- Consider a smooth function $f \in C^{\infty}, \ f: \mathbb{R} \to \mathbb{R}$.

- Taylor series of f at x_0 is defined as

$$T_{\infty}(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

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- Taylor polynomial of degree n:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k, \qquad f(x) \approx T_n(x) \text{ around } x_0.$$

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Taylor Series - Example

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$$T_\infty(x) = \frac{1}{0!} + \frac{1}{1!} \cdot x - \frac{1}{2!} x^2 - \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \frac{1}{5!} x^5 - \cdots$$

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$$= \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} + \sum_{k=0}^{\infty}(-1)^{k+1}\frac{x^{2k+1}}{(2k+1)!} = \sin x + \cos x\,.$$

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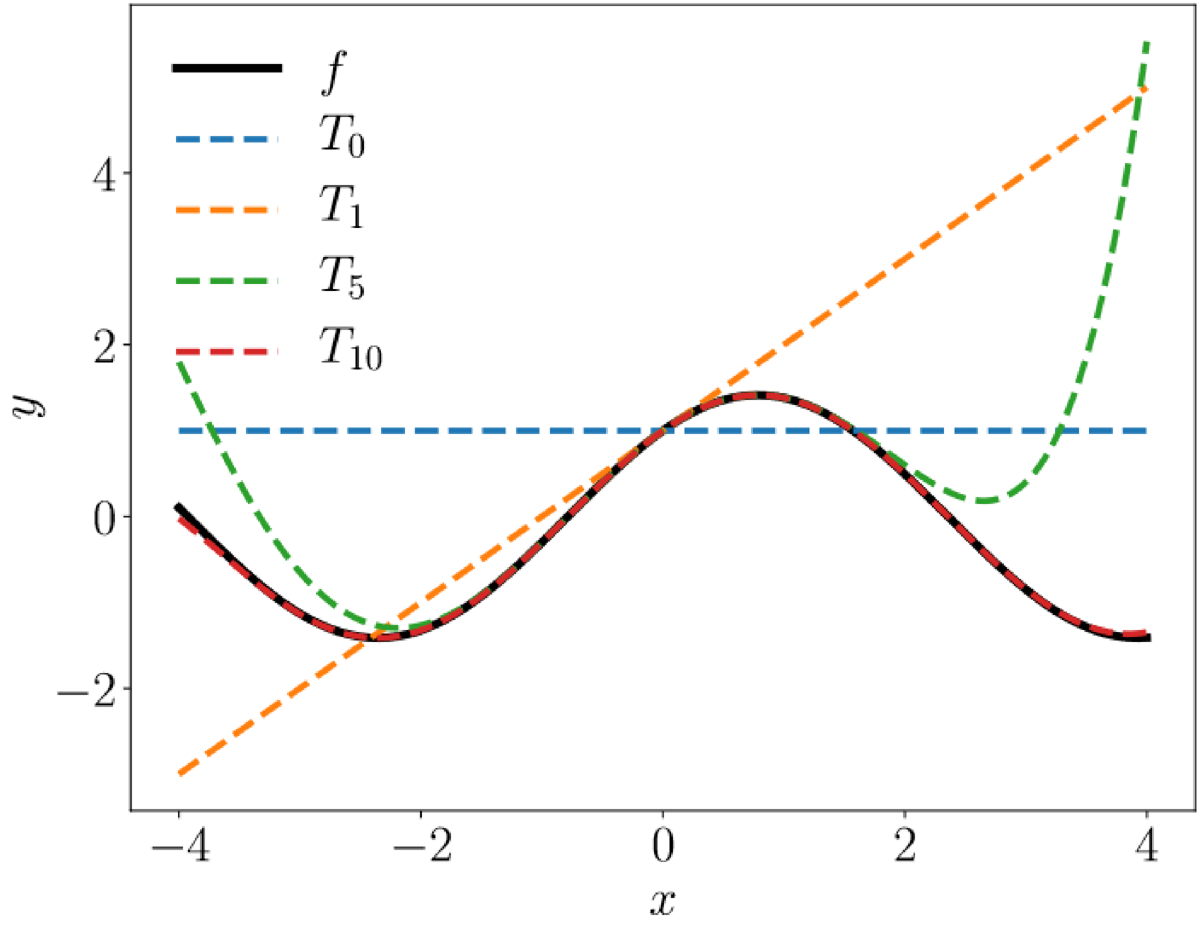

To sum up

- Univariate functions
- Basic properties
  - Continuity
  - Monotonicity
  - Convexity
- Limits
- Derivatives
- Extrema