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Math Refresher for DS

Practical Session 4

Today

- Least Squares (continued)
- More on coordinates change

Graded Assignment 1 is OUT

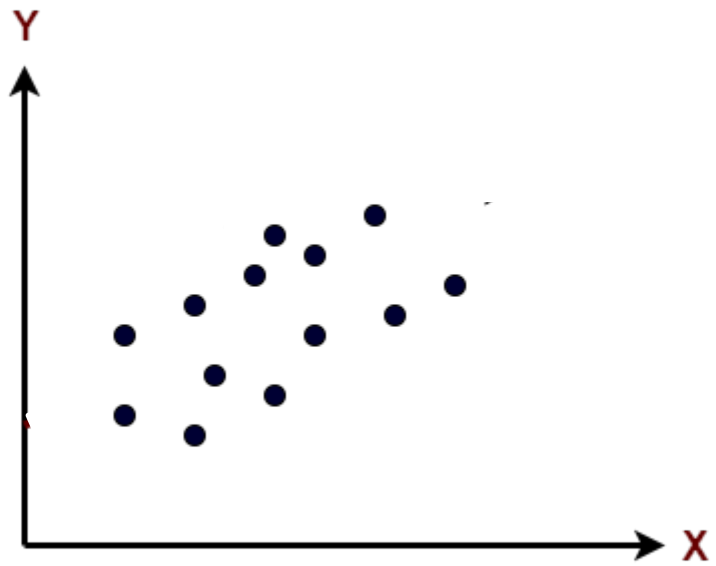
- Google-form, link in repo and on Telegram.
- Submit answers <u>and</u> detailed solutions.

- Submission deadline: Monday, October 17, 23:59 AoE
- Late submissions won’t be accepted.

Where we stopped last time…

Method of Least Squares

- Our goal: fit a hyperplane through the data (x^i, y^i).



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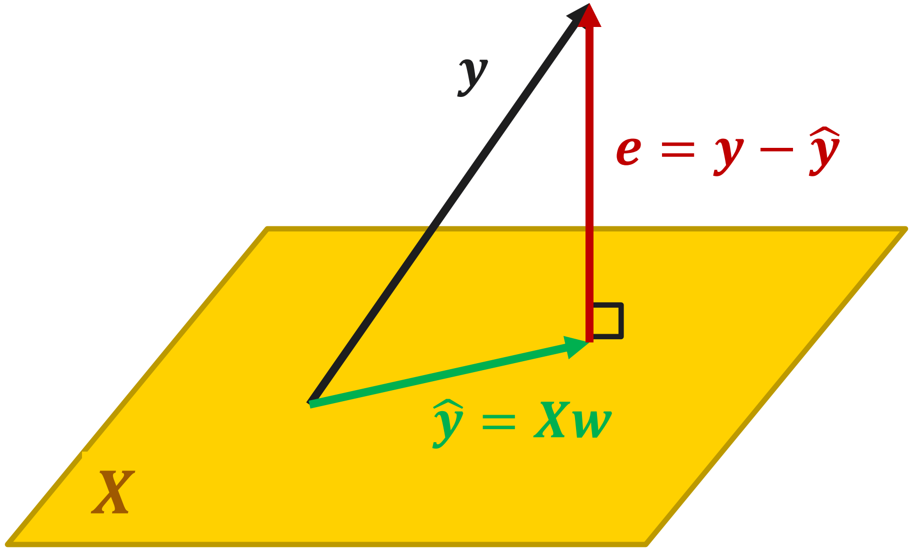
- What do $\hat{y}$ and e look like?



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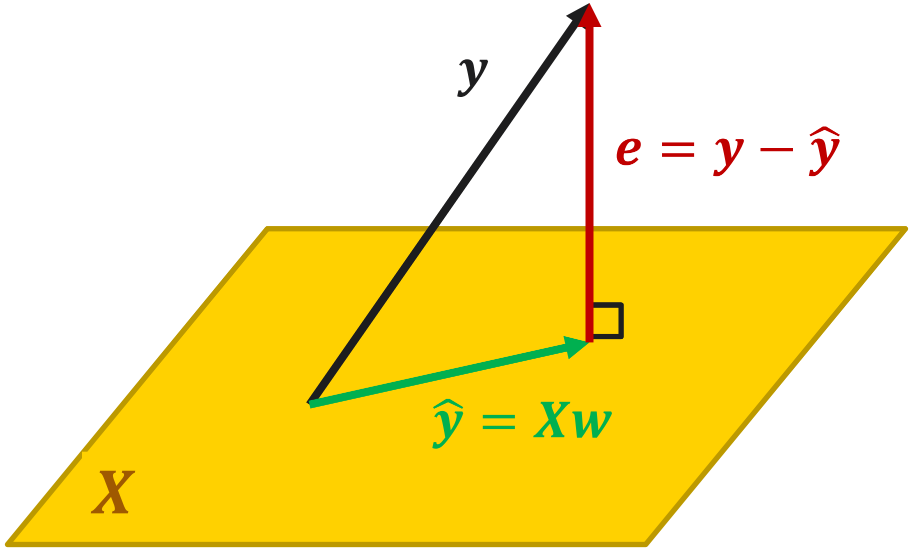

- What do $\hat{y}$ and $\boldsymbol{e}$ look like?
  $\hat{y}$ is the orthogonal projection of $\boldsymbol{y}$ onto the column space of $\boldsymbol{X}$!
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Orthogonal Projections

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- Consider a vector space V and a subspace W.
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- Example:

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- Example:

$$\mathrm{V} = \mathbb{R}^3, \qquad \mathrm{W} = span\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}, \qquad \mathrm{W}_{\perp} = span\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

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- x_W is the closest vector to x in W.

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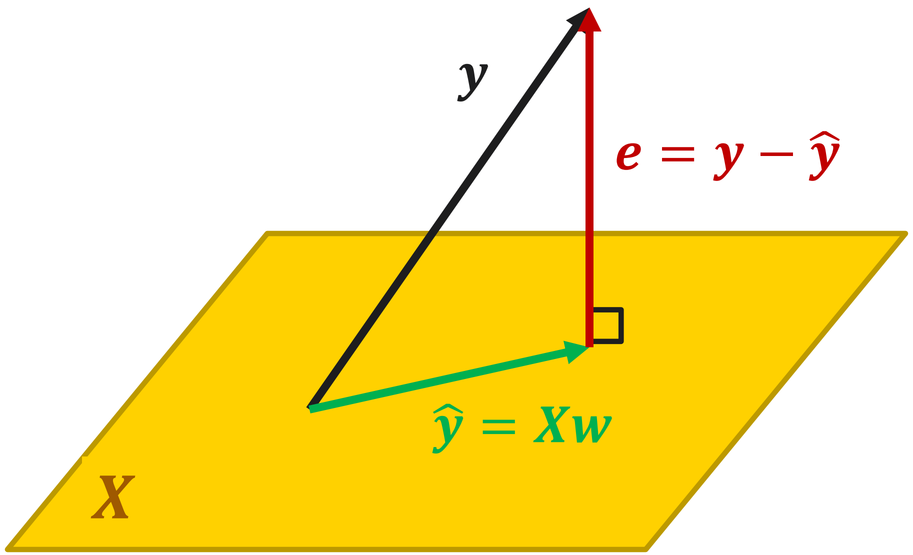

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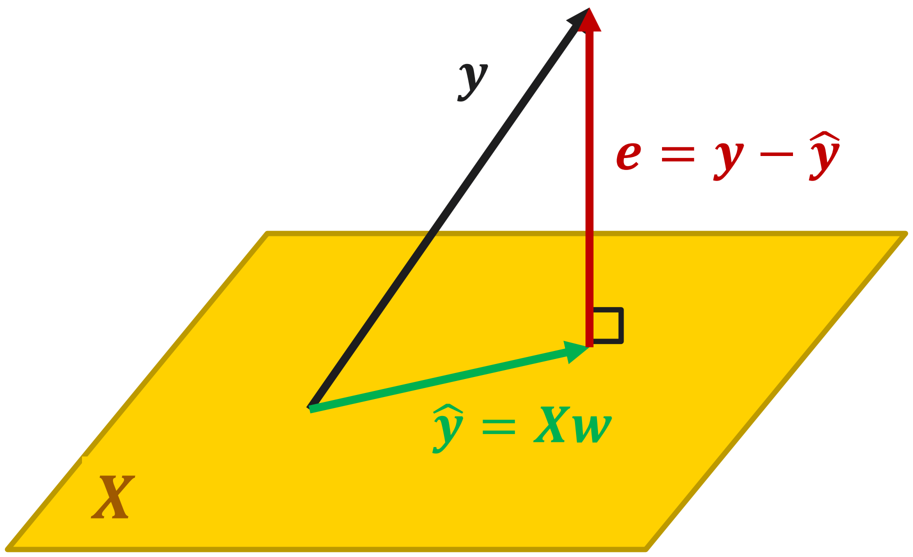

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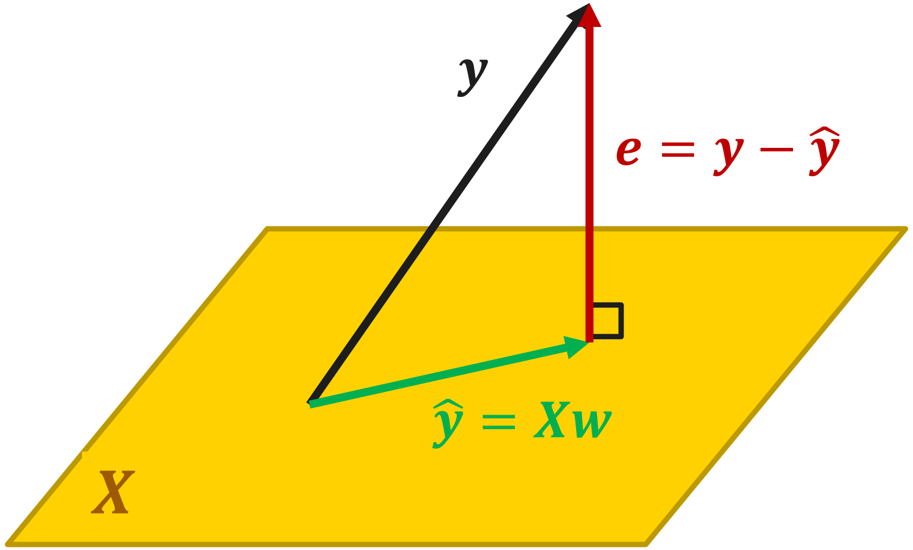

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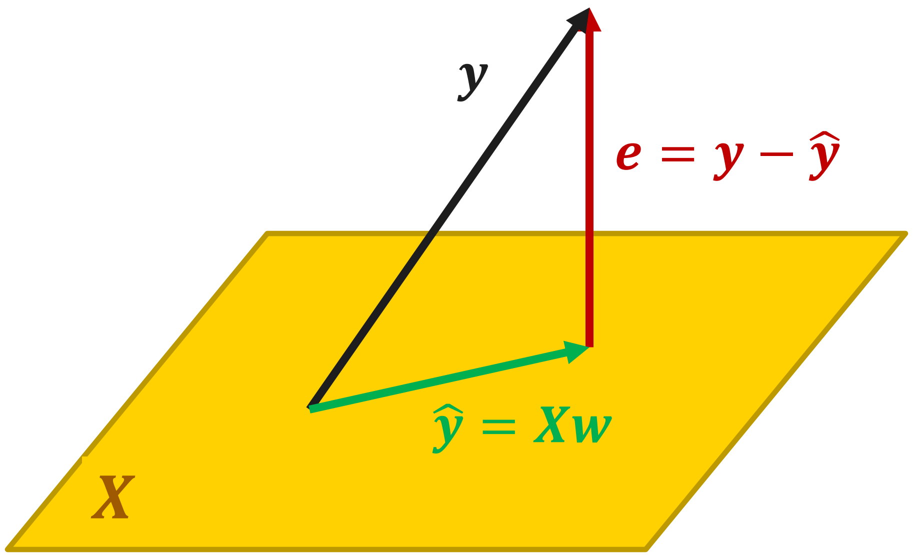
$$Xw^* = \hat{y} = y - e$$

$\hat{y}$ - orthogonal projection of y onto $col(X)$
$w^* = ?$ – optimal weights



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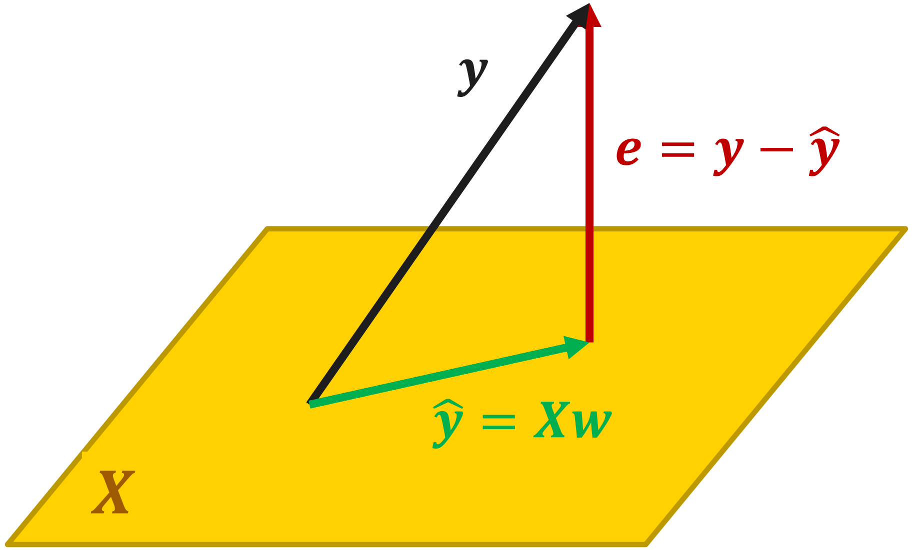

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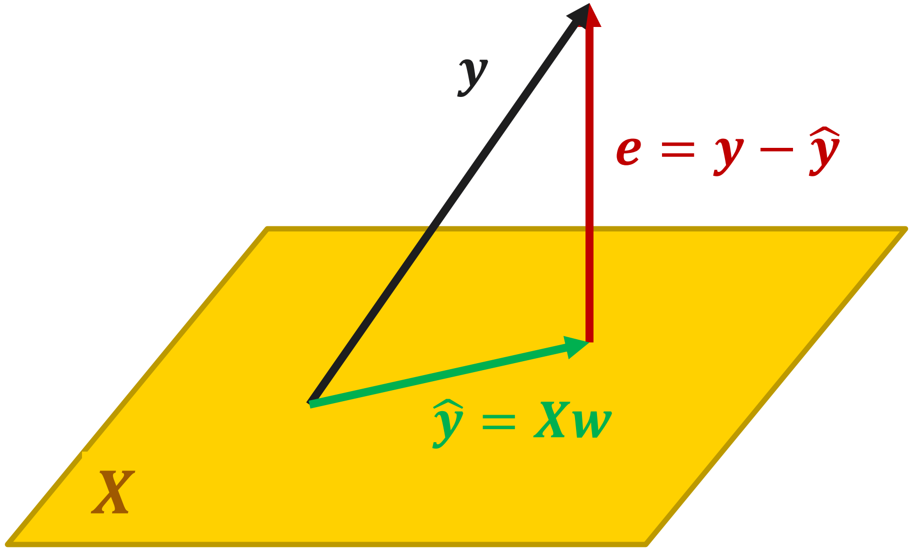

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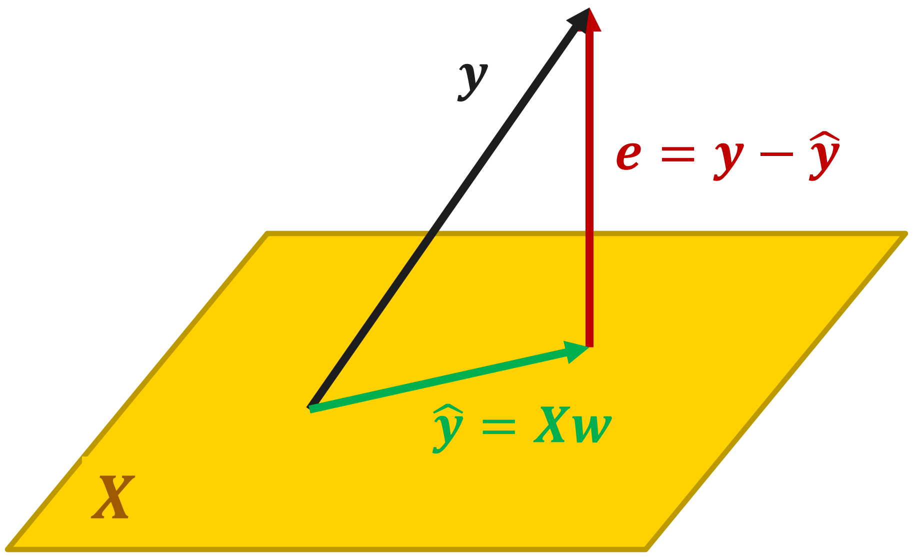

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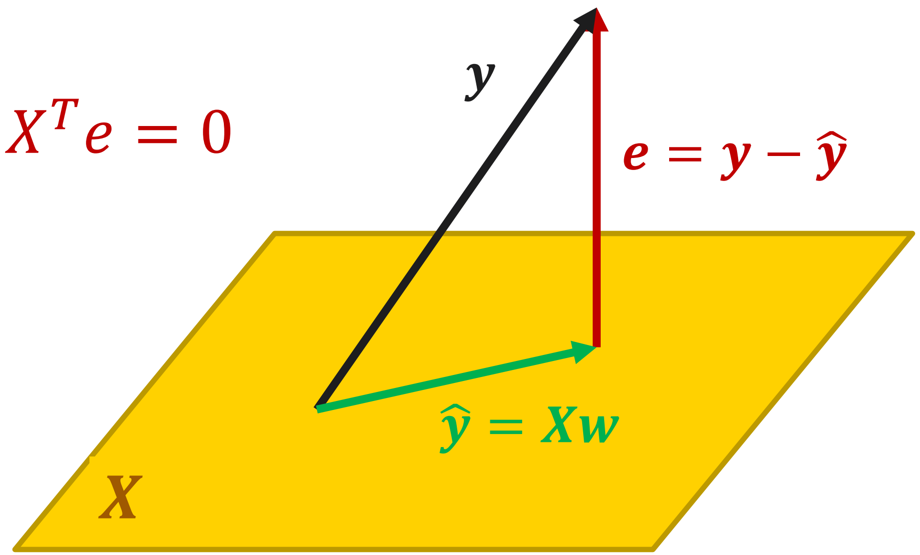

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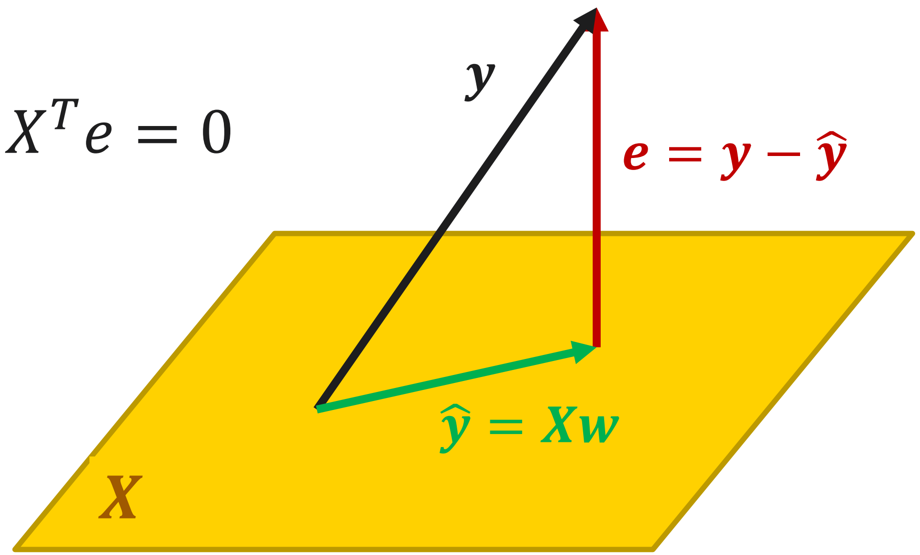

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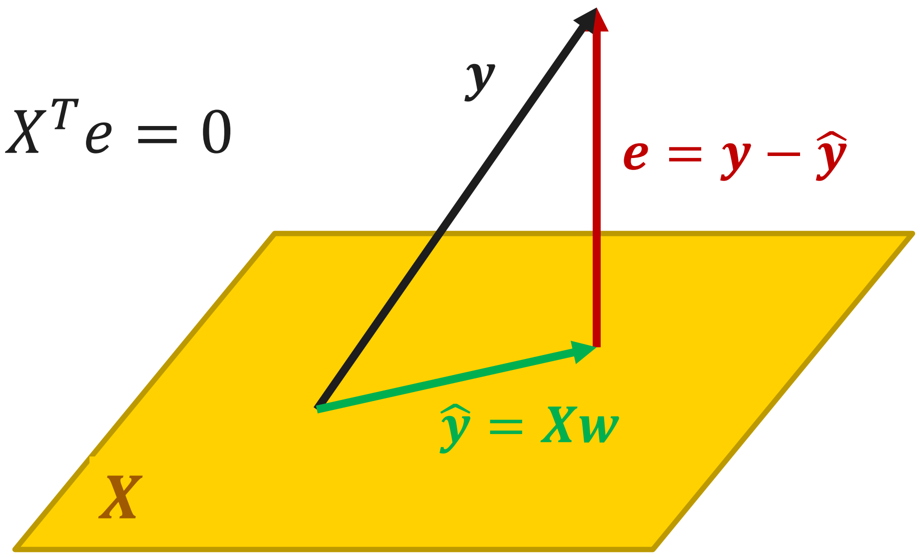

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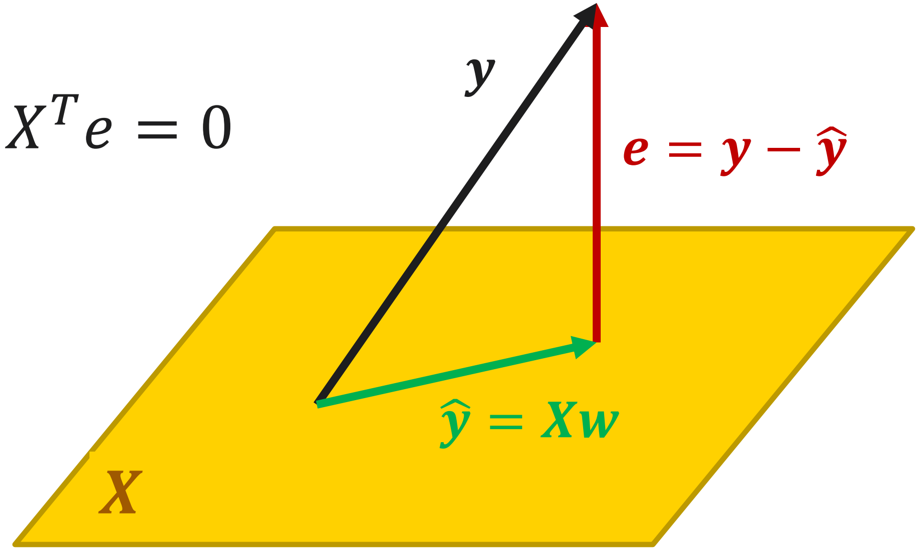

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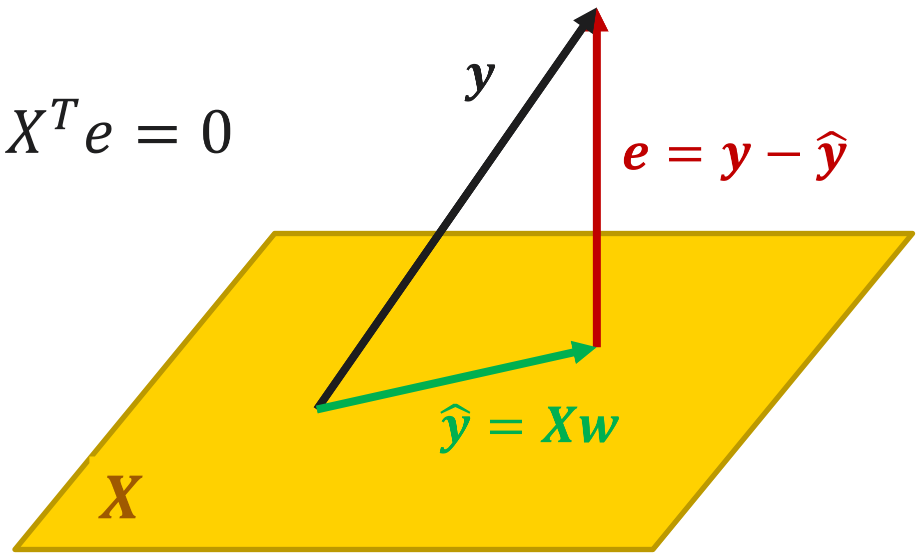

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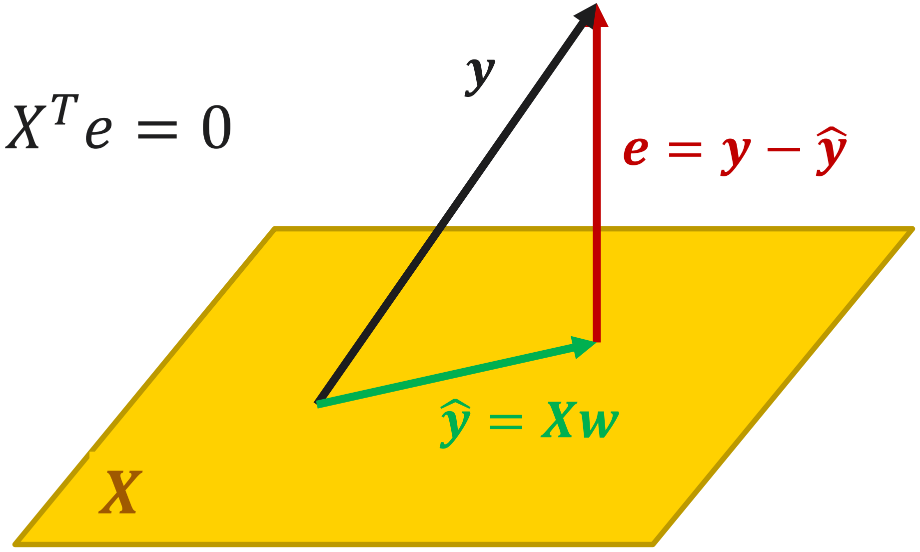

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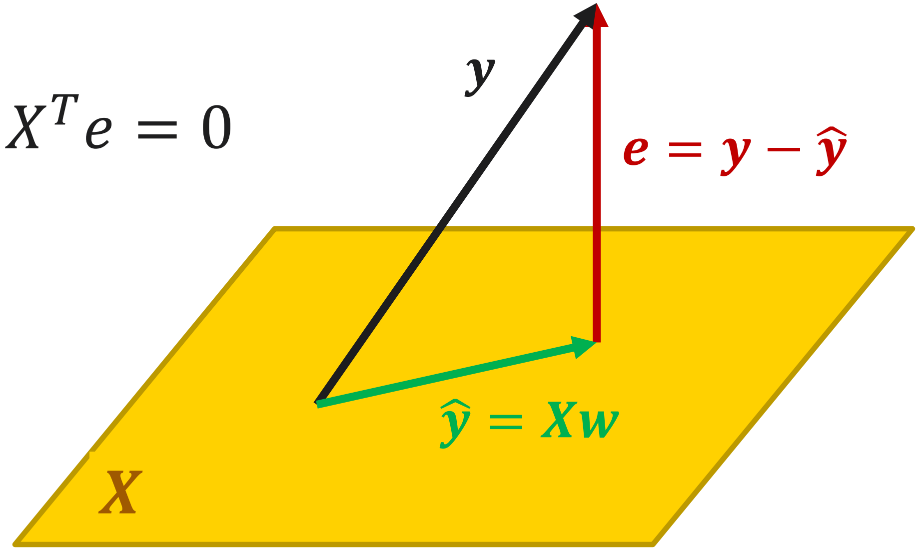
$$\hat{\boldsymbol{y}} = X w^* = \underbrace{\boldsymbol{X}(\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X})^{-1}\boldsymbol{X}^{\boldsymbol{T}}}_{\text{projection matrix}}\boldsymbol{y}$$



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$\Longrightarrow$ contradiction.

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Why Does $(X^TX)^{-1}$ Exist?

- We derived that $w = (X^TX)^{-1}X^Ty$. But can we be sure $(X^TX)^{-1}$ exists?
- Suppose it doesn't.

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$$X^TXv = 0$$

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So, $(X^TX)^{-1}$ exists.

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Toy Example

- Observations (x_i, y_i):

$$(1, 1), \qquad (2, 3), \qquad (3, 2)$$

- With least squares, fit a line $y = w_0 + w_1 x$ through these points.

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$X^T X =$, $(X^T X)^{-1} =$, $(X^T X)^{-1} X^T =$

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$$X^T X = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}, \qquad (X^T X)^{-1} = \qquad\qquad , \qquad (X^T X)^{-1} X^T =$$

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$$X^T X = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}, \qquad (X^T X)^{-1} = \begin{bmatrix} 7/3 & -1 \\ -3/2 & 1 \end{bmatrix}, \qquad (X^T X)^{-1} X^T =$$

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$$w = (X^T X)^{-1} X^T y = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \rightarrow \text{regression line}$$

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- Observations (x_i, y_i):

$$(1,1), \qquad (2,3), \qquad (3,2)$$

- With least squares, fit a line $y = w_0 + w_1 x$ through these points.

Reminder: $w^* = (X^T X)^{-1} X^T y$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$ – no exact solutions.

$$X^T X = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}, \qquad (X^T X)^{-1} = \begin{bmatrix} 7/3 & -1 \\ -3/2 & 1 \end{bmatrix}, \qquad (X^T X)^{-1} X^T = \begin{bmatrix} 4/3 & 1/3 & -2/3 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

$w = (X^T X)^{-1} X^T y = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$ → regression line $y = 1 + 0.5x$.

Let’s practice more!

https://colab.research.google.com/drive/1v_dDH5aSx9pQG4SSCLzNjKuCk6rwNWzf?usp=sharing

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Coordinates Change

Change of Basis for Vectors

- V – a vector space.
- $B = \{b_1, \dots, b_n\}$ – current basis, $S = \{s_1, \dots, s_n\}$ – new basis.

- $x \in V$ – some vector.

- We know already that

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$$\exists M^{-1} = M_{S\to B}, \qquad x_S = M^{-1}x_B$$

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- But vectors aren’t the only things with coordinates…

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Change of Basis for Linear Transforms

- Consider a linear transform A.
- It's defined by its matrix: columns = what happens to basis vectors.

- Example: rotation

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

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- $S = \left\{ s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ – another basis.

How would A look like in this basis?

Change of Basis for Linear Transforms

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$$\mathbf{[A]_S = M^{-1}[A]_B M}$$

Change of Basis for Linear Transforms

- Back to our example:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad S\left\{s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} -$$ new basis.

$$[A]_S = M^{-1}AM =$$

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$$= \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix}$$

Change of Basis for Linear Transforms

$[A]_E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $[A]_S = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix}$, $S = \left\{ s_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ – new basis;

$$x_S = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad x'_S = [A]_S \cdot x_S =$$

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$$x_E =$$

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$$x_E = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \qquad x'_E =$$

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Change of Basis for Linear Transforms

- Another example:

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, \qquad S = \left\{ s_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} - \text{ new basis}$$

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We get a diagonal matrix, it's easier to work with it!