Math Refresher for DS

Practical Session 3

Plan for Today

- A short quiz
- SLE with no solutions
- Practice in Python

We (Finally) Have a Course Repo!

- https://github.com/girafe-ai/math-basics-for-ai
    - Slides
    - Links to colab-notebooks
    - Links to lectures / practical session recordings
    - Additional material

Short Quiz Lectures 1 - 3

https://forms.gle/Mw28SUSTwohWWv9d7

Solving Systems of Linear Equations

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- $A - m \times n$ matrix (= m equations, n variables).

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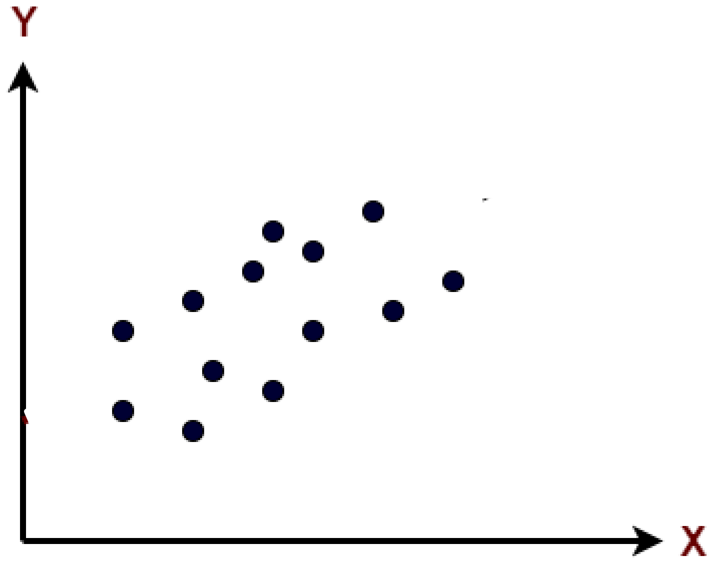
How to find a reasonable approximate solution?

Least Squares

Motivating Example

- Imagine that you have m observations:

$$(x_1, y_1), \quad (x_2, y_2), \quad \ldots, \quad (x_m, y_m)$$



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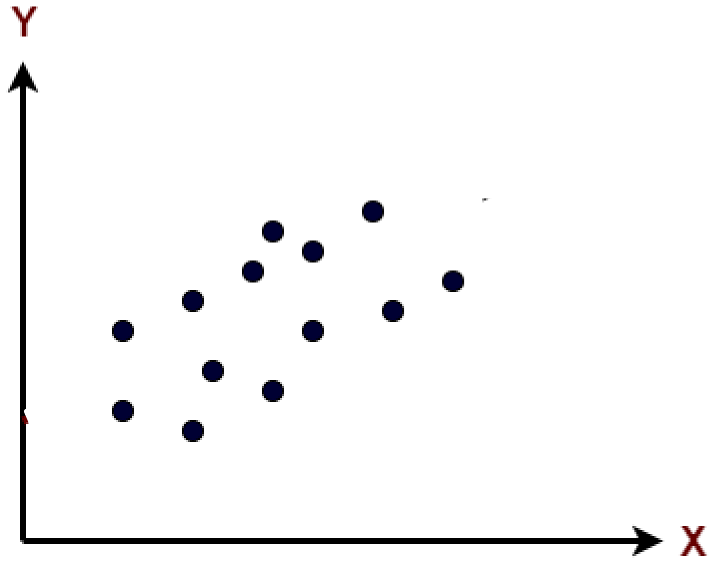

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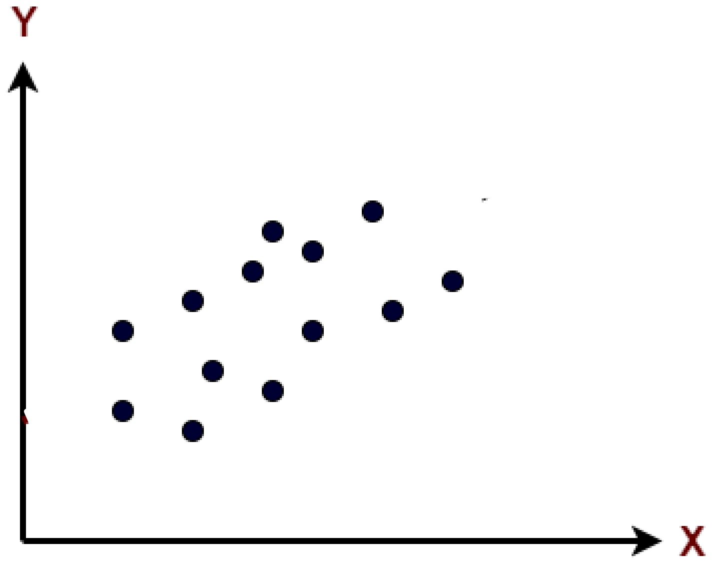

$$\begin{matrix} \boldsymbol{w_0} + \boldsymbol{w_1} x_1 = y_1 \\ \boldsymbol{w_0} + \boldsymbol{w_1} x_2 = y_n \\ \vdots \\ \boldsymbol{w_0} + \boldsymbol{w_1} x_m = y_m \end{matrix} \quad \Leftrightarrow$$

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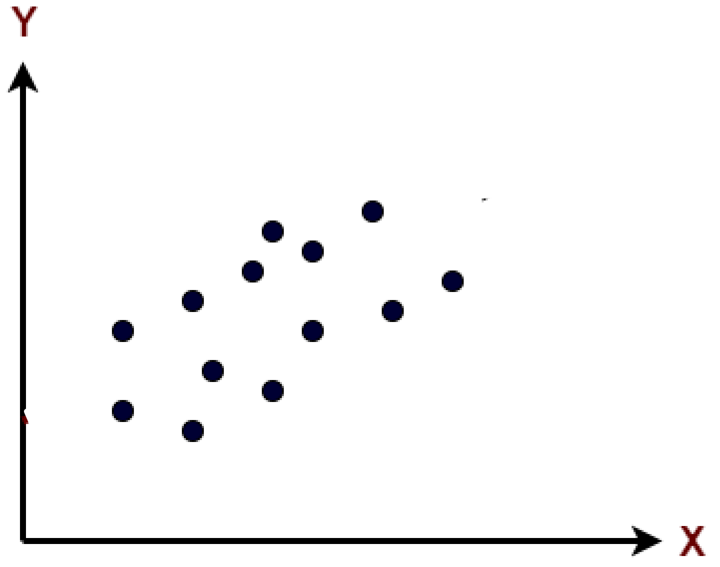

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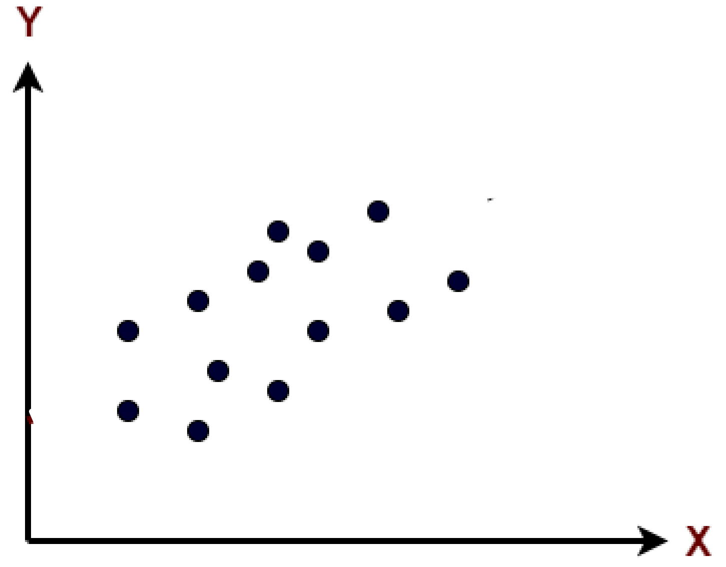

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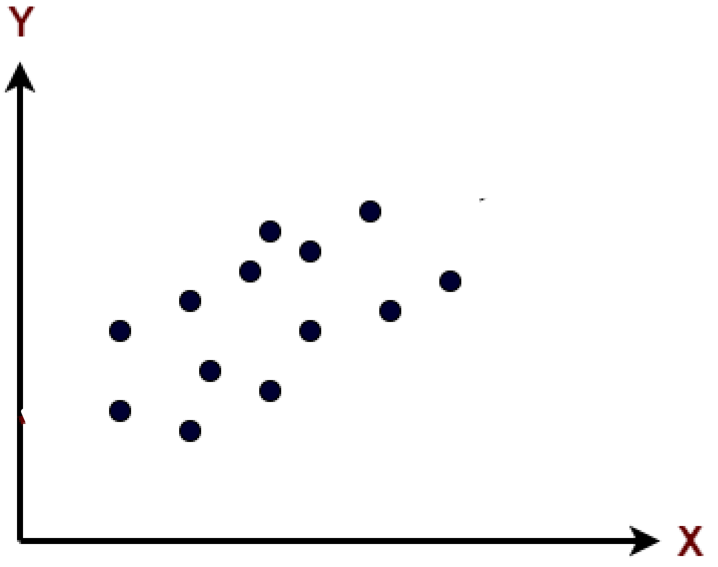

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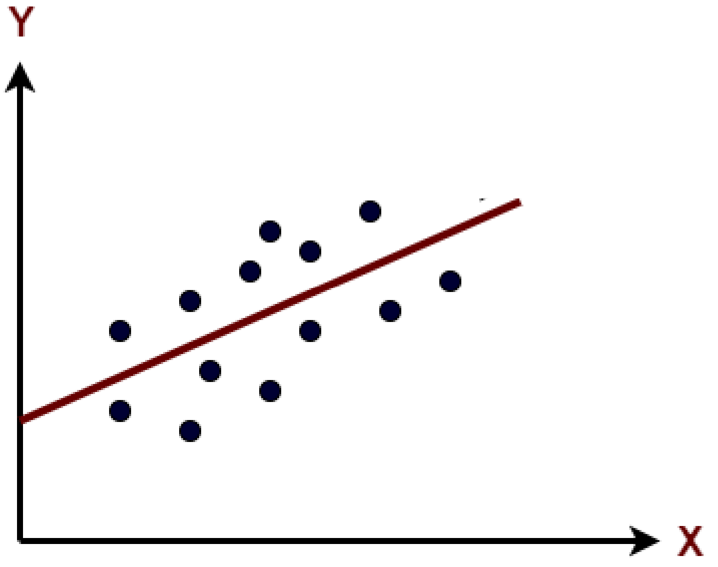

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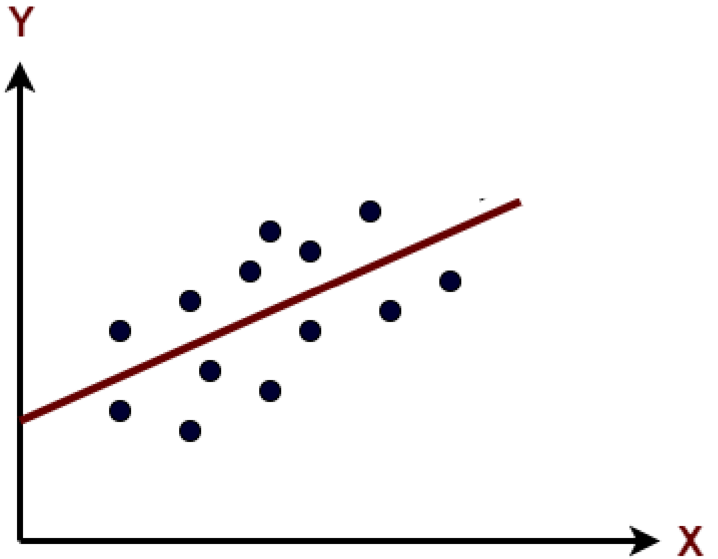

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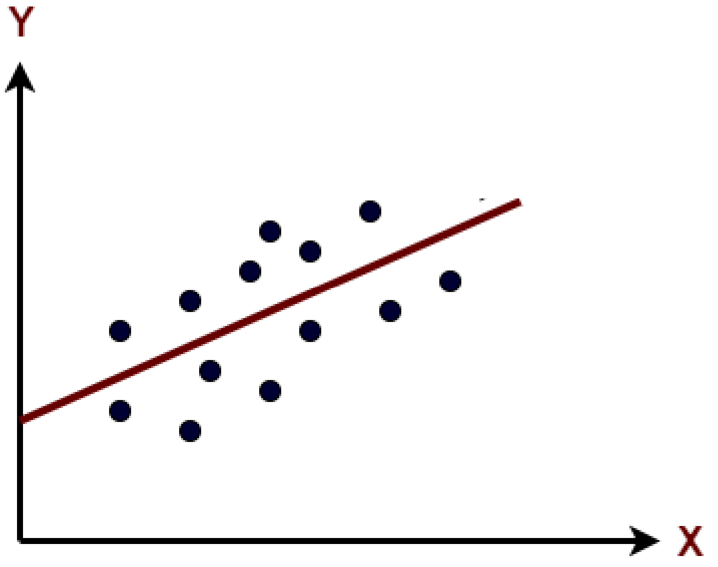

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- Let’s look at it from the Linear Algebra perspective.

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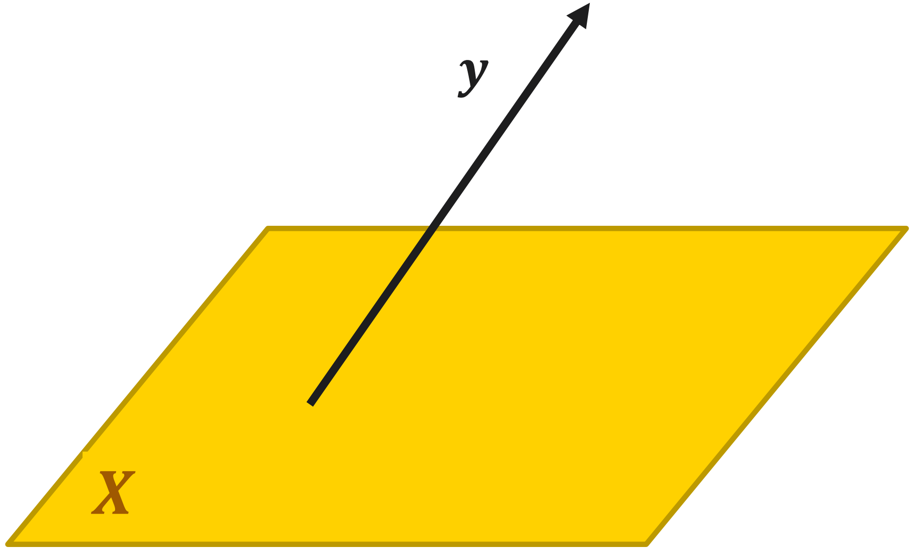
- $Xw = y$ has no solutions $\Leftrightarrow$ y is not in the column space of X.



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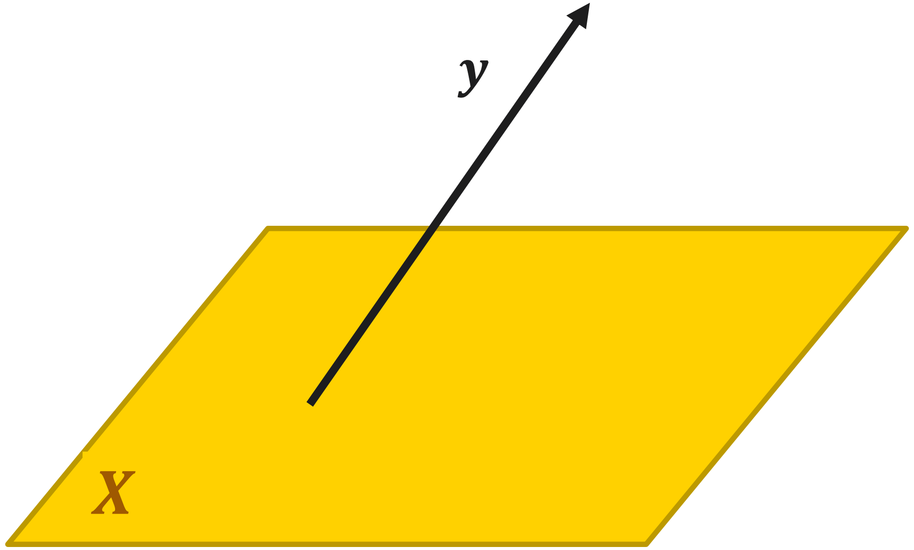

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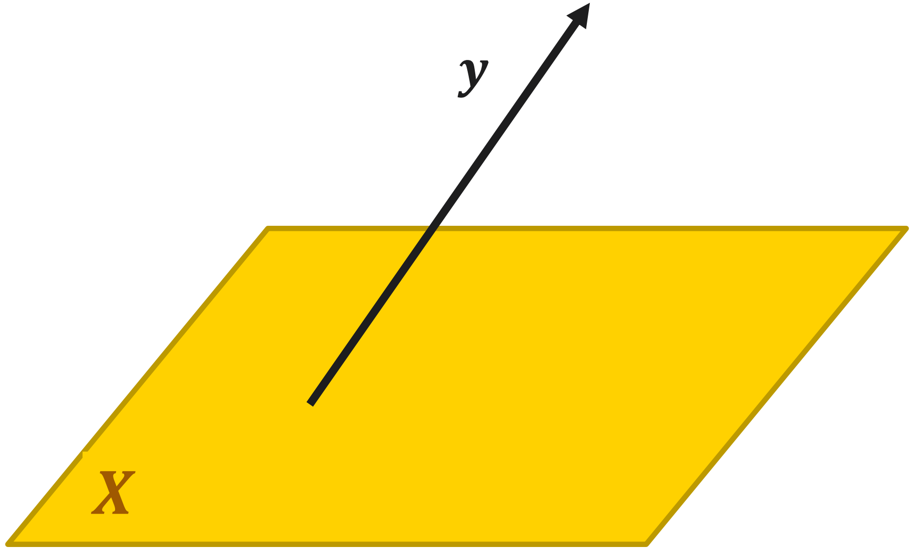

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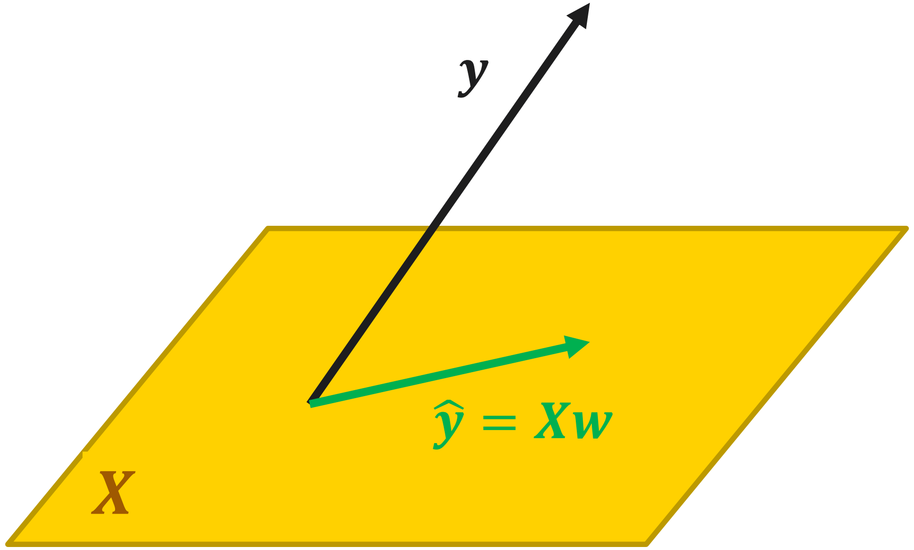

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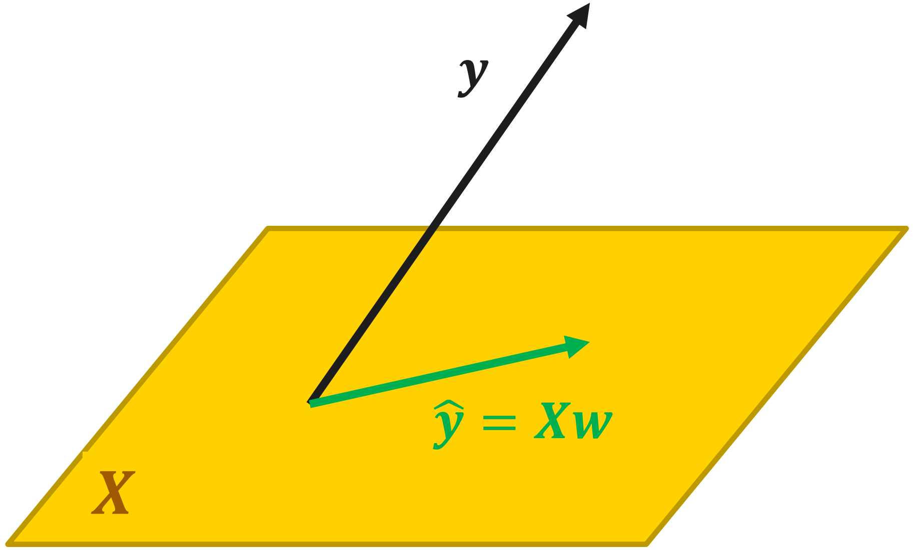

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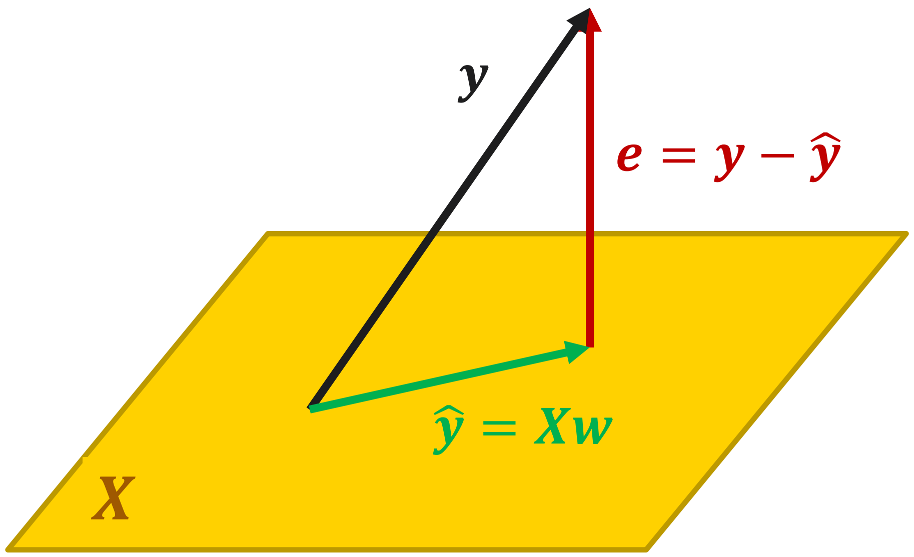

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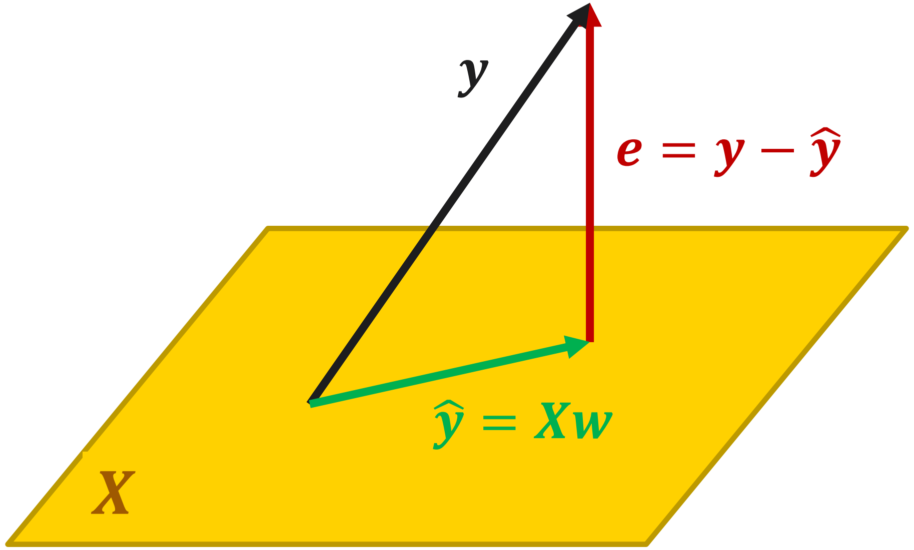

Orthogonal Projections

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- Consider a vector space V and a subspace W.

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- Example:

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Orthogonal Decomposition

- Consider a vector space V and a subspace W.
- $x \in V$ and $x \notin W$.

- x can be decomposed into a sum of two vectors:

$$x = x_W + x_{W^\perp}, \qquad x_W \in W, x_{W^\perp} \in W^\perp$$

- x_W – **orthogonal projection** of x onto W.

- x_W is the closest vector to x in W.

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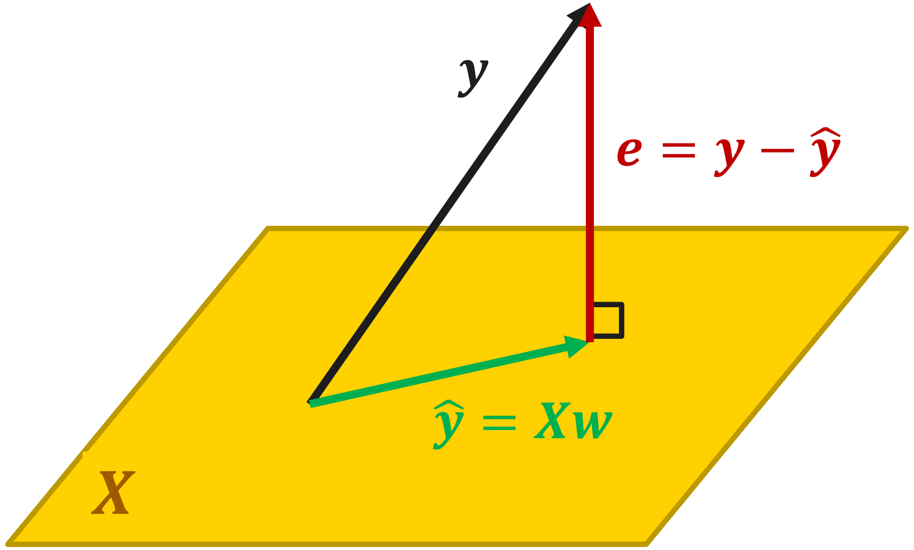

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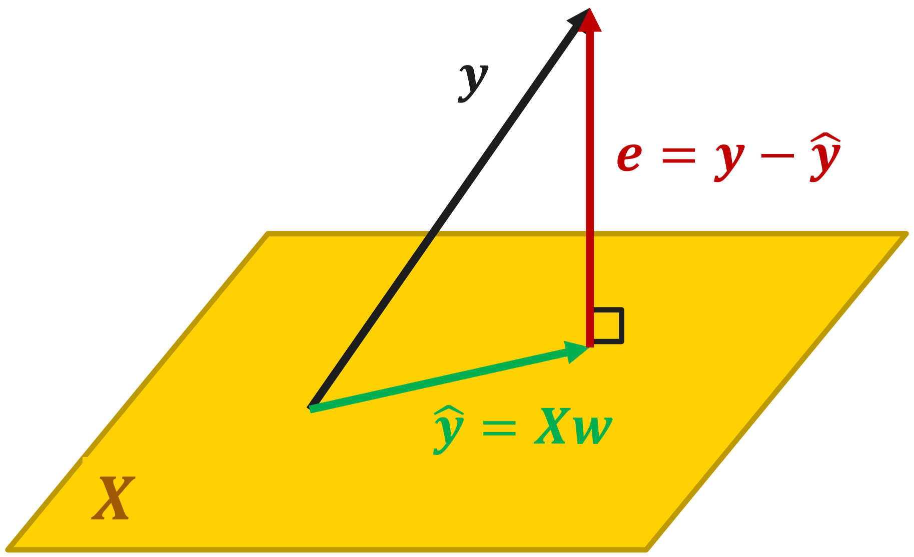

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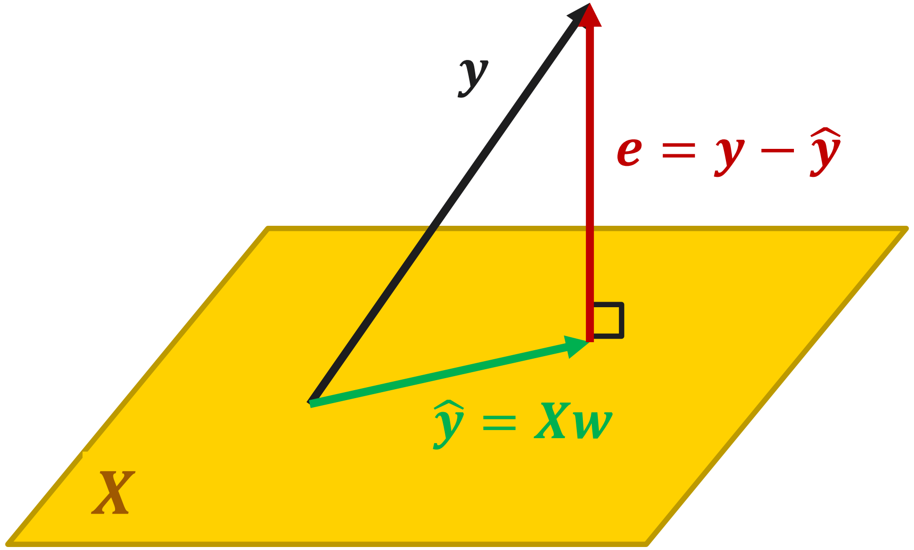

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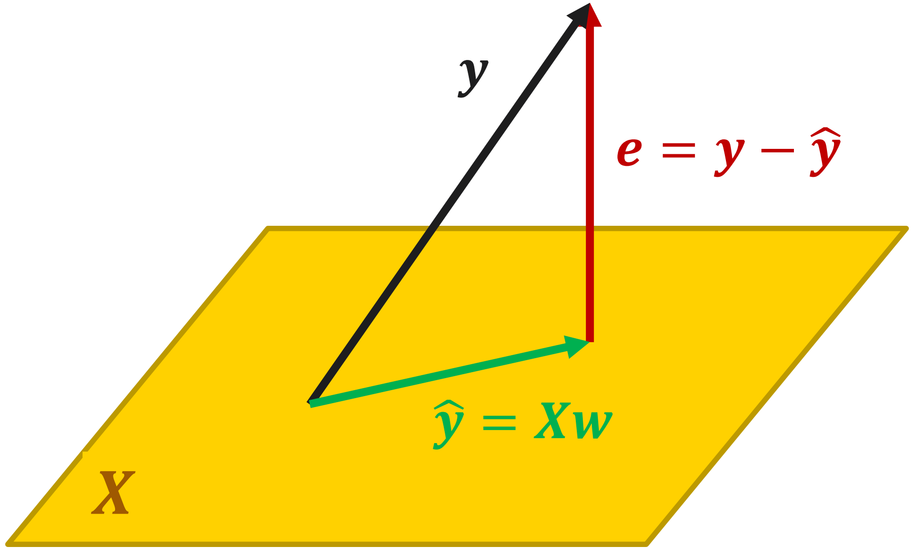
$$Xw = \hat{y} = y - e$$



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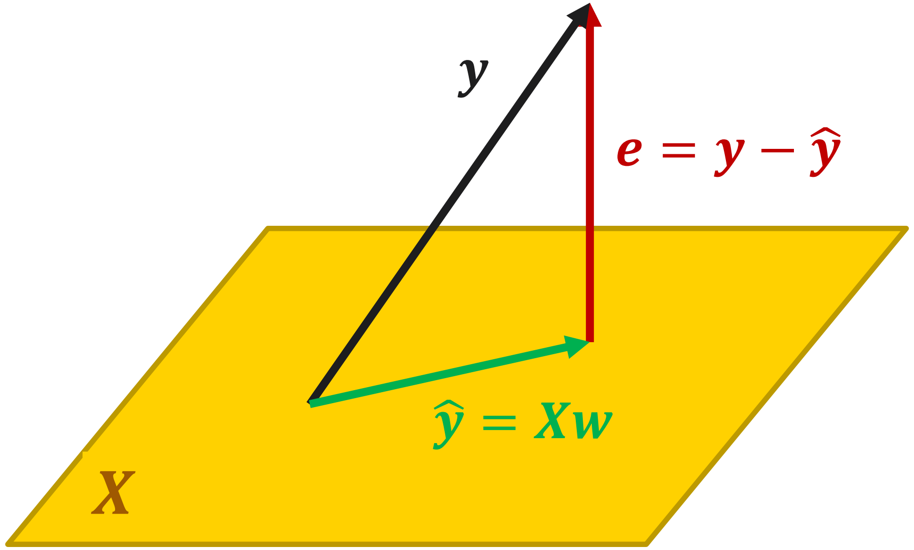

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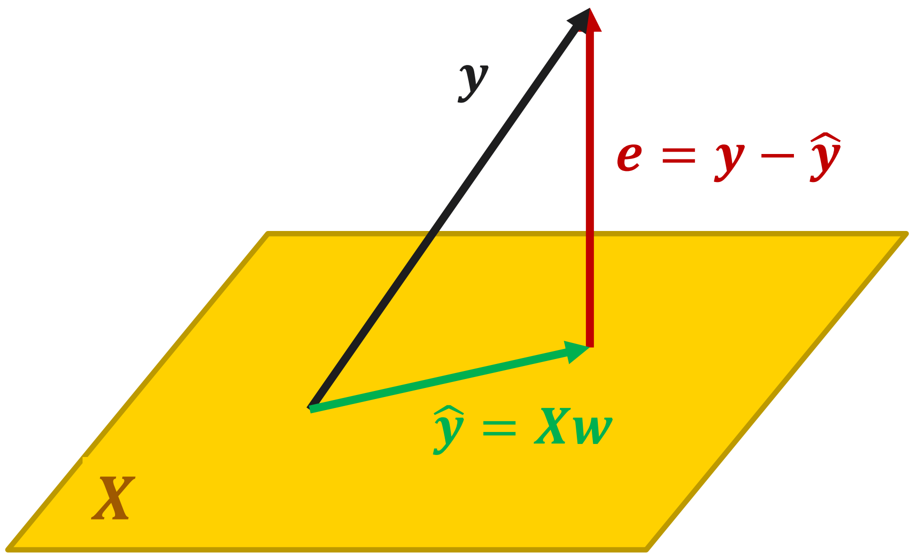

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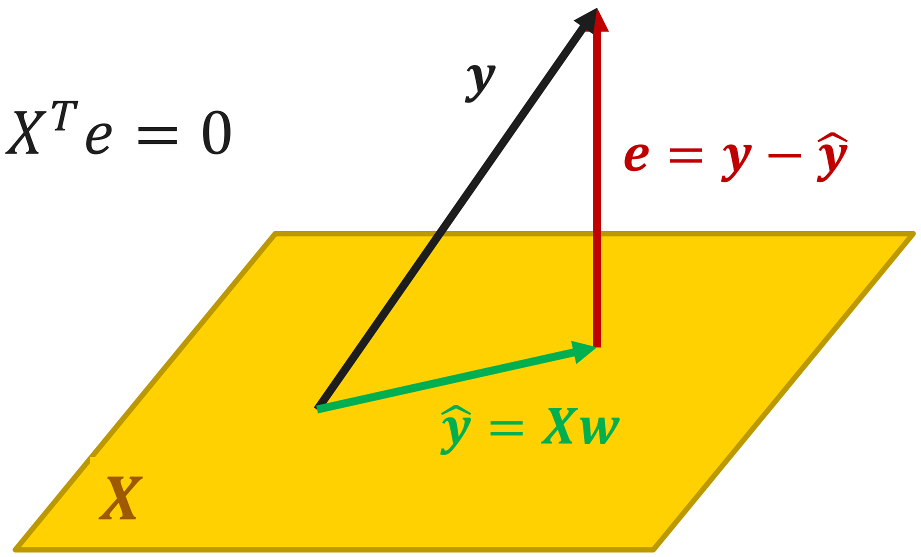

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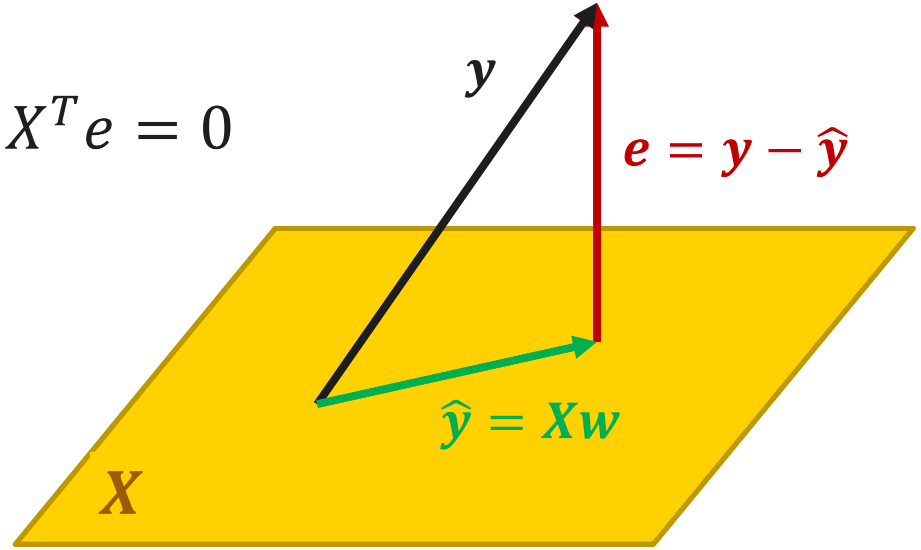

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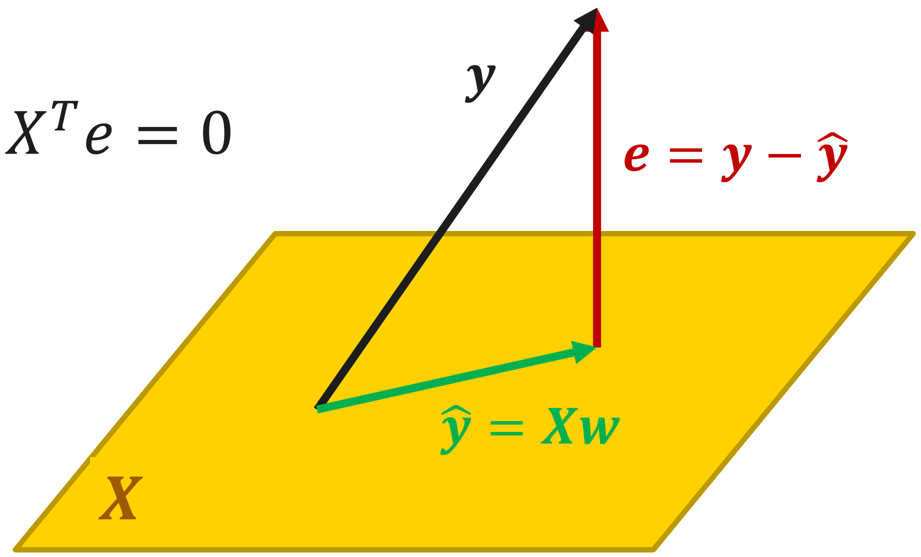

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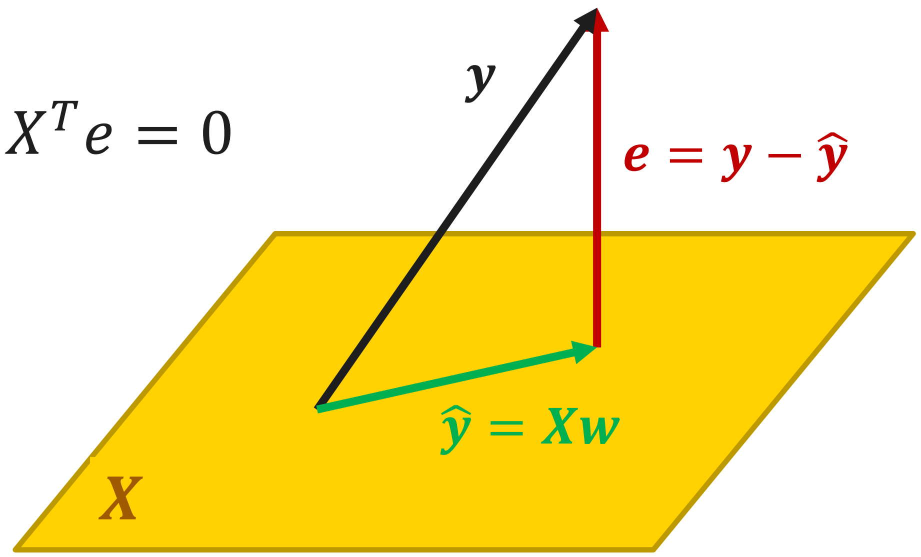

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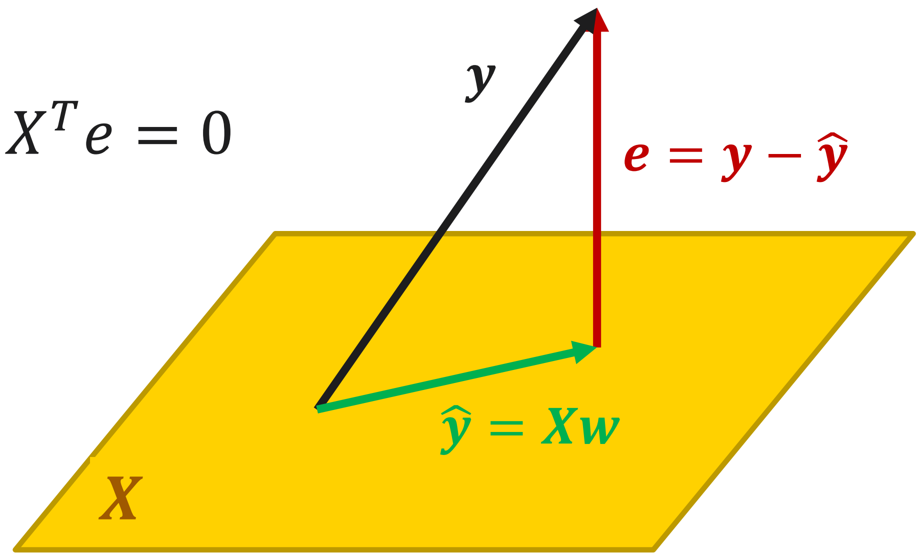

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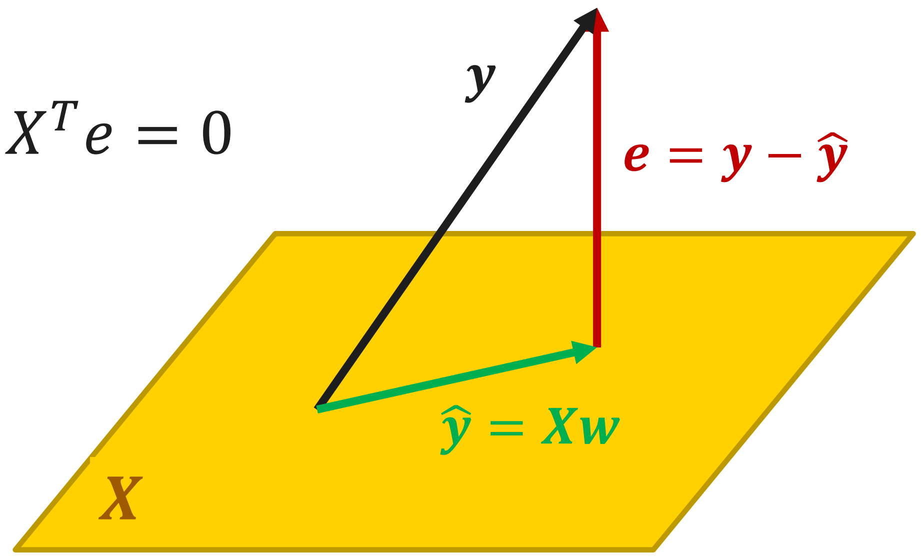
$$\hat{\boldsymbol{y}} = Xw = \underbrace{\boldsymbol{X}(\boldsymbol{X^T X})^{-1}\boldsymbol{X^T}}_{\text{projection matrix}}\boldsymbol{y}$$



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So, $(X^TX)^{-1}$ exists.

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Toy Example

- Observations (x_i, y_i):

$$(1,1), \qquad (2,3), \qquad (3,2)$$

- With least squares, fit a line $y = w_0 + w_1 x$ through these points.

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$$w = (X^T X)^{-1} X^T y = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$ → regression line

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$$w = (X^T X)^{-1} X^T y = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \rightarrow \text{regression line } y = 1 + 0.5x.$$

Let’s practice more!

https://colab.research.google.com/drive/16UqY0p5h5324atAIQ3kWilF7AZTa5mbX?usp=sharing

We (Finally) Have a Course Repo!

- https://github.com/girafe-ai/math-basics-for-ai
  - Slides
  - Links to colab-notebooks
  - Links to lectures / practical session recordings
  - Additional material

Graded Assignment 1 Will Be Out Tomorrow

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