Math Refresher for DS

Practical Session 4

Plan for Today

- A short quiz
- SLE with no solutions
- Practice in Python

We (Finally) Have a Course Repo!

- https://github.com/girafe-ai/math-basics-for-ai
    - Slides
    - Links to colab-notebooks
    - Links to lectures / practical session recordings
    - Additional material

Short Quiz Lectures 1 - 3

https://forms.gle/Mw28SUSTwohWWv9d7

Solving Systems of Linear Equations

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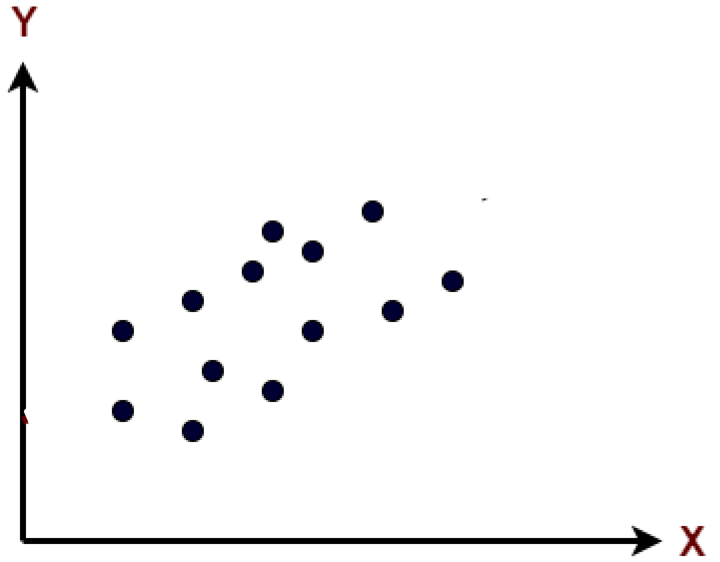
How to find a reasonable approximate solution?

Least Squares

Motivating Example

- Imagine that you have m observations:

$$(x_1, y_1), \qquad (x_2, y_2), \qquad \dots, \qquad (x_m, y_m)$$



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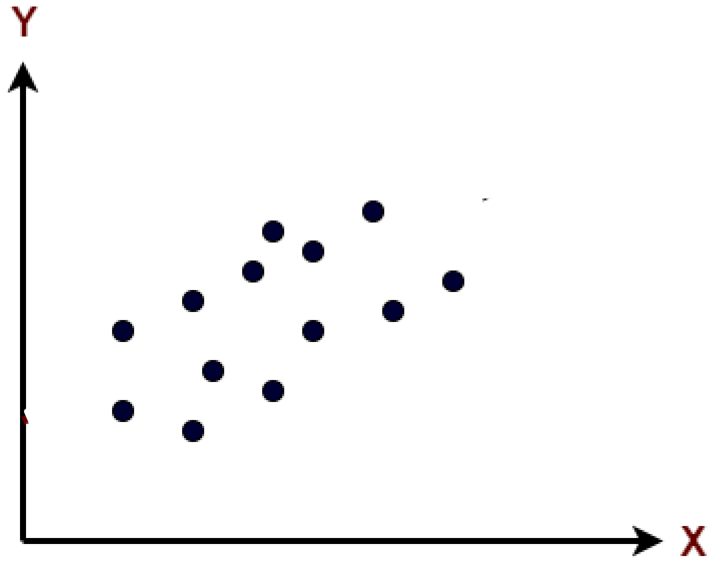

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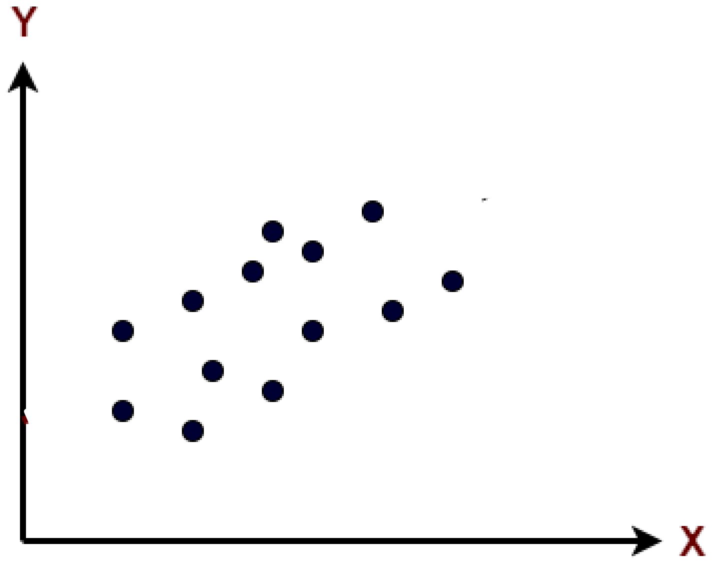

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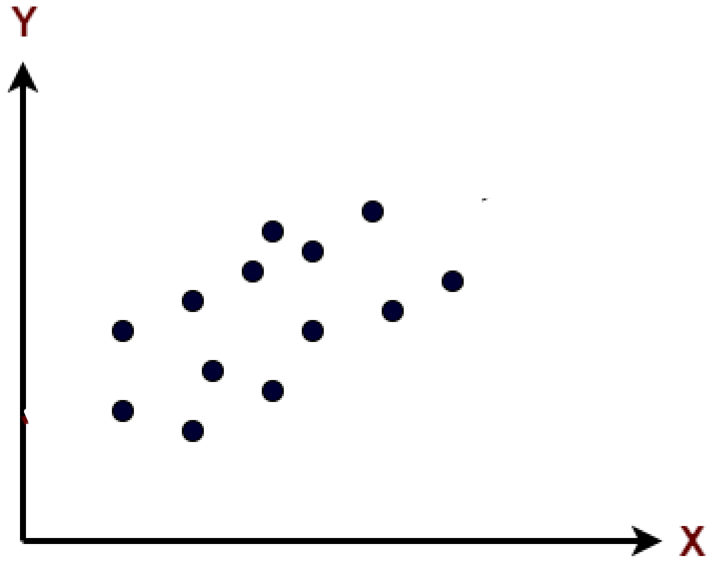

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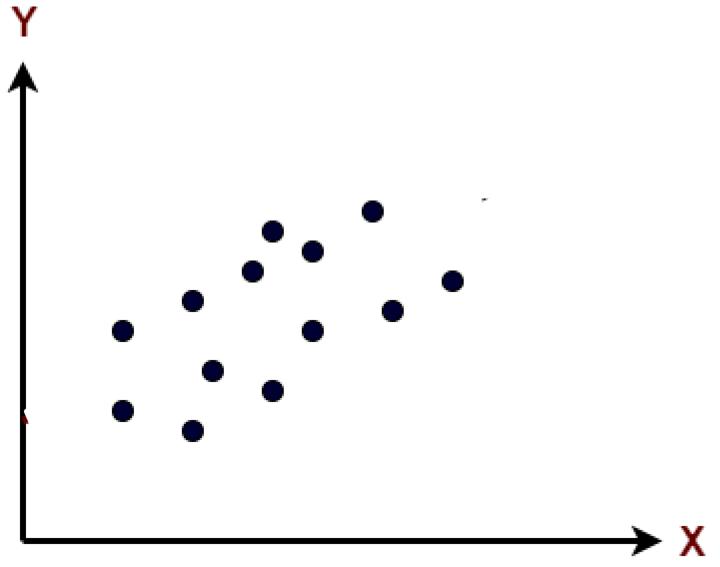

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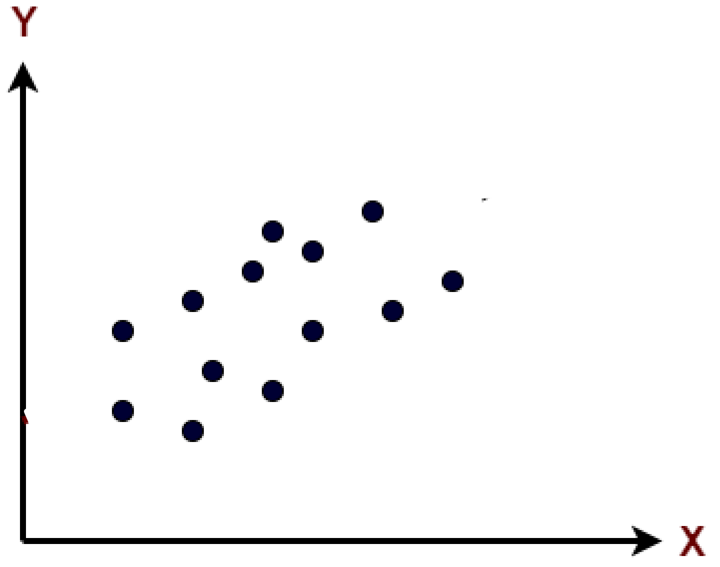

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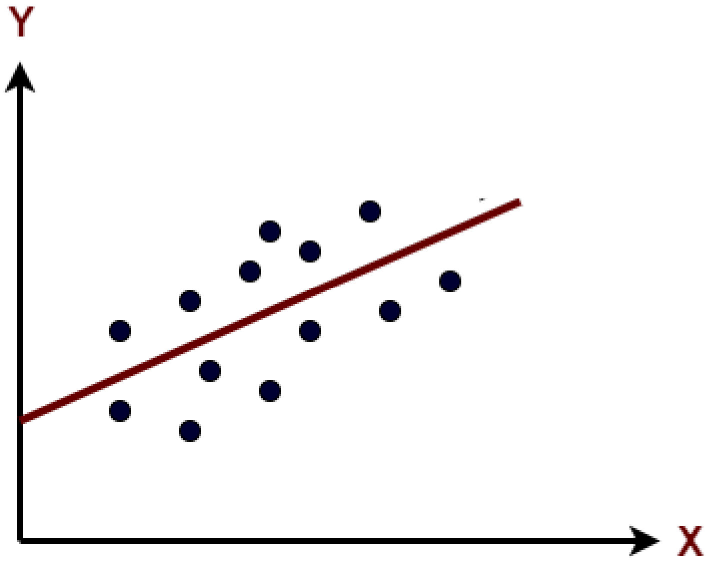

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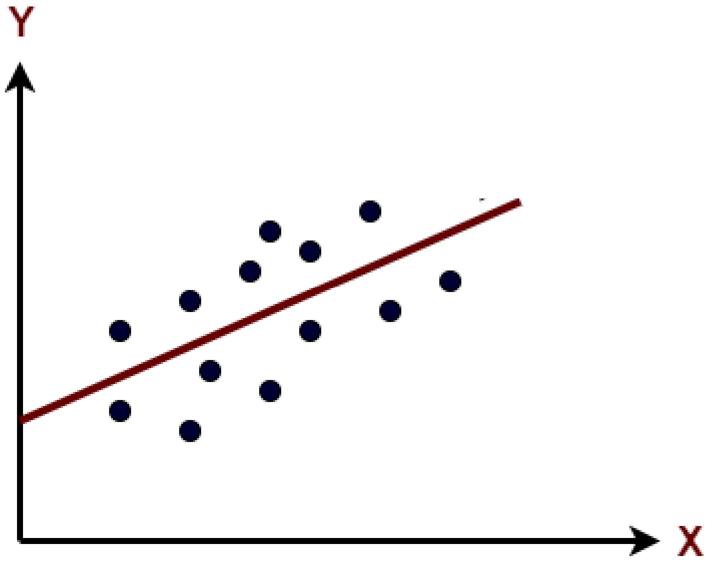

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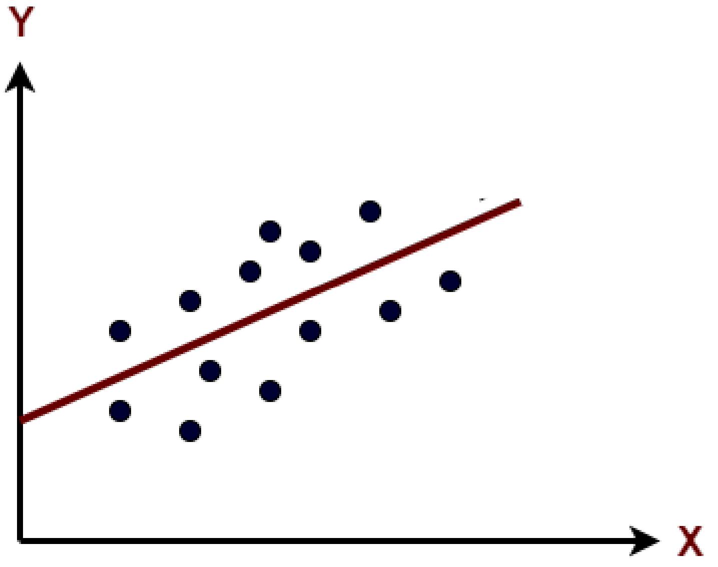

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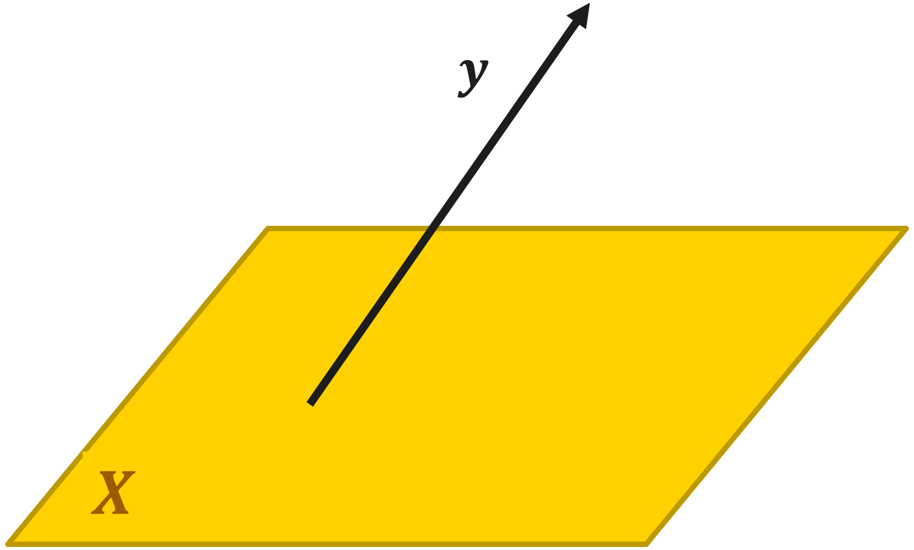
- Let's look at it from the Linear Algebra perspective.

Method of Least Squares

- $Xw = y$ has no solutions $\Leftrightarrow$ y is not in the column space of X.



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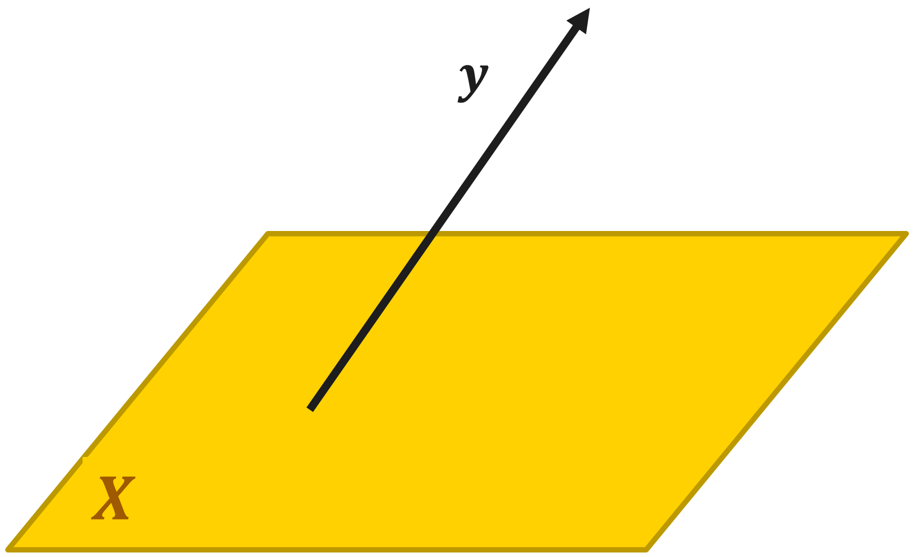

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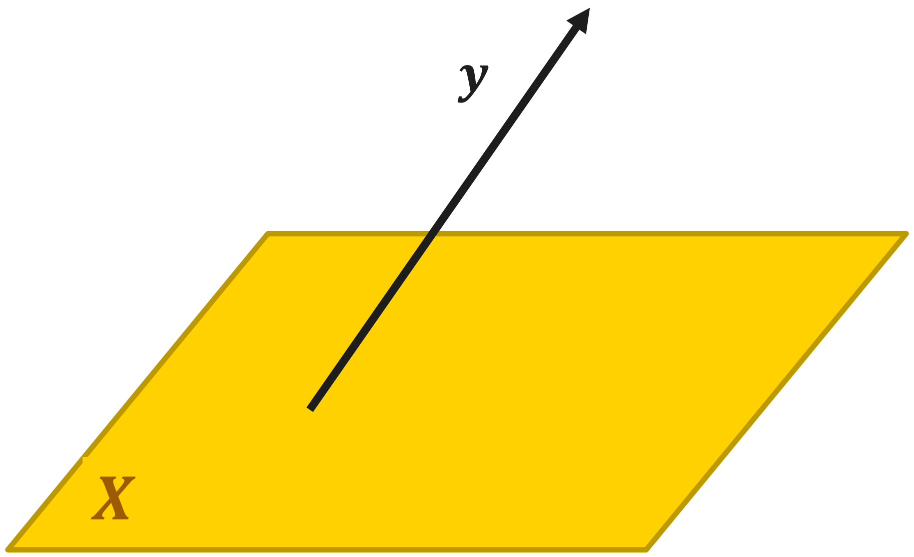

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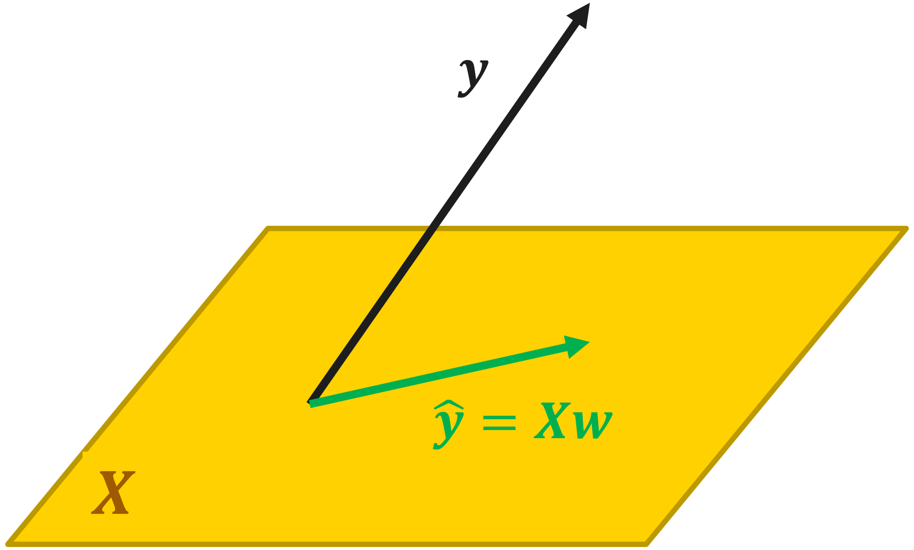

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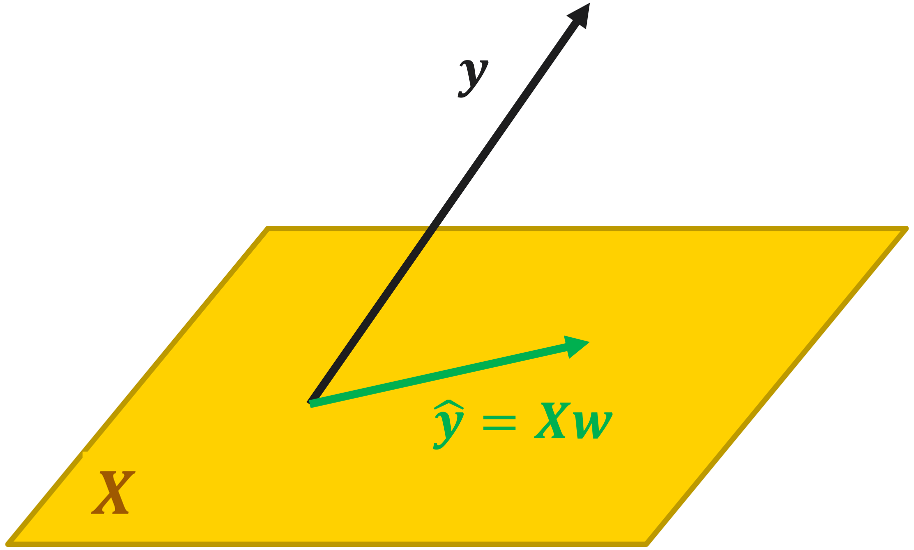

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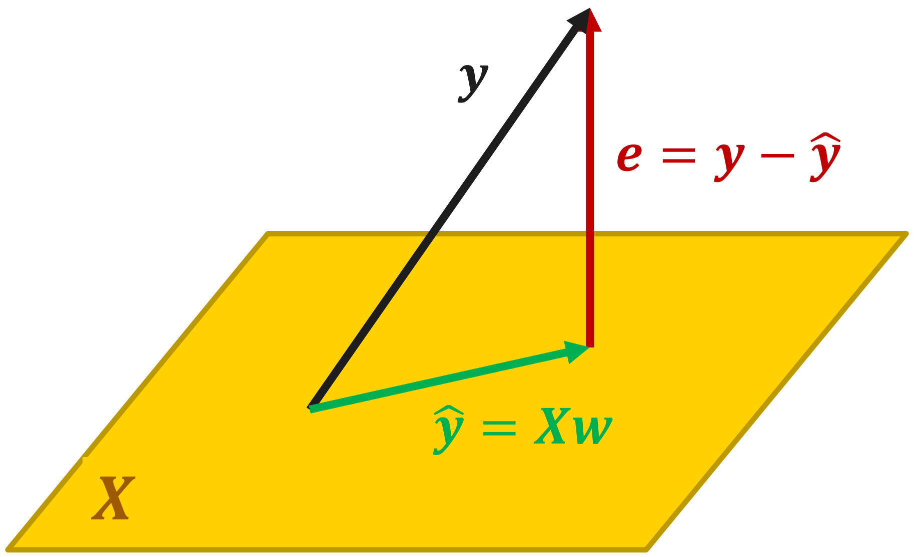

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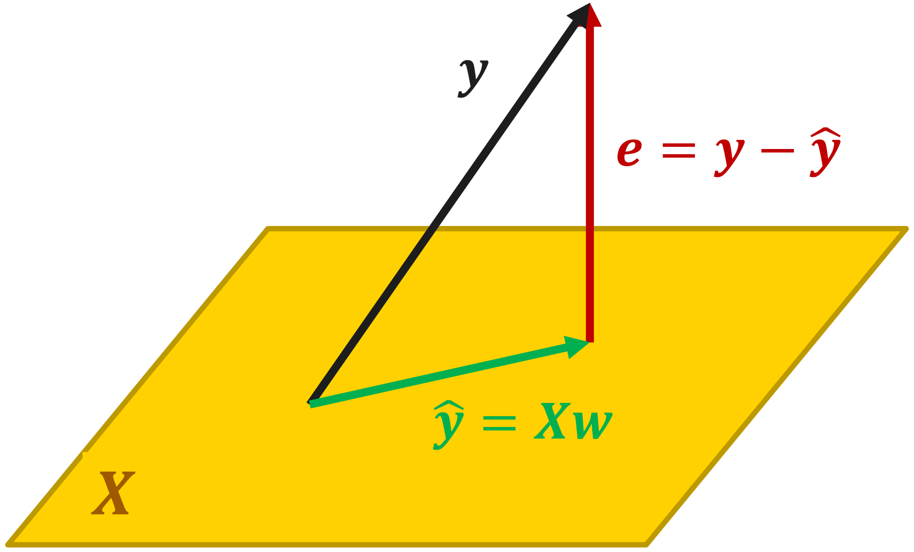
- What do $\hat{y}$ and $\boldsymbol{e}$ look like?



Orthogonal Projections

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- Consider a vector space V and a subspace W.

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- <u>Example:</u>

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$$\mathrm{W} = span\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}, \qquad \mathrm{W}_{\perp} = span\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

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- x_W is the closest vector to x in W.

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