Inverse problems: lecture 1

Alen Alexanderian

Department of Mathematics

NC State University

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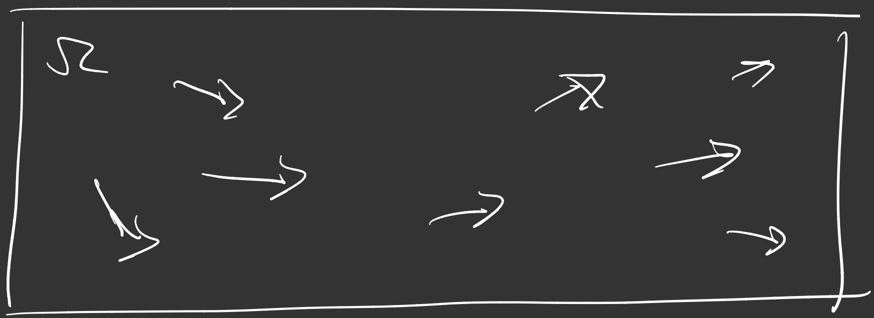
Let us start with a motivating example:

Flow in a porous medium



Forward problem:

Given f, k, μ, and B.Cs can solve the following equation for u

$$\begin{cases} -\nabla\cdot\left(\frac{k}{\mu}\nabla u\right)=f \\ \quad + \text{ BCs} \end{cases}$$

u — fluid pressure

$k(x)$ — permeability field

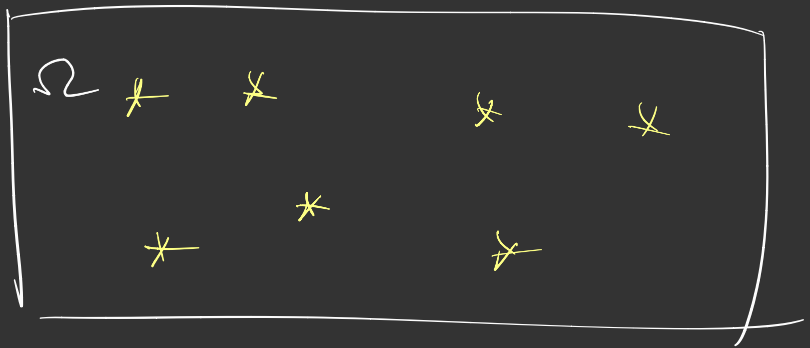
μ — viscosity

f — source term

Note: k is typically unknown.

Inverse problem:

use measurements of the pressure to estimate k



points where measurements are taken

measurement points: $\{x_1, x_2, \dots, x_m\} \subseteq \Omega$

How to do this?

$$d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}$$

d_i — pressure measurement at x_i

B — observation operator

$$(Bu)_i = u(x_i), \quad i = 1, \dots, m$$

Rough idea:

find k that minimizes

use the Euclidean norm here

$$J(k) := \| B u(k) - d \|^2$$

where $$-\nabla \cdot \left(\frac{k}{\mu} \nabla u \right) = f$$

$$+ \text{ BCs}$$

challenges:

- sparse/noisy data
- evaluation of J requires solving a PDE — expensive (FD / FEM)

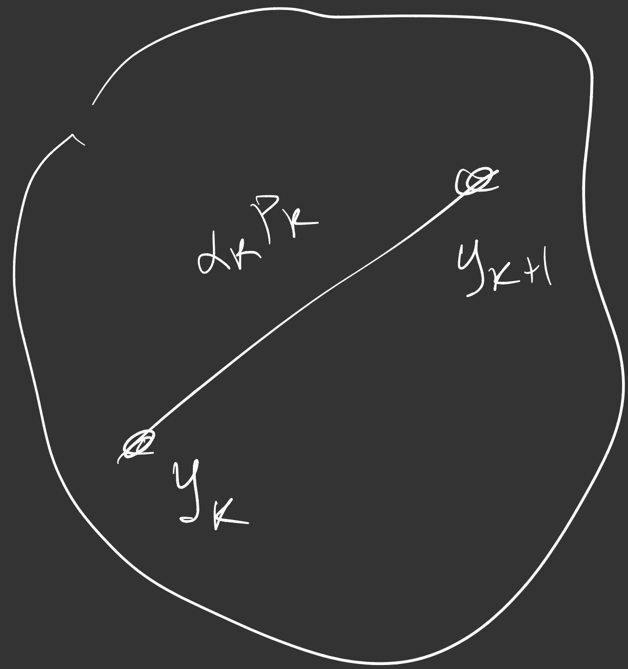
- problem is "ill-posed"
  (no unique sol, small
  change in data ⇒ large change in
  parameter estimate)
  — Need regularization.

- Optimization in function space
  + need tools from calculus of variations
  + optimality conditions: in terms of PDEs
  + need efficient derivative computation
  + large scale optimization
  — will use Newton-like
  methods

Consider Newton's method for

$$\min_{y \in \mathbb{R}^n} f(y)$$

$$y_{k+1} = y_k + \underbrace{\alpha_k}_{\text{step length}} \underbrace{p_k}_{\text{search direction}}$$



$$p_k = -H^{-1}(y_k)\, g(y_k)$$

H : Hessian , g : gradient

In each Newton Step: Need to solve

$$H(y_k)\, p_k = -g(y_k)$$

- linear solve
- will use iterative methods
- Need preconditioning

Key idea: recognize and exploit problem structure

Further Challenges:

- model is often imperfect / data is noisy
  → uncertainty in parameters
    → can consider statistical formulations

- In some cases, we have severe limitations in amount of data we can collect
  — design of experiments is important.

- Model often has additional parameters in it that are not being estimated, but not known exactly either.

  Can we assess sensitivity of parameter estimate to additional unknowns?

Basic setup of an inverse problem:

$$d = F(m) + \eta$$

data

model (ODE/PDE/DAE)

parameter (scalar / vector / function)

measurement error

F —— forward model (aka parameter-to-observable map)

Note: In science applications F is often an approximation of a real world system

Remarks:

- We focus mainly on the case where F is defined by a PDE

- discrepancy between $F(m)$ and d due to
  - + experimental / measurement error

    Control by improving measurement process
  - + model error / discrepancy

    "Control" by improving the model / understand the shortcomings of the model used / quantify model error
  - + numerical / round off errors

    Do good numerical analysis!

- η ——— not known, use statistical model

Two main strategies for solving inverse problems:

- Deterministic
  - regularization / optimization
  - Solution: a single parameter estimate
  - will be our main focus.

- Statistical / Bayesian:
  - model m as a random variable
  - Solution: a statistical distribution of P
  - enables quantifying uncertainty in the solution
  - For PDE-based problems: probability in function space
  - we will discuss this towards the end