Contract Theory

Lecture 10: Hidden Action Extensions

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Winter Semester 2024–25

??.1. Risk Neutrality and Limited Liability

- We keep the assumption that the agent is risk neutral, but now we impose a limited liability constraint such that the payment can never fall below $-l$:

$$\begin{aligned} \bar{t} &\geq -l && (1) \\ \underline{t} &\geq -l. && (2) \end{aligned}$$

- The principal's maximization problem is now given by:

$$\max_{\bar{t},\underline{t}} \ \pi_1(\bar{S} - \bar{t}) + (1 - \pi_1)(\underline{S} - \underline{t})$$

subject to (**??**), (**??**), (1) and (2).

- The maximization problem is again linear in the transfers.

Second-best optimal contract

- Again, we approach the solution by careful thinking.
- **Step 1:** Whenever the wealth of the agent is sufficiently high, the limited liability constraint has no bite.
    - Recall that in the case of no limited liability optimal payments were given by (**??**) and (**??**) with $\bar{t} > \underline{t}$.
    - So, whenever we have:

$$\underline{t} \geq -l \Leftrightarrow \frac{\pi_0}{\Delta\pi}\psi \leq l$$

    the limited liability constraint is irrelevant and the optimal contract is given by (**??**) and (**??**).
    - In this case the first-best solution is achieved.

- **Step 2:** For all relevant values of l (thus for $0 \leq l \leq \frac{\pi_0}{\Delta\pi}\psi$) constraint (2) must be binding. Otherwise, the principal could increase his expected utility by reducing $\bar{t}$ and $\underline{t}$ in a way such that (**??**) remains satisfied.
- **Step 3:** Constraint (**??**) must be binding at the optimum too, since inducing $e^* = 1$ must occur at the lowest possible cost for the principal.

- Solving the system of equations (**??**) and (2) yields:

$$\underline{t}^{SB} = -l \quad \text{and} \quad \bar{t}^{SB} = -l + \frac{\psi}{\Delta\pi}$$

- At this point we can easily check that conditions (**??**) and (1) are satisfied at the optimum.
- Thus, for $l = 0$ the optimal payment is an *option contract* .

Comparison with the first-best solution

Case $0 \leq l \leq \frac{\pi_0}{\Delta\pi}\psi$

The agent receives a rent because the payment $\underline{t}$ cannot be pushed below $-l$:

$$\begin{aligned} U &= \pi_1 \bar{t}^{SB} + (1-\pi_1)\underline{t}^{SB} - \psi \\ &= \pi_1(-l + \frac{\psi}{\Delta\pi}) + (1-\pi_1)(-l) - \psi \\ &= -l + \frac{\pi_0}{\Delta\pi}\psi \geq 0. \end{aligned}$$

Comparison with the first best solution

Case $l = 0$

- When does the principal want to induce $e = 1$?
- She can easily induce $e = 0$ by $\underline{t} = \overline{t} = 0$, so that her expected utility is given by:

$$\pi_0 \overline{S} + (1 - \pi_0)\underline{S}.$$

- If she induces $e = 1$, her expected payoff is:

$$\pi_1 \overline{S} + (1 - \pi_1)\underline{S} - \frac{\pi_1}{\Delta\pi}\psi.$$

- Accordingly, the principal prefers $e = 1$ if:

$$\begin{aligned} \pi_1 \overline{S} + (1-\pi_1)\underline{S} - \frac{\pi_1}{\Delta\pi}\psi &\geq \pi_0 \overline{S} + (1-\pi_0)\underline{S} \\ \Leftrightarrow \Delta\pi\Delta S &\geq \psi + \frac{\pi_0}{\Delta\pi}\psi \end{aligned}$$

- Compared to the first-best, for $e = 1$ to be optimal for the principal, the expected social utility gain $\Delta\pi\Delta S$ must not only exceed the social cost increase ψ, but it must also cover the information rent of the agent $\frac{\pi_0}{\Delta\pi}\psi$.

- Therefore, for all $\Delta\pi\Delta S \in (\psi, \psi + \frac{\pi_0}{\Delta\pi}\psi)$, the high effort $e = 1$ is not induced although this would be socially efficient.

??.2. Risk Aversion

- Utility function of the agent with $u' > 0$ and $u'' < 0$.
- "Standard case" in the moral hazard literature.

Second-best optimal contract

- The principal's problem is given by:

$$
\begin{aligned}
&\max_{\overline{t},\underline{t}}\ \pi_1(\overline{S}-\overline{t})+(1-\pi_1)(\underline{S}-\underline{t})\\
&\text{s.t.}\\
&\pi_1 u(\overline{t})+(1-\pi_1)u(\underline{t})-\psi \geq \pi_0 u(\overline{t})+(1-\pi_0)u(\underline{t}),\ \text{and}\\
&\pi_1 u(\overline{t})+(1-\pi_1)u(\underline{t})-\psi \geq 0
\end{aligned}
$$

- Here restrictions are no longer linear, so that we need to apply the methods of non-linear programming to find the solution (Kuhn-Tucker conditions).

- The objective function is linear in $\overline{t}$ and $\underline{t}$, whereas the concave function $u(\cdot)$ appears in the constraint.
- We make a change of variables to check if the problem is well-defined.
- Set $h = u^{-1}$, $\overline{u} = u(\overline{t})$ and $\underline{u} = u(\underline{t})$.
- $\overline{t} = h(\overline{u})$: the payment that leads to utility $\overline{u}$ for the agent.
- $\underline{t} = h(\underline{u})$: the payment that that leads to utility $\underline{u}$ for the agent.

- By this change of variables, the objective function becomes concave and the constraints become linear in $\overline{u}$ and $\underline{u}$:

$$\max_{\overline{u},\underline{u}} \pi_1(\overline{S} - h(\overline{u})) + (1 - \pi_1)(\underline{S} - h(\underline{u}))$$

- Subject to:

$$\pi_1\overline{u} + (1 - \pi_1)\underline{u} - \psi \geq \pi_0\overline{u} + (1 - \pi_0)\underline{u}$$

$$\pi_1\overline{u} + (1 - \pi_1)\underline{u} - \psi \geq 0$$

- Therefore, the Kuhn-Tucker conditions are necessary and sufficient to solve the problem.

- After putting the constraints into the correct form ($\leq$), we arrive at the Lagrange function:

$$L = \pi_1(\overline{S} - h(\overline{u})) + (1-\pi_1)(\underline{S} - h(\underline{u}))$$

$$-\lambda(\pi_0\overline{u} + (1-\pi_0)\underline{u} - \pi_1\overline{u} - (1-\pi_1)\underline{u} + \psi)$$

$$-\mu(-\pi_1\overline{u} - (1-\pi_1)\underline{u} + \psi)$$

- The first-order conditions are:

$$\begin{aligned} -\pi_1 h'(\overline{u}) + \lambda\Delta\pi + \mu\pi_1 &= 0 \\ -(1-\pi_1)h'(\underline{u}) - \lambda\Delta\pi + \mu(1-\pi_1) &= 0 \end{aligned}$$

- From $h = u^{-1}$ it follows that $h' = \frac{1}{u'}$, and thus:

$$\frac{-\pi_1}{u'(\bar{t})} + \lambda \Delta\pi + \mu\pi_1 = 0 \Leftrightarrow \lambda\Delta\pi = \frac{\pi_1}{u'(\bar{t})} - \mu\pi_1$$

- And:

$$\begin{aligned} \frac{-(1-\pi_1)}{u'(\underline{t})} - \lambda\Delta\pi + \mu(1-\pi_1) &= 0 \\ \Leftrightarrow \lambda\Delta\pi &= \frac{-(1-\pi_1)}{u'(\underline{t})} + \mu(1-\pi_1) \end{aligned}$$

- And so:

$$\mu = \frac{\pi_1}{u'(\bar{t})} + \frac{(1-\pi_1)}{u'(\underline{t})} > 0$$

- It follows from the latter inequality that the participation constraint needs to be binding at the optimum.

- Is the incentive compatibility constraint also binding at the optimum?

- Inserting $\mu = \frac{\pi_1}{u'(\bar{t})} + \frac{(1-\pi_1)}{u'(\underline{t})}$ into the first-order condition yields:

$$\frac{1}{u'(\bar{t})} = \frac{\pi_1}{u'(\bar{t})} + \frac{(1-\pi_1)}{u'(\underline{t})} + \lambda \frac{\Delta\pi}{\pi_1}.$$

- Solving for λ yields:

$$\lambda = \frac{\pi_1(1-\pi_1)}{\Delta\pi}\left(\frac{1}{u'(\bar{t})} - \frac{1}{u'(\underline{t})}\right) > 0$$

because $u'' < 0$ and $\bar{t} > \underline{t}$.

- Hence, at the optimum, the incentive-compatibility constraint is binding as well.
- Therefore, the payments $\bar{t}$ and $\underline{t}$ can directly be derived from the participation and incentive-compatibility constraints:
- **Step 1:** One can write the two constraints in the following way:

$$\begin{aligned} \pi_1 u(\bar{t}) + (1-\pi_1)u(\underline{t}) - \psi &= \pi_0 u(\bar{t}) + (1-\pi_0)u(\underline{t}) \\ \pi_1 u(\bar{t}) + (1-\pi_1)u(\underline{t}) - \psi &= 0 \end{aligned}$$

- Note that the LHS are equal and can be eliminated.

- It follows that:

$$\begin{aligned}\pi_0 u(\bar{t}) + (1-\pi_0)u(\underline{t}) &= 0 \\ \Leftrightarrow u(\bar{t}) &= \frac{-(1-\pi_0)u(\underline{t})}{\pi_0}.\end{aligned} \tag{3}$$

- Moreover:

$$\begin{aligned}\pi_1(\frac{-(1-\pi_0)u(\underline{t})}{\pi_0}) + (1-\pi_1)u(\underline{t}) - \psi &= 0 \\ \Leftrightarrow u(\underline{t}) &= \frac{-\pi_0}{\Delta\pi}\psi.\end{aligned}$$

- By substituting into (3), we get:

$$u(\bar{t}) = \frac{(1-\pi_0)}{\Delta\pi}\psi.$$

- **Step 2:** Invert

$$
\begin{aligned}
u(\underline{t}) &= \frac{-\pi_0}{\Delta\pi}\psi \Rightarrow \underline{t} = h(\psi - \frac{\pi_1\psi}{\Delta\pi}) \\
u(\overline{t}) &= \frac{(1-\pi_0)}{\Delta\pi}\psi \Rightarrow \overline{t} = h(\psi + \frac{(1-\pi_1)\psi}{\Delta\pi})
\end{aligned}
$$

- The above argument holds because $-\pi_0 = \Delta\pi - \pi_1$ and $1 - \pi_0 = \Delta\pi + (1 - \pi_1)$.
- Thus, we have $\underline{t}^{SB} < \overline{t}^{SB}$, and the agent carries part of the risk.

Comparison with the first-best solution

- We have:

$$\begin{aligned}\underline{t}^{SB} &= h(\psi - \frac{\pi_1\psi}{\Delta\pi}) < \underline{t}^{*}, \text{ and} \\ \overline{t}^{SB} &= h(\psi + \frac{(1-\pi_1)\psi}{\Delta\pi}) > \overline{t}^{*}\end{aligned}$$

- What does that mean for the agent's expected utility?

- Both in the first-best and the present case, the participation constraint is binding, so, in both cases, the agent's expected utility is given by:

$$\begin{aligned}U^{*} &= \pi_1 u(\overline{t}) + (1-\pi_1)u(\underline{t}) = \psi, \text{ and} \\ \\ U^{SB} &= u(\pi_1\overline{t} + (1-\pi_1)\underline{t}) > \psi\end{aligned}$$

(because of Jensen's inequality).

- The agent is compensated for bearing some of the risk because his utility is higher than in the first-best (“risk-premium”).
- Hence, the expected payment to the agent is strictly higher than in the first-best:

$$C^{SB} = \pi_1 h(\psi + \frac{(1-\pi_1)\psi}{\Delta\pi}) + (1-\pi_1)h(\psi - \frac{\pi_1\psi}{\Delta\pi}) > h(\psi) = C^*$$

because $h'' > 0$.

- When does the principal want to induce $e = 1$?
- His gain from doing so is still given by $B = \Delta\pi\Delta S$, whereas the costs C^{SB} are higher than in the first-best, so that for any $B \in (C*, C^{SB})$, $e = 1$ is not induced although this would be socially efficient.