Game Theory

Lecture 4: Static Games of Incomplete Information

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- The players in the games we have considered so far were well informed: Any player knew the utility functions (payoffs) of all players, i.e., the game was a game of **complete information**.
- We will now relax the assumption of complete information because (a) in reality it is frequently not satisfied, **and** (b) in general predictions differ depending on whether one assumes complete or incomplete information.

Examples

- Firms usually do not know the cost functions of their competitors. E.g., a firm that contemplates to enter a market may not know the cost function of the incumbent firm.
- In an auction the bidders usually do not know the other bidders' valuation for the object that is auctioned off.
- An employer usually does not observe the effort of a worker but only observes a noisy signal about the worker's effort.
- In card games such as Poker, Skat, Doppelkopf or Schafkopf, a player usually does not know the cards held by the other players.

Bayesian Games

We start with considering **static games with incomplete information**, which are also called **Bayesian Games** (where we restrict attention to the case of **private values**; as opposed to **common values**).

A Bayesian game can be thought of as proceeding through the following steps:

1. Nature chooses a profile of types $(\theta_1, \theta_2, ..., \theta_n)$, where θ_i is the type of player i.
2. Each player i learns his own type θ_i, which is his private information, and then uses his prior ϕ_i to form posterior beliefs over the other types of players.
3. Players simultaneously choose actions $a_i \in A_i$ for $i \in N$.
4. Given the players' choices $a = (a_1, ..., a_n)$, the payoffs $v_i(a; \theta_i)$ are realized for each player $i \in N$.

Definition

A **Bayesian Game** consists of:

- A finite set of **players** $N = \{1, \ldots, n\}$.
- A set of **actions** A_i for each player $i \in N$.
- A set of **types** Θ_i for each player $i \in N$ (which can be finite or infinite).
- A **utility function** $v_i : A \times \Theta_i \to \mathbb{R}$ for each player $i \in N$, where $A \equiv A_1 \times A_2 \times ... \times A_n$. That is, $v_i(a, \theta_i)$ is player i's utility at the action profile $a = (a_1, \ldots, a_n)$, if i has type θ_i.
- A **belief** ϕ_i for each player with respect to the uncertainty over the other players' types, that is $\phi_i(\theta_{-i} \mid \theta_i)$ is the (posterior) conditional distribution on θ_{-i} (all other types but i), given that i knows that his type is θ_i (based on a common prior, i.e., the unconditional joint distribution of types is common knowledge).

Strategies in a Bayesian Game

Definition

A **pure strategy** for player i in a Bayesian game is a function

$$s_i : \Theta_i \to A_i,$$

i.e., $s_i(\theta_i)$ is the action chosen by player i if he is of type θ_i. A **mixed strategy** is a probability distribution over a player's pure strategies.

Bayesian Nash equilibria

We now turn to Nash equilibria in Bayesian games, also referred to as Bayesian Nash equilibria:

Definition

A strategy profile $s^* = (s_1^*(\cdot), \ldots, s_n^*(\cdot))$ is a (pure-strategy) **Bayesian Nash equilibrium**, if for every player $i \in N$, for each of player i's types $\theta_i \in \Theta_i$, and for every $a_i \in A_i$, $s_i^*(\cdot)$ solves

$$\begin{aligned} & \sum_{\theta_{-i} \in \Theta_{-i}} \phi(\theta_{-i}|\theta_i) v_i\Big(\big(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})\big), \theta_i \Big) \\ \geq \quad & \sum_{\theta_{-i} \in \Theta_{-i}} \phi(\theta_{-i}|\theta_i) v_i\Big(\big(a_i, s_{-i}^*(\theta_{-i})\big), \theta_i \Big), \end{aligned}$$

where $\theta_{-i} = (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots \theta_n)$ and $s_{-i}^*(\theta_{-i}) = \big(s_1^*(\theta_1), \ldots, s_{i-1}^*(\theta_{i-1}), s_{i+1}^*(\theta_{i+1}), \ldots, s_n^*(\theta_n)\big)$.

Correlated versus independent types

Note:

- These definitions allow for correlated types.
- However, in the following, we restrict attention to the simpler case of independent types, i.e. player i's beliefs ϕ_i is equal to his priors with respect to the other players' types.

2. Application: First-Price Sealed-Bid Auction

We only scratch the huge research area on auctions. Interested readers are referred to Chapters 13 of Tadelis (2013) or the (more advanced) textbook "Auction Theory" by V. Krishna, 2002.

- One unit of an indivisible good is to be auctioned.
- There are two bidders, $i = 1, 2$, each with valuation (type) θ_i for the good (i.e., private values), where $\theta_i \sim i.i.d.\ U[0, 1]$. i.e., types are independent.
- Each bidder's θ_i is her private information.
- Each bidder secretly submits a bid p_i, the highest bid wins the auction, and (only) the winner pays his bid (first-price sealed-bid auction)
- Payoffs:

$$u_i = \begin{cases} \theta_i - p_i & if \quad p_i > p_j \\ \frac{\theta_i - p_i}{2} & if \quad p_i = p_j \\ 0 & if \quad p_i < p_j \end{cases} .$$

- We search for a symmetric Bayesian Nash equilibrium $p_i(\theta_i) = p_{-i}(\theta_{-i}) = p(\cdot)$, where $p(\cdot)$ is non-negative, strictly increasing and differentiable.
- Suppose player $-i$ plays strategy $p(\theta_{-i})$. Hence, the best response for player i satisfies

$$p_i^*(\theta_i) \in \arg\max_{p_i} (\theta_i - p_i) Prob\left(p_i > p(\theta_{-i})\right)$$

- Note that a tie ($p_i = p_{-i}$) has probability zero here, and can be neglected.
- For the uniform distribution, we have

$$Prob\left(p_i > p(\theta_{-i})\right) = Prob\left(p^{-1}(p_i) > \theta_{-i}\right) = p^{-1}(p_i)$$

where $p^{-1}(p_i)$ is the valuation bidder $-i$ must have in order to bid p_i (when playing strategy $p(\cdot)$).
- The first order condition for the objective function is

$$-p^{-1}(p_i) + (\theta_i - p_i)\frac{d}{dp_i}(p^{-1}(p_i)) = 0$$

- In a symmetric equilibrium, $p_i = p(\theta_i)$ so that

$$-\theta_i + (\theta_i - p(\theta_i))\frac{1}{p'(\theta_i)} = 0 \Leftrightarrow p'(\theta_i)\theta_i + p(\theta_i) = \theta_i$$

- Note that the last expression is a differential equation which, however, admits an easy solution as the LHS is just the derivative of $p(\theta_i) \cdot \theta_i$. Hence, taking the anti-derivative on both sides, we get

$$p(\theta_i)\theta_i = \frac{\theta_i^2}{2} + K$$

- In a last step, the boundary condition is given by $p(\theta_i) \leq \theta_i$ for all θ_i which leads to $K = 0$, since bids are restricted to be non-negative.
- The symmetric equilibrium strategy is therefore $p(\theta_i) = \frac{\theta_i}{2}$.

Example: Private information can be detrimental

- Two players $i = 1, 2$.
- Types: Player 1 has only one possible type, while player 2 is either of type A or of type B with probability 0.5 each.
- Actions: Player 1's action set is $A_1 = \{T, B\}$ and player 2's action set is $A_2 = \{L, C, R\}$
- Utility payoffs: Player 1's payoffs are

			Player 2	
		L	**C**	**R**
Player 1	**T**	4	4	4
	B	8	0	0

Player 2's payoffs are type-dependent:

Type A: **Player 1**

	Player 2		
	L	**C**	**R**
T	2	0	3
B	8	0	12

Type B: **Player 1**

	Player 2		
	L	**C**	**R**
T	2	3	0
B	8	12	0

Then $s_2^* = (s_2^*(A), s_2^*(B)) = (R, C)$ is a strictly dominant strategy for player 2.

$s_1^* = T$ is the unique best response of player 1 to s_2^*.

Hence, (s_1^*, s_2^*) is the unique Bayesian Nash equilibrium of the game.

Observe that player 2's equilibrium payoff is 3 independent of his type.

Player 2 would be better off if he had to choose an action before he learns about his type:

If player 2 does not know his type, then expected payoffs of the players are

		Player 2		
		L	**C**	**R**
Player 1	**T**	$4, 2$	$4, \frac{3}{2}$	$4, \frac{3}{2}$
	B	$8, 8$	$0, 6$	$0, 6$

and the unique Nash equilibrium is (B, L) with payoff 8 for both players.