Game Theory

Lecture 3: Dynamic Games of Complete Information, Applications

Alex Alekseev

Winter Semester 2024–25

Application I: Bargaining Games

1. Application I: Bargaining Games

- Two players are bargaining over the division of a pie of size one.
- A division (x_1, x_2) with $0 \leq x_i \leq 1$ for $i = 1, 2$, is feasible, if $x_1 + x_2 = 1$.
- The players alternate in making feasible offers (x_1, x_2), one per period, which the other player can accept or reject.
- If in period t player i makes an offer that is rejected by player $j \neq i$, then the game moves to period $t + 1$, where it is player j's turn to make an offer.
- If an offer (x_1, x_2) is accepted in period t, then the game ends and player i's payoff is $\delta_i^t x_i$, where $0 < \delta_i < 1$ is player i's discount factor.
- Offers are accepted in case of indifference.
- If no offer is ever accepted, then both players' payoff is 0.

Finite Horizon $T < \infty$

Suppose first that there is a maximum number of bargaining periods $T < \infty$. Consider different cases:

$\underline{T = 1}$ **(Ultimatum Game)**

Let player 1 be the first (and unique) proposer.

The ultimatum game has a unique subgame perfect Nash equilibrium, where player 1 proposes $(1, 0)$ and player 2 accepts any offer (x_1, x_2) with $x_2 \geq 0$.

$T = 2$

Using backwards induction and the result for $T = 1$, we derive the following unique subgame perfect Nash equilibrium for $T = 2$:

- In period 2, player 2 proposes $x^2 = (0, 1)$ and player 1 accepts any offer (x_1, x_2) with $x_1 \geq 0$.
- In period 1, player 1 proposes $x^1 = (1 - \delta_2, \delta_2)$ and player 2 accepts any offer (x_1, x_2) with $x_2 \geq \delta_2$.

$T = 3$

Using backwards induction and the result for $T = 2$, we derive the following unique subgame perfect Nash equilibrium for $T = 3$:

- In period 3, player 1 proposes $x^3 = (1, 0)$ and player 2 accepts any offer (x_1, x_2) with $x_2 \geq 0$.
- In period 2, player 2 proposes $x^2 = (\delta_1, 1 - \delta_1)$ and player 1 accepts any offer (x_1, x_2) with $x_1 \geq \delta_1$.
- In period 1, player 1 proposes $x^1 = (1 - \delta_2(1 - \delta_1), \delta_2(1 - \delta_1))$ and player 2 accepts any offer (x_1, x_2) with $x_2 \geq \delta_2(1 - \delta_1)$.

Continuing in this manner we can derive the unique subgame perfect Nash equilibrium for any $T < \infty$.

What happens for $T \to \infty$?

$T = \infty$

In order to derive the subgame-perfect Nash equilibrium for $T = \infty$, for the moment, we restrict attention to equilibria in **stationary strategies**:

A player's strategy is **stationary**, if the player always makes the same proposal and always accepts the same set of proposals.

Hence, stationary strategies take the following form:

- Player 1 always proposes (x_1^*, x_2^*) and accepts a proposal (y_1, y_2) by player 2 if and only if $y_1 \geq t_1^*$.
- Player 2 always proposes (y_1^*, y_2^*) and accepts a proposal (x_1, x_2) by player 1 if and only if $x_2 \geq t_2^*$.

Suppose that offers (of either Player 1 or 2) are immediately accepted in equilibrium (we have to verify that later); implying: $t_2^* = x_2^*$ and $t_1^* = y_1^*$.

Hence, our candidate equilibrium strategies are

- Player 1 always proposes (x_1^*, x_2^*) and accepts a proposal (y_1, y_2) by player 2 if and only if $y_1 \geq y_1^*$.
- Player 2 always proposes (y_1^*, y_2^*) and accepts a proposal (x_1, x_2) by player 1 if and only if $x_2 \geq x_2^*$.

Subgame perfection then requires that

$$x_2^* = \delta_2 y_2^* \quad \text{and} \quad y_1^* = \delta_1 x_1^*$$

(which ensures that offers are immediately accepted)

Since $x_1^* + x_2^* = 1$ and $y_1^* + y_2^* = 1$ it follows that

$$\begin{aligned} x_1^* &= \frac{1-\delta_2}{1-\delta_1\delta_2} \\ y_1^* &= \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2} \end{aligned}$$

Hence, the above strategies form a subgame-perfect equilibrium in stationary strategies. As pointed out on the next slide, one can even show that the above strategies form the unique subgame-perfect equilibrium of the game (see e.g., Fudenberg and Tirole, Ch. 4.4).

Theorem

The alternating offer bargaining game has a unique subgame perfect Nash equilibrium, in which

- *Player 1 always proposes* (x_1^*, x_2^*) *and accepts a proposal* (y_1, y_2) *by player 2 if and only if* $y_1 \geq y_1^*$.
- *Player 2 always proposes* (y_1^*, y_2^*) *and accepts a proposal* (x_1, x_2) *by player 1 if and only if* $x_2 \geq x_2^*$.

where

$$\begin{aligned} x_1^* &= \frac{1-\delta_2}{1-\delta_1\delta_2} \\ y_1^* &= \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2} \end{aligned}$$

Some Observations:

- The subgame perfect Nash equilibrium is efficient, since player 1's offer in period 1 is immediately accepted (in general, this is quite different in bargaining games under asymmetric information).
- If we fix player i's discount factor δ_i, then player j's equilibrium share goes to one if $\delta_j \to 1$.
- If both players have the same discount factor δ, then the equilibrium payoffs are $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$ and player 1 has a first-mover advantage, since $\frac{1}{1+\delta} > \frac{1}{2}$. If $\delta \to 1$, then the equilibrium payoffs converge to $(\frac{1}{2}, \frac{1}{2})$.

Application II: Repeated Games

2. Application II: Repeated Games

Until now we have only studied games which were played once.

Often we are confronted with the same strategic situation again and again, i.e., the game is played repeatedly.

In the following we will analyze to what extent the players' equilibrium play is influenced by the repetition.

- In a repeated game a player's strategy can depend on observed play in the past.
- Hence, a repeated game allows for "punishment".

For illustration consider the following version of the prisoner's dilemma:

Player 2

Player 1		C	D
	C	$3, 3$	$0, 5$
	D	$5, 0$	$1, 1$

Suppose we repeat the game, i.e., in each period t player $i, i = 1, 2$, chooses an action $a_i^t \in \{C, D\}$.

The players observe the actions of both players in all past periods before they choose an action in any given period.

We assume that players discount future payoffs with a common discount factor $0 < \delta < 1$.

If the game is repeated infinitely many times, player i's discounted average payoff, i.e., the constant payoff $\bar{u}_i$ received each period which gives the same NPV as the stream of payoffs $u_i(a_1^t, a_2^t)$ (which might vary over time) is

$$(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_i(a_1^t, a_2^t)$$

- If the game is repeated finitely many times (T times), then in any Nash equilibrium the equilibrium outcome is (D, D) in each period $t = 1, \dots, T$.
- If the game is repeated infinitely many times and if δ is sufficiently large, then there are Nash equilibria, where the outcome is (C, C) in each period. For example, if δ is sufficiently large, then there exists a Nash equilibrium, in which both players play the following (trigger) strategy:

  Strategy T1: Play C until the **other** player plays D in some period t. Then, play D in all periods $t' > t$.
- Is the Nash equilibrium induced by strategy T1 subgame perfect? What is the set of subgame perfect Nash equilibria?

2.2. Nash Equilibria in the Infinitely Repeated Prisoner's Dilemma

Here are some examples for Nash equilibrium strategies:

- **(Defect always)** Each player plays D in every period.

  In this case the equilibrium outcome is (D, D) in every period.

- **(Modified trigger strategy)** Each player plays a modified trigger strategy, where the punishment is for k periods only:

  Strategy T2: Player i plays C until the other player j plays D in some period t. Then i plays D in periods $t+1, t+2, \ldots, t+k$ and returns to C in period $t+k+1$ independent of player j's behavior in the periods $t+1, \ldots, t+k$.

  For $k \geq 2$ and δ sufficiently large, this is a Nash equilibrium and the equilibrium outcome is (C, C) in each period.

- **(Tit-for-tat)** Each player plays **tit-for-tat**, i.e., plays C in $t = 1$ and in all periods $t > 1$ the player chooses the action of the other player in period $t - 1$.

  If both players play tit-for-tat, then the outcome is (C, C) in each period.

  Attention: If both players play tit-for-tat, then this is a Nash equilibrium for the payoffs in our example of the prisoner's dilemma if δ is sufficiently large. However, there exist other payoffs, for which both players playing tit-for-tat is no Nash equilibrium in the infinitely repeated prisoner's dilemma.

Consider the following version of the prisoner's dilemma:

		Player 2	
		C	D
Player 1	C	x, x	$0, y$
	D	$y, 0$	$1, 1$

with $y > x > 1$.

Then, the tit-for-tat strategy is not consistent with Nash equilibrium if $y > 2x$.

There are also other Nash equilibria with outcomes different from (C, C) in each period:

Definition (Feasible Payoffs)

The payoff profile (x_1, x_2) is **feasible** in the prisoner's dilemma, if it is a weighted average of the payoffs in the prisoner's dilemma.

Hence, in the version of the prisoner's dilemma considered before all payoffs profiles (x_1, x_2) with

$$(x_1, x_2) = \alpha_1(3,3) + \alpha_2(0,5) + \alpha_3(5,0) + \alpha_4(1,1)$$

where $0 \leq \alpha_i \leq 1$ for all i and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$, are feasible.

Theorem (Nash Folk Theorem for the Infinitely Repeated Prisoner's Dilemma)

Let $G = (\{1,2\}, \{C,D\}, \{C,D\}; u_1, u_2)$ *be an arbitrary prisoner's dilemma. Then:*

(i) *For any discount factor* $0 < \delta < 1$ *the discounted average payoff of player* i *in any Nash equilibrium of the infinitely repeated game* G *is at least* $u_i(D,D)$.

(ii) *For any discount factor* $0 < \delta < 1$ *the infinitely repeated game* G *has a Nash equilibrium, where the discounted average payoff of each player* i *is* $u_i(D,D)$.

(iii) *Let* (x_1, x_2) *be a feasible payoff profile in* G *with* $x_i > u_i(D,D)$ *for* $i = 1,2$. *Then there exists a* $\bar{\delta} < 1$ *such that for* $\delta > \bar{\delta}$ *the infinitely repeated game* G *has a Nash equilibrium, in which the discounted average payoff of player* i *is* x_i, $i = 1,2$.

(i) and (ii) are easy to prove.

Sketch of the Proof of (iii)

- Any feasible payoff profile (x_1, x_2) can be approximated by the average payoff in a finite sequence of action profiles $(a^1, a^2, \ldots, a^k)$.
- The Nash equilibrium strategy of player i then is such that player i repeats the sequence of actions $(a_i^1, a_i^2, \ldots, a_i^k)$ until the other player deviates. If the other player deviates, player i plays D in all future periods.
- If both players play these strategies, then their discounted average payoff is approximately (x_1, x_2). If δ is sufficiently large, there is no profitable deviation for any player, because the maximum payoff player i can achieve after deviation is $u_i(D, D) < x_i$.

2.3. Subgame Perfect Nash Equilibria in the Infinitely Repeated Prisoner's Dilemma

Not all Nash equilibria in the infinitely repeated prisoner's dilemma are subgame perfect.

For example, both players playing trigger strategy T1 is not consistent with subgame perfect Nash equilibrium:

If player 1 plays T1, then T1 is not a best-response to T1 in the subgame that is reached after a deviation of player 2.

- In this subgame player 2 can improve by playing D immediately (Player 1 always plays D in this subgame).

The following modification of the trigger strategy yields a subgame perfect Nash equilibrium if played by both players:

Strategy T3: A player plays C in the first period and in all periods $t > 1$, if both players have always played C in the past. The player plays D, if some player (**the player himself or the other player**) has played D in some previous period.

Using a similar argument we can show that any other feasible payoff profile (x_1, x_2) with $x_i > u_i(D, D)$ for $i = 1, 2$, can be supported as a subgame perfect Nash equilibrium of the infinitely repeated prisoner's dilemma. The proof is similar to the proof of the Nash-Folk-Theorem, the only difference being that a player always plays D if there was a deviation by one of the players in the past.

Theorem (Subgame Perfect Nash Folk Theorem for the Infinitely Repeated Prisoner's Dilemma)

Let $G = (\{1,2\},\{C,D\},\{C,D\};u_1,u_2)$ *be an arbitrary prisoner's dilemma. Then:*

(i) *For any discount factor* $0 < \delta < 1$ *the discounted average payoff of player* i *in any subgame perfect Nash equilibrium of the infinitely repeated game* G *is at least* $u_i(D,D)$.

(ii) *For any discount factor* $0 < \delta < 1$ *the infinitely repeated game* G *has a subgame perfect Nash equilibrium, where the discounted average payoff of each player* i *is* $u_i(D,D)$.

(iii) *Let* (x_1,x_2) *be a feasible payoff profile in* G *with* $x_i > u_i(D,D)$ *for* $i = 1,2$. *Then there exists a* $\bar{\delta} < 1$ *such that for* $\delta > \bar{\delta}$ *the infinitely repeated game* G *has a subgame perfect Nash equilibrium, in which the discounted average payoff of player* i *is* x_i, $i = 1,2$.

Summarizing, we see that cooperative behavior can be supported as a subgame perfect Nash equilibrium in the infinitely repeated game, since deviations from cooperation can be punished.

However, there are also many other subgame perfect Nash equilibria (infinitely many!).

2.4. Folk Theorems for Other Infinitely Repeated Games

Similar folk theorems hold for general repeated strategic games as well.

Consider a general strategic game $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$ and define player i's **minmax payoff** by

$$M_i = \min_{s_{-i} \in S_{-i}} \left(\max_{s_i \in S_i} u_i(s_i, s_{-i}) \right).$$

M_i is the minimum payoff the other players $j \neq i$ can force upon player i.

Then, any feasible payoff profile $(x_1, x_2, \ldots, x_n)$ such that $x_i > M_i$ for all i, can be supported as a subgame perfect Nash equilibrium of the infinitely repeated game if δ is sufficiently large.