Game Theory

Lecture 5: Dynamic Games of Incomplete Information

Alex Alekseev

Winter Semester 2024–25

Basic Theory

1. Basic Theory

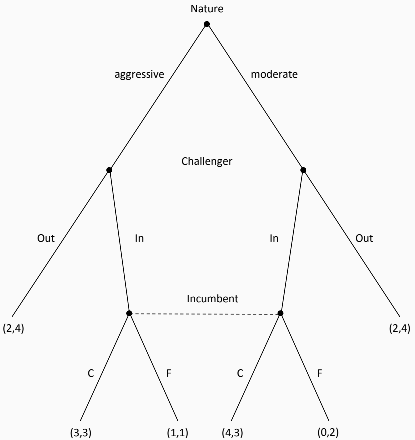
In a dynamic game of incomplete information, there is asymmetric information with respect to the payoffs of players.

As in Bayesian games, this can be modeled by assuming that a player is one of several types.

We can capture this in the extensive-form by transforming the game into one of imperfect information where moves of nature are assumed to be unobservable by some players. This is often referred to as **Harsanyi transformation**.

As a result, information sets might be non-singleton, such that a player may not know the node he is at when it is his turn to take an action.

Example: Market Entry



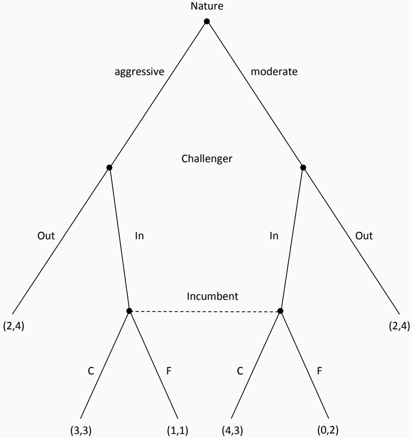
The incumbent does not know whether the challenger is of the aggressive or moderate type if the challenger enters the market. We indicate this lack of knowledge by a dotted line between the incumbent's respective decision nodes (which leads to an information set with two nodes).

Strategies and equilibrium concept

Recall that a **pure strategy** of player i in an extensive-game is a function that assigns an action to each of his information sets. (Recall also the set of possible actions is the same at each node in a given information set.)

As we will see, if an extensive game has non-singleton information sets, not even subgame perfection rules out implausible equilibria. Hence, we need a further refinement of the concept of subgame perfect Nash equilibrium, which will lead to the notion of **perfect Bayesian equilibrium**.

The problem with subgame perfection



This game has no proper subgames, and hence every Nash equilibrium of this game is also subgame perfect. What are the (subgame perfect) Nash equilibria of this game?

Two pure strategy (subgame perfect) Nash equilibria

1. Suppose, the incumbent chooses C (“cooperate”). Then, if the challenger is aggressive (moderate), playing “In” yields him a payoff of 3 (4) which is better than staying out (which independent of the type yields 2). Hence, “In” is a best response to C.
   Is C also a best response to “In”? Yes, because the incumbent gets 3 irrespective of the challenger's type, while playing F (“fight“) would yield a payoff of either 1 or 2 (depending on the challenger's type). Hence, $\{(In, In), C\}$ is a subgame perfect Nash equilibrium.

2. Similarly, if the incumbent chooses F, then either challenger type gets less upon entering (1 or 0) compared to staying out (2). Moreover, if the challenger stays out, any action of the incumbent (F in particular) is a best response. Hence, $\{(Out, Out), F\}$ is also a subgame perfect Nash equilibrium.

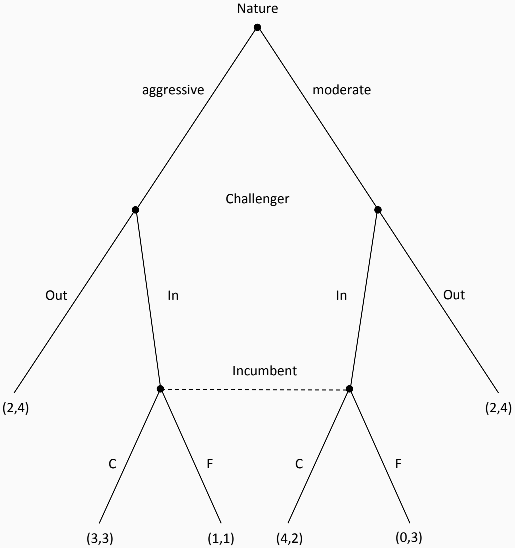
The second Nash equilibrium is implausible since it is supported by the incredible threat that the incumbent fights if the challenger does not stay out.

Hence, we are looking for a new notion of optimal decisions at information sets which contain more than one decision node.

In the example above this is straightforward, because C is the optimal action of the incumbent if his information set is reached, independent of the challenger's type. Hence, it seems unreasonable to predict that the incumbent would fight.

This is different once we slightly change the payoffs in this game:

The modified game



Now C is only optimal for the incumbent if the challenger's type is "aggressive."

In this case everything depends on the incumbent's belief about whether the entering challenger's type is "aggressive" or "moderate".

If the incumbent believes that the challenger's type is "aggressive" with probability $p > \frac{1}{3}$ then "cooperate" is optimal, if $p < \frac{1}{3}$, then "fight" is optimal and if $p = \frac{1}{3}$, then the incumbent is indifferent between "cooperate" and "fight."

Moving towards a refined equilibrium concept

Given these observations, we impose the following additional requirement that an equilibrium must satisfy.

- At every (non-singleton) information set, the respective player forms expectations (beliefs) about the probabilities of being at the different nodes of the information set.
- Given these beliefs, the player chooses an optimal action.
- Beliefs are formed rationally.

In the following, we will formalize this idea.

Beliefs

Definition

A system of **beliefs** μ of an extensive-form game assigns a probability distribution over decision nodes to every information set. That is, for every information set $h \in H$ and every decision node $x \in h$, $\mu(x) \in [0, 1]$ is the probability that player i who moves in information set h assigns to his being at x, where $\sum_{x \in h} \mu(x) = 1$ for every $h \in H$.

Requirements on refined equilibrium concept

1. Every player has a well-defined belief over where he is in each of his information sets. That is, the game will have a system of beliefs.

2. In all information sets that are **on** the equilibrium path, beliefs must be consistently derived from the equilibrium strategies using Bayes' rule.

3. In all information sets that are **off** the equilibrium path, beliefs must be consistently derived from the equilibrium strategies using Bayes' rule, where possible.

4. Given their beliefs, players' strategies must be sequentially rational. That is, in every information set, players will play a best response to their beliefs and the strategies of the other players.

Perfect Bayesian equilibrium

Definition

A strategy profile σ^* together with a system of beliefs μ^* constitute a **perfect Bayesian equilibrium** if they satisfy requirements 1 to 4.

Remark:

- There also exists in the literature the notion of a "weak" perfect Bayesian equilibrium. Under this concept, no restrictions are placed on off-equilibrium beliefs.
- Unfortunately, the term in not used consistently in the literature (not even in Tadelis, 2013...).

Applications

- **Signaling** games
- A first-mover has different types, e.g., students can have different productivities
- They can take an action (go to college) to signal their type to the second-mover (employer)
- The employer anticipates that only high-productivity types would go to college
- So the employer can offer a higher wage to those who have a college degree
- Pooling vs. separating equilibria
- Spence (1973)