Game Theory

Lecture 2: Dynamic Games of Complete Information, Theory

Alex Alekseev

Winter Semester 2024–25

Motivation

- In many strategic situations in real life the players do not choose their actions simultaneously, but rather sequentially.
- Examples:
  - Bargaining: offers and counteroffers
  - Firm decisions such as market entry, R&D or other investment choices, which are often made before competition in the product market occurs.
  - Legislative decisions by the government (e.g., taxes and subsidies, tariffs), followed by firms' choices of product and pricing policies.
  - (Credible?) Promises of politicians to rule out coalitions with certain other parties, followed by voters deciding for which party to vote.
- This sequential structure cannot be captured by a game in normal form.

Extensive form of dynamic games

Definition

The extensive-form of a dynamic game consists of:

1. a set of players (N)
2. players' payoff as function of final outcomes ($v_i(\cdot)$)
3. order of moves
4. actions of players when they can move
5. the knowledge that players have when they can move
6. probability distributions over exogenous events
7. common knowledge of this structure among all players

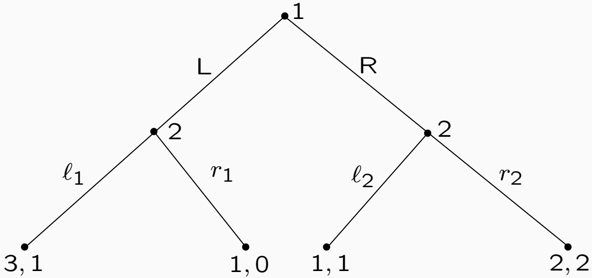
Game trees

It is convenient to represent an extensive game by a tree:

Definition

A **game tree** is a set of nodes $x \in X$ with a precedence relation $x > x'$, which means "x precedes x'". Every node in a game tree has only one predecessor. The precedence relation is **transitive** ($x > x', x' > x'' \Rightarrow x > x''$), **asymmetric** ($x > x' \Rightarrow$ not $x' > x$), and **incomplete** (not every pair of nodes x and x' can be ordered). There is special node called the **root** of the tree, denoted by x_0, that precedes any other $x \in X$. Nodes that do not precede other nodes are called **terminal nodes**. Terminal nodes denote the final outcomes of the game with which payoffs are associated. Every node x that is not a terminal node is assigned either to a player $i(x)$, with the action set $A_i(x)$, or to Nature.

Example



- The nodes in the tree correspond to the histories of the game.
- A player is assigned to each non-terminal node of the tree. This is the player who takes an action at this node.
- The action chosen by the player at a given node leads to a new node which either is a terminal node or a node where another player (or nature) is supposed to take an action.
- Players payoffs (in utilities) are indicated at the terminal nodes of the tree.

Information sets

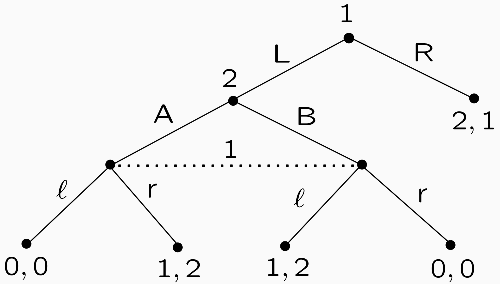
Which information do players have when it is their turn to move?

Definition

Every player i has a collection of information sets $h_i \in H_i$ that partition the nodes of the game at which player i moves with the following properties:

1. If h_i is a singleton that includes only x, then player i who moves at x knows that he is at x.
2. If $x \neq x'$ and if both $x \in h_i$ and $x' \in h_i$ then player i who moves at x does not know whether he is at x or at x'.
3. If $x \neq x'$ and if both $x \in h_i$ and $x' \in h_i$, then $A_i(x') = A_i(x)$.

Example



Perfect versus imperfect information

Furthermore, complete information games can distinguished in more detail with respect to their information structure:

Definition

A game of complete information is called a game of

1. perfect information, if every information set is a singleton and there are no moves of nature
2. imperfect information otherwise.

Imperfect information may result from **exogenous** or **endogenous** uncertainty.

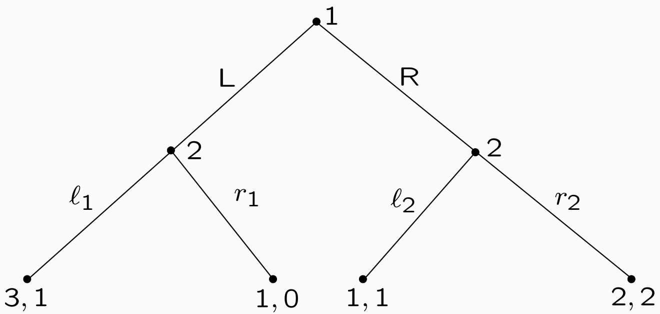
Strategies in extensive games

Definition

A **(pure) strategy** of player i is a **complete plan of play** that describes which action player i chooses at each of his information sets. That is, a pure strategy for player i is a mapping s_i that assigns an action $s_i(h_i) \in A_i(h_i)$ for every information set $h_i \in H_i$. (Observe that each player's feasible actions are the same for each node in a given information set.)

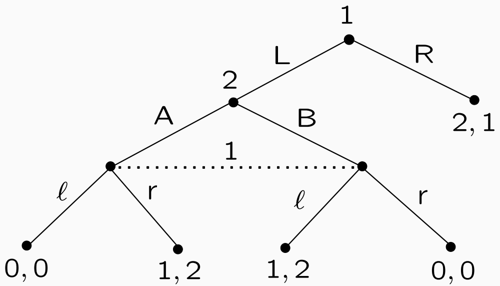
$\Rightarrow$ **Important**: In extensive games, actions and strategies **are not the same!**

Example



In the example above, s_1 with $s_1(\emptyset) = L$ is a strategy for player 1, and s_2 with $s_2(L) = \ell_1, s_2(R) = \ell_2$ is a strategy for player 2.

Another example



The set of pure strategies for player 1 is $\{(L, \ell), (L, r), (R, \ell), (R, r)\}$.

Mixed versus Behavioral Strategies

Definition

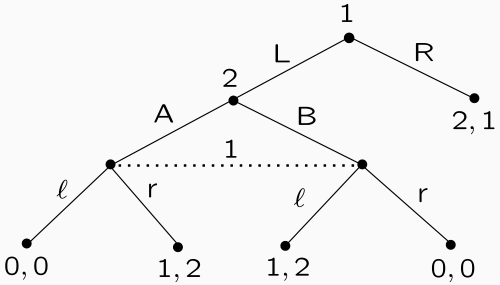
A **mixed strategy** in an extensive game is a probability distribution over the set of pure strategies as defined above.

Definition

A **behavioral strategy** specifies for each information set $h_i \in H_i$ an independent probability distribution over $A_i(h_i)$ and is denoted by σ_i, where $\sigma_i(a_i(h_i))$ is the probability that player i plays action $a_i(h_i) \in A_i(h_i)$ in information set h_i.

As shown in Kuhn (1953), any randomization over play can be represented by either mixed or behavioral strategies (under the assumption of **perfect recall**, i.e., a player never forgets anything she once knew.).

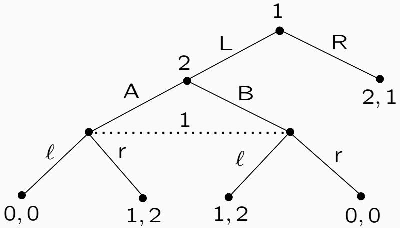
Mixed versus Behavioral Strategies: Example



A mixed strategy of player 1 is a probability distribution over the set of pure strategies $\{(L, \ell), (L, r), (R, \ell), (R, r)\}$.

A behavioral strategy of player 1 are two independent probability distributions over $\{L, R\}$ and over $\{\ell, r\}$.

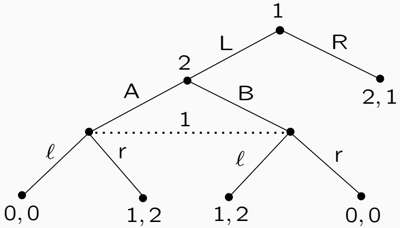
Mixed versus Behavioral Strategies: Example



At first sight this does not seem to be the same, because, e.g., the mixed strategy, where player 1 plays (L, ℓ) with probability $\frac{1}{2}$ and (R, r) with probability $\frac{1}{2}$ is not induced by any behavioral strategy: for any non-degenerate behavioral strategy player 1 also plays (L, r) and (R, ℓ) with positive probability.

Nevertheless, there is a behavioral strategy that is equivalent to the mixed strategy in the sense that it induces the same distribution over terminal nodes, independent of player 2's strategy.

Mixed versus Behavioral Strategies: Example



If player 2 plays A, then for the given mixed strategy of player 1 ((L, ℓ) and (R, r) with probability $\frac{1}{2}$ each) the terminal nodes (L, A, ℓ) and (R) are reached with probability $\frac{1}{2}$.

The same distribution over terminal nodes is obtained if player 1 plays the behavioral strategy, where he chooses L with probability $\frac{1}{2}$ and ℓ with probability 1.

A similar argument applies if player 2 plays strategy B.

Mixed versus Behavioral Strategies: Example

Conversely, it is easy to see that there is an equivalent mixed strategy for any behavioral strategy:

Consider the behavioral strategy, where player 1 plays L with probability p and ℓ with probability q.

These probabilities induce the following distribution over the set of pure strategies $\{(L,\ell),(L,r),(R,\ell),(R,r)\}$ and hence a mixed strategy for player 1:

$$
\begin{aligned}
&(L,\ell) \text{ with probability } pq\\
&(L,r) \text{ with probability } p(1-q)\\
&(R,\ell) \text{ with probability } (1-p)q\\
&(R,r) \text{ with probability } (1-p)(1-q)
\end{aligned}
$$

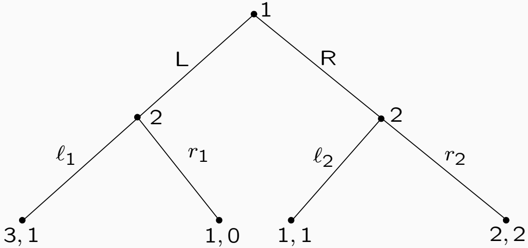
Nash equilibrium

The Nash equilibria of extensive-form games can be found by looking at their **normal-form representation**, where

- each player's strategy set S_i is the set of her strategies in the extensive game
- a player's utility at a strategy profile s is her utility under the resulting outcome in the extensive game

Note: Every extensive-form game has a unique normal-form representation, but not vice versa.

Example revisited



Player 2

		(ℓ_1, ℓ_2)	(ℓ_1, r_2)	(r_1, ℓ_2)	(r_1, r_2)
Player 1	L	$\boxed{3, 1}$	$\boxed{3, 1}$	$1, 0$	$1, 0$
	R	$1, 1$	$2, 2$	$1, 1$	$\boxed{2, 2}$

		Player 2			
		(ℓ_1, ℓ_2)	(ℓ_1, r_2)	(r_1, ℓ_2)	(r_1, r_2)
Player 1	L	$\boxed{3, 1}$	$\boxed{3, 1}$	$1, 0$	$1, 0$
	R	$1, 1$	$2, 2$	$1, 1$	$\boxed{2, 2}$

We see that the strategy profiles $(L, (\ell_1, \ell_2)), (L, (\ell_1, r_2))$ are outcome-equivalent as both lead to the same terminal node.

The outcome-equivalence is due to the fact that player 2's strategy is only different at the node (history) R that is not reached during the actual play of the game since player 1 chooses action L.

Hence, the extensive game has 3 Nash equilibria: $(L, (\ell_1, \ell_2)), (L, (\ell_1, r_2))$ and $(R, (r_1, r_2))$.

When we look only at the normal-form representation of the game, all Nash equilibria seem to be equally plausible.

However, if we look at the extensive form, the Nash equilibrium $(R, (r_1, r_2))$ does not seem plausible:

- This strategy profile is a Nash equilidbrium because player 2 "threatens" to play r_1 if player 1 chooses L instead of R. Thus, player 1's best-response is to play R.
- **But:** Player 2's threat to play r_1 if player 1 deviates to L is not credible as player 2 can achieve a higher payoff by playing ℓ_1 in that case. In other words: r_1 is not a Nash equilibrium in the subgame that follows the history L.
- Given that player 1 gets a higher payoff from playing L if player 2 plays ℓ_1 than from playing R given that player 2 plays r_2, any Nash equilibrium, where player 1 plays R seems implausible.

Hence, this Nash equilibrium relies on an empty threat off the equilibrium path.

Definition

Let σ^* be a Nash equilibrium profile of behavioral strategies in an extensive-form game. We say that an information set is **on (off) the equilibrium path** if, given σ^*, it is reached with positive (zero) probability.

Ruling out such implausible Nash equilibria that are supported by incredible threats off the equilibrium path leads to the notion of a **subgame perfect equilibrium** (Selten, 1965).

Subgame perfection requires that a player's strategy is a best-response to the strategies of the other players in **all subgames** of the game, not only in those subgames that are actually reached in equilibrium.

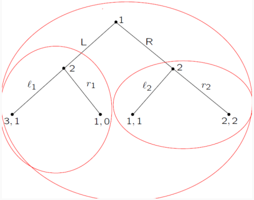
Subgames

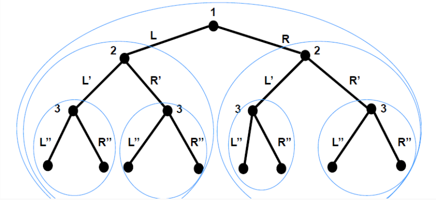
Definition

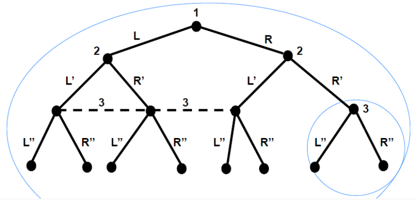
A proper **subgame** G of an extensive form game Γ consists of only a single node (i.e., an information set that is a singleton) and all its successors in Γ with the property that if $x \in G$ and there is some x' which is in the same information set as x (i.e. $x' \in h(x)$), then $x' \in G$. The subgame G is itself a game tree with its information sets and payoffs inherited from Γ.

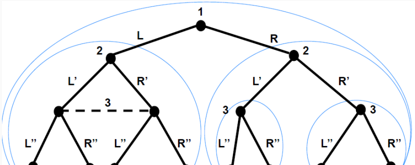
By convention, the whole game is typically also counted as a subgame.

Examples


1
L
R
2
ℓ_1
r_1
2
ℓ_2
r_2
3, 1
1, 0
1, 1
2, 2


1
L
R
2
2
L'
R'
L'
R'
3
3
3
3
L”
R”
L”
R”
L”
R”
L”
R”


1
L
R
2
2
L'
R'
L'
R'
3
3
3
L”
R”
L”
R”
L”
R”
L”
R”


1
L
R
2
2
L'
R'
L'
R'
3
3
3
L”
R”
L”
R”
L”
R”
L”
R”

Subgame Perfect Equilibrium

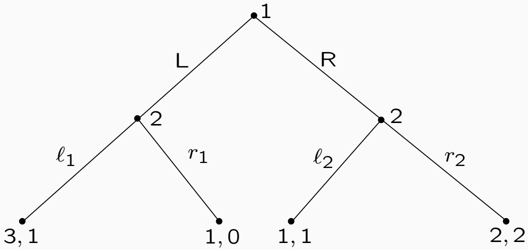
Definition

Let Γ be an n-player extensive-form game. A behavioral strategy profile σ^* is a **subgame-perfect (Nash) equilibrium** if for every proper subgame G of Γ, the restriction of σ^* to G is a Nash equilibrium in G.

In other words: A subgame perfect equilibrium is a strategy profile that induces a Nash equilibrium in every subgame.

Every subgame perfect equilibrium is a Nash equilibrium.

Example



Backward Induction

In an extensive game that is finite in the sense of having terminal nodes, subgame perfect equilibria can easily be found by **backward induction**.

Simply work backwards in finding optimal behavior, starting at the end of the game tree.

Observe that an extensive game may have several subgame perfect equilibria: This is the case, if at some stage of the backward induction procedure a player has several optimal actions.

Existence of subgame perfect equilibria

The backward induction procedure also delivers a simple proof of existence of a subgame perfect equilibrium for finite extensive games:

Theorem (Existence of subgame perfect equilibria)

Every extensive game with perfect information that is finite in the sense of having terminal nodes has at least one subgame perfect equilibrium.