Game Theory

Lecture 1

Alex Alekseev

Winter Semester 2024–25

Game theory deals with **strategic situations**, i.e., situations

- where several (at least two but usually few) players (e.g. individuals or firms) can choose between different strategies
- where each players's utility (payoff) may depend on the actions chosen by **other** players involved
- where each player is aware of the fact that
    - her own behavior may influence the utility of other players, and
    - the behavior of other players may influence her own utility

In this course, when analyzing a "game" we assume that players are **rational**, i.e., choose the best strategy given their preferences (often represented by a utility or profit function).

But what does it mean to "choose the best strategy" if your utility depends on the strategies chosen by other players?

- You can only maximize your utility for given strategies of the other players: best response

But which strategies do the others choose?

- You have to predict the strategies of the other individuals and ideally your prediction should be correct.

It is obvious that the analysis of such strategic situations can easily become very difficult and it may be no surprise that the first game theorists were mathematicians.

Any economist using game theory has to learn to describe a strategic situation in a very precise way and to use logical arguments (sometimes a long chain of such arguments) to derive a prediction for the outcome of the problem at hand.

The assumption of rationality is tricky because in a strategic situation any decision maker has to predict the behavior of other players and this will again depend on her own strategy.

Even if you are rational, is it reasonable to assume that all others are rational as well?

Traditionally, game theorists have assumed **common knowledge of rationality**, i.e., every player is rational and knows that all other players are rational and knows that all other players know that all players are rational and so on ...

Importantly, note that game theory per se does not make assumptions on the preferences or rationality of the players. Rather, game theory is a tool-kit that allows to analyze strategic interactions.

Simplifying assumptions throughout the course:

1. Common knowledge of rationality
2. Selfish preferences

However, game theory can also be employed to study settings where players have social preferences or are boundedly rational (see e.g., the field of "Behavioral Game Theory" and the section on "Behavioral Economics" in this course).

Origins

Game theory as a field started with the seminal book *Theory of Games and Economic Behavior* by von John von Neumann and Oskar Morgenstern (1944).

However, the idea of a Nash equilibrium dates back at least to Cournot (1801-1877) who studied the competition between firms in an oligopoly.

In today's economic research, game theoretic models are widely used in many fields such as industrial organization, political economy, organizational economics, and mechanism design, as well as in other social and behavioral sciences such as law, political science, or even biology.

We will focus on **non-cooperative game theory** (as opposed to cooperative game theory), where it is assumed that individuals act independently of each other (which is the most widely used branch of game theory). Cooperative game theory can, for example, be used to analyze bargaining games (see e.g., the "Nash bargaining solution").

Some Famous Game Theorists

- John von Neumann (1903-1957)
- Oskar Morgenstern (1902-1977)
- John Nash (1928-2015, Nobel Prize 1994)
- John C. Harsanyi (1920-2000, Nobel Prize 1994)
- Reinhard Selten (1930-2016, Nobel Prize 1994)
- William Vickrey (1914-1996, Nobel Prize 1996)
- James Mirlees (Nobel Prize 1996)
- Robert Aumann (Nobel Prize 2005)
- Thomas Schelling (Nobel Prize 2005)
- Eric Maskin (Nobel Prize 2007)
- Leonid Hurwicz (Nobel Prize 2007)
- Roger Myerson (Nobel Prize 2007)
- Lloyd Shapley (1923-2016, Nobel Prize 2012)
- Jean Tirole (Nobel Prize 2014)
- Paul Milgrom and Charles Wilson (Nobel Prize 2020)
- Ken Binmore, Ariel Rubinstein, Drew Fudenberg, etc...

Preliminaries

Definition

A **static game** can be thought of having two distinct steps:

1. Each player simultaneously and independently chooses an action.
2. Conditional on the players' choices of actions, payoffs are distributed to each player.

Definition

Complete information means that the following information is common knowledge among all players:

- all the possible actions of all players,
- all the possible outcomes,
- how each combination of actions of all players affects which outcome will materialize,
- the preferences of each player over outcomes (no private information).

Definition

A **pure strategy** s_i for player i is a deterministic plan of action. The set of all pure strategies of player i is denoted S_i. A **profile of pure strategies** $s = (s_1, s_2, ..., s_n) \in S$ describes a particular combination of pure strategies chosen by the n players of the game, where $S = S_1 \times S_2 \times ... \times S_n$.

In the following, we refer to a static game of complete information as a "normal-form game" (sometimes also referred to as "strategic game").

Definition

A **normal-form game** (in pure strategies) consists of

- a finite set of **players** $N = \{1, \dots, n\}$,
- for each player $i \in N$ a nonempty set of **strategies** S_i (can be finite, e.g., $S_i = \{0, 1\}$, or infinite, e.g., $S_i = [0, 1]$),
- for each player $i \in N$ a **payoff function** ("utility function") v_i, assigning a payoff (utility) to player i for each possible profile of pure strategies.

Hence, a normal-form game can simply be written as

$$(N, (S_i)_{i \in N}, (v_i)_{i \in N}).$$

Remarks:

1. A game where not only the number of players, but also their strategy sets S_i are finite is referred to as a **finite game**.
2. Note that we assume that the preferences of each player i can be represented by a **utility** (or payoff) **function** v_i, although v_i is not necessarily a payoff in monetary terms!

   (Recall that there exists a utility function that represents the preference relation if the latter is **complete** (the player can express her preference between any two strategy profiles), **transitive** (if $(s_1, s_2, \dots, s_n) \succsim_i (s_1', s_2', \dots, s_n') \succsim_i (s_1'', s_2'', \dots, s_n'')$, then $(s_1, s_2, \dots, s_n) \succsim_i (s_1'', s_2'', \dots, s_n'')$) and **continuous**, whenever the strategy sets are infinite.)

Some famous normal-form games

P 2

P 1		Cooperate	Defect
	Cooperate	$3, 3$	$0, 5$
	Defect	$5, 0$	$1, 1$

(a) Prisoners' Dilemma

P 2

P 1		Football	Opera
	Football	$2, 1$	$0, 0$
	Opera	$0, 0$	$1, 2$

(b) Battle of the Sexes

P 2

P 1		Heads	Tails
	Heads	$1, -1$	$-1, 1$
	Tails	$-1, 1$	$1, -1$

(c) Matching Pennies

P 2

P 1		Swerve	Straight
	Swerve	$0, 0$	$-1, 1$
	Straight	$1, -1$	$-10, -10$

(d) Chicken Game

Movie time

Dark Knight boat scene

Predicting the outcome(s) of normal-form games

Can we make predictions concerning the outcome of a normal form game, i.e., which strategies will rational players choose?

Evaluating solution concepts:

- **Existence**: A concept has no predictive power if we cannot guarantee existence under quite general conditions.
- **Uniqueness**: If there are multiple predictions, which strategy should a player choose?

Different concepts require different degrees of sophistication on the part of players:

- Dominance
- Iterative elimination
- Nash equilibrium

Dominance and Iterative Elimination

Denote by s_{-i} a strategy profile of all players $j \neq i$, i.e.,

$$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n).$$

Definition

Strategy s_i is a **strictly dominant** strategy for player i, if

$$v_i(s_i, s_{-i}) > v_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i,\ s_i' \neq s_i, \text{ and all } s_{-i} \in S_{-i}.$$

A player will always optimally choose a dominant strategy (if it exists). If every player has a strictly dominant strategy, then we get a unique prediction for the outcome of the game (example: Prisoners' Dilemma).

Note: This only requires rationality, but not common knowledge of rationality.

Unfortunately, only few games exhibit dominant strategies...

Prisoner's dilemma

		P 2	
		Cooperate	Defect
P 1	Cooperate	$3, 3$	$0, 5$
	Defect	$5, 0$	$1, 1$

Battle of the sexes

		P 2	
		Football	Opera
P 1	Football	$2, 1$	$0, 0$
	Opera	$0, 0$	$1, 2$

Dominated strategies

Definition

A strategy s_i of player i is **strictly dominated** by strategy s_i', if

$$v_i(s_i', s_{-i}) > v_i(s_i, s_{-i}) \text{ for all strategy profiles } s_{-i} \text{ of players } j \neq i.$$

Obviously, no rational player will choose a strictly dominated strategy.

If in addition to rationality, we assume common knowledge of rationality, then **iterative elimination of strictly dominated strategies (IESDS)** might be helpful in predicting the outcome of the game (for example, see Tadelis (2013, p. 64).

Again, many games do not exhibit dominated strategies...

Strictly dominated strategies

P 2

P 1		Cooperate	Defect
	Cooperate	$3, 3$	$0, 5$
	Defect	$5, 0$	$1, 1$

(e) Prisoners' Dilemma

P 2

P 1		Football	Opera
	Football	$2, 1$	$0, 0$
	Opera	$0, 0$	$1, 2$

(f) Battle of the Sexes

Weak dominance

There is also a notion of weak dominance:

Definition

Strategy s_i of player i is **weakly dominated** by strategy s_i', if

$$v_i(s_i', s_{-i}) \geq v_i(s_i, s_{-i}) \text{ for all strategy profiles } s_{-i} \text{ of players } j \neq i$$

and

$$v_i(s_i', s_{-i}) > v_i(s_i, s_{-i}) \text{ for at least one strategy profile } s_{-i} \text{ of players } j \neq i.$$

Note: Under weak dominance, the outcome prediction might depend on the order in which weakly dominated strategies are eliminated!

Hence, this concept is often not satisfactory...

Nash Equilibrium

Definition

Define by $B_i(s_{-i})$ the **best-response** correspondence of player i to a given strategy profile of all players s_{-i}, i.e.,

$$B_i(s_{-i}) = \{s_i \in S_i \mid v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i\}.$$

Observe that B_i is a correspondence, i.e., a set-valued mapping, since the best-response to s_{-i} need not be unique in general.

If all players choose a strategy that is a best-response to the strategies of all other players, i.e., if we have a strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ with

$$s_i^* \in B_i(s_{-i}^*) \text{ for all } i,$$

then we have reached a stable situation, where no player wants to change her strategy any more.

Such a strategy profile s^* is called a **Nash equilibrium**.

Prisoner's dilemma and best response

P 2

P 1		Cooperate	Defect
	Cooperate	$3, 3$	$0, 5$
	Defect	$5, 0$	$1, 1$

We can define a Nash equilibrium also without using the concept of best responses as follows:

Definition (Nash equilibrium)

A **Nash equilibrium** of a normal form game $(N, (S_i)_{i \in N}, (v_i)_{i \in N})$ is a strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in S$ such that for every player $i \in N$

$$v_i(s_i^*, s_{-i}^*) \geq v_i(s_i, s_{-i}^*) \text{ for all } s_i \in S_i$$

In words: A Nash equilibrium is a strategy profile s^* such that no player can do better by choosing a strategy different from s_i^* if all other players $j \neq i$ adhere to s_j^*. Hence, player i correctly anticipates that all other players also play their equilibrium strategies s_{-i}^*.

Discussion of the concept of Nash equilibrium

- Once reached, no single player has an incentive to deviate. In this sense, a Nash equilibrium exhibits (strategic) stability.
- Observe, however, that a Nash equilibrium is not necessarily Pareto efficient as a group of players may improve by a joint deviation (e.g., Prisoners' Dilemma). However, non-cooperative game theory assumes that such coordinated deviations are not feasible.
- If $s^* = (s_1^*, \ldots, s_n^*)$ is a Nash equilibrium, then for all players i, s_i^* survives the iterated elimination of strictly dominated strategies.

Discussion of the concept of Nash equilibrium (continued)

- If there is a unique strategy profile $s^* = (s_1^*, \dots, s_n^*)$ that survives the iterated elimination of strictly dominated strategies, then s^* is the unique Nash equilibrium of the game.
- Observe also that players may well use weakly dominated strategies in a Nash equilibrium. Hence, by iterative elimination of weakly dominated strategies, you may miss some Nash equilibria of the game.
- In general, uniqueness of Nash equilibrium is not guaranteed, see e.g., Battle of the Sexes.
- Existence: will be discussed next ...

Multiple Nash equilibria

P 2

P 1		Football	Opera
	Football	$2, 1$	$0, 0$
	Opera	$0, 0$	$1, 2$

No Nash equilibria*

		P 2	
		Heads	Tails
P 1	Heads	$1, -1$	$-1, 1$
	Tails	$-1, 1$	$1, -1$

Existence of a Nash equilibrium

Theorem (Existence of a Nash equilibrium)

Let $(N, (S_i)_{i\in N}, (v_i)_{i\in N})$ be a normal form game such that for all i

- *S_i is a compact and convex subset of some euclidean space $\mathbb{R}^K$,*
- *v_i is continuous,*
- *v_i is quasi-concave on S_i, i.e., for all strategy profiles s_{-i}, for all $s_i, s_i' \in S_i$ and for all $\lambda \in [0,1]$,*

$$v_i(\lambda s_i + (1-\lambda)s_i', s_{-i}) \geq \min\{v_i(s_i, s_{-i}), v_i(s_i', s_{-i})\}.$$

Then there exists at least one Nash equilibrium for $(N, (S_i)_{i\in N}, (v_i)_{i\in N})$.

Idea of Proof: Any fixed point of the best-response correspondence $B : S \twoheadrightarrow S$, where $B(s) = (B_1(s_{-1}), \ldots, B_n(s_{-n}))$ for all $s \in S$, is a Nash equilibrium of $(N, (S_i)_{i \in N}, (v_i)_{i \in N})$ since

$$\begin{aligned} s^* \in B(s^*) \quad &\iff \quad s_i^* \in B_i(s_{-i}^*) \text{ for all } i \\ &\iff \quad s^* \text{ is a Nash equilibrium} \end{aligned}$$

The conditions in the theorem guarantee the existence of a fixed point of B by an application of Kakutani's fixed point theorem.

$\square$

Obviously, we can still have a Nash equilibrium if some of the conditions in the theorem are violated.

Hence, the conditions are sufficient but not necessary for existence!

In particular, in the examples discussed above, each player only has a finite number of strategies (which violates the convexity assumption on S_i), but some of them still have a Nash equilibrium (in pure strategies).

Mixed Strategies

Many games do not have a Nash equilibrium in **pure strategies**, which is what we have considered so far. (Recall that the conditions of the existence theorem are not satisfied for all games with finite strategy sets.)

A simple example is the game "Matching Pennies".

If we allow players to choose probability distributions (lotteries) over their strategy sets we obtain **mixed strategies**.

Here, we restrict attention to finite games (i.e., finite number of players and finite strategy sets S_i).

Definition

Let $(N, (S_i)_{i \in N}, (v_i)_{i \in N})$ be a finite normal form game. A **mixed strategy** is a vector p_i (i.e. a "lottery") that assigns a probability to each of player i's pure strategies (i.e., numbers between 0 and 1), such that these probabilities sum up to 1.

We can interpret a mixed strategy as follows: If p_i is player i's mixed strategy, then player i plays strategy $s_i \in S_i$ with probability $p_i(s_i)$, i.e., the player uses some randomization device and chooses the strategy that is drawn by the device.

Mixed strategies and expected utility theory

If players choose lotteries over their strategy sets the outcome of a game is a lottery over strategy profiles.

How do players evaluate such lotteries, i.e., what is the utility of a mixed strategy profile $(p_1, \ldots, p_n)$?

In applied work (and we will do the same) it is often assumed that players' preferences over lotteries satisfy the von Neumann-Morgenstern axioms, so that they can be represented by an expected utility function. However, note that this theory is far from uncontroversial and alternative (though more cumbersome) theories of decision-making under risk are studied in behavioral economics.

If each player i plays a mixed strategy p_i, then player i's **expected utility** under a mixed strategy profile $(p_1, \dots, p_n)$ is given by

$$V_i(p_1, \dots, p_n) = \sum_{(s_1, s_2, \dots, s_n) \in S} p_1(s_1) \cdot p_2(s_2) \cdot \ldots \cdot p_n(s_n) \cdot v_i(s_1, s_2, \dots, s_n).$$

Observe that we assume that the mixed strategies of the players are independent, i.e., the probability of the strategy profile $(s_1, \dots, s_n)$ is given by the product $p_1(s_1) \cdot p_2(s_2) \cdot \ldots \cdot p_n(s_n)$.

Mixed-Strategy Nash Equilibrium

Definition

A probability profile $p^* = (p_1^*, \dots, p_n^*)$ is a **Nash equilibrium in mixed strategies**, if for all players $i = 1, \dots, n$,

$$V_i(p_i^*, p_{-i}^*) \geq V_i(p_i, p_{-i}^*)$$

for all mixed strategies p_i of player i.

Observe that a Nash equilibrium in pure strategies $s^* = (s_1^*, \dots, s_n^*)$ is a special case of a mixed-strategies Nash equilibrium, where player i plays the strategy s_i^* with probability 1.

Fortunately, for finite games there always exists a Nash equilibrium in mixed strategies. This simply follows from the above existence theorem.

In this case the strategy sets are easily seen to be compact and convex and the expected utility function is linear in the probabilities, hence it is continuous and quasi-concave in the probabilities.

Existence result

Theorem

Every finite normal-form game has a mixed strategy Nash equilibrium.

If p^* is a mixed strategy Nash equilibrium, then each strategy that player i plays with positive probability must be a best-response to the mixed strategies p^*_{-i} of the other players.

Also, each strategy that player i does not play (i.e., plays with zero probability) gives at most the expected utility to player i as any strategy that player i plays with positive probability.

Characterization result

Theorem (Characterization of mixed strategy Nash equilibria of finite games)

A mixed strategy profile p^* in a finite normal form game $(N, (S_i)_{i \in N}, (v_i)_{i \in N})$ is a mixed strategy Nash equilibrium if and only if the following two conditions are satisfied for all players i:

- *Given p^*_{-i}, player i's expected utility of every strategy s_i with $p^*_i(s_i) > 0$ is the same.*
- *Given p^*_{-i}, player i's expected utility of every strategy s_i with $p^*_i(s_i) = 0$ is at most the expected utility of any strategy s'_i with $p^*_i(s'_i) > 0$.*

Mixed strategies equilibrium

P 2

P 1		Heads	Tails
	Heads	$1, -1$	$-1, 1$
	Tails	$-1, 1$	$1, -1$