Introductory Econometrics

Lecture 6: Multiple Regression, Model Selection

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Previously on *Introductory Econometrics*...

Regression anatomy formula

$$\hat{\beta}_j = \frac{\widehat{Cov}(Y, U_j)}{\widehat{Var}(U_j)},$$

where U_j is the error term from the regression of X_j on all other predictors:

$$X_j = \gamma_0 + \gamma_1 X_1 + \ldots + \gamma_{j-1} X_{j-1} + \gamma_{j+1} X_{j+1} + \ldots + \gamma_k X_k + U_j.$$

Previously on *Introductory Econometrics*...

Omitted variables bias formula

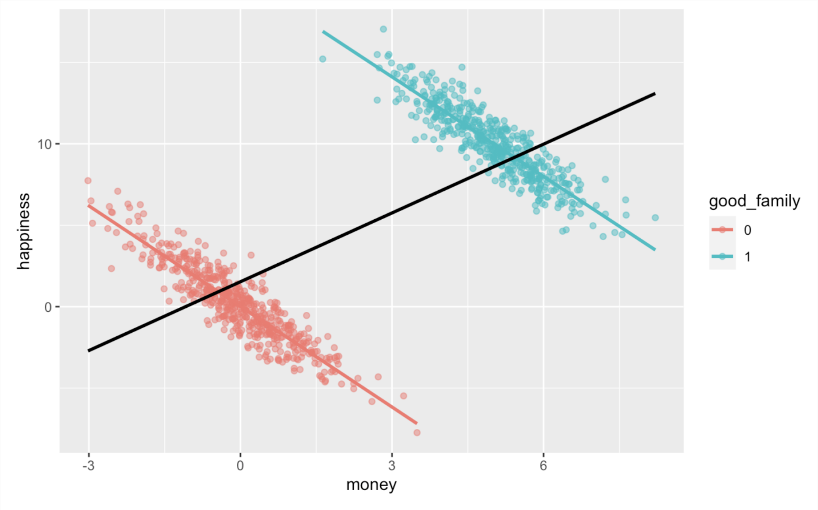
The expectation of the estimate of the slope coefficient from a simple regression of Y on Z is

$$\mathbb{E}\left[\frac{\widehat{Cov}(Y, Z)}{\widehat{Var}(Z)} \mid Z, \mathbf{x}\right] = \gamma + \beta_1\rho_1 + \ldots + \beta_k\rho_k,$$

where ρ_j are the slope coefficients from simple regressions of X_j on Z

OVB and the Simpson paradox

- Suppose we are studying the causal effect of money on happiness
- Let's assume that money has a truly negative effect on happiness
- Let's assume that we also have a third variable: whether a person is from a "good" family
- This variable will affect both the amount of money a person has (people from good families have more money than people from not so good families) and a person's happiness (people from good families are happier on average)
- Now we will illustrate graphically what happens when you estimate the "naive," unconditional effect of money on happiness and when you condition on the family background



happiness
10
0
-3
0
3
6
money
good_family
0
1

Variance of OLS

Assumptions

1. **Linear CEF** The CEF of Y given $X_1, X_2, \ldots, X_k$ is linear:

$$\mathbb{E}[Y \mid X_1, X_2, \ldots, X_k] = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_k X_k,$$

2. **Random Sampling** Our sample $(x_{i1}, x_{i2}, \ldots, x_{ik}, y_i)_{i=1}^n$ is a random sample from the population, i.e., the observations are pairwise independent and identically distributed.
3. **No Perfect Collinearity** None of the predictors is a linear combination of other predictors.
4. **Homoskedasticity**

$$Var(u_i \mid x_{i1}, \ldots, x_{ik}) = \sigma^2, \quad i = 1, \ldots, n$$

Gauss-Markov Assumptions

Covariance of error terms

- Note that the **Random Sampling** assumptions implies that the error terms for each observation are uncorrelated

$$Cov(u_i, u_j \mid \mathbf{X}) = Cov(u_i, u_j) = 0, \quad i \neq j$$

Covariance matrix (aka variance-covariance matrix)

- If you have vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

  then its covariance matrix is

$$Var(\mathbf{x}) = \begin{pmatrix} Cov(x_1, x_1) & Cov(x_1, x_2) & \dots & Cov(x_1, x_n) \\ \vdots & \vdots & \vdots & \vdots \\ Cov(x_n, x_1) & Cov(x_n, x_2) & \dots & Cov(x_n, x_n) \end{pmatrix}$$

- Notice that its diagonal elements are

$$Var(x_1), \dots, Var(x_n)$$

Covariance matrix of the error term

- The covariance matrix of the error term is then

$$
\begin{aligned}
Var(\mathbf{u} \mid \mathbf{X}) &= \begin{pmatrix}
Var(u_1 \mid \mathbf{X}) & Cov(u_1, u_2 \mid \mathbf{X}) & \dots & Cov(u_1, u_n \mid \mathbf{X})) \\
Cov(u_2, u_1 \mid \mathbf{X}) & Var(u_2 \mid \mathbf{X}) & \dots & Cov(u_2, u_n \mid \mathbf{X})) \\
\vdots & \vdots & \vdots & \vdots \\
Cov(u_n, u_1 \mid \mathbf{X}) & Cov(u_n, u_2 \mid \mathbf{X})) & \dots & Var(u_n \mid \mathbf{X})
\end{pmatrix} \\
&= \begin{pmatrix}
\sigma^2 & 0 & \dots & 0 \\
0 & \sigma^2 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & \sigma^2
\end{pmatrix} \\
&= \sigma^2 \mathbf{I}
\end{aligned}
$$

- where $\mathbf{I}$ is the **identity** matrix

Variance of individual coefficients

- Under the **Gauss-Markov** assumptions

$$Var(\hat{\beta}_j \mid \mathbf{X}) = \frac{\sigma^2}{SST_j(1 - R_j^2)},$$

where SST_j is the total sum of squares for the j-th predictor

$$SST_j = \sum_{i=1}^{n}(x_{ij} - \bar{x}_j)^2$$

and R_j^2 is the R-squared from the auxilliary regression of X_j on all other predictors

$$X_j = \gamma_0 + \gamma_1 X_1 + \ldots + \gamma_{j-1}X_{j-1} + \gamma_{j+1}X_{j+1} + \ldots + \gamma_k X_k + U_j.$$

Variance of individual coefficients explained

- The variance of coefficient j

$$Var(\hat{\beta}_j \mid \mathbf{X}) = \frac{\sigma^2}{SST_j(1 - R_j^2)}$$

- increases in σ^2 (not affected by the sample size)
- decreases in SST_j (i.e., we want to have as much variation in X_j as possible)
- increases in R_j^2 (if X_j is explained well by other predictors, then its variance will be high)

Variance inflation factors

- Write the variance of coefficient j as

$$Var(\hat{\beta}_j \mid \mathbf{X}) = \frac{\sigma^2}{SST_j} \frac{1}{1 - R_j^2}$$

- The term on the right

$$\frac{1}{1 - R_j^2}$$

  is called the **variance inflation factor** (VIF)

- It is used in practice to assess whether there are multicollinearity issues in the data
- The smallest possible value of VIF is 1
- Values that exceed 5 or 10 indicate the presence of multicollinearity

Special cases

- $R_j^2 = 0$: ideal case, X_j is uncorrelated with any other predictors
- $R_j^2 = 1$: perfect collinearity
- $0 < R_j^2 < 1$: typical case, multicollinearity

Estimation of the error variance

- The unbiased estimate of the error variance σ^2 is

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} \hat{u}_i^2}{n - (k+1)}$$

- Then the estimate of standard deviation of β_j is

$$\hat{sd}(\hat{\beta}_j \mid X) = \frac{\hat{\sigma}}{\sqrt{SST_j(1 - R_j^2)}}$$

- This term is called the **standard error** of $\hat{\beta}_j$

Covariance matrix of the OLS

- The covariance between individual estimates is

$$Cov(\hat{\beta}_j, \hat{\beta}_l \mid \mathbf{X}) = \mathbb{E}[(\hat{\beta}_j - \beta_j)(\hat{\beta}_l - \beta_l) \mid \mathbf{X}]$$

- Stacking all the terms together, we get the **covariance matrix** of $\hat{\beta}$

$$Var(\hat{\beta} \mid \mathbf{X}) = \mathbb{E}[(\hat{\beta} - \beta)(\hat{\beta} - \beta)' \mid \mathbf{X}]$$

The covariance matrix of the OLS estimator

$$Var(\hat{\beta} \mid \mathbf{X}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$$

Proof

- Start with the OLS formula and substitute for **y**

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{X}\beta + \mathbf{u}) = \beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}$$

- Then

$$\hat{\beta} - \beta = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}$$

Proof

- Plug this expression into the formula for the covariance matrix

$$
\begin{aligned}
\mathbb{E}[(\hat{\beta}-\beta)(\hat{\beta}-\beta)' \mid \mathbf{X}] &= \mathbb{E}[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u})' \mid \mathbf{X}] \\
&= \mathbb{E}[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}\mathbf{u}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \mid \mathbf{X}] \\
&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbb{E}[\mathbf{u}\mathbf{u}' \mid \mathbf{X}]\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\
&= \sigma^2(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\
&= \sigma^2(\mathbf{X}'\mathbf{X})^{-1}
\end{aligned}
$$

Efficiency of OLS

Recall the assumptions

1. **Linear CEF** The CEF of Y given $X_1, X_2, \ldots, X_k$ is linear:

$$\mathbb{E}[Y \mid X_1, X_2, \ldots, X_k] = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_k X_k,$$

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$$Var(u_i \mid x_{i1}, \ldots, x_{ik}) = \sigma^2, \quad i = 1, \ldots, n$$

Gauss-Markov Assumptions

OLS: unbiased and linear

- We have already established that OLS is **unbiased** under our assumptions
- It is also a **linear** estimator of the coefficients, in the sense that it can be written as a linear function of the outcome variable

OLS: unbiased and linear

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Note

The linearity of the **estimator** is different from the linearity of the **model**

OLS: unbiased and linear

- We have already established that OLS is **unbiased** under our assumptions
- It is also a **linear** estimator of the coefficients, in the sense that it can be written as a linear function of the outcome variable
- Recall the regression anatomy formula

$$\hat{\beta}_j = \frac{\widehat{Cov}(Y, U_j)}{\widehat{Var}(U_j)} = \frac{\sum_{i=1}^{n} \hat{u}_{ij} y_i}{\sum_{i=1}^{n} \hat{u}_{ij}^2} = \sum_{i=1}^{n} w_{ij} y_i,$$

where

$$w_{ij} \equiv \frac{\hat{u}_{ij}}{\sum_{i=1}^{n} \hat{u}_{ij}^2}$$

- Each $\hat{\beta}_j$ is a linear function of y_i

Gauss-Markov theorem

Gauss-Markov theorem

Under the Gauss-Markov assumptions, the OLS estimator is the best linear unbiased estimator (BLUE)

- The **best** in the theorem means that OLS has the smallest variance among any other linear unbiased estimators $\tilde{\beta}_j$

$$Var(\hat{\beta}_j) \leqslant Var(\tilde{\beta}_j), \quad \text{for all } j$$

Model Selection Criteria

Goal of model selection

- In principle: find the population model
- In practice: find the "best" model for the purpose of the analysis
- More specific: Under the assumption that the population model is a multiple linear regression model find all regressors that are included in the regression and their appropriate transformations. Avoid omitting variables and including irrelevant variables

A brief theory of model selection

- There are two issues
  - variable (model) choice
  - estimation variance
- For the first issue, we choose an objective function to evaluate different models
- A popular objective function is the mean squared error (MSE) or root mean squared error (RMSE)

$$RMSE \equiv \sqrt{\frac{SSR}{n}}$$

- In-sample vs. out-of-sample

RMSE decomposition

- If the model includes all the relevant variables, the population model is a multiple linear regression, and MSE is minimized with respect to the parameters, then

$$MSE = Var(U) = \sigma^2$$

- If some relevant variables are missing, it can be shown that the *MSE* can be decomposed into variance and a squared bias
- For simplicity, suppose there are k relevant predictors but we use just one

$$Y = \beta_0 + \beta_1 X_1 + U$$

RMSE decomposition

- Then

$$MSE_1 = \sigma^2 + \mathbb{E}[(\mathbb{E}[Y \mid X_1, \ldots, X_k] - \mathbb{E}[Y \mid X_1])^2]$$

- The first term represents the deviation of the observed outcome from its conditional expectation in the true population model
- The second term captures the deviation of the conditional expectation of the **true** model from the conditional expectation of the **misspecified** model

MSE and estimation

- If parameters have to estimated, we have to add another term to MSE

$$
\begin{aligned}
MSE = &\ \text{variance of population model} \\
&+ \text{squared bias} \\
&+ \text{estimation variance}
\end{aligned}
$$

- The estimation variance in general increases with the number of variables
- It might be that minimizing MSE means choosing a model that omits some variables

Bias-variance trade-off

- When we start adding predictors to the model, a natural question arises of whether these extra predictors add value
- We run into an issue of how to compare different models
- Including more predictors typically improves a model's fit (reduces **variance**)
- However, the fit could improve even if predictors are truly irrelevant
- Hence more predictors could increase the **bias** of the model
- When the bias is high, taking our model to a new data set could lead to bad performance.

Model selection criteria

- One way to balance this trade-off is to use **model selection criteria**
- The two most popular model selection criteria are the Akaike Information Criterion (AIC) and the Bayesian (Schwartz) Information Criterion (BIC)
- The AIC formula is given by

$$\text{AIC} = 2k + n \ln \frac{SSR}{n} + n \ln(2\pi) + n$$

  where k is the number of estimated coefficients, n is the number of observations, and SSR is the residual sum of squares

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Note

Some formulas use the estimated number of coefficients + 1 for k when computing the AIC for the OLS. The function AIC in R does that.

Trade-off

$$\text{AIC} = 2k + n \ln \frac{SSR}{n} + n \ln(2\pi) + n$$

- When a new predictor is added to the model (k increases)
  - the first term $2k$ increases
  - the second term $n \ln \frac{SSR}{n}$ (weakly) decreases
  - the third term $n \ln(2\pi) + n$ remains constant
- This leads to a trade-off
- One should include an additional predictor if the corresponding decrease in the first term is larger than the increase in the second term
- When we compare different models, we pick the model with the **smallest** value of the AIC.

Note

- When computing the AIC for the OLS models, some prefer to omit the constant term $n \ln(2\pi) + n$
- This omission does not affect the comparison between the models but does change the absolute value of the AIC
- The function ‘extractAIC‘ in ‘R‘ does that. It also uses the number of coefficients for k, not the number of coefficients $+ 1$.

BIC

- The formula for the BIC slightly differs from the AIC in the first term:
$$\text{BIC} = k \ln n + n \ln \frac{SSR}{n} + n \ln(2\pi) + n$$
- The BIC penalizes extra predictors more than the AIC, hence the values of the BIC are typically higher than the AIC for a given model

Note

- In case you are wondering where the $n\ln(2\pi)+n$ term comes from, the formulas for the AIC and the BIC are actually derived for models estimated using the **Maximum Likelihood Estimation** method
- A model estimated using this method has a **likelihood**, L
- The original AIC formula is

$$\mathrm{AIC} = 2k - 2\ln(L).$$

- However, for the linear regression, one can show that the logarithm of the likelihood can be written in terms of SSR as follows:

$$\ln L = -\frac{n}{2}\ln\frac{SSR}{n} - \frac{n}{2}\ln(2\pi) - \frac{n}{2},$$

  which, after substitution, yields the formula in the beginning.

Using the information criteria

- When using the information criteria, there are a few of points worth keeping in mind
- First, the criteria are used to **compare** different models, they do not tell you about a model's fit
- You need a measure like R-squared for this purpose
- Second, you **should not** compare two models that use different transformations of the outcome variable
- For example, you should not compare a model where the outcome variable is not transformed with the model where the outcome variable is logged
- Third, it is a good idea to check both criteria, however, they **do not** always give the same results

Trade example

- The first model will be our very first simple linear regression

$$\ln(imports_i) = \beta_0 + \beta_1 \ln(gdp_i) + u_i.$$

- The second model will be the so-called gravity model that adds distance as a predictor

$$\ln(imports_i) = \beta_0 + \beta_1 \ln(gdp_i) + \beta_2 \ln(distance_i) + u_i.$$

- The third model will add another predictor to the gravity model: the degree of a country's liberalization:

$$\ln(imports_i) = \beta_0 + \beta_1 \ln(gdp_i) + \beta_2 \ln(distance_i) + \beta_3 liberal + u_i.$$

- Finally, the fourth model will add a country's area to the previous model

$$\ln(imports_i) = \beta_0 + \beta_1 \ln(gdp_i) + \beta_2 \ln(distance_i) + \beta_3 liberal + \beta_4 \ln(area) + u_i.$$

Results

	simple	gravity	add liberal	add area
(Intercept)	−5.785	4.670	2.774	2.451
	(2.199)	(2.181)	(2.183)	(2.132)
log(gdp)	1.078	0.976	0.941	1.030
	(0.088)	(0.064)	(0.062)	(0.077)
log(distance)		−1.075	−0.973	−0.888
		(0.157)	(0.153)	(0.156)
liberal			0.497	0.333
			(0.193)	(0.207)
log(area)				−0.159
				(0.086)
Num.Obs.	48	48	48	48
AIC	166.333	134.103	129.337	127.634
BIC	171.947	141.588	138.693	138.861
R2	0.767	0.886	0.901	0.908
R2 Adj.	0.762	0.881	0.894	0.900

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Variable transformations