Lecture 7 Non-linear relationships

Linear models and non-linear relationships

Let's start with the following linear regression

$$Y = \beta_0 + \beta_1 X + \beta_2 Z + U.$$

We call this model *linear* because the outcome is written as a linear function of the predictors. Sometimes it is said the the model is linear in coefficients. Essentially, what makes a model linear is that the only way you can combine predictors is to multiply them by a constant and add together. However, it does *not* mean that a linear model cannot model non-linear relationships.

Consider the Cobb-Douglas production function

$$P = AK^{\beta_1} L^{\beta^2}.$$

Taking the natural logs on both sides yields

$$\ln P = \ln A + \beta_1 \ln K + \beta_2 \ln L.$$

Let's denote $\beta_0 \equiv \ln A$ and add an error term U to make it an econometric model

$$\ln P = \beta_0 + \beta_1 \ln K + \beta_2 \ln L + U.$$

Does it look similar to the model we started with? Denote $Y \equiv \ln P$, $X_1 \equiv \ln K$, $X_2 \equiv \ln L$. We started with a non-linear model (Cobb-Douglas) and ended up with a linear regression. Transforming the outcome and predictor variables in a linear regression allows one to model non-linear relationships between variables. However, remember that while you can transform the outcome and predictors using whatever functions you want, after all these transformations you can only multiply the variables by a constant and add them together to preserve the linear structure of the regression.

All of the examples below are linear models, even though they model non-linear relationships between the outcome and predictors.

Logs:

$$Y = \beta_0 + \beta_1 \ln(X) + \beta_2 Z + U.$$

Polynomials:

$$Y = \beta_0 + \beta_1 X + \beta_2 X^2 + U.$$

Interaction terms:

$$Y = \beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ + U.$$

However, there are many more models that are not linear and cannot be transformed to be linear, e.g.,

$$Y = \beta_0 + \beta_1 X^{\beta_2} + U$$

or

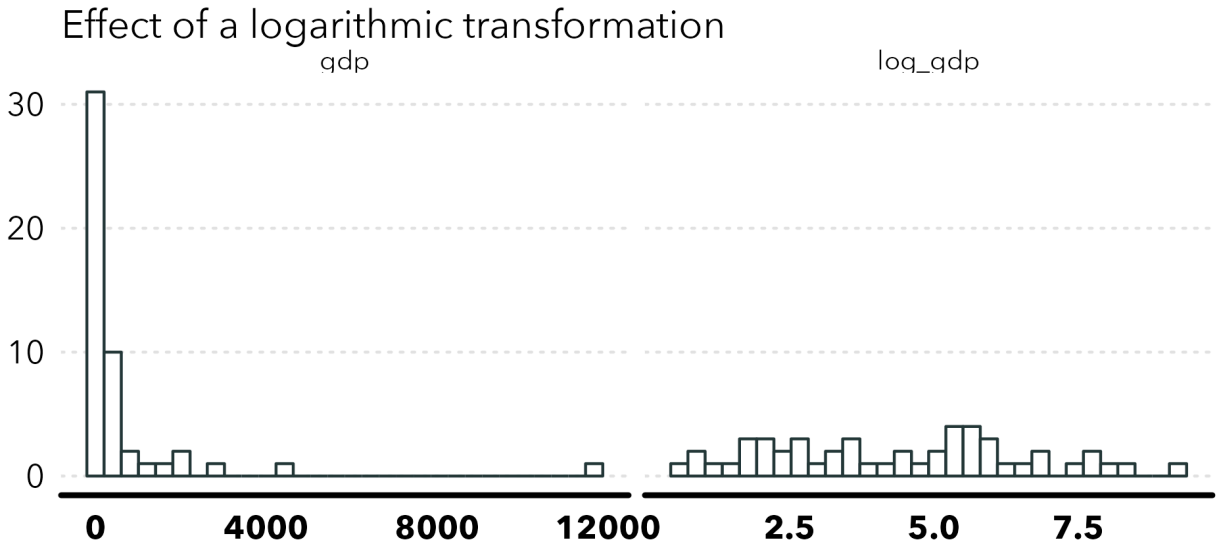
$$Y = \beta_0 + \frac{1}{\beta_1 X + \beta_2 Z} + U$$

Variable transformations

Logarithms

There are several reasons for using a logarithmic transformation on your variables. We have already seen the first reason. Logs arise when we want to transform a multiplicative relationship into an additive one, because logs transform products into sums. The Cobb-Douglas production function we considered is an example of using logs to linearize a model.

A second reason for using logs is a technical one. The logarithmic function is concave, it squishes down the value of its argument. Thus if your predictor has a lot of small values and a few very large values, using the log will make all the values closer together. As a consequence, very large values (often called outliers) will not have such a big effect on the estimates. The log transformation works well for variables that are heavily right skewed, like the income distribution or GDP.



Using logs changes the interpretation of a unit change in a predictor. To understand the interpretation of a unit change in logs, first let's recall the interpretation of the slope coefficient in a simple linear regression without logs. Let's start with the model

$$Y = \beta_0 + \beta_1 X + U.$$

The CEF of Y is

$$\mathbb{E}[Y \mid X] = \beta_0 + \beta_1 X.$$

If we increase X by c units, we get

$$\mathbb{E}[Y \mid X + c] = \beta_0 + \beta_1(X + c) = \beta_0 + \beta_1 X + \beta_1 c.$$

Subtracting from this expression the previous one, we get

$$\mathbb{E}[Y \mid X + c] - \mathbb{E}[Y \mid X] = c\beta_1.$$

Thus, if we increase X by c units, the conditional expectation of Y changes by $c\beta_1$ units. For example, if X increases by 1 unit, the CEF of Y changes by β_1 units.

Now let's consider a model where we transform X as $\ln X$:

$$Y = \beta_0 + \beta_1 \ln X + U.$$

As before, we have that

$$\mathbb{E}[Y \mid \ln X + c] - \mathbb{E}[Y \mid \ln X] = c\beta_1,$$

i.e., if $\ln X$ increases by c units, the CEF of Y changes by $c\beta_1$ units. But what does it mean that $\ln X$ increases by c units?

We need to recall the following property of logarithms. For small c, we have

$$\ln(1 + c) \approx c.$$

Therefore

$$\ln X + c \approx \ln X + \ln(1 + c) = \ln(X(1 + c)).$$

This means that increasing $\ln X$ by c units is approximately equivalent to increasing X by $100c\%$. For example, increasing $\ln X$ by 0.01 is equivalent to increasing X by 1%. Combining this result with the previous one, we conclude that in this model increasing X by $100c\%$ changes the CEF of Y by $c\beta_1$ units. For example, increasing X by 1% changes the CEF of Y by $0.01\beta_1$ units.

We can apply a similar logic to the model in which we transform Y instead:

$$\ln Y = \beta_0 + \beta_1 X + U.$$

In this case, we have that

$$\mathbb{E}[\ln Y \mid X + c] - \mathbb{E}[\ln Y \mid X] = c\beta_1.$$

In words, increasing X by c units changes the CEF of $\ln Y$ by $c\beta_1$ units. And from the previous analysis we know that changing $\ln Y$ by $c\beta_1$ units is approximately equivalent to changing Y by $100c\beta_1\%$. For example, increasing X by 1 unit changes Y by $100\beta_1\%$.

Finally, let's consider the case when both the outcome and the predictor are log-transformed:

$$\ln Y = \beta_0 + \beta_1 \ln X + U.$$

As before, we have that

$$\mathbb{E}[\ln Y \mid \ln X + c] - \mathbb{E}[\ln Y \mid \ln X] = c\beta_1.$$

In words, increasing $\ln X$ by c units changes the CEF of $\ln Y$ by $c\beta_1$ units. Translating this back to the raw variables, this means that increasing X by $100c\%$ changes Y by $100c\beta_1\%$. For example, increasing X by 1% changes Y by $\beta_1\%$. In this case β_1 represents the *elasticity* of Y with respect to X. Another way to see this is by taking the derivative

$$\frac{d\ln Y}{d\ln X} = \frac{dY/Y}{dX/X} = \beta_1.$$

Alternatives to logarithms

An issue with using a logarithmic transformation is that it is not defined for zeros. If the variable you plan to transform has zero values, you cannot take the log of it. There are several practical ways of dealing with zeros. One commonly used option is to simply add a small value to all of the values of your variable, so that none of the values is exactly zero. It is an easy fix, although somewhat crude and arbitrary.

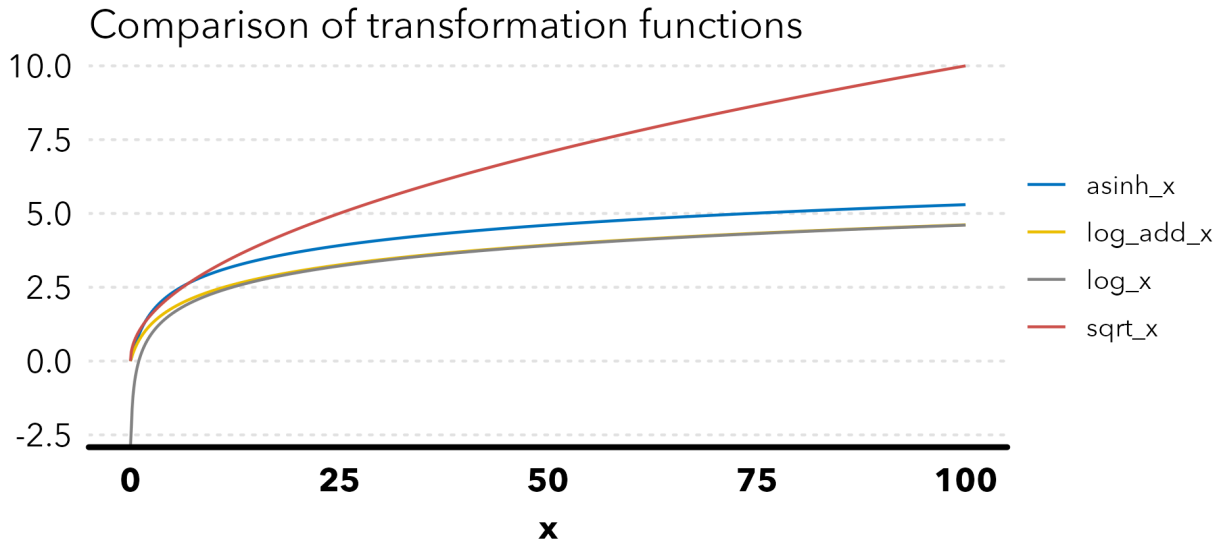
Another option is to use a square root transformation. It is also concave (squishes down large values) and it can handle zeros. However, it does not squish down large values as much as the logarithm does. It also becomes harder to interpret a unit change in the transformed variable.

A more sophisticated approach to dealing with zeros is to use the *inverse hyperbolic sine* (or *asinh*) transformation defined as

$$\text{asinh}(x) = \ln(x + \sqrt{x^2 + 1}).$$

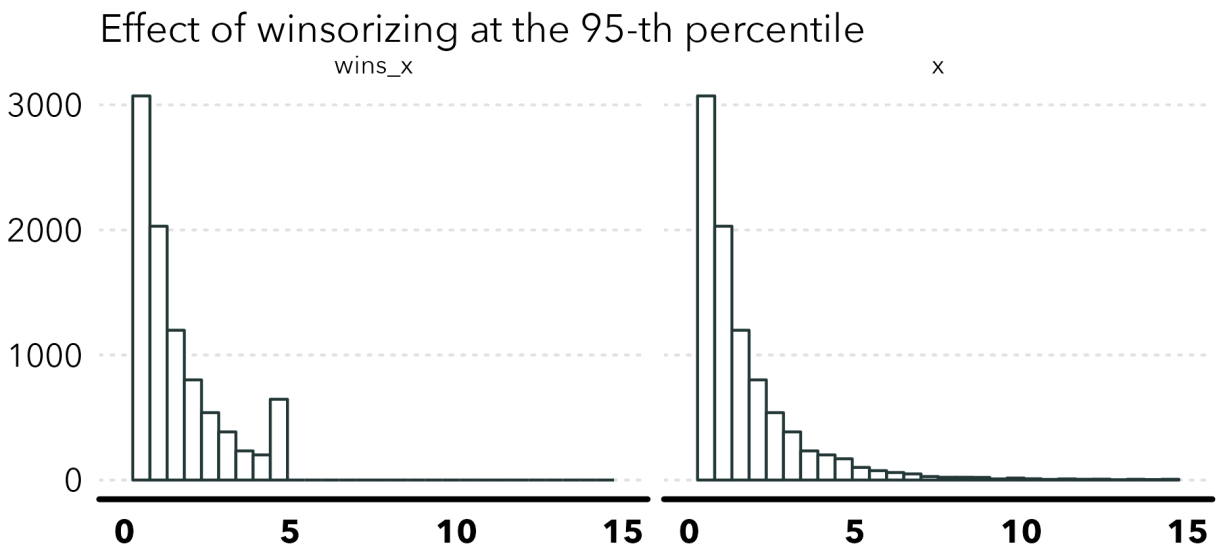
This transformation handles zeros and behaves almost like the logarithm for large values of x. However, it still does not have an interpretation of percentage changes.

The graph below compares different types of transformations.



Winsorizing

An alternative way of dealing with outliers is *winsorizing*. Instead of squishing down all the values of a variable using a concave function, winsorizing works by limiting extreme values. A fist step in winsorizing is to define outliers. For example, we can say that the values that are above the 95-th percentile of the distribution of our variable are outliers. To winsorize, we simply take all the outliers, as defined above, and replace them with the value of our variable at the 95-th percentile. Note that winsorizing is different from *truncating* (or *trimming*). Winsorizing does not discard outliers, it replaces them with other values.



Standardizing

To standardize a variable means to subtract its mean and divide by the standard deviation:

$$\frac{X - \mathbb{E}X}{sd(X)}.$$

Standardizing simply rescales a variable by making it have a zero mean and a standard deviation of one (compare to a *standard normal distribution*). Standardizing does not do anything about outliers. The reason for standardizing variables is to make their effects on an outcome comparable. If all the predictors are standardized, then each slope coefficient tells us by how many units an outcome changes if a given predictor increases by one standard deviation. In this case we do not have to worry about the units of measurement of individual predictors.

Let's call the standardizing transformation $std(X)$. Our model is then

$$Y = \beta_0 + \beta_1 std(X) + U.$$

As before

$$\mathbb{E}[Y \mid std(X) + c] - \mathbb{E}[Y \mid std(X)] = c\beta_1.$$

Then increasing $std(X)$ by c units means

$$std(X) + c = \frac{X - \mathbb{E}X}{sd(X)} + c = \frac{X + c \cdot sd(X) - \mathbb{E}X}{sd(X)} = std(X + c \cdot sd(X)).$$

In words, increasing $std(X)$ by c units is equivalent to increasing X by c standard deviations. Hence, increasing the standardized X by one unit is equivalent to increasing X by one standard deviation. And increasing X by one standard deviation will change the CEF of Y by β_1 units.

In addition to standardizing predictors, we can also standardize an outcome. In this case, the interpretation of each slope coefficient will be by how many standard deviations the outcome changes if a given predictor increases by one standard deviation. The coefficients in a linear regression in which all the variables (an outcome and predictors) are standardized are called *standardized* (or *beta*) coefficients.