Introductory Econometrics

Lecture 14: Classification

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Motivation

Binary and categorical variables

- We worked with the outcome and predictor variables that can be called **continuous**, e.g., GDP, education, wages, test scores
- In Lecture 8 we have introduced **binary** predictors, as well as more general **categorical** predictors
- These variables take on two or more possible values that are **qualitative** in nature
- We call their values **categories** or **classes**
  - A person's gender can take values such as "male" or "female"
  - A person's race can take values such as "black," "hispanic," and "white"
- We learned that we can work with these qualitative categorical predictors by encoding them as binary **indicator** variables that take values of 0 and 1

Categorical outcome

- But what if our **dependent** variable is binary or categorical?
- We are trying to explain why a given observation belongs into a given category
- This type of problems is called the **classification problem**
- The methods for dealing with it are different from the methods for dealing with the **regression problem**
- Here we will only consider the classification problem for **binary** outcomes

Encoding a binary outcome

- Our first step would be to encode our outcome as a binary **indicator** variable
- Let's say our outcome $\tilde{Y}$ can take two possible values class A and class B
- For example, $\tilde{Y}$ could be whether a person has a college education, and the two classes are "has a college degree" and "does not have a college degree"
- We can convert our outcome into a numeric variable Y as follows:

$$Y = \begin{cases} 1, \tilde{Y} = \text{class A}, \\ 0, \tilde{Y} = \text{class B}. \end{cases}$$

- Whether we assign a value of 1 to class A or B is completely **arbitrary**
- But the **interpretation** of the results will depend on that

Linear probability model

Regression

- Our first approach to the classification problem might be to use the **linear regression**

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_k X_k + U$$

- Let's consider the **conditional expectation** of the outcome Y given predictors $\mathbf{x} \equiv (X_0, X_1, \ldots, X_k)$:

$$\mathbb{E}[Y \mid \mathbf{x}] = \beta_0 + \beta_1 X_1 + \ldots + \beta_k X_k.$$

- Recall the expectation of a binary random variable:

$$\mathbb{E}[Y \mid \mathbf{x}] = 1 \times \mathbb{P}(Y = 1 \mid \mathbf{x}) + 0 \times \mathbb{P}(Y = 0 \mid \mathbf{x}) = \mathbb{P}(Y = 1 \mid \mathbf{x}).$$

- The conditional expectation of our binary Y is the **probability** that it equals 1

Linear probability model

- If we use the linear regression model on our binary outcome, we are effectively modeling the **probability** of $Y = 1$ conditional on predictors
- Since our regression is linear, the probability will be modeled as a **linear** function of the predictors
- The resulting model is called the **linear probability model** (LPM)
- Even though this is not the most appropriate tool for the classification problem, the LPM is often used in economics research

Estimation and interpretation

- We can estimate the coefficients of the LPM using OLS, as usual
- The interpretation of each individual coefficient β_j is the change in the probability of Y being 1 when X_j increases by 1 unit while keeping other variable fixed
- To interpret the value of β_j correctly you need to remember which class corresponds to $Y = 1$
- For example, if $Y = 1$ corresponds to a class "has a college degree", then a positive coefficient β_j would imply that the predictor X_j has a positive effect on the probability of having a college degree

First issue with LMP

- There are a few **issues** with the LPM
- The biggest one is that it is possible to generate predicted values $\hat{Y}$ that are either **less than zero** or **greater than one**

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- The biggest one is that it is possible to generate predicted values $\hat{Y}$ that are either **less than zero** or **greater than one**
- We are predicting **probabilities** that are by definition bounded **between zero and one**
- It does not make sense to have predictions like that

Second issue with LMP

- The second issue is that the estimated coefficients will be **constant** regardless of the values of the predictors
- Suppose that the values of the predictors for an observation i are such that the predicted probability of class A is 0.99
- Suppose also that some estimated coefficient $\hat{\beta}_j$ is 0.1

Second issue with LMP

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- Suppose that the values of the predictors for an observation i are such that the predicted probability of class A is 0.99
- Suppose also that some estimated coefficient $\hat{\beta}_j$ is 0.1
- The model would predict that increasing the value of X_j by one unit would increase the predicted probability by 0.1
- However, the predicted probability is already at 0.99 and can only increase by 0.01 at most
- Thus the estimated coefficients from the linear model may be **wrong**

Third issue with LPM

- There is a third issue, which can be fixed relatively easily
- When the outcome is binary, its variance is, by definition

$$\text{Var}(Y \mid \mathbf{x}) = \mathbb{P}(Y = 1 \mid \mathbf{x})(1 - \mathbb{P}(Y = 1 \mid \mathbf{x}))$$

- The variance of the error term cannot be constant, which leads to **heteroskedasticity**

The fix

- The good news is that we at least know its **shape** and hence can use FGLS to correct for it
- The estimated $\hat{h}_i$ for each observation will be

$$\hat{h}_i = \hat{y}_i(1 - \hat{y}).$$

- The bad news is that, the **first issue** with the LMP can mess up this strategy
- If our predicted probabilities fall outside of the unit interval, this will generate **negative weights**
- The first fix would be to abandon FGLS and simply use heteroskedasticity robust standard errors
- The second fix would be to force all of the $\hat{h}_i$ to be between 0 and 1
- For example, we can set $\hat{y}_i = 0.01$ if $\hat{y}_i < 0$ and $\hat{y}_i = 0.99$ if $\hat{y}_i > 1$

Logit model

Generalized linear models

- The **logit model** is a tool specifically designed for classification problems
- It is an example of a **generalized linear model** (GLM)
- As a starting point, recall that in the LPM we are modeling the probability of $Y = 1$ conditional on predictors

$$\mathbb{P}(Y = 1 \mid \mathbf{x}) = \beta_0 + \beta_1 X_1 + \ldots + \beta_k X_k = \mathbf{x}'\beta.$$

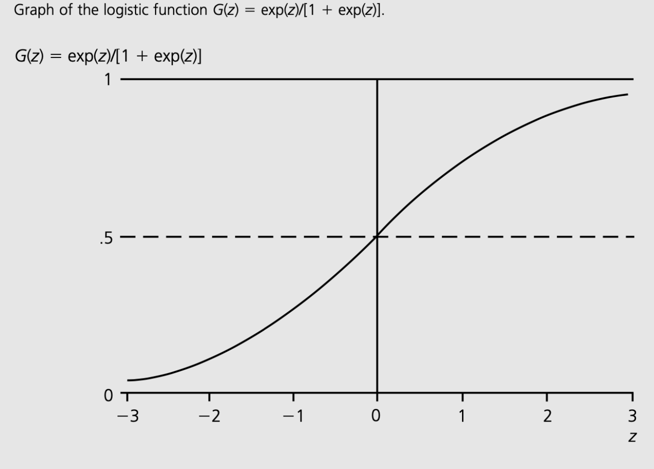
- The main issue with the LPM is that the linear function $\mathbf{x}'\beta$ technically allows the predicted values to be outside of the unit interval
- The idea of a GLM is to put some function G, called a **link function**, on top of the linear function $\mathbf{x}'\beta$ to ensure that the predicted values have the desired properties

GLM for classification

- In a binary classification problem we are predicting **probabilities**
- Hence, the G function should produce values strictly between 0 and 1
- It should also be defined for any value between minus and plus infinity and be monotonically increasing
- All of these properties are satisfied by a G function that is a **CDF**
- One of the most commonly used G functions is the **logistic CDF** (denoted by the capital letter Λ):

$$G(z) = \Lambda(z) = \frac{e^z}{1 + e^z}$$

Illustration



The logit model

- The logistic CDF looks similar to the standard normal CDF
- The logistic CDF is also symmetric, meaning that

$$\Lambda(z) = 1 - \Lambda(-z)$$

- Thus our model becomes

$$\mathbb{P}(Y = 1 \mid \mathbf{x}) = \Lambda(\mathbf{x}'\beta)$$

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Probit model

Another popular link function is the standard normal CDF. Using this link function results in a **probit** model.

Latent variable model

- The logit model can be derived using the so-called **latent variable model** that satisfies the classical linear model assumptions
- A latent variable model has **two parts**
- The first part specifies a model for the latent variable Y^*
- The second part specifies how the observed variable Y is related to the latent Y^*
- For the logit model, we assume that

$$\begin{aligned} Y^* &= \mathbf{x}'\beta + U, \\ Y &= \mathbb{I}[Y^* > 0], \\ U \mid \mathbf{x} &\sim \Lambda(\cdot) \end{aligned}$$

Interpretation

- The latent variable U^* can be thought of as the unobserved **utility** associated with choosing class A over class B
- Notice that this means that we model the utility as a **linear** function of predictors
- When the utility of class A is positive, we will choose it over class B, hence the observed variable Y will become 1
- For example, if our outcome variable is whether a person has a college degree, the latent variable will be the utility of going to college
- If this utility is positive, the person will choose to go to college
- If the utility is negative, the person will choose not to go to college

Derivation

- Now, the probability of $Y = 1$ is

$$\mathbb{P}(Y = 1 \mid \mathbf{x}) = \mathbb{P}(Y^* > 0 \mid \mathbf{x})$$

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&= 1 - \mathbb{P}(U \leqslant -\mathbf{x}'\beta \mid \mathbf{x}) \\
&= 1 - \Lambda(-\mathbf{x}'\beta) \\
&= \Lambda(\mathbf{x}'\beta)
\end{aligned}$$

We derived the logit model!

Maximum likelihood estimator

- Estimating the logit model cannot be done using OLS
- Instead, we need to use a different kind of estimator called the **Maximum Likelihood Estimator** (MLE)
- The **likelihood** of observing an outcome $Y_i = y_i$, conditional on the predictors $\mathbf{x}_i$ and parameters β is

$$f(Y_i = y_i \mid \mathbf{x}_i, \beta) = \Lambda(\mathbf{x}_i'\beta)^{y_i}(1 - \Lambda(\mathbf{x}_i'\beta))^{1-y_i}$$

- If we observe a random sample of size n, then the **joint likelihood** of observing this random sample is the product of individual likelihoods

$$\Pi_{i=1}^{n}\Lambda(\mathbf{x}_i'\beta)^{y_i}(1 - \Lambda(\mathbf{x}_i'\beta))^{1-y_i},$$

where $\Pi_{i=1}^{n} a_i$ denotes the product of all the elements from 1 to n: $a_1 \times a_2 \times \ldots \times a_n$

Likelihood function

- We can view this expression as the **likelihood function** of the parameters β given the data $\mathbf{y}, \mathbf{X}$:

$$L(\beta \mid \mathbf{y}, \mathbf{X}) = \Pi_{i=1}^{n} \Lambda(\mathbf{x}_i'\beta)^{y_i}(1 - \Lambda(\mathbf{x}_i'\beta))^{1-y_i}$$

- Taking the logs, we get the **log-likelihood function**

$$\mathcal{L}(\beta \mid \mathbf{y}, \mathbf{X}) = \sum_{i=1}^{n} [y_i \ln \Lambda(\mathbf{x}'\beta) + (1 - y_i) \ln(1 - \Lambda(\mathbf{x}'\beta))]$$

- Then the maximum likelihood estimator of β is defined as the value that **maximizes** the log-likelihood function:

$$\widehat{\beta}^{\text{MLE}} = \arg\max_{\beta} \mathcal{L}(\beta \mid \mathbf{y}, \mathbf{X})$$

MLE properties

- Unfortunately, there is typically **no closed-form solution** for the ML estimator, unlike for OLS
- The value of $\widehat{\beta}^{\text{MLE}}$ is usually found through **numeric** optimization, although, it is often quite fast
- In general, it can be shown that the MLE is consistent, asymptotically normal, and asymptotically efficient

Interpretation of the logit estimates

Odds

- The interpretation of the coefficients in the logit model is trickier than in the OLS
- We first note that the logistic function has the following property

$$\frac{\Lambda(z)}{1-\Lambda(z)} = e^z.$$

- We also need to define a statistic called the **odds**, which in our context equals

$$\frac{\mathbb{P}(Y=1 \mid \mathbf{x})}{1-\mathbb{P}(Y=1 \mid \mathbf{x})}.$$

- The odds tell one how **likely** a given outcome is by computing the ratio of the probability of that outcome happening (in our case, $Y=1$) versus the probability of the outcome not happening (in our case, $Y=0$)
- If the odds are greater than one, then the outcome is more likely to happen than not

Log odds

- Since in the logit model, $\mathbb{P}(Y = 1 \mid \mathbf{x}) = \Lambda(\mathbf{x}'\beta)$, we have that

$$\frac{\mathbb{P}(Y = 1 \mid \mathbf{x})}{1 - \mathbb{P}(Y = 1 \mid \mathbf{x})} = e^{\mathbf{x}'\beta}.$$

- Taking logs, we get

$$\ln \frac{\mathbb{P}(Y = 1 \mid \mathbf{x})}{1 - \mathbb{P}(Y = 1 \mid \mathbf{x})} = \mathbf{x}'\beta.$$

- The term on the left is called the **log-odds** or **logit**
- For the logit model, the log-odds turns out to be a **linear** function of the predictors
- Therefore, each coefficient β_j has the interpretation of the **marginal effect** of the predictor X_j on the **log-odds**

Odds ratio

- Thinking about the marginal effect on the log-odds can be a little unintuitive
- Instead, we can make use of **exponentiated** coefficients, e^{β_j}
- Let's denote the odds as $\text{Odds}(\mathbf{x})$

$$
\begin{aligned}
\text{Odds}(X_1, \ldots, X_j + 1, \ldots, X_k) &= e^{\beta_0 + \beta_1 X_1 + \ldots + \beta_j (X_j + 1) + \ldots + \beta_k X_k} \\
&= e^{\beta_0 + \beta_1 X_1 + \ldots + \beta_j X_j + \ldots + \beta_k X_k} e^{\beta_j} \\
&= \text{Odds}(\mathbf{x}) e^{\beta_j}.
\end{aligned}
$$

- The exponentiated coefficient e^{β_j} tells one by how much the odds change (in multiplicative terms) when X_j increases by one unit
- If $e^{\beta_j} > 1$, then X_j has a positive effect on the probability of $Y = 1$, if $e^{\beta_j} < 1$, then X_j has a negative effect on the probability of $Y = 1$, and if $e^{\beta_j} = 1$, then X_j has no effect

Marginal effect

- Even the odds ratio can often be a little hard to interpret
- Instead, we might want to work directly with the **marginal effect** of X_j on the probability of $Y = 1$

$$\frac{\partial \mathbb{P}(Y = 1 \mid \mathbf{x})}{\partial X_j} = \frac{\partial \Lambda(\mathbf{x}'\beta)}{\partial X_j} = \beta_j \Lambda'(\mathbf{x}'\beta).$$

- Clearly, β_j is **not** the marginal effect of X_j on the probability of $Y = 1$
- Instead, the marginal effect of X_j now depends on the values of **all** other predictors, it is not constant

Computing marginal effects

- It is easy to show that the logistic CDF has the following property:

$$\Lambda'(z) = \Lambda(z)(1 - \Lambda(z))$$

- Therefore, the marginal effect of X_j can be written as

$$\frac{\partial \mathbb{P}(Y = 1 \mid \mathbf{x})}{\partial X_j} = \beta_j \Lambda(\mathbf{x}'\beta)(1 - \Lambda(\mathbf{x}'\beta))$$

- Once we estimate the parameters, we have that the predicted probability of class A is $\hat{y}_i = \Lambda(\mathbf{x}_i'\hat{\beta})$, and therefore

$$\frac{\partial \mathbb{P}(Y = 1 \mid \mathbf{x}_i)}{\partial X_j} = \hat{\beta}_j \hat{y}_i (1 - \hat{y}_i)$$

Interpretation

$$\frac{\partial \mathbb{P}(Y = 1 \mid \mathbf{x}_i)}{\partial X_j} = \hat{\beta}_j \hat{y}_i (1 - \hat{y}_i)$$

- This expression shows that the marginal effect of X_j depends on the predicted probability in a reasonable way
- When the predicted probability is already high with $\hat{y}_i$ close to 1, then the marginal effect of X_j on the probability will be close to zero
- The marginal effect of X_j on the probability will be largest when the predicted probability is close to 0.5

Average marginal effect

- The marginal effect of X_j on the probability $\mathbb{P}(Y = 1 \mid \mathbf{x})$ is not a single number, it is a function of the values of **all** the predictors
- But often we would like to get a single number that **summarizes** the effect of a given predictor
- In this case, we can compute the **average marginal effect** (AME) by computing the marginal effects of X_j on the probability for each observation and then compute the average of those effects:

$$\text{AME}(X_j) = \frac{1}{n} \sum_{i=1}^{n} \hat{\beta}_j \hat{y}_i (1 - \hat{y}_i).$$