Introductory Econometrics

Lecture 10: Multiple Regression Analysis - Inference Pt. 2

Alex Alekseev

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University of Regensburg, Department of Economics

Previously on *Introductory Econometrics*...

- t test
- p values
- Confidence intervals


Reject
Do
not reject
WHAT WAS THE NULL HYPOTHESIS?
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Single linear restriction

Set up

- Let's begin with our population model

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_k X_k + U.$$

- Previously, we have learned how to conduct hypotheses testing about a single coefficient, e.g., $\beta_j = b, j = 0, \ldots, k$
- However, often we would like to test hypotheses involving more than one coefficient
- The simplest case of that would be when we would like to test the equality of any two coefficients, say j and l ($j \neq l$):

$$H_0 : \beta_j = \beta_l.$$

- Suppose our alternative hypothesis is two-sided

$$H_1 : \beta_j \neq \beta_l.$$

New test statistic

- We can no longer use t statistics for individual coefficients to conduct hypothesis testing
- What we can do, is construct a new test statistic of the following form

$$t = \frac{\hat{\beta}_j - \hat{\beta}_l}{\text{se}(\hat{\beta}_j - \hat{\beta}_l)}.$$

- Once we compute this test statistic, we can proceed as before:
  - by either comparing the t-statistic with the appropriate critical value
  - or computing the p-value of the statistic and comparing it with the significance level
  - or using a confidence interval

Difficulty

- There is one difficulty involved in computing this new test statistic
- While we can easily compute the numerator, the standard error in the denominator is not readily available
- We would have to use the following property of variances:

$$\mathrm{Var}(\hat{\beta}_j - \hat{\beta}_l) = \mathrm{Var}(\hat{\beta}_j) + \mathrm{Var}(\hat{\beta}_l) - 2\mathrm{Cov}(\hat{\beta}_j, \hat{\beta}_l).$$

- Then the standard error we are looking for is the squared root of the expression above:

$$\mathrm{se}(\hat{\beta}_j - \hat{\beta}_l) = \mathrm{Var}(\hat{\beta}_j - \hat{\beta}_l)^{1/2}.$$

- Notice that we would need to know the covariance between $\hat{\beta}_j$ and $\hat{\beta}_l$, which we can obtain from the full covariance matrix of β.

Alternative way

- There is an alternative way to conduct this hypothesis test
- Instead of modifying the t statistic, we can instead modify the **model** and use an existing t statistic
- How can we achieve that?
- We know how test simple hypothesis of a kind $H_0 : \theta = 0$
- If we can somehow re-write the model so that $\theta = \beta_j - \beta_l$, then testing the hypothesis about θ would be equivalent to testing the hypothesis about the equality of β_j and β_l

Re-writing the model

- Let's re-write the equality $\theta = \beta_j - \beta_l$ as $\beta_j = \theta + \beta_l$
- Then we can re-write our population model as

$$\begin{aligned} Y &= \beta_0 + \beta_j X_j + \beta_l X_l + \ldots + U \\ &= \beta_0 + (\theta + \beta_l) X_j + \beta_l X_l + \ldots + U \\ &= \beta_0 + \theta X_j + \beta_l (X_j + X_l) + \ldots + U \end{aligned}$$

Using the new model

$$Y = \beta_0 + \theta X_j + \beta_l(X_j + X_l) + \ldots + U$$

- The resulting model is identical to the previous one, except that now we use X_j and $X_j + X_l$ as predictors (plus whatever the remaining predictors are) instead of X_j and X_l
- Estimating a model with this modified set of predictors will then allow us to easily test whether θ is statistically significant
- This test is exactly equivalent to our original test of whether β_j equals β_l

Multiple linear restrictions, F test

Exclusion restrictions

- Often we would like to test hypotheses about more than one linear restriction
- In these cases, we cannot use a clever re-writing of the original model to reduce it to a simple t test
- For example, we might be interested in testing whether more than two coefficients, say are **jointly** statistically significant.

Hypothesis

- Recall our population model, we will call it the **unrestricted** model

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_k X_k + U.$$

- Suppose we are interested in testing the following hypothesis:

$$H_0 : \beta_{k-q+1} = 0, \ldots, \beta_k = 0.$$

- The hypothesis we make imposes q **exclusion restrictions** on the coefficients
- We are interested in whether the last q coefficients are all zero

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- Taking the last coefficients is without the loss of generality, of course, because the order of variables and coefficients is arbitrary
- The alternative hypothesis is that at least of these coefficients is non-zero

How can we test it

- One possibility would be to conduct q corresponding t tests of significance
- However, recall that a single t statistic assumes that only one restriction is placed and that there are no further restrictions on other coefficients
- We need to find a way to test for all the restrictions together instead of one by one

Restricted model

- If our null hypothesis is true and the last q coefficients are in fact zero, we can write a **restricted** model as

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_{k-q} X_{k-q} + U.$$

Comparing the models

- The idea behind our joint test is based on the residual sums of squares (SSR) from the unrestricted and restricted models
- Let's call them SSR_{ur} and SSR_{r}, respectively
- We know that removing predictors from a model mechanically increases the SSR
- Thus the SSR from the restricted model will always be greater than the SSR from the unrestricted model
- The question is whether this increase in the SSR is **big enough**

F statistic

- We can answer this question formally by constructing the following F statistic
$$F \equiv \frac{(\text{SSR}_r - \text{SSR}_{ur})/q}{\text{SSR}_{ur}/(n-k-1)}.$$
- Since SSR_r is always no smaller than SSR_{ur}, the resulting test statistic will always be non-negative
- In essence, the F statistic is measuring the relative increase in the SSR when going from the unrestricted to restricted models
- However, we scale the numerator and the denominator by the appropriate degrees of freedom, which is q, the number of imposed restriction, for the numerator and $n-k-1$ for the denominator
- You may recall that the denominator is also an unbiased estimator of the variance of the error term

Distribution of the F statistic

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- The following result holds

Sampling distribution of the F statistic

Under the CLM assumptions and if H_0 holds, the F statistic follows an F distribution with $(q, n-k-1)$ degrees of freedom:

$$F \sim F_{q,n-k-1}$$

Decision rule

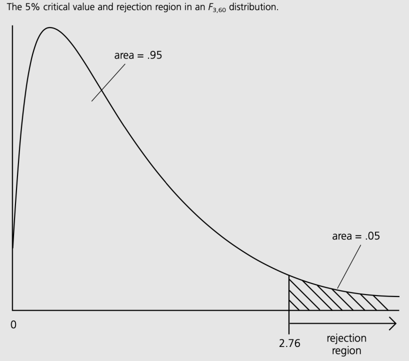
- Since the F statistic is non-negative, to reject the null hypothesis, the value of the statistic has to be **large enough**
- The general decision rule is

$$\begin{cases} F > c, & \text{reject } \mathrm{H}_0 \\ F \leqslant c, & \text{do not reject } \mathrm{H}_0. \end{cases}$$

- The critical value c will be the $1 - \alpha$ percentile of the F distribution with $(q, n - k - 1)$ df, where α is a chosen significance level

Example

- For example, for a 5% significance level, 3 restrictions and 60 *df* in the unrestricted model, the critical value is the 95th percentile of the F distribution with (3,60) *df* $(= 2.76)$
- We will reject the null hypothesis wherever the F statistic is above 2.76 and we will not reject the null wherever the F statistic is below 2.76.



Joint significance

- If we reject the null hypothesis, we would say that $\beta_{k-q+1}, \ldots, \beta_k$ are **jointly statistically significant**
- If we do not reject the null, we say that the coefficients are **jointly insignificant**
- However, we cannot tell which coefficient exactly is significant and which one is not

Applications

- Testing for joint significance can be useful when we are working with categorical predictors
- Since we create a bunch of indicator variables for those predictors, we can test whether the variable on the whole is statistically significant by conducting a joint test on the coefficients of those indicator variables

Relationship between F and t statistics

- While we motivated the F statistic as a way to test for multiple linear restrictions, nothing prevents us from setting q to 1 and testing a single restriction
- It turns out that the results of an F test in this case will be exactly the same as the result of a corresponding t test
- In fact, one can show that the F statistic in this case of a single restriction is exactly the square of the t statistic
- In other words, t^2_{n-k-1} has an $F_{1,n-k-1}$ distribution
- However, the t statistic is more flexible in this case because we can test a one-sided alternative, too.

R-squared form of F statistic

- The F statistic can be re-written using R-squared from the unrestricted and restricted models instead of the SSR
- Recall that we can write $\text{SSR} = \text{SST}(1 - R^2)$
- Plugging this expression into the formula for the F statistic and doing the substitution for the unrestricted and restricted models, we get

$$\begin{aligned}
F &\equiv \frac{(\text{SSR}_r - \text{SSR}_{ur})/q}{\text{SSR}_{ur}/(n-k-1)} \\
&= \frac{(\text{SST}(1-R_r^2) - \text{SST}(1-R_{ur}^2))/q}{\text{SST}(1-R_{ur}^2)/(n-k-1)} \\
&= \frac{(R_{ur}^2 - R_r^2)/q}{(1-R_{ur}^2)/(n-k-1)}.
\end{aligned}$$

p-value of F test

- Instead of using critical values to conduct an F test, we can also use p-values
- The p-value of an F test is defined as

$$p \equiv \mathbb{P}(\mathcal{F} > F),$$

where $\mathcal{F}$ is a random variable following an F distribution with $(q, n-k-1)$ df

Interpretation

- The interpretation of the p-value for an F test is identical to its interpretation for a t test
- It is the probability of observing the value of an F statistic at least as large as we did, given that the null hypothesis is true
- To reject the null, the p-value has to be **low enough**

Decision rule

- Our decision rule for a chosen α significance level will be

$$\begin{cases} p \leqslant \alpha, & \text{reject } H_0, \\ p > \alpha, & \text{do not reject } H_0. \end{cases}$$

- For example, suppose we calculate the p-value of our F test to be 0.024
- Then we would reject the null at a 5% significance level and would not reject it at a 1% level

Overall significance

- We can the F test for a hypothesis about the overall significance of our regression
- The null hypothesis takes the form

$$H_0 : \beta_1 = 0, \ldots, \beta_k = 0,$$

- If the null hypothesis is true than none of the predictors in our model has any effect on the outcome
- The alternative hypothesis is here is that at least one coefficient is non-zero

Conditional expectation

- Another way to think about this is that under the null, the conditional expectation of the outcome is the same as its **unconditional** expectation:

$$\mathbb{E}[Y \mid X_1, \ldots, X_K] = \mathbb{E}[Y].$$

F statistic for overall significance

- Imposing all of these k restrictions results in the following restricted model

$$Y = \beta_0 + U,$$

which includes only the constant term

- In this model, the R-squared is trivially zero, since there are no predictors
- Therefore, we can write the F statistic in the R-squared form as

$$F = \frac{R^2/k}{(1 - R^2)/(n - k - 1)},$$

where R^2 is the R-squared from the unrestricted model

General linear restrictions

- While testing for exclusion restrictions is the most common use of an F test, the test can be applied to more general restrictions
- These restrictions are often suggested by a theoretical model
- For a concrete example, suppose we have a regression model with three predictors

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + U.$$

- We hypothesize that all of the regression coefficients are zero except for β_1, which we assume equals b:

$$H_0 : \beta_1 = b, \beta_2 = 0, \beta_3 = 0.$$

Re-writing the model

- To test this hypothesis, we first write down the restricted model

$$Y = \beta_0 + bX_1 + U.$$

- We can re-write it as

$$Y - bX_1 = \beta_0 + U.$$

- The right-hand side of this expression is the same as in the restricted model in which all the coefficients are zero
- The left-hand side, however, can be thought of as a transformed outcome
- Instead of using Y, we create a new outcome variable defined as $Y - bX_1$
- Then we use this new outcome variable in the restricted model

Note

We cannot use the R-squared form of an F statistic in this case because the outcome (and hence SST) now differs between the two models. We have to use the original formula with SSR instead.

Example

Set up

- Let's return to our trade data and estimate the following model with four predictors

$$\ln(imports_i) = \beta_0 + \beta_1 \ln(gdp_i) + \beta_2 \ln(distance_i) + \beta_3 liberal_i + \beta_4 \ln(area_i) + u_i$$

- And a small model with two predictors

$$\ln(imports_i) = \beta_0 + \beta_1 \ln(gdp_i) + \beta_2 \ln(distance_i) + u_i$$

Estimation results

	full model	small model
(Intercept)	2.451	4.670
	(2.132)	(2.181)
log(gdp)	1.030	0.976
	(0.077)	(0.064)
log(distance)	-0.888	-1.075
	(0.156)	(0.157)
liberal	0.333	
	(0.207)	
log(area)	-0.159	
	(0.086)	
Num.Obs.	48	48
R2	0.908	0.886
Statistics	106.317	174.507
DF	4	2
DF Resid	43	45
p	0.000	0.000

Computations explained

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- F statistic from full model

$$F = \frac{0.908/4}{(1-0.908)/43} = 106.317$$

- F statistic from small model

$$F = \frac{0.886/2}{(1-0.886)/45} = 174.507$$

- F statistic for excluding liberal and area

$$F = \frac{(0.908-0.886)/2}{(1-0.908)/43} = 5.240$$

- p-value $= 0.009$

Next Time on *Introductory Econometrics*...

Happy Holidays!