Introductory Econometrics

Lecture 8: Variable transformations and interpretation, part 2

Alex Alekseev

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University of Regensburg, Department of Economics

Previously on *Introductory Econometrics*...

- Linear models and non-linear relationships
- Variable transformations (logarithms, winsorizing, standardizing)

Binary predictors

Binary predictors, examples

- So far we have been dealing with predictors that we might call "continuous"
- GDP, country size, education, ability
- Many predictors, however, are **binary** in nature
  - did a person get a treatment or not
  - is a person a man or a woman
  - is a person married or not
  - did a person go to college or not
- Binary variables take a central part in causal analysis since treatment variables are often binary, i.e., did a person get a treatment or not.

Modeling binary variables

- These binary variables can be included in a regression just like continuous variables
- They only take two values: 0 or 1
- Sometimes we call these variables **indicator** or **dummy** variables
- For example, if our variable is whether someone went to college or not, then values of 0 would correspond to people who did not go to college, and values of 1 would correspond to people who did go to college
- Suppose our outcome is a person's wage, then our regression would look like

$$\text{Wage} = \beta_0 + \beta_1 \text{Went to college} + U$$

Interpretation

- Start with the CEF of Y:

$$\mathbb{E}[Y \mid X] = \beta_0 + \beta_1 X.$$

- Since X takes only two possible values, 0 or 1, we can write down the CEF of Y for each possible value of X

$$\mathbb{E}[Y \mid X = 0] = \beta_0$$
$$\mathbb{E}[Y \mid X = 1] = \beta_0 + \beta_1.$$

- Take the difference to get

$$\mathbb{E}[Y \mid X = 1] - \mathbb{E}[Y \mid X = 0] = \beta_1.$$

- The slope coefficient β_1 is simply the difference between the conditional means of the outcome when the predictor is "on" ($X = 1$) and "off" ($X = 0$).

Back to college example

$$\text{Wage} = \beta_0 + \beta_1 \text{Went to college} + U.$$

- The slope coefficient would tell us by how much the wage of a person who went to college is different from the wage of a person who did not go to college
- The calculations above also tell us the meaning of the intercept term: it is the conditional mean of the outcome when the predictor is "off" ($X = 0$).
- In this example, it is the mean wage of people who did not go to college

Only one variable, not two

- When we work with binary predictors, we only include one variable for each binary predictor, not two
- For example, we do not (and cannot) estimate a model like

$$\text{Wage} = \beta_0 + \beta_1 \text{Went to college} + \beta_2 \text{Did not go to college} + U,$$

  where Went to college would equal 1 if a person went to college and 0 if they did not, and Did not go to college would equal 1 if a person did not go to college and 0 if they did
- Having those two variables together would violate the assumption of **no perfect collinearity** because
  $\text{Went to college} + \text{Did not go to college} = 1$

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  $\text{Went to college} + \text{Did not go to college} = 1$
- And 1 is our constant variable with the associated coefficient β_0

Explanation

$$\text{Wage} = \beta_0 + \beta_1 \text{Went to college} + \beta_2 \text{Did not go to college} + U$$

- Suppose that the mean wage of a person who went to college is 20 euros an hour and the mean wage of a person who did not go to college is 15 euros an hour
- Then if include both variables in the model we could get those predictions with $\beta_0 = 15, \beta_1 = 5, \beta_2 = 0$
- ...or we could get those predictions with $\beta_0 = 20, \beta_1 = 0, \beta_2 = -5$
- ...or in infinitely many other different ways
- The OLS would not be able to give us a unique result

Reference category

When we have a binary predictor we always exclude one category from the model, and this category becomes the **reference** category

Categorical predictors

Examples

- A logical extension of a binary variable is a variable that has more than two categories
- For example, a person's highest level of education could be "high school", "college", "graduate degree"
- A person's race or ethnicity could be "black", "white", "hispanic"
- A person's marital status could be "single", "married", "divorced", "in a relationship"
- These variables are called **categorical** variables

Modeling categorical predictors

- The way to include these categorical variables in a regression is to give each category of a variable its own binary variable
- Instead of having a variable "highest level of eduction" we will have several binary variables of a kind
  - "whether a person's highest level of education is high school or not"
  - "whether a person's highest level of education is college or not"
  - "whether a person's highest level of education is graduate degree or not", etc
- However, just like for binary variables, we always have to exclude one category that would become a **reference** category
- All the coefficients on the binary variables associated with a given categorical predictor then will be relative to that reference category

An example of a model

- For example, suppose our outcome is a person's wage and our predictor is the highest level of education
- The predictor can take three possible values: "high school", "college", "graduate degree"
- Our regression model would look like this

$$\text{Wage} = \beta_0 + \beta_1 \text{College} + \beta_2 \text{Graduate} + U.$$

- In this model, we exclude the "high school" category, which becomes the reference category

Interpretation

- Consider the conditional mean of the outcome for each possible level of the predictor

$$\mathbb{E}[\text{Wage} \mid \text{College} = 0, \text{Graduate} = 0] = \beta_0$$
$$\mathbb{E}[\text{Wage} \mid \text{College} = 1, \text{Graduate} = 0] = \beta_0 + \beta_1$$
$$\mathbb{E}[\text{Wage} \mid \text{College} = 0, \text{Graduate} = 1] = \beta_0 + \beta_2.$$

- The intercept term is the mean wage of a person with only a high school degree

Interpretation

- Consider the conditional mean of the outcome for each possible level of the predictor

$$\mathbb{E}[\text{Wage} \mid \text{College} = 0, \text{Graduate} = 0] = \beta_0$$
$$\mathbb{E}[\text{Wage} \mid \text{College} = 1, \text{Graduate} = 0] = \beta_0 + \beta_1$$
$$\mathbb{E}[\text{Wage} \mid \text{College} = 0, \text{Graduate} = 1] = \beta_0 + \beta_2.$$

- Taking the difference between the second and first expression, we get

$$\mathbb{E}[\text{Wage} \mid \text{College} = 1, \text{Graduate} = 0] - \mathbb{E}[\text{Wage} \mid \text{College} = 0, \text{Graduate} = 0] = \beta_1$$

- Therefore, β_1 is the difference between mean wages of a person who went to college (but not to graduate school) and a person with only a high school degree

Interpretation

- Consider the conditional mean of the outcome for each possible level of the predictor

$$\mathbb{E}[\text{Wage} \mid \text{College} = 0, \text{Graduate} = 0] = \beta_0$$
$$\mathbb{E}[\text{Wage} \mid \text{College} = 1, \text{Graduate} = 0] = \beta_0 + \beta_1$$
$$\mathbb{E}[\text{Wage} \mid \text{College} = 0, \text{Graduate} = 1] = \beta_0 + \beta_2.$$

- Taking the difference between the third and first expression, we get $\mathbb{E}[\text{Wage} \mid \text{College} = 0, \text{Graduate} = 1] - \mathbb{E}[\text{Wage} \mid \text{College} = 0, \text{Graduate} = 0] = \beta_2.$
- Therefore, β_2 is the difference between mean wages of a person who went to a graduate school and a person with only a high school degree.

Reference category

- A reference category can be **anything**
- You can pick whatever category makes sense for your research question
- But then all of the slope coefficients will be interpreted relative to that reference category
- For example, if in our example we make "college" the reference category, all our coefficients will change and we will interpret them relative to college

Polynomials

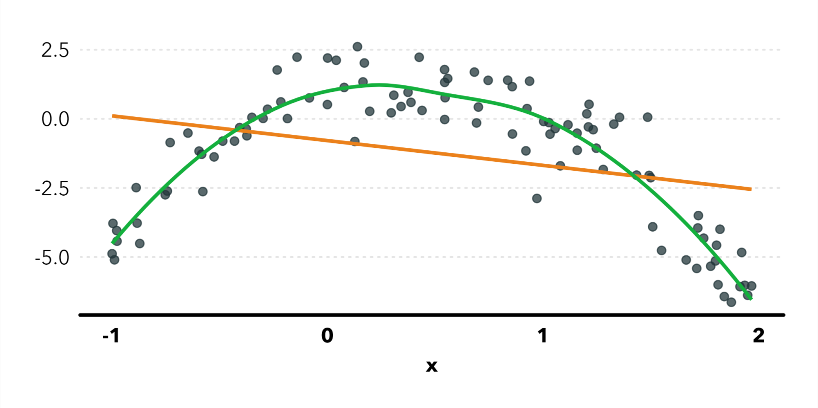
Example

- We have seen that by using the logarithmic transformation we can model non-linear relationships between variables
- An even more flexible approach is to use **polynomials**
- An example of a regression that includes polynomials is

$$Y = \beta_0 + \beta_1 X + \beta_2 X^2 + \beta_3 X^3 + U.$$

- This regression includes the third degree polynomial of X
- Each of the three terms is usually referred to as a linear term (X), a quadratic term (X^2), and a cubic term (X^3)
- When using polynomials of a given degree, we include **all** of the terms up to that degree, without skipping
- In economics we usually use either second- or third-degree polynomials, but not higher

Linear and quadratic fits



Interpretation, challenges

- Polynomials allow us to model very flexible relationships
- However, the coefficients become trickier to interpret
- Previously the interpretation of each individual slope coefficients in a multiple regression was as a **partial** effect
- That is, the effect on the outcome of changing one variable while keeping the rest constant
- In case of polynomials, this interpretation is no longer possible
- If we change X we cannot keep X^2 constant

Interpretation

- Recall that an alternative derivation of the slope coefficient was to take a partial derivative of the conditional mean of an outcome with respect to the corresponding predictor
- We can use this logic with polynomials
- For example, for our third-degree polynomial the conditional mean of Y is
$$\mathbb{E}[Y \mid X] = \beta_0 + \beta_1 X + \beta_2 X^2 + \beta_3 X^3$$
- and the derivative of the conditional mean of Y with respect to X is
$$\frac{d\mathbb{E}[Y \mid X]}{dX} = \beta_1 + 2\beta_2 X + 3\beta_3 X^2$$

Non-constant effect

$$\frac{d\mathbb{E}[Y \mid X]}{dX} = \beta_1 + 2\beta_2 X + 3\beta_3 X^2$$

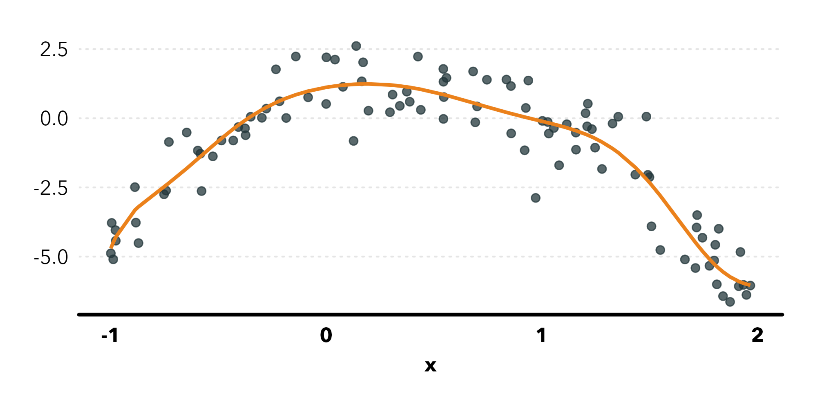
- Now the effect of X on the mean outcome depends on the value of X, it is not constant
- For example, when $X = 0$, increasing X by a small amount changes the conditional mean of Y by β_1 units
- However, when $X = 1$, increasing X by a small amount changes the conditional mean of Y by $\beta_1 + 2\beta_2 + 3\beta_3$ units
- Each individual coefficient has little meaning on its own, they have to be interpreted together

Order of polynomials

- How can we determine the order of polynomials?
- Why do we have to stop at the cubic term?
- Why not take higher terms?
- There are a couple of reasons why in economics we typically stop at the quadratic term or at the cubic term
- First, a model with too many polynomial terms becomes **harder to interpret**
- The second-order polynomial can have a nice interpretation of a decreasing marginal effect of a predictor, for example, when the linear term has a positive coefficient but a quadratic term has a negative coefficient
- But adding a cubic term makes the results much harder to interpret

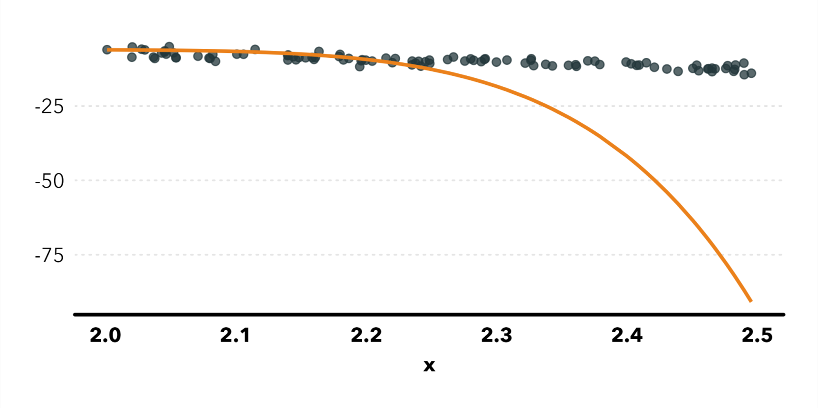
Overfitting

- Second, even if we do not care too much about interpretation having too many polynomial terms can lead to **overfitting**
- In the picture below we fit a 10th degree polynomial when the true relationship is quadratic



Problems with overfitting

- If we try to use this overly complex model on a new data set generated using the same underlying process where the true relationship is quadratic, we will get some bizarre predictions



Interaction terms

Example

- Interaction terms let the effect of a predictor depend on the value of **another** predictor
- For example, let's say we are interested in the effect of years of schooling on wage
- Does this effect depend on a person's gender or race?
- Modeling an interaction effect would allow us to answer this question
- An example of a model with an interaction term is

$$Y = \beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ + U$$

- Here we have two main effects, X and Z, and an interaction term XZ, which is simply a product of the two variables

Main and interaction effects

When we work with interactions, we **always include** the main effects in addition to the interaction terms

Polynomials and interactions

Notice that polynomial terms can be thought of as interacting a variable with itself: $X^2 = X \times X$

Interpretation, challenges

- Including an interaction term in a model complicates the interpretation of individual coefficients
- There are two related questions we have to answer
    1. What is the effect of X when the model includes an interaction term?
    2. What is the interpretation of the interaction term itself?

Effect of X

- We can no longer interpret β_1 as the partial effect of X because changing X also changes XZ
- The partial derivative of the conditional mean of Y with respect to X

$$\frac{\partial \mathbb{E}[Y \mid X, Z]}{\partial X} = \beta_1 + \beta_3 Z$$

- Now the partial effect of X depends not only on β_1 but also on the value of Z
- For example, when $Z = 0$, the partial effect of X on the conditional mean of Y is β_1
- But when $Z = 1$ the partial effect of X on the conditional mean of Y is $\beta_1 + \beta_3$, and so on
- Therefore, individual slope coefficients have little meaning by themselves, we have to interpret them together with other coefficients

Interaction term

- What is then the interpretation of the interaction term?
- One way to deal with it is to take the cross-partial derivative of the conditional mean of Y to get

$$\frac{\partial^2 \mathbb{E}[Y \mid X, Z]}{\partial X \partial Z} = \beta_3$$

- Recall that the partial effect of X on the conditional mean of Y is

$$\frac{\partial \mathbb{E}[Y \mid X, Z]}{\partial X} = \beta_1 + \beta_3 Z$$

- Then we can interpret β_3 as the change in the partial effect of X when Z increases by one unit

Interaction term

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Symmetry

The interaction effect is **symmetric**. It also tells us by how much the partial effect of Z changes when X increases by one unit.

Interactions with categorical predictors

- What if one of the variables is binary or categorical?
- This would be the case in our opening examples of the effect of schooling on wage
- Suppose we want to let this effect vary by gender
- Our model would look like

$$\text{Wage} = \beta_0 + \beta_1 \text{Education} + \beta_2 \text{Male} + \beta_3 \text{Education} \times \text{Male} + U,$$

where Education is years of education and Male equals 1 if a person is male and 0 if not

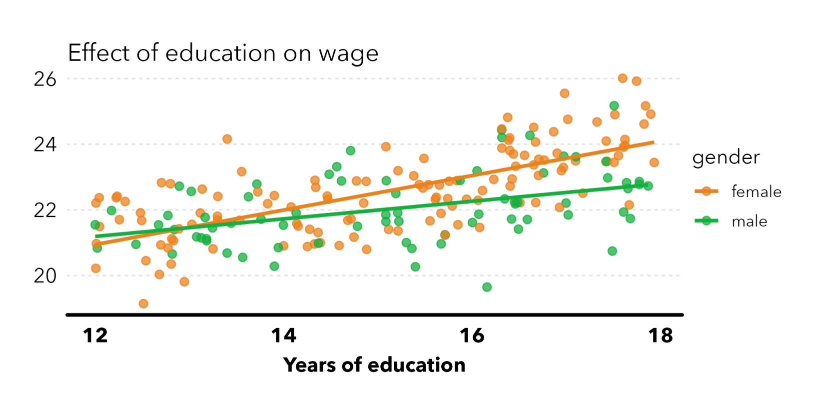
Effect of education

- Then the partial effect of education on wage is

$$\frac{\partial \mathbb{E}[\text{Wage} \mid \text{Education}, \text{Male}]}{\partial \text{Education}} = \beta_1 + \beta_3 \text{Male}$$

- For males, the effect of schooling on wage is $\beta_1 + \beta_3$
- For females the effect of schooling on wage is β_1
- And β_3 is the difference between the effect of schooling on wage for males vs. females

Illustration



Higher-order interactions

- Interactions are not limited to two-way interactions, i.e., products between only two variables
- You can have three-way interactions and higher, too
- For example

$$
\begin{aligned}
Y = {} & \beta_0 + \beta_1 X + \beta_2 Y + \beta_3 Z \\
& + \beta_4 XY + \beta_5 XZ + \beta_5 YZ \\
& + \beta_6 XYZ
\end{aligned}
$$

- If we include, say, a three-way interaction term in a model (XYZ above) we typically **include all** the lower interaction terms, as well as the main effects

Estimation of interactions

- Interactions are a powerful modeling tool
- However, it is difficult to estimate the interaction effects in practice
- You need a lot of observations to precisely estimate them

Next Time on *Introductory Econometrics*...

Hypotheses testing