Introductory Econometrics

Lecture 11: Multiple Regression Analysis - OLS Asymptotics

Alex Alekseev

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University of Regensburg, Department of Economics

Recap

Model and assumptions

- Let's recall our population model

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_k X_k + U.$$

- We made several assumptions about it
    1. Linear CEF
        1.1 Linear model
        1.2 Error term is mean-independent of predictors
    2. Random Sampling
    3. No Perfect Collinearity
    4. Homoskedasticity
    5. Normality

Assumptions and OLS properties

1. Linear CEF
    1.1 Linear model
    1.2 Error term is mean-independent of predictors
2. Random Sampling
3. No Perfect Collinearity
4. Homoskedasticity
5. Normality

- We call the first four (or five if you break down the first assumption into two separate ones) assumptions the **Gauss-Markov** assumptions
- We call the full set of assumptions the **classical linear model** assumptions

Assumptions and OLS properties

1. Linear CEF
   1.1 Linear model
   1.2 Error term is mean-independent of predictors
2. Random Sampling
3. No Perfect Collinearity
4. Homoskedasticity
5. Normality

- We showed that under the first three assumptions the OLS estimator is **unbiased**
- We showed that under the full set of Gauss-Markov assumptions the OLS estimator is the best linear unbiased estimator (the **Gauss-Markov theorem**)

Assumptions and OLS properties

1. Linear CEF
    1.1 Linear model
    1.2 Error term is mean-independent of predictors
2. Random Sampling
3. No Perfect Collinearity
4. Homoskedasticity
5. Normality

- Once we added the normality assumption, we derived the **exact sampling distribution** of the OLS estimator, as well as the exact distributions of the various test statistics
- This opened the way for **hypotheses testing**

Finite sample properties

- These properties of the OLS estimator are often called **finite sample** or **small sample** or **exact** properties
- In this lecture, we will study the **asymptotic** (or **large sample**) properties of the OLS estimator and test statistics
- The asymptotic properties are not defined for a particular sample size, they are defined as the sample size **grows without bound**
- The main takeaway from this analysis will be that even without the normality assumption, the test statistics that we discussed (t and F) will **approximately** follow their corresponding distributions

Consistency

A must-have property

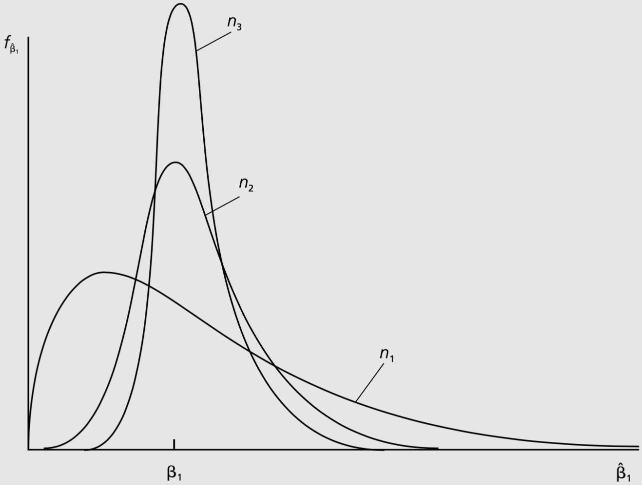
- Recall that under our assumptions, the OLS estimator is unbiased
- Unbiasedness is a nice property to have, although it cannot always be achieved
- On the other hand, there is another property that an estimator **must have** to be useful
- This property is called **consistency**
- The famous econometrician Clive W.J. Granger once remarked: "If you can't get it right as n goes to infinity, you shouldn't be in this business."
- The idea is that if an estimator is not consistent, it is unlikely to be useful
- But what is consistency?

Intuition behind consistency

- Suppose we have an OLS estimator, $\hat{\beta}_j$ of some population coefficient β_j
- For each sample size n, $\hat{\beta}_j$ has a probability distribution
- Because $\hat{\beta}_j$, is unbiased under our assumptions, this distribution has mean β_j
- **Consistency** means that the distribution of $\hat{\beta}_j$ becomes more and more tightly distributed around β_j as the sample size grows
- As n goes to infinity, the distribution of $\hat{\beta}_j$ collapses to the single point β_j
- In other words, if an estimator is consistent, we can make our estimator arbitrarily close to β_j if we can collect as much data as we want

Illustration

Sampling distributions of $\hat{\beta}_1$ for sample sizes $n_1 < n_2 < n_3$.



Formal definition

- Let $\hat{\theta}_n$, be an estimator of some population parameter θ based on a sample of size n
- Then $\hat{\theta}_n$ is a **consistent estimator** of θ, if for every $\epsilon > 0$

$$\lim_{n \to \infty} \mathbb{P}(|\hat{\theta}_n - \theta| > \epsilon) = 0$$

- When $\hat{\theta}_n$ is consistent, we also say that θ is the **probability limit** of $\hat{\theta}_n$, written as

$$\text{plim } \hat{\theta}_n = \theta$$

Unbiasedness and consistency

- Unbiasedness and consistency are related
- Unbiased estimators whose **variances shrink to zero** as the sample size grows are consistent
- Formally, if $\hat{\theta}_n$ is an unbiased estimator of θ and $\text{Var}(\hat{\theta}_n) \to 0$ as $n \to \infty$, then plim $\hat{\theta}_n = \theta$

Consistency of OLS

Theorem: Consistency of OLS

Under assumptions 1-3, the OLS estimator $\hat{\beta}_j$ is consistent for β_j, for all $j = 0, \ldots, k$

Proof

- It is easy to show consistency using the relationship between unbiasedness and consistency
- We know that the OLS estimator is unbiased
- We also know that the variance of each $\hat{\beta}_j$ is

$$\mathrm{Var}(\hat{\beta}_j \mid \mathbf{X}) = \frac{\sigma^2}{SST_j(1 - R_j^2)}.$$

- Notice that $SST_j \equiv \sum_{i=1}^{n}(x_{ij} - \bar{x}_j)^2 = n\mathrm{Var}(X_j)$
- Therefore,

$$\mathrm{Var}(\hat{\beta}_j \mid \mathbf{X}) = \frac{\sigma^2}{n\mathrm{Var}(X_j)(1 - R_j^2)}.$$

Proof

$$\mathrm{Var}(\hat{\beta}_j \mid \mathbf{X}) = \frac{\sigma^2}{n\mathrm{Var}(X_j)(1 - R_j^2)}.$$

- Recall that the assumption of no perfect collinearity implies that $1 - R_j^2$ is non-zero
- We also need to assume that $\mathrm{Var}(X_j)$ is finite
- The assumption of no perfect collinearity also guarantee that it is non-zero
- It is now easy to see that as n grows, the variance shrinks to zero, which in turn implies that $\hat{\beta}_j$ is consistent for β_j

Another proof

- We can also show consistency of the OLS more directly
- For example, recall that in the simple linear regression the slope coefficient is

$$\hat{\beta}_1 = \frac{\widehat{Cov}(Y, X)}{\widehat{Var}(X)}.$$

- Using the **law of large numbers**, we can re-write this expression as

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The law of large numbers

Let $Y_1, Y_2, \ldots, Y_n$ be independent, identically distributed random variables with mean μ. Then,

$$\text{plim } \bar{Y}_n = \mu$$

Another proof

- Now substitute for Y

$$\text{plim } \hat{\beta}_1 = \frac{\text{Cov}(\beta_0 + \beta_1 X + U, X)}{\text{Var}(X)}.$$

- We have that $\text{Cov}(\beta_0, X) = 0$, $\text{Cov}(\beta_1 X, X) = \beta_1 \text{Var}(X)$, therefore

$$\text{plim } \hat{\beta}_1 = \beta_1 + \frac{\text{Cov}(X, U)}{\text{Var}(X)}$$

- Since $\text{Cov}(X, U) = 0$ by our first assumption, $\text{plim } \hat{\beta}_1 = \beta_1$

Inconsistency of the OLS

Failure of assumptions

- Just as failure of the first assumption causes bias in the OLS estimators, correlation between the error term and any of the predictors generally causes **all** of the OLS estimators to be inconsistent
- The bias will persist even as the **sample size grows**

Deriving the inconsistency

- In the simple regression case, we can obtain the inconsistency of the OLS from the derivations we made before
- The inconsistency (sometimes called the **asymptotic bias**) is

$$\text{plim } \hat{\beta}_1 - \beta_1 = \frac{\text{Cov}(X, U)}{\text{Var}(X)}.$$

- Because $\text{Var}(X) > 0$, the inconsistency in $\hat{\beta}_1$ is positive if X and U are positively correlated, and the inconsistency is negative if X and U are negatively correlated
- If the covariance between X and U is small relative to the variance in X, the inconsistency can be negligible
- However, we cannot estimate how big the covariance is because U is unobserved

Omitted variables bias

- We can use this formula to derive the asymptotic analogue of the omitted variables bias. For example, suppose the true population model is
$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + V.$$
- Now suppose that we instead estimate a model without X_2
$$Y = \beta_0 + \beta_1 X_1 + U,$$
where $U \equiv V + \beta_2 X_2$

Omitted variables bias

$$Y = \beta_0 + \beta_1 X_1 + U$$

- Then the inconsistency in the OLS estimator of $\hat{\beta}_1$ from the incorrect model is

$$\begin{aligned}
\text{plim } \tilde{\beta}_1 &= \beta_1 + \frac{\text{Cov}(X_1, U)}{\text{Var}(X_1)} = \beta_1 + \frac{\text{Cov}(X_1, V + \beta_2 X_2)}{\text{Var}(X_1)} \\
&= \beta_1 + \beta_2 \frac{\text{Cov}(X_1, X_2)}{\text{Var}(X_1)} \\
&= \beta_1 + \beta_2 \rho_2,
\end{aligned}$$

where $\rho_2 \equiv \frac{\text{Cov}(X_1, X_2)}{\text{Var}(X_1)}$.

Inconsistency is (almost) bias

- Thus, for practical purposes, we can view the **inconsistency** as being the same as the **bias**
- The difference is that the inconsistency is expressed in terms of the **population** variance of X_1 and the population covariance between X_1 and X_2, while the bias is based on their **sample** counterparts

Signing the inconsistency

$$\text{plim } \tilde{\beta}_1 = \beta_1 + \beta_2 \frac{\text{Cov}(X_1, X_2)}{\text{Var}(X_1)}$$

- If X_1 and X_2 are uncorrelated (in the population), then $\tilde{\beta}_1$ is a consistent estimator of β_1 (although not necessarily unbiased)
- If X_2 has a positive partial effect on Y ($\beta_2 > 0$), and X_1 and X_2 are positively correlated ($\rho_2 > 0$), then the inconsistency is positive, and so on
- If the covariance between X_1 and X_2 is small relative to the variance of X_1 the inconsistency can be small

Asymptotic normality

Sampling distribution

- Consistency of an estimator is an important property, but consistency by itself does not allow us to perform **statistical inference**
- For testing, we need the **sampling distribution** of the OLS estimator
- Under the classical linear model assumptions, we showed that the sampling distribution of the OLS is **normal**
- We then used this result to derive the distributions of the t and F statistics

Normality assumption

- The exact normality of the OLS estimator hinges crucially on the normality of the of the error
- But normality is a **strong assumption**
- If it fails, the the OLS will not be normally distributed, which means that the t statistics will not have t distributions and the F statistics will not have F distributions
- This is a potentially serious problem because our inference relies on obtaining the critical values of p values from those specific distributions
- Notice, however, that normality will not affect the **unbiasedness** or **consistency** of OLS, nor does it affect the conclusion that OLS is the best linear unbiased estimator

Non-normal outcome variables

- Recall that the normality assumption is equivalent to saying that the distribution of an **outcome given predictors** is normal
- Since the outcome is **observed** (unlike the error term), it is much easier to think about whether the distribution of the outcome is likely to be normal
- A normally distributed random variable is symmetrically distributed about its mean, it can take on any positive or negative value (but with zero probability), and more than 95% of the area under the distribution is within two standard deviations
- Many outcome variables **do not** have these properties
- Does this mean that we cannot use the t and F tests for models with these outcome variables?

Theorem: Asymptotic normality of OLS

Under the Gauss-Markov assumptions,

1. $\sqrt{n}(\hat{\beta}_j - \beta_j) \dot{\sim} \mathcal{N}(0, \sigma^2/a_j^2)$, where $a_j^2 \equiv \text{plim } \sum_{i=1}^{n} \hat{u}_{ij}^2/n = \text{plim } SSR_j/n$, and $\hat{u}_{ij}$ are the residuals from the regression of X_j on all other predictors
2. $\hat{\sigma}^2$ is a consistent estimator of the $\text{Var}(U)$
3. For each j,
$$\frac{\hat{\beta}_j - \beta_j}{\text{se}(\hat{\beta}_j)} \dot{\sim} \mathcal{N}(0, 1).$$

Asymptotic normality

Let $Z_1, Z_2, \ldots, Z_n$ be a sequence of random variables, such that for all numbers z

$$\mathbb{P}(Z_n \leqslant z) \to \Phi(z) \text{ as } n \to \infty,$$

where $\Phi(z)$ is the standard normal CDF. Then Z_n is said to have an **asymptotic standard normal distribution**. We could write this as

$$Z_n \dot{\sim} \mathcal{N}(0, 1)$$

Theorem explained

- The fact that the expected value of $\sqrt{n}(\hat{\beta}_j - \beta_j)$ is zero follows from the unbiasedness of the OLS
- The asymptotic variance follows from the formula for the variance of $\hat{\beta}_j$:

$$\mathrm{Var}(\hat{\beta}_j) = \frac{\sigma^2}{SST_j(1-R_j^2)} = \frac{\sigma^2}{SST_j \frac{SSR_j}{SST_j}} = \frac{\sigma^2}{SSR_j}.$$

- Then

$$\mathrm{Var}(\sqrt{n}(\hat{\beta}_j - \beta_j)) = n\mathrm{Var}(\hat{\beta}_j) = \frac{\sigma^2}{SSR_j/n}.$$

Theorem explained

- The proof of the asymptotic normality is somewhat involved and is omitted
- It is based on the **central limit theorem**

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Central limit theorem

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample with mean μ and variance σ^2, then

$$Z_n \equiv \frac{\bar{Y}_n - \mu}{\sigma/n}$$

has an asymptotic standard normal distribution

Theorem explained

- The proof of the asymptotic normality is somewhat involved and is omitted
- It is based on the **central limit theorem**
- The theorem above is useful because the normality assumption has been dropped
- The only restriction on the distribution of the error is that it has finite variance

Standard normal distribution

$$\frac{\hat{\beta}_j - \beta_j}{\text{se}(\hat{\beta}_j)} \dot{\sim} \mathcal{N}(0, 1)$$

- Notice how the **standard normal distribution** appears in the theorem, as opposed to the t_{n-k-1} distribution as before
- This is because the distribution is only **approximate**
- By contrast before the distribution of the ratio in was exactly t_{n-k-1} for any sample size
- In fact, it is just as legitimate to write

$$\frac{\hat{\beta}_j - \beta_j}{\text{se}(\hat{\beta}_j)} \dot{\sim} t_{n-k-1}$$

  since t_{n-k-1} approaches the standard normal distribution as the degrees of freedom gets large

Hypotheses testing

- This result tells us that t testing and the construction of confidence intervals are carried out **exactly** as under the classical linear model assumptions
- If the sample size is not very large, then the t distribution can be a poor approximation to the distribution of the t statistics when U is not normally distributed
- Unfortunately, there are no general prescriptions on **how big** the sample size must be before the approximation is good enough
- It is important to realize that the theorem above does require the **homoskedasticity** assumption
- Without it, the usual t statistics and confidence intervals are invalid no matter how large the sample size is

F tests

- The asymptotic normality of the OLS estimators also implies that the F statistics have **approximate** F distributions in large samples
- Thus, for testing exclusion restrictions or other multiple hypotheses, nothing changes from what we have done before