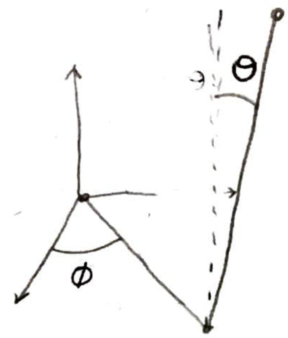

Pendulum: m_p, J_p, L_p, B_p

Arm: J_r, L_r, B_r

Using Lagrange Method,

$$L = T - V$$

$$V = m_p\left(\frac{L_p}{2}\right)g\left(\cos\theta - 1\right)$$

To find T, we use position of the pendulum center of mass and take time derivative:

$$T = \left(\frac{1}{2}J_r\right)(\dot{\phi})^2 + \left(\frac{1}{2}J_p\right)(\dot{\theta})^2 + \frac{1}{2}m_p\left(\left[\frac{d}{dt}\left(L_r\sin\phi + \frac{L_p}{2}\sin\theta\cos\phi\right)\right]^2 + \left[\frac{d}{dt}\left(L_r\cos\phi - \frac{L_p}{2}\sin\theta\sin\phi\right)\right]^2 + \left[\frac{d}{dt}\left(\frac{L_p}{2}\cos\theta\right)\right]^2\right)$$

We look at the velocity term:

$$\frac{1}{2}m_p\Bigg(\left[L_r\cos\phi(\dot{\phi}) + \frac{L_p}{2}\left(\cos\theta\cos\phi(\dot{\theta}) - \sin\theta\sin\phi(\dot{\phi})\right)\right]^2 +$$

$$\left[-L_r\sin\phi(\dot{\phi}) - \frac{L_p}{2}\left(\cos\theta\sin\phi(\dot{\theta}) + \sin\theta\cos\phi(\dot{\phi})\right)\right]^2 +$$

$$\left[-\frac{L_p}{2}\sin\theta(\dot{\theta})\right]^2$$

Now we square:

$$\frac{1}{2}m_p\left(L_r^2\cos^2\phi(\dot\phi)^2+2\left(\frac{1}{2}L_rL_p\cos^2\phi\cos\theta(\dot\theta)(\dot\phi)\right)\right.$$

$$+2\left(\frac{-1}{2}L_rL_p\cos\phi\sin\phi\sin\theta(\dot\phi)^2\right)+2\left(\frac{-L_p^2}{4}\cos\theta\cos\phi\sin\theta\sin\phi(\dot\theta)(\dot\phi)\right)$$

$$+\frac{L_p^2}{4}\cos^2\theta\cos^2\phi(\dot\theta)^2+\frac{L_p^2}{4}\sin^2\theta\sin^2\phi(\dot\phi)^2+L_r^2\sin^2\phi(\dot\phi)^2$$

$$+2\left(\frac{1}{2}L_rL_p\sin^2\phi\cos\theta(\dot\phi)(\dot\theta)\right)+2\left(\frac{1}{2}L_rL_p\sin\theta\sin\phi\cos\phi(\dot\phi)^2\right)$$

$$+2\left(\frac{L_p^2}{4}\cos\theta\cos\phi\sin\theta\sin\phi(\dot\theta)(\dot\phi)\right)+\frac{L_p^2}{4}\cos^2\theta\sin^2\phi(\dot\theta)^2$$

$$+\frac{L_p^2}{4}\sin^2\theta\cos^2\phi(\dot\phi)^2+\frac{L_p^2}{4}\sin^2\theta(\dot\theta)^2$$

$$=\frac{1}{2}m_p\left(L_r^2(\dot\phi)^2+L_rL_p\cos\theta(\dot\phi)(\dot\theta)+\frac{L_p^2}{4}\cos^2\theta(\dot\theta)^2\right.$$

$$\left.+\frac{L_p^2}{4}\sin^2\theta(\dot\phi)^2+\frac{L_p^2}{4}\sin^2\theta(\dot\theta)^2\right)$$

$$=\frac{1}{2}m_p\left(L_r^2(\dot\phi)^2+L_rL_p\cos\theta(\dot\phi)(\dot\theta)+\frac{L_p^2}{4}(\dot\theta)^2+\frac{L_p^2}{4}\sin^2\theta(\dot\phi)^2\right)$$

Bringing back,

$$T=(\dot\phi)^2\left(\frac{1}{2}J_r+\frac{1}{2}m_pL_r^2+\frac{1}{8}m_pL_p^2\sin^2\theta\right)+\frac{1}{2}m_pL_rL_p\cos\theta(\dot\phi)(\dot\theta)$$
$$+(\dot\theta)^2\left(\frac{1}{8}m_pL_p^2+\frac{1}{2}J_p\right)$$

So $L=T-V$

$$L=(\dot\phi)^2\left(\frac{1}{2}J_r+\frac{1}{2}m_pL_r^2+\frac{1}{8}m_pL_p^2\sin^2\theta\right)+\frac{1}{2}m_pL_rL_p\cos\theta(\dot\phi)(\dot\theta)$$
$$+(\dot\theta)^2\left(\frac{1}{8}m_pL_p^2+\frac{1}{2}J_p\right)+\frac{1}{2}m_pL_pg(1-\cos\theta)$$

Now we can calculate $\frac{\partial L}{\partial \dot{\phi}}, \frac{\partial L}{\partial \phi}, \frac{\partial L}{\partial \dot{\theta}}, \frac{\partial L}{\partial \theta}$

$$\frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}\left(J_r + m_p L_r^2 + \frac{1}{4} m_p L_p^2 \sin^2\theta\right) + \frac{1}{2} m_p L_r L_p \cos\theta (\dot{\theta})$$

$$\frac{\partial L}{\partial \phi} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m_p L_r L_p \cos\theta (\dot{\phi}) + \dot{\theta}\left(\frac{1}{4} m_p L_p^2 + J_p\right)$$

$$\frac{\partial L}{\partial \theta} = (\dot{\phi})^2\left(\frac{1}{4} m_p L_p^2 (\sin\theta \cos\theta)\right) - \frac{1}{2} m_p L_r L_p \sin\theta (\dot{\phi})(\dot{\theta})$$
$$+ \frac{1}{2} m_p L_p g \sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \ddot{\phi}\left(J_r + m_p L_r^2\right) + \frac{1}{4} m_p L_p^2\left(\ddot{\phi} \sin^2\theta + \dot{\phi}(2\sin\theta\cos\theta\,\dot{\theta})\right)$$
$$+ \frac{1}{2} m_p L_r L_p\left(-\sin\theta (\dot{\theta})^2 + \cos\theta (\ddot{\theta})\right)$$

Equations of Motion:

$$\boxed{\tau - B\dot{\phi} = \ddot{\phi}\left(J_r + m_p L_r^2 + \frac{1}{4} m_p L_p^2 \sin^2\theta\right) + \frac{1}{2} m_p L_p^2 \dot{\theta}\dot{\phi} \sin\theta\cos\theta - (\dot{\theta})^2\left(\frac{1}{2} m_p L_r L_p \sin\theta\right) + \ddot{\theta}\left(\frac{1}{2} m_p L_r L_p \cos\theta\right)}$$

$$\frac{\partial}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{1}{2} m_p L_r L_p\left(\ddot{\phi}\cos\theta - (\dot{\phi})(\dot{\theta})\sin\theta\right) + \ddot{\theta}\left(\frac{1}{4} m_p L_p^2 + J_p\right)$$

$$-B\dot{\theta} = \ddot{\theta}\left(\frac{1}{4} m_p L_p^2 + J_p\right) + \ddot{\phi}\frac{1}{2} m_p L_r L_p \cos\theta - \dot{\theta}\dot{\phi}\frac{1}{2} m_p L_r L_p \sin\theta$$
$$-(\dot{\phi})^2 \frac{1}{4} m_p L_p^2 \sin\theta\cos\theta + \frac{1}{2}\dot{\theta}\dot{\phi} m_p L_r L_p \sin\theta - \frac{1}{2} m_p L_p g \sin\theta$$

$$\boxed{-B\dot{\theta} = \ddot{\theta}\left(\frac{1}{4} m_p L_p^2 + J_p\right) + \ddot{\phi}\left(\frac{1}{2} m_p L_r L_p \cos\theta\right) - (\dot{\phi})^2 \frac{1}{4} m_p L_p^2 \sin\theta\cos\theta - \frac{1}{2} m_p L_p g \sin\theta}$$