Final Project:
Euler Totient Function and Sturmian Words

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1 Introduction

This final paper is focused on exploring a conjecture that we are currently working to find a proof. The conjecture is that the number of lists of all balanced finite Sturmian words of length n, where n is a natural number, that are generated over the binary alphabet converge to the sum of Euler Totient function. In this paper we will explore the concepts of the Euler Totient function, Sturmian words, and the relationship between the two. Then, we will define the conjecture, along with related terminology, and explore visualizations using Mathematica.

2 Background

2.1 Euler Totient Function

The Euler Totient function, also commonly known as Euler's Phi, defines a function of a positive integer n, where Euler Phi of n counts the number of positive integers up to n that are relatively prime to that n (3). Note that the first 100 terms of the Euler Phi function are listed in Figure 1.

This beautifully simple function has been found to be generated in several ways, as it is a powerful function that's pattern defines several things in mathematics, some that are not even expected. For example, the sum of Euler Phi defines the number of complex numbers satisfying $x^1 = 1, x^2 = 1, x^3 = 1, \ldots, x^n = 1$ [3]. I decided to explore this function more along with some of the different patterns the sum of Euler Phi sequence generates.

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xaminemus, valores huius characteris πD pro omnibus numeris centenario non maioribus apponamus:

$\pi\ 1 = 0$	$\pi 21 = 12$	$\pi 41 = 40$	$\pi 61 = 60$	$\pi\ 81 = 54$
$\pi\ 2 = 1$	$\pi 22 = 10$	$\pi 42 = 12$	$\pi 62 = 30$	$\pi\ 82 = 40$
$\pi\ 3 = 2$	$\pi 23 = 22$	$\pi 43 = 42$	$\pi 63 = 36$	$\pi\ 83 = 82$
$\pi\ 4 = 2$	$\pi 24 = 8$	$\pi 44 = 20$	$\pi 64 = 32$	$\pi\ 84 = 24$
$\pi\ 5 = 4$	$\pi 25 = 20$	$\pi 45 = 24$	$\pi 65 = 48$	$\pi\ 85 = 64$
$\pi\ 6 = 2$	$\pi 26 = 12$	$\pi 46 = 22$	$\pi 66 = 20$	$\pi\ 86 = 42$
$\pi\ 7 = 6$	$\pi 27 = 18$	$\pi 47 = 46$	$\pi 67 = 66$	$\pi\ 87 = 56$
$\pi\ 8 = 4$	$\pi 28 = 12$	$\pi 48 = 16$	$\pi 68 = 32$	$\pi\ 88 = 40$
$\pi\ 9 = 6$	$\pi 29 = 28$	$\pi 49 = 42$	$\pi 69 = 44$	$\pi\ 89 = 88$
$\pi 10 = 4$	$\pi 30 = 8$	$\pi 50 = 20$	$\pi 70 = 24$	$\pi\ 90 = 24$
$\pi 11 = 10$	$\pi 31 = 30$	$\pi 51 = 32$	$\pi 71 = 70$	$\pi\ 91 = 72$
$\pi 12 = 4$	$\pi 32 = 16$	$\pi 52 = 24$	$\pi 72 = 24$	$\pi\ 92 = 44$
$\pi 13 = 12$	$\pi 33 = 20$	$\pi 53 = 52$	$\pi 73 = 72$	$\pi\ 93 = 60$
$\pi 14 = 6$	$\pi 34 = 16$	$\pi 54 = 18$	$\pi 74 = 36$	$\pi\ 94 = 46$
$\pi 15 = 8$	$\pi 35 = 24$	$\pi 55 = 40$	$\pi 75 = 40$	$\pi\ 95 = 72$
$\pi 16 = 8$	$\pi 36 = 12$	$\pi 56 = 24$	$\pi 76 = 36$	$\pi\ 96 = 32$
$\pi 17 = 16$	$\pi 37 = 36$	$\pi 57 = 36$	$\pi 77 = 60$	$\pi\ 97 = 96$
$\pi 18 = 6$	$\pi 38 = 18$	$\pi 58 = 28$	$\pi 78 = 24$	$\pi\ 98 = 42$
$\pi 19 = 18$	$\pi 39 = 24$	$\pi 59 = 58$	$\pi 79 = 78$	$\pi\ 99 = 60$
$\pi 20 = 8$	$\pi 40 = 16$	$\pi 60 = 16$	$\pi 80 = 32$	$\pi 100 = 40$

Figure 1: This is a list of the first 100 terms of the Euler Totient Function. This specific list is the first time the terms of the Totient function were ever seen, as this table was taken from an article published by Leonhard Euler in 1780, who was the first person to define the Euler Totient Function [2]. Euler uses the notation $\pi(n)$ to define the terms of the function.

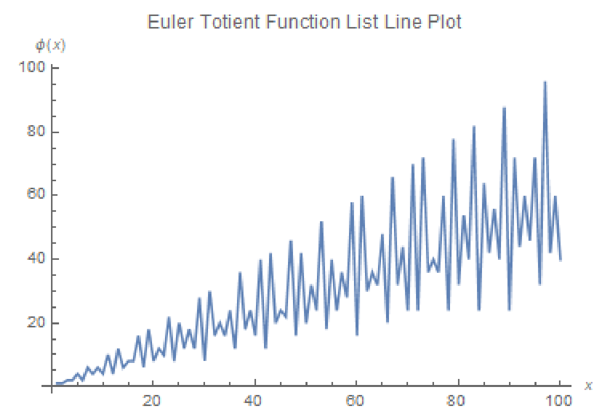

Figure 2: This is a list line plot of the first 100 terms of the Euler Totient Function.

2.2 Sturmian words

Fred Lunnon claims that another place that the sum of Euler phi function appears is that, $\sum \phi(n+2)$ is equal to the number of Sturmian words of length n that are special [3]. Sturmian words are defined as infinite words consisting of the elements: $\{0, 1\}$, that have $n+1$ factors of length n for each $n >= 0$ [1]. A factor of a Sturmian word S is defined as a finite binary word w, such that $S = uS*$, for some Sturmian word $S*$. The length of a factor of Sturmian words is defined as the number of one digits in that factor, and let the length be notated using the magnitude notation. For example let the factor u be defined as $u = 00111$, then $|u| = 3$. A Sturmian word S is considered balanced if and only if for any two factors of S, say u, and v, that are of the same length, then $||u| - |v|| <= 1$ [1]. A factor u of a Sturmian word S is defined as special if for any factors a, and b in S, ua, au, ub, and bu are also factors of S. A Sturmian word is defined as special if it contains special factors.

2.2.1 Fibonacci words

An example of Sturmian words is the Fibonacci word [1]. The Fibonacci word is similarly defined as the popular Fibonacci sequence. The Fibonacci word is defined over the binary alphabet and begins with $f_0 = 0$, and $f_1 = 10$, and then is recursively defined as $f_n = f_{n-1}f_{n-2}$. Thus, $f_2 = 010, f_3 =$

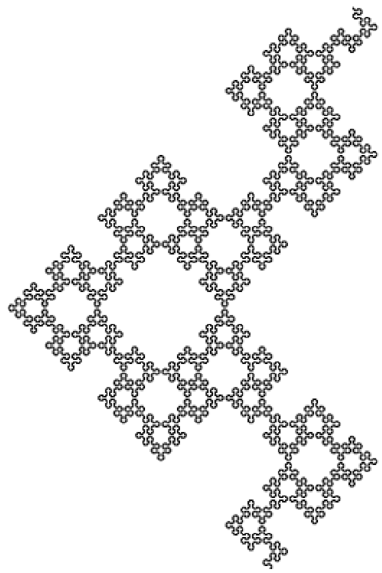

Figure 3: This is an example of a fractal that is generated by the Fibonacci word.

$01001, f_4 = 01001010, \ldots$. An interesting application of this recursive pattern can be seen in the fractal show in 3.

3 Balanced Word Lists

In order to connect the concepts of Euler's Phi and Sturmian words, we must first define what is a balanced word list. A word denotes the finite factors of Sturmian words. That is a single word consists of a finite combination of length n of elements from the alphabet $\{0, 1\}$. A list is just a finite set of different words, and for our purposes, all lists considered will be of words of the same length n. A list L, is considered balanced if and only if, for any two elements of L, call them u and v, then $||u| - |v|| <= 1$. For example, the list $L = \{000, 001, 100\}$, is a balanced list for $n = 3$. The list, $T = \{000, 001, 111\}$, is an example of a word list that is not balanced.

We will be considering all possible balanced word lists of words of length n, for any given n in the natural numbers. That is for $n = 1$, we only have

n	Exclusions	Word Lists
1	{}	{0, 1}
2	{11}	{00, 01, 10}
2	{00}	{11, 10, 01}
3	{000, 111}	{001, 010, 100, 101}
3	{101, 111}	{000, 001, 010, 100}
3	{111, 000}	{110, 101, 011, 010}
3	{010, 000}	{111, 110, 101, 011}
4	{0000, 1111}	{0010, 0100, 0101, 1001, 1010}
4	{0000, 0101, 111}	{0001, 0010, 0100, 1000, 1001}
4	{1001, 0101, 111}	{0000, 0001, 0010, 0100, 1000}
4	{1111, 0000}	{1101, 1011, 1010, 0110, 0101}
4	{1111, 1010, 0000}	{1110, 1101, 1011, 0111, 0110}
4	{0110, 1010, 1111}	{1111, 1110, 1101, 1011, 0111}
5	{00100, 000, 11}	{00101, 01001, 01010, 10010, 10100, 10101}
5	{10101, 000, 11}	{00100, 00101, 01001, 01010, 10010, 10100}
5	{00000, 101, 11}	{00010, 00100, 01000, 01001, 10001, 10010}
5	{00000, 1001, 101, 11}	{00001, 00010, 00100, 01000, 10000, 10001}
5	{10001, 1001, 101, 11}	{00000, 00001, 00010, 00100, 01000, 10000}
5	{11011, 111, 00}	{11010, 10110, 10101, 01101, 01011, 01010}
5	{01010, 111, 00}	{11011, 11010, 10110, 10101, 01101, 01011}
5	{1111, 010, 00}	{11101, 11011, 10111, 10110, 01110, 01101}
5	{11111, 0110, 010, 00}	{11110, 11101, 11011, 10111, 01111, 01110}
5	{01110, 0110, 010, 00}	{11111, 11110, 11101, 11011, 10111, 01111}

Figure 4: This is a table of all balanced word lists up to length five, also including the exclusions for each list.

one possible balanced word list, $\{0, 1\}$. For $n = 2$, we have two possible balanced words lists $\{00, 10, 01\}$, and $\{10, 01, 11\}$. For $n = 3$, we have four word lists, $\{001, 010, 100, 101\}$, $\{000, 001, 010, 100\}$, $\{110, 101, 011, 010\}$, and $\{111, 110, 101, 011\}$, ect. Notice that in order to be balanced, there are exclusions in certain lists that need to be made. An exclusion occurs when two words are of more than a length of one apart, and thus causes a split between lists, where one unbalanced list generates two smaller balanced lists. For example, the unbalanced word list $\{00, 01, 10, 11\}$, forms two balanced lists, $\{00, 01, 10\}$ which has the exclusion 11, and $\{01, 10, 11\}$ which has exclusion 00, as seen in 4. Once an element is excluded in a list, it is also excluded in all lists of greater n that have sub-words in the excluded list. Where a sub-word is simply a word that has roots in another word, (i.e 10 is a sub-word of 100 and 101). The word 11 is an exclusion for the list $\{00, 10, 01\}$, which are all sub-words of the list $\{001, 010, 100, 101\}$, thus the word 11 is also not included in this larger list and is also labeled an exclusion, (in this case of the form 110, which has the sub-word 11).

Our research conjecture is that the number of balanced word lists of length

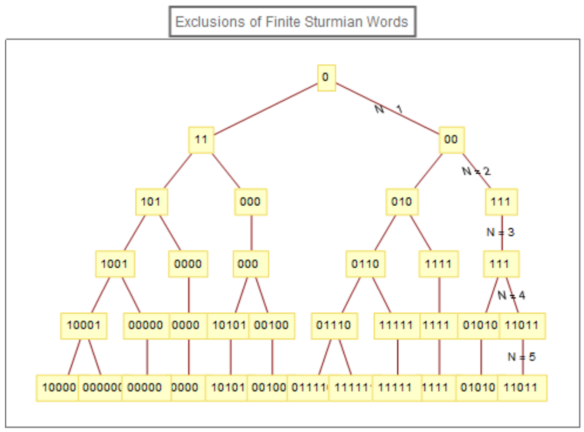

Figure 5: The tree plot below defines the number of word exclusions of all possible balanced word lists up to length $n = 5$, where n defines the length of the words in that lists. Note that the number of branches for each n are exactly the first five terms of the sum of Euler Totient function : $2, 4, 6, 10, 12$.

n, which also corresponds to the number of exclusions, generate the sum of Euler Phi function for a given n. The number of splits in the tree are exactly the values of Euler $\Sigma\phi(n)$ where n is the nth level of the tree, as seen in Figure 5. The conjecture is based on research that we have found that these splits are exact up for at least to n = 70.

4 Future Goals

Future goals include continuing research for this project, by proving the conjecture that the number of balanced word lists for each n, (also the number of exclusions), define the sum of the Euler Totient function. It is likely that further exploration on different ways to represent this data through Mathematica can help to grant insight in solving this conjecture.

References

[1] P. Baláži. Various properties of sturmian words. *Acta Polytechnica*, 45(5), 2005.

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[3] N. J. A. Sloane. Sum of totient function: $a(n) = \sum_{k=1\ldots n} \phi(k)$. *The On-Line Encyclopedia of Integer Sequences*, 1964.