Quantum Mechanics and Tunneling

The behaviour of most objects that we deal with in our day-to-day life can be explained by classical mechanics. Classical physics is causal, which means that outcomes are completely predictable given full knowledge of the system's past. But what about at the smallest levels of the universe? The behaviour of particles is not similarly predictable. To study particles, quantum mechanics is what helps us understand the behaviour of matter at the subatomic level.

One of many interesting quantum phenomena is quantum tunneling. Quantum tunneling describes a particle penetrating through a potential barrier. A potential barrier is "a region in which the potential is significantly higher than at points on either side of it" and higher than the energy of the particle [1]. This behaviour is impossible for a particle from the perspective of classical physics, as the particle simply doesn't have the energy to do so.

Quantum mechanics describes particles as being wavelike, also known as the wave nature of particles. Quantum physics is also probabilistic. Predictions of outcomes take the form of probabilities. In the last section of this paper we will find the probability that a particle penetrates through a potential barrier, i.e. solve the barrier tunneling problem. To do this however, we need to first establish the relevant mathematical concepts (specifically the method for solving second-order differential equations) and classical wave concepts. We will then derive Schrödinger's equation, which is the first step into looking at the quantum ideas.

1. Solving second-order differential equations with constant coefficients

Solving wave equations often involves solving second order differential equations. The most common form that will be relevant throughout the following sections is

$$\frac{d^2x}{dt^2} - A^2x = 0. \tag{1}$$

Rearranging, we notice that the second derivative of x is equal to the square of the constant A multiplied by x. This relationship is found in exponential functions of x. Therefore we can make an assumption that $x = e^{\lambda t}$. Substituting this into equation (1) we obtain

$$\lambda^2 e^{\lambda t} - A^2 e^{\lambda t} = 0. \tag{2}$$

Cancelling the exponential component and finding λ in terms of A we find

$$\lambda = \pm A \tag{3}$$

and by substituting back into our assumed form for $x(t)$, we find the two solutions

$x = C_1 e^{At}$ and $x = C_2 e^{-At}$. (4)

Now there appear to be two solutions - this suggests that we would need to take one of the two, or both (as independent solutions). However, we can use a useful property of second order linear differential equations to obtain one solution depending on two parameters. This property is called linearity and it says that for linear differential equations, the sum of its solutions will generate another solution [2]. This property is very relevant when finding solutions to wave equations as adding the two solutions allows us to generate one solution that incorporates the effect of both of the independent motions occurring in a specific system.

For example, in a simple harmonic oscillator, the solutions come in the forms $\sin(t)$ and $\cos(t)$. The $\sin(t)$ solution starts with some velocity but no displacement, which could occur when you push a mass from rest at the equilibrium. On the other hand the $\cos(t)$ solution begins at a position greater than zero, but no velocity, which corresponds to pulling the mass away from equilibrium. There are two types of motion here. We could combine these by pulling the mass away from equilibrium to the same position and then pushing it away with some velocity. Linearity says that you would just add the two trigonometric solutions to get a solution representing the motion for all values of time t.

We can verify that the sum of the solutions is also a solution using the general form of a second order linear differential equation

$A\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Cx = 0,$ where A, B and C are constant coefficients.
(5)

We can consider $x_1(t)$ and $x_2(t)$to be the solutions of eq (1). Writing the equations in terms of these solutions gives us

$A\frac{d^2x_1}{dt^2} + B\frac{dx_1}{dt} + Cx_1 = 0$ and $A\frac{d^2x_2}{dt^2} + B\frac{dx_2}{dt} + Cx_2 = 0.$
(6)

Since these are both equal to zero, we can add the equations, giving

$A\frac{d^2(x_1+x_2)}{dt^2} + B\frac{d(x_1+x_2)}{dt} + C(x_1 + x_2) = 0.$
(7)

Hence $(x_1 + x_2)(t)$ is also a solution.

Going back to the solutions in equation (4) we can use linearity to get the solution

$x = C_1 e^{At} + C_2 e^{-At}$.

When the second-order differential equation involves a negative function of x, the solution will differ slightly. For example, consider the equation

$$\frac{d^2x}{dt^2} = -A^2 x. \tag{8}$$

Now if we were to substitute $x = e^{\lambda t}$ there is an imaginary component introduced because when we solve for λ we get $\lambda = \pm iA$. This makes mathematical sense since the relationship in equation (4) can only be true when the power of e contains i (in order for the negative sign to exist).

Hence the two solutions are complex exponentials; $x = Ce^{\pm iAt}$. Using linearity we can write the general solution

$$x = C_1 e^{iAt} + C_2 e^{-iAt}. \tag{9}$$

For this second case, we could also use a sine or cosine function, as the negative of these two types of functions is also proportional to the second derivative. Further, we can derive the trigonometric form using the complex exponential already found.

We can look at the solution as the summation of two complex numbers, since complex exponentials represent complex numbers (E.g $Be^{i\theta}$ is a complex number with magnitude B and argument θ). With this, (9) can be expanded into the polar form of the complex numbers to give

$$C_1 \cos(At) + iC_1 \sin(At) + C_2 \cos(At) - iC_2 \sin(At). \tag{10}$$

This simplifies to

$$(C_1 + C_2)(\cos(At)) + i(C_1 - C_2)(\sin(At)) \tag{11}$$

We will find this understanding very useful as second-order differential equations will come up continuously throughout the rest of the paper.

2. Concepts from Classical Physics

2.1. Oscillators and waves

In 1801, Thomas Young's 'double slit' experiment showed us that unlike the classical expectation that the particles would end up at one of two locations on a detector, the particles sent through a double slit actually behaved as waves do. On the detector, the particles formed an interference pattern in the way that two coherent waves would. So quantum physics describes that "particles have wavelike properties" [2] and the wave-like behaviour of the particles is determined by a wave equation, namely Schrödinger's equation. Given this, it is important to understand classic properties of waves in order to explain quantum tunneling of particles. First, we will look at simple harmonic motion (SHM), specifically solving for the position $x(t)$ in an oscillating mass on a spring (as this also provides the necessary prior understanding for looking at the behaviour of waves and solving the wave equation).

2.1.1 SHM and oscillations

Solving for $x(t)$

An object undergoing simple harmonic motion has a restoring force that is directly proportional to the object's displacement acting towards the equilibrium of the system. In this section we will look at a mass on a spring. Determining the position $x(t)$ can be done by writing Newton's second law of motion in terms of x, with the aid of Hooke's law force $F(x) = -kx$, where k is the spring constant.

Let's look at Newton's second law of motion, $F = ma$.

We can substitute the force given in Hooke's law, giving

$$-kx = ma. \qquad (1)$$

Now we have two variables, x and a (acceleration). In order to solve for x, acceleration should also be written in terms of x. Acceleration is the second derivative of the position, therefore equation (1) becomes

$$-kx = m\frac{d^2x}{dt^2}.$$
(2)

This now resembles the second-order differential equations we looked at in section one so we can apply the same method to solve it. Plugging in $x(t) = e^{\beta t}$ we get

$$-ke^{\beta t} = m\beta^2 e^{\beta t}. \qquad (3)$$

Cancelling the common term $e^{\beta t}$ shows us that $\beta^2 = \frac{-k}{m}$.

We can now introduce the parameter of angular velocity of the SHM system, ω. It satisfies $\omega = \sqrt{\frac{k}{m}}$.

Since $\beta^2 = \frac{-k}{m}$ and $\omega = \sqrt{\frac{k}{m}}$, we see that $\beta = \pm i\omega$.

Therefore the solutions are

$$x(t) = Ce^{\pm i\omega t}, \tag{4}$$

which (using linearity) can be written as one solution,

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}. \tag{5}$$

The coefficients C_1 and C_2 can take any value in order to satisfy equation (2), complex or real. However, we know that $x(t)$ is equal to its own complex conjugate so it can be assumed that the two terms are complex conjugates of each other. For the complex numbers C_1 and C_2, we can use the exponential form of complex numbers which will make it easier to simplify later. Moreover, two parameters of the SHM system (phase, φ and amplitude, A) are now introduced into the solution since the exponential form is the product of the magnitude (which can be called A) and the phase, $e^{i\varphi}$ of the complex number. The complex conjugate has equal magnitude and equal but negative power.

Hence, equation (5) now expands to

$$x(t) = C_0 e^{i\varphi} e^{i\omega t} + C_0 e^{-i\varphi} e^{-i\omega t}.$$
(6)

Now expanding equation (6) using polar form as we did in equation (10) of section one and then simplifying by collecting like terms, we get

$$x(t) = 2C_0 \cos(\omega t + \phi).$$
(7)

This method is important as it shows how to take the real part of the complex solution in order to get a solution that is consistent with the physical context of the mass-spring system. Looking back at (2), where x is proportional to its second derivative, we can see that the solution in (7) is consistent, as the cosine function also possesses this relationship with its second derivative.

2.1.2 Transverse and standing waves

Transverse waves on a string are governed by the wave equation, which we will derive in this section. We will also look at reflection and transmission of waves and examine wave behaviour when reaching different types of boundaries.

As mentioned previously, particles in quantum mechanics are governed by a wave equation, Schrödinger's equation. Before we look at particles as described by quantum physics, let us derive the wave equation that describes wave phenomena within classical physics. Similar to the equations we have solved up to this point, the wave equation is a second order linear differential equation.

Deriving the wave equation

First let us consider a string, as illustrated in the figure (1), with tension T and mass per unit length μ. For the sake of this derivation, we will assume that the string extends infinitely in both directions. Let's examine small transverse displacements of the string. The coordinate along the string we will call x, and the transverse displacement ψ. Restricting this case to small displacements or vibrations means that the angle θ between the string and the x direction is much smaller than one. Therefore we can use the small-angle approximation;$\sin(\theta) \approx \theta$ and $\cos(\theta) \approx \theta$.

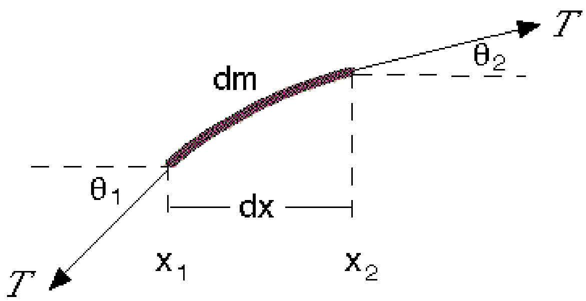

Figure (1) [3]

Let's start by writing the transverse Newton's second law equation

$$F_{\psi} = ma_{\psi} \tag{8}$$

The forces acting in the ψ direction are

$$F_{\psi} = T\sin\theta_2 - T\sin\theta_1. \tag{9}$$

Using the small-angle approximation, $\sin\theta \approx \tan\theta = \frac{d\psi}{dx}$, so (9) becomes

$$F_{\psi} = T\{(\frac{d\psi}{dx})_2 - (\frac{d\psi}{dx})_1\}. \tag{10}$$

We can see now that the total force acting in the transverse direction is dependent on the difference in slope between the two ends of the string. On the right hand side of equation (8) we have mass and acceleration. We know μ is the mass per unit length, therefore the mass $dm = \mu dx.$ The acceleration in the transverse direction is the rate of change in the ψ direction. This is simply the second time derivative of ψ, so we have

$$F_{\psi} = T\{(\frac{d\psi}{dx})_2 - (\frac{d\psi}{dx})_1\} = \mu dx\, \frac{d^2\psi}{dt^2}.$$
(11)

Rearranging to get the dx terms on one side gives

$$\frac{d^2\psi}{dt^2} = \frac{T}{\mu}\frac{(\frac{d\psi}{dx})_2 - (\frac{d\psi}{dx})_1}{dx}$$
(12)

The numerator of the last term on the right hand side of (12) is the difference between the derivatives at points x and $x + dx$, denoted by the subscripts 1 and 2 respectively. This numerator is being divided by dx, so this term is essentially the rate of change of the first derivative with respect to x. In other words: The second derivative of ψ with respect to x. So we now have

$$\frac{d^2\psi}{dt^2} = \frac{T}{\mu}\frac{d^2\psi}{dx^2}.$$
(13)

ψ is a function of position and time, so we should write ψ as $\psi(x, t)$. Consequently, we must now use partial derivatives since we have a function of two variables so (13) becomes

$$\frac{\partial^2\psi(x,t)}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2\psi(x,t)}{\partial x^2}$$
(14)

This is the wave equation!

The solutions for the wave equation are in the form

$$\psi(x, t) = Ae^{i(\pm kx \pm \omega t)} \text{ where } \frac{\omega}{k} = \sqrt{\frac{T}{\mu}} \equiv c. \quad (15)$$

We can find the relationship between ω and k by substituting (15) into the wave equation and subsequently cancelling the exponential term. We will get

$$-\omega^2 = -\frac{T}{\mu}k^2 \qquad (16)$$

Which simplifies to

$$\frac{\omega}{k} = \sqrt{\frac{T}{\mu}}, \qquad \text{where } \sqrt{\frac{T}{\mu}} \equiv c \qquad (17)$$

c is the speed of the wave and k and ω can take any value as long as they satisfy the relationship $\frac{w}{k} = c$. Wavelength is $\lambda = \frac{2\pi}{k}$ and the period of the oscillation is $\tau = \frac{2\pi}{\omega}$.

Boundary behaviour and boundary conditions.

When a wave reaches a boundary, different outcomes will occur depending on the boundary. For our purpose, understanding quantum tunneling, we will look specifically at waves with two fixed ends and waves on a string with a change in density. Understanding the wave functions and wave behaviour with these two cases will be very useful to us in later sections.

Reflection and transmission:

If a string changes density at the point $x = 0$, this point will act as a boundary. At $x = 0$ the incident wave travelling through the string will undergo reflection and transmission. As we will discuss later, the same occurs with regards to quantum physics when a particle encounters a boundary in the form of potential change.

Consider an infinite string with uniform tension but changing density: μ_1 for $-\infty < x < 0$ and μ_2 for $0 < x < \infty$. Given this property, $x = 0$ can be considered as a boundary which a wave travelling through the string will encounter. Let's say a wave of the form

$$\psi_i(x, t) = f_i(t - \frac{x}{\nu_1}) \qquad (18)$$

(i stands for incident) starts at the left and travels towards $x = 0$. Equation (15) tells us that $\nu_1 = \sqrt{T/\mu_1}$. At the boundary, $x = 0$, there will be a reflected (r) wave and transmitted (t) wave.

Reflected wave: $\psi_r(x, t) = f_r(t + \frac{x}{\nu_1})$. (19)

This wave moves leftwards, towards $-\infty$ and therefore has a positive sign in its argument.

Transmitted wave: $\psi_t(x, t) = f_t(t - \frac{x}{\nu_2})$ where $\nu_2 = \sqrt{T/\mu_2}$. (20)

This wave moves to the right, towards ∞.

Now if we were to look at the resultant expressions for the waves on the left (L) and right (R) of $x = 0$ we can write

$$\psi_L(x, t) = \psi_i(x, t) + \psi_r(x, t) = f_i(t - \frac{x}{v_1}) + f_r(t + \frac{x}{v_1})$$
$$\psi_R(x, t) = \psi_t(x, t) = f_t(t - \frac{x}{v_2}).$$
(21)

We have three different wave functions here, but if we find the reflected and transmitted waves in terms of the incident wave, we can simplify equation (21) to obtain a complete wave function that we can understand further. This can be done using the *boundary conditions* at $x = 0$.

Condition 1: The string is continuous therefore

$$\psi_L(0, t) = \psi_R(0, t) \Rightarrow f_i(t) + f_r(t) = f_t(t).$$
(22)

Condition 2: The slope is continuous. If there was a difference in slope between the left and right side of $x = 0$ it would imply that there is a resultant force acting on the atom at $x = 0$. But a nonzero force implies that acceleration, $a = \frac{F}{\mu dx}$, would be infinite since we are taking dx to be infinitesimal. Since this is not possible, the slope must be constant and therefore

$$\frac{\partial\psi_L(x,t)}{\partial x} = \frac{\partial\psi_R(x,t)}{\partial x} \Rightarrow -\frac{1}{v_1}f'_i(t) + \frac{1}{v_1}f'_r(t) = -\frac{1}{v_2}f'_t(t).$$
(23)

Integrating and rearranging to remove the fractions we get

$$v_2 f_i(t) - v_2 f_r(t) = v_1 f_t(t) \quad (24)$$

We will assume the string has zero displacement before the wave passes so we do not need to consider an integration constant.

Now we can find the relationships between both $f_r(t)$and $f_t(t)$, and $f_i(t)$ by solving equations (22) and (24). We get

$$f_r(t) = \frac{v_2 - v_1}{v_2 + v_1} f_i(t) \text{ and } f_t(t) = \frac{2v_2}{v_2 + v_1} f_i(t), \quad (25)$$

We have written t as the argument of the functions here but this relationship holds true regardless of the argument. So we can rewrite (21), the full wave, as

$$\psi_L(x,t) = \psi_i(x,t) + \psi_r(x,t) = f_i(t - \frac{x}{\nu_1}) + \frac{\nu_2-\nu_1}{\nu_2+\nu_1} f_i(t + \frac{x}{\nu_1})$$

$$\psi_R(x,t) = \psi_t(x,t) = \frac{2v_2}{v_2+v_1} f_i(t - \frac{x}{\nu_2}). \quad (26)$$

Reflection and transmission coefficients:

Reflection and transmission coefficients (R and T, respectively) represent what amount of the incident wave is reflected/transmitted.

What we can infer from (25) is the reflection and transmission coefficients. They are simply the coefficients of $f_i(t)$, the amplitudes of the reflected and transmitted waves relative to the incident wave. So the reflection and transmission coefficients are

$R \equiv \frac{\nu_2-\nu_1}{\nu_2+\nu_1}$ and $T \equiv \frac{2\nu_2}{\nu_2+\nu_1}$, respectively.
(27)

Note that the coefficients must always satisfy the relation $1 + R = T$.

To look at the cases involving a change in density in a string, it will be useful to bring in the variable μ into our analysis so that the boundaries can be clearly represented by the density, or change in density.

We know that $\nu = \sqrt{T/\mu}$ and that the tension T of the string remains constant. Therefore $\nu_1 \propto \frac{1}{\sqrt{\mu_1}}$ and $\nu_2 \propto \frac{1}{\sqrt{\mu_2}}$. With this, we can write R and T in terms of the densities μ_1 and μ_2:

$$R \equiv \frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_1}+\sqrt{\mu_2}} \text{ and } T \equiv \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1}+\sqrt{\mu_2}} \quad (28)$$

Now let's look at applications of the formulas we have derived to different cases involving density change:

1. *Light string on left, heavy string on right:* $\mu_1 < \mu_2 < \infty$ $(\nu_2 < \nu_1) \Rightarrow$
$-1 < R < 0,\ 0 < T < 1.$

We have already discussed it in this section; both reflection and transmission will occur once the wave reaches the heavy side of the string. Note that R is negative which means that the wave will be inverted as it moves back to the left. This happens for the following reason: As the wave reaches $x = 0$, the end of the waves on the left and right exert upwards and downwards forces, respectively. This is explained by Newton's third law that 'for every action there is an equal and opposite reaction.' Due to the greater density on the right side relative to the left, the upward force becomes a downward force “causing the upward displacement to become downward displacement, hence the inverted reflected wave.” [4]

2. *Heavy string on left, light string on right:* $0 < \mu_2 < \mu_1$ $(\nu_1 < \nu_2) \Rightarrow$ $0 < R < 1,\ 1 < T < 2.$

This case is similar to case 1 as there is partial reflection and partial transmission. However in this case there will be more transmission than reflection. It is also worth noting that the reflection coefficient is positive, therefore the reflected wave is not inverted.

3. *Zero mass string on the right:* $\mu_2 = 0$ $(\nu_2 = \infty) \Rightarrow R = 1,\ T = 2.$

Since the right part of the string is massless, it cannot carry any energy. Therefore the wave is completely reflected. However, there is no inversion of the reflected wave since the zero-mass string will not exert any downward force on the string.

4. *Brick wall on the right:* $\mu_2 = \infty$ $(\nu_2 = 0) \Rightarrow R = -1,\ T = 0$.

No part of the wave will be transmitted, meaning that all of the wave is reflected. However R is negative which means that the wave will be inverted as it moves back to the left. As the wave reaches the wall, the string experiences a downwards force. Since energy must be conserved and the wall cannot absorb any energy being infinitely massive, the wave ends up inverted.

<u>*Standing waves*</u>

We have used the boundary conditions at $x = 0$ but what if we require that the wave function satisfies the condition $\psi(0, t) = \psi(L, t) = 0$? What sort of wave function would we get?

Let us look back at the brick-wall case in more detail. We will decide that the wall is located at $x = 0$. First, the general solution for the wave equation from (15) is

$$\psi(x, t) = e^{i(kx+\omega t)}.$$

Consider a sinusoidal wave moving towards the left where it will be incident on the wall. We can take the real part of the above function, which gives

$$\psi_i(x, t) = \cos(\omega t + kx). \tag{29}$$

As we discovered previously, the reflection coefficient of $\psi(x, t)$ when it encounters a wall is -1. So the reflected wave that travels in the opposite direction is

$$\psi_r(x, t) = -\cos(\omega t - kx). \tag{30}$$

So the total wave is

$$\psi(x,t) = \psi_i(x,t) + \psi_r(x,t) = \cos(\omega t + kx) - \cos(\omega t - kx)$$
$$= -2\sin(\omega t)\sin kx. \qquad (31)$$

Equation (31) satisfies the first boundary condition $\psi(0, t)$. As we can see, the total wave function is made up of a product of a function of x and a function of t. This is a standing wave. All points on the string will be at rest at exactly the same times, unlike travelling waves where the peaks and troughs are instantaneously at rest.

We now need to consider the second boundary condition: $\psi(L, t) = 0$. If we substitute $x = L$ into equation (31), the condition would only be satisfied if $\sin(kL) = 0$. Hence $kL = n\pi$ where n is an integer, giving

$$k_n = \frac{n\pi}{L}.$$

The method we have gone through is relevant to quantum tunneling. Here we have looked at $\psi(0, t) = \psi(L, t) = 0$, which gives a standing wave within two fixed ends, but in quantum mechanics this is the same condition that arises when there is an infinite potential barrier.

Attenuation

In quantum tunneling, the wave function in the area of high potential energy (i.e within the barrier) takes a decaying exponential form. How does this happen? To understand, let us look at our classical system and explore what happens when a string experiences *attenuation,* which causes $\psi(x, t)$ to decay. Attenuation can occur when a string experiences an additional force such as drag force from damping if it were submerged underwater. Here we will explore a different case in which the string, instead of vibrating freely, is attached to a bed of springs, which will add an additional force.

When there are no springs, the wave equation from equation (15) is

$$\frac{\partial^2\psi(x,t)}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2\psi(x,t)}{\partial x^2}.$$

To incorporate the additional force from the springs, let's get this equation into a more familiar form. If we multiply through by the length of a bit of string Δx and rearrange we can obtain the $F = ma$ equation

$$T\Delta x\frac{\partial^2\psi}{dx^2} = \mu\Delta x\frac{\partial^2\psi}{dt^2}. \qquad (27)$$

Now let's add the force from the springs. Take σ to be the spring constant per unit length. The spring constant is therefore $\sigma\Delta x$. The force on a spring is $F =- kx$, so the force from the bed of springs is

$$F =- \sigma\Delta x\psi. \tag{28}$$

Placing this appropriately into (27) gives

$$T\Delta x\frac{\partial^2\psi}{dx^2} - \sigma\Delta x\psi = \mu\Delta x\frac{\partial^2\psi}{dt^2}. \tag{29}$$

If we divide the equation by Δx we will get a differential equation for $\psi(x, t)$. To solve it let's substitute

$$\psi(x, t) = e^{i(kx-\omega t)}. \tag{30}$$

This will give the relationship between ω and k, which comes out as

$$- Tk^2e^{i(kx-wt)} - \sigma e^{i(kx-wt)} =- \mu\omega^2e^{i(kx-wt)}. \tag{31}$$

Now let's cancel out the exponential term and solve for k. We get

$$- Tk^2 - \sigma =- \mu\omega^2. \tag{32}$$

Solving gives

$$k = \pm\sqrt{\frac{\mu\omega^2-\sigma}{T}}.$$
(33)

Depending on whether the discriminant $\mu\omega^2 - \sigma$ is positive or negative, k can take a real or imaginary value. If $\mu\omega^2 < \sigma$, k is imaginary and if $\mu\omega^2 > \sigma$, k is real. It can also take the positive or negative value of the square root. The question now is which value of k should we use? For now, we will use the positive and imaginary value.

Lets define

$k = i\kappa$, where κ is real. (34)

If we substitute this into (30) we get

$$\psi(x, t) = e^{i(i\kappa x-\omega t)}. \tag{35}$$

Expanding the power of e then gives

$$\psi(x, t) = e^{-\kappa x - i\omega t}. \tag{36}$$

Looking carefully at the above function, we can see that x is now being multiplied by a negative factor: $-\ \kappa$. This shows us that $\psi(x, t)$ decays with distance. The envelope of the function as it decays is equal to $\psi(x, t) = e^{-\kappa x}$. This wave can now be classified as an attenuated wave because it narrows as x grows.

Now we can also see why taking the positive value of k is necessary, if we took the negative value, x in the function for $\psi(x, t)$ would be multiplied by a positive factor, κ. Instead of decaying, the wave function would increase with time, which is not possible since the string doesn't have the source of energy to account for this behavior.

2.2 Symmetry and conservation

The final concept from classical physics that will be useful to us comes from Noether's Theorem, which states that '*every symmetry of a system gives rise to a dynamically conserved quantity*.' This means that the conservation laws can be related with a symmetry of a certain system.

Let us look at how the above statement is true for momentum. We will begin with the question: *When is momentum conserved?* We can use Newton's second law and find an equation involving momentum for a ball being thrown. We have

$$F = ma. \tag{37}$$

Since acceleration is the rate of change in velocity with respect to time, giving

$$F = m\frac{dv}{dt}. \tag{38}$$

Assuming the mass of the ball stays constant we can write

$$F = \frac{d}{dt}(mv). \tag{39}$$

Momentum is equal to mv so

$$F = \frac{dp}{dt}. \tag{40}$$

Equation (40) shows us that force is the change of momentum. So the answer to the original question is momentum is conserved when the force is zero, as there will be no change in momentum over time.

Taking energy to be conserved, let us now find the potential, $V(x)$ of the system, specifically the amount of potential energy used when an object moves from the ground to a certain height.

The equation for the gravitational force is

$$F = - mg\hat{y}. \tag{41}$$

The potential is the work done in taking the object to a certain y value and work is force times distance. However force is changing over time, so we must find the work done by adding the individual values of work, which can be represented by the integral

$$V(x) = - \int_{0}^{x} F(x')dx'. \tag{42}$$

Because the force is acting against gravity the dot product will be negative, but since we want a positive answer we put a negative sign in front of the integral.

Now let's take the derivative of equation (42). We get

$$\frac{dV}{dx} = - F(x). \tag{43}$$

Since we have taken $F(x)$ to be zero we can see using (43) that when force is zero, the potential should be constant with position in the x direction. This means that if the system was shifted to the left or right, the potential would be the same. In other words, there is *symmetry of translation.* Note that equation (43) is a general representation, as the use of gravity here is just an example.

To summarise, we now have

$$\frac{dp}{dt} = 0 = F(x) = - \frac{dV}{dx}. \tag{44}$$

This aligns with Noether's theorem as it shows that the conserved quantity (momentum) corresponds to a symmetry (translation). The same relation holds true for energy and time translation as well as angular momentum and rotation.

3. Schrödinger's Wave Equation

3.1 Deriving Schrödinger's Equation starting from Symmetry and Conservation.

Now we transition to quantum mechanics. As mentioned previously, particles as described by quantum physics act like waves. The wave-like behaviour is represented by Schrödinger's equation, which determines a wavefunction $\psi(x, t)$. To derive Schrödinger's equation we must introduce and use quantum principles. In order to do this we can begin with the classical concept of symmetry and conservation and see how it relates to quantum mechanics. We can then find relationships between conserved quantities, namely energy and momentum, using our knowledge of differential equations. Incorporating the wavefunction ψ into these relationships, we will be able to find Schrödinger's equation.

First, here is a summary of the symmetry and conservation relationships:

	Momentum ⟶	**Translation**
Units	$kg\ m\ s^{-1}$	m
	Energy ⟶	**Time**
Units	$kg\ m^2\ s^{-2}$	s
	Angular Momentum ⟶	**Rotation**
Units	$kg\ m^2\ s^{-1}$	1 (unitless)

Angular momentum is not strictly relevant to Schrödinger's equation, but it has been added to make the following observation more clear.

If we multiply the units of the conserved quantity with the units of the symmetrical aspect of the system, we get the exact same units $kg\ m^2\ s^{-1}$. This now leads us to an a connection to quantum physics as these units are also the units of an important constant: Planck's constant, h. In 1900 Max Planck was trying to find the spectrum of colours emitted by a thermal radiator. The results couldn't be explained by the existing physics at the time, so Planck postulated that energy of light must come in packets called quanta, and that the energy is proportional to the light's frequency by the constant h, now known as Planck's constant. By dimensional analysis, it could be found that this constant of proportionality has these same units, $kg\ m^2\ s^{-1}$. Its value is $6.62607015 \times 10^{34}\ kg\ m^2\ s^{-1}$. This constant is important to quantum physics.

To derive Schrödinger's equation let's look back at the solution to the wave equation from section 2. The function is

$$\psi(x, t) = e^{i(kx-\omega t)}. \tag{1}$$

We can use Planck's constant

$$h = 6.62607015 \times 10^{34}\, kg\, m^{2}\, s^{-1} \tag{2}$$

to generate a relationship from (1) between ωand energy because it relates a photon's energy to its frequency. Using Planck's constant we can write

$$E = hf. \tag{3}$$

Multiplying and dividing the RHS by 2π gives

$$E = \frac{h}{2\pi} \cdot 2\pi f. \tag{4}$$

We define $\frac{h}{2\pi} = \hbar$ and $2\pi f = \omega$, so we now have an equation relating angular frequency and energy as

$$E = \hbar\omega. \tag{5}$$

Now let's see how we can find a relationship between equation (1) and (5) so that we involve ψ. If we took the first derivative with time, we would bring down ω from the power. But it also brings down $-\ i$. Furthermore, it doesn't involve ħ. However, if we multiplied the first derivative by i and ħ, we would get

$$i\hbar\frac{\partial}{\partial t}\psi = \hbar\omega\psi = E\psi. \tag{6}$$

This links back to Noether's theorem as the time derivative gives us the energy. This would suggest that the x derivative will give you momentum. We can begin by writing

$$\frac{\partial}{\partial x}\psi \sim p\psi \tag{7}$$

The derivative of $\psi(x, t)$ with respect to x is actually

$$\frac{\partial}{\partial x}\psi = ik\psi. \tag{8}$$

The units for momentum are $kgms^{-1}$, but the unit for k is just m^{-1}. Therefore we need to multiply the derivative by some constant with the units $kgm^{2}s^{-1}$. As we discovered, ħ has these exact units. So we can modify (7) to

$$\hbar\frac{\partial}{\partial x}\psi \sim p\psi. \tag{9}$$

Since we prefer not to have an i, we can multiply the LHS of (9) by $-\ i$, which gives us a more definitive equation

$$-\ i\hbar\frac{\partial}{\partial x}\psi = \hat{p}\psi.$$
(10)

Here we have derived an equation that works when $\psi(x, t)\ =\ e^{i(kx-\omega t)}$. However, there are many different possible functions which we need the equation to account for so that we can find their momentum as well. If we consider p not simply as the classical quantity momentum but rather as the act of multiplying ψ by $-\ i\hbar\frac{\partial}{\partial x}$, it becomes more widely usable. The concept of doing something to a function in mathematics is called an operator, where the operator is marked with a 'hat'. Going forward when we derive Schrödinger's equation we will be exchanging numbers for operators, eventually deriving an equation of operators.

Now we have an equation for both energy and momentum in terms of ψ.

To bring it all together, we can find an equation to relate energy and momentum.

Energy can be defined as

$$E\ =\ KE\ +\ PE. \tag{10}$$

$$KE\ = \frac{1}{2}mv^2 \text{ and } p\ =\ mv \tag{11}$$

Therefore

$$KE\ = \frac{1}{2}mv^2 = \frac{p^2}{2m}. \tag{12}$$

We can substitute this into (10) to get

$$E\ = \frac{p^2}{2m} +\ V(x).$$

(13)

Let us assume equations (6) and (10) are true at the same time and see what we can get for the wavefunction. Multiplying both of equation (13) by ψ gives

$$E\psi\ =\ \{\frac{p^2}{2m} +\ V(x)\}\psi.$$

Now, instead of continuing to solve the equation with the classical quantities p and E, we are going to do something different. That is *replacing the classical quantities by their operators*. We will replace the quantity p by the value of the $\hat{p}$ operator we found in equation (6). For E we will substitute by the operator suggested in equation (10). In doing this we get

$$i\hbar\frac{\partial}{\partial t}\psi = \frac{(-i\hbar)^2\frac{\partial^2\psi}{\partial x^2}}{2m} + V(x)\psi. \quad (14)$$

Expanding gives

$$i\hbar\frac{\partial}{\partial t}\psi = \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi. \quad (15)$$

This is Schrödinger's equation! More specifically, it is the time-dependent Schrödinger equation.

As we have seen, the Schrödinger equation is linked to equation (13) $E = \frac{p^2}{2m} + V(x)$, which is based on classical mechanics. This equation can be extended to include the classical Hamiltonian function $H(p_x, x)$ by writing it as

$$E = KE(p_x) + V(x) = H(p_x, x).$$

(16)

Now replacing p_x by the operator found in (10) and E with the operator found in (6), we get

$$\hat{H}(-i\hbar\frac{\partial}{\partial x}, x)\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = i\hbar\frac{\partial}{\partial t}\psi.$$

(17)

This is identical to equation (15), but we can now see that the wave equation can be written as

$$\hat{H}\psi = i\hbar\frac{\partial}{\partial t}\psi. \quad (18)$$

3.2 Solving Schrödinger's equation

The Time-Independent Equation

Let us now look at how to solve Schrödinger's equation (15). As we have done in previous sections with partial differential equations, we will first deal with the solutions of ψ. We can attempt to express the wavefunctions as the product of two function; one involving time alone and the other only involving the coordinate:

$$\Psi(x,t) = \psi(x)\varphi(t). \tag{19}$$

Rewriting Schrödinger's equation with this and dividing through by $\psi(x)\varphi(t)$ gives

$$\frac{1}{\psi(x)}\left\{-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x)\right\} = i\hbar \cdot \frac{1}{\varphi(t)}\frac{d\varphi(t)}{dt},$$
(20)

where the right-hand side is a function of time alone and the left side is a function of the position alone. Consequently, both of the sides must be equal to a constant, since it is the only quantity that is independent of x and t. We will call this constant E. We can now write equation (20) as two equations. We have

$$\frac{d\varphi(t)}{dt} = \frac{-i}{\hbar}E\varphi(t) \tag{21}$$

and

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x), \tag{22}$$

which can also be written as

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V(x))\psi = 0. \tag{23}$$

Equation (23) is generally known as the time-independent Schrödinger's equation, or the amplitude equation since the position function $\psi(x)$ determines the amplitude of the $\Psi(x,t)$. The equation provides a number of solutions that correspond to various values of E. These values can be represented with the subscript n. Consequently we will denote the corresponding functions as $\psi_n(x)$. The corresponding time function, $\varphi(t)$ can be integrated to give

$$\varphi_n(t) = e^{-iE_n t/\hbar}.$$
(24)

The general solution of Schrödinger's equation will be the sum of all n solutions. So the general expression for the wave function can be written as

$\Psi(x,t) = \sum_n a_n \Psi_n(x,t) = \sum_n a_n \psi_n(x) e^{-iE_n t/\hbar}$, where a_n denotes an arbitrary coefficient. (25)

We will find later that the constant E_n is required to represent the energy of various stationary states of the system.

Equation (25) contains the unknowns $\psi_n(x)$ and E_n for which we need to find values for. The functions $\psi_n(x)$ that satisfy Schrödinger's equation, equation (23), are called wavefunctions or *eigenfunctions*. These wavefunctions exist for certain values of E_n. These certain values are the "characteristic energy values or *eigenvalues* of the wave equation" (Pauling). A wavefunction must have certain properties in order to be considered a satisfactory solution of Schrödinger's equation. The wavefunction must be:

1. Continuous
2. Single valued
3. Finite for all values of x.

The eigenvalues E_n can occur as a set of only discrete values, or as a set of continuous values, or as both.

In order to find $\psi_n(x)$ and E_n for equation (25), we can use the time-independent Schrödinger's equation - equation (23). Since it is a second order differential equation, we are familiar with the method used to solve it.

Solving the Time-Independent Schrödinger's Equation

Let's take the example of a particle within an infinite potential well given by

$$V(x) = \begin{cases} \infty & \text{if } |x| > \frac{L}{2}, \\ 0 & \text{if } |x| \leq \frac{L}{2} \end{cases}$$

Since the potential is infinite beyond $L/2$, we have the boundary condition $\psi(\pm L/2) = 0$. Now we will split the system into three regions - the left, right and middle regions which we will call regions I, II and III respectively. They will each have different time-independent Schrödinger equations. For our purpose, we will just look at the equation for region II:

Region II: $-\frac{L}{2} \leq x \leq \frac{L}{2} \Rightarrow \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} \cdot E\psi = 0.$

(26)

Position-wise, region II describes the middle of the potential well, so we will just write $E\psi$ since the potential between $-\frac{L}{2}$ and $\frac{L}{2}$ is zero. Looking at equation (26), we can see that it will be easier to solve for ψ and E if we let

$$k = \sqrt{\frac{2m}{\hbar^2} \cdot E}.$$

Now equation (26) looks like

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0$$
(27)

which now more closely resembles the second differential equations we have solved in previous sections. Since the second derivative of ψ is equal to its negative multiplied by k^2, we can deduce that ψ could be the complex exponential

$$\psi = e^{\pm ikx}, \quad (28)$$

In its full trigonometric form, (28) becomes

$\psi = A\cos kx + B\sin kx$, where $A = C_1 + C_2$ and $B = C_1 - C_2$.
(29)

In order to find the value of k, we will need to utilise the boundary conditions. Substituting the values $x = \frac{L}{2}$ and $x = -\frac{L}{2}$ and equating to zero gives the expressions

$$\psi(\tfrac{L}{2}) = A\cos\frac{kL}{2} + B\sin\frac{kL}{2} = 0,$$
(30)

$$\psi(-\tfrac{L}{2}) = A\cos\frac{kL}{2} - B\sin\frac{kL}{2} = 0. \quad (31)$$

Solving equations (30) and (31) simultaneously gives the following two possibilities:

$$2A\cos\frac{kL}{2} = 0,\ B = 0 \quad (32)$$

or

$$2B\sin\frac{kL}{2} = 0,\ A = 0.$$
(33)

So we have two sets of values for k. Let's solve (32) and (33) to see what type of values we get. Solving (32) gives

$\frac{kL}{2} = \pi(n + \frac{1}{2})$ where n is any integer

and solving (33) gives

$\frac{kL}{2} = \pi n.$

This tells us that k has discrete values. We can then infer that there are discrete energy E values corresponding to the wavefunction solutions since $E_n \sim k$. For each of the energy values or eigenvalues E there will be a wavefunction/eigenfunction containing the value E. These fixed values of energy tell us that the oscillation of the particle in region II is uniform. As defined previously, $E = hf$, so the frequency is also fixed. So, these solutions, or eigenfunctions are also called stationary states, as the energy for each solution is fixed. This is how we solve the time-independent Schrödinger equation, and its results allow us to solve the full Schrödinger equation (equation (25)).

For visual reference, here are rough diagrams showing the graph forms for the wavefunctions for the first few values of n:

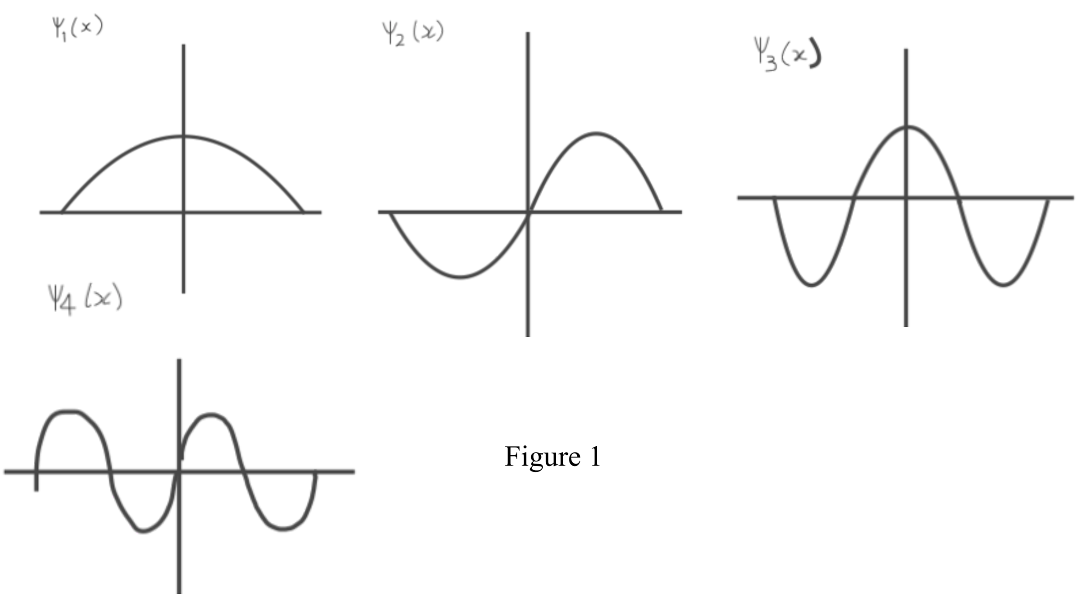

Figure 1

Note that the diagrams are rough and that these graphs actually have equal amplitudes. However as n increases by 1, the same scale fits another half wavelength. As time changes, these graphs will oscillate up and down, keeping their exact form.

Significance of the Wavefunctions/Eigenfunctions

Wavefunctions are useful because they are directly linked to probability. Specifically, *the square of a wavefunction is proportional to the "probability of finding the particle at a given point and time"* [6].

Looking back at the boundary condition from the infinite well problem we see that the condition $\psi(\pm L/2) = 0$ must be satisfied since the particle cannot enter the area where potential is infinite. The wavefunction is equal to zero because the probability of the particle existing at $x = \pm L/2$ is also zero.

The complex conjugate of $\Psi(x, t)$ is the wavefunction $\Psi^*(x, t)$. Let us consider the product of a solution $\Psi(x, t)$ and its conjugate $\Psi^*(x, t)$. This function, $\Psi(x, t)\Psi^*(x, t)$ is real and defined for all x values from $-\infty$ through to $+\infty$. We can postulate that the quantity $\Psi(x, t)\Psi^*(x, t)$is the *probability distribution function for the system.* $\Psi(x, t)\Psi^*(x, t)dx$ can therefore be used to represent the probability that the particle is in “the region between x and $x + dx$ at the time t” [5]. The probability density can also be written as $|\psi(x)|^2$.

4. Quantum Tunneling

Solving One-Dimensional Quantum Problems: Barrier tunneling

Now that we have gone through solving second order differential equations, classical wave concepts and Schrödinger's equation, we can finally look at how to solve the quantum problem of a particle tunneling through a potential barrier!

We looked at reflection and transmission at boundaries for classic waves. Since particles behave like waves, we can apply the same ideas about transmission and reflection to particles. Previously we discussed the boundaries being points on a string at which the string mass changes. In the case of particles, the boundaries are the positions where the potential changes. At these points there could be transmission and/or reflection. It acts similar to light going through a medium.

Unlike classical physics however, we have the possibility of quantum tunneling, where a particle penetrates a potential barrier to reach a classically inaccessible reigon. The probability of the particle existing in this third region decays, but doesn’t reach zero. Although the region is classically inaccessible, we will find that the behavior of the particle in quantum mechanics is analogous to attenuation in classical physics; a system we looked at in section two in which there is a mechanism for energy dissipation.

Tunneling is possible when the “region where $V > E$ is only of finite extent,” in which case “a particle may ‘leak’ through a potential barrier” [Bohm]. Now let’s look at the tunneling problem. We will take the potential

$$V(x) = \begin{cases} V_0 & \text{if } 0 \leq x \leq L, \\ 0 & \text{otherwise} \end{cases}$$

We will consider particles coming in from the left with $E < V_0$. Using the information given, we can represent the potential in three different regions with figure 1.

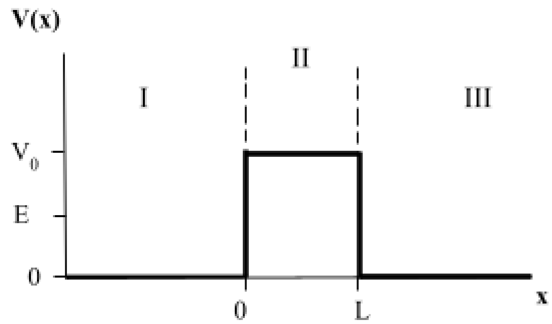

Figure 1

It is worth noting that the potential represented in Figure 1 is called a 'square potential' because of the 90 degree corners in the graph, however this type of potential is an approximation. In nature there wouldn't be square potentials as that would "imply an infinite force at the points of discontinuity in the potential." However, these graphs do help us illustrate actual systems with enough accuracy.

Now that we have our system set up, we want to find the *probability that particles tunnel* through to region three. Because of the wave nature of matter, there is a probability of transmission through the barrier and penetration of the boundary at $x = L$. The probability of tunneling is equal to the ratio of the rate of outgoing particles to incoming particles. Once we find the coefficients of the waves at each region, we will be able to solve for the transmission coefficient, T. So we first need to find out the solutions for each region's Schrödinger equation.

First let's write the time independent Schrödinger equation for each region as we did with the infinite well example in section three. We have

Region I: $x < 0 \Rightarrow \frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(-E)\psi,$ (1)

Region II: $0 < x < L \Rightarrow \frac{d^2\psi}{dx^2}(V_0 - E)\psi,$

(2)

Region III: $x > L \Rightarrow \frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(-E)\psi.$

(3)

Now let us find the two independent solutions for each of these regions.

Equations (1), (2) and (3) are all second order differential equations. At this point we are more familiar with this form of equation. So, we can straight away see that equations (1) and (3) will have solutions in the form of complex exponentials since there must be an imaginary number in the power for ψ to satisfy the negative relationship with its second derivative. Furthermore, solutions to equation (2) will be exponentials, but not complex.

For equations (1) and (3) lets define

$$k^2 = \frac{2mE}{\hbar^2}.$$

So for region I we have

$$\psi = Ee^{\pm ikx} \tag{4}$$

which we can then write as two independent solutions

$$\psi = De^{ikx}$$
$$\psi = Fe^{-ikx}. \tag{5}$$

Similarly, the independent solutions for region III are

$$\psi = Ae^{ikx}$$
$$\psi = A_1e^{-ikx}. \tag{6}$$

For region two, we will define

$$\kappa^2 = \frac{2m(V_0-E)}{\hbar^2}.$$

Now we have the equation

$$\frac{d^2\psi}{dx^2} = \kappa^2\psi, \tag{7}$$

which has the two independent solutions

$$\psi = Be^{\kappa x}$$
$$\psi = Ce^{-\kappa x}. \tag{8}$$

A, A_1, B, C, D and F are all undetermined coefficients. We will leave the solutions as complex exponents for simplicity in the consequent steps.

For region III, $x > L$, we can actually eliminate one of the solutions. The wavefunction in region III can only represent transmitted particles moving towards the right, as the particles enter the first region only from the left. Therefore the solution for region III, $\psi = A_1 e^{-ikx}$ can be rejected as a solution.

Now we can move on to finding the coefficients A, B, C, D and F. We will not be able to find actual values, but we can form equations in order to express the coefficients in terms of each other. The first two equations we can form come from using two important conditions of the wavefunction: ψ and $\frac{d\psi}{dx}$ must be continuous at $x = L$. These are the same conditions that we had for the transverse wave on a string. We can now write equations relating the solutions of region II and III.

So we have

$$Be^{\kappa L} + Ce^{-\kappa L} = Ae^{ikL} \tag{9}$$

and

$$\kappa Be^{\kappa L} - \kappa Ce^{-\kappa L} = ikAe^{ikL}. \tag{10}$$

Solving (9) and (10) simultaneously gives us

$$2\kappa Be^{\kappa L} = (\kappa + ik)Ae^{ikL}.$$
(11)

Rearranging for B gives

$$B = \frac{(\kappa+ik)}{2\kappa}Ae^{(ik-\kappa)L}. \tag{12}$$

We also get

$$2\kappa Ce^{-\kappa L} = (\kappa - ik)Ae^{ikL} \tag{13}$$

so

$$C = \frac{(\kappa-ik)}{2\kappa}Ae^{(\kappa+ik)L}. \tag{14}$$

For a range of parameters, we can actually ignore the smaller of B and C. Comparing B and C we see that the only difference in terms of magnitude is the exponential terms; $e^{(ik-\kappa)L}$ and $e^{(\kappa+ik)L}$. The negative sign in the power of the exponential term of B indicates that C must have the greater magnitude. The factor $e^{(ik-\kappa)L}$ will be very small when either V_0 is much greater than E or when L is very large. So B is actually negligible for these range of parameters.

Now let us write the equations for continuity of ψ and $\frac{d\psi}{dx}$ at $x = 0$ so that we can find the constants D and F in terms of B and C. We have

$$\psi\text{: } D + F = B + C. \tag{15}$$

$$\frac{d\psi}{dx}\text{: } \frac{i\sqrt{2mR}}{h}D - \frac{i\sqrt{2mE}}{h}E = \frac{\sqrt{2m(v_0-E)}}{h}B - \frac{\sqrt{2m(v_0-E)}}{h}C.$$
(16)

If we multiply equation (15) by $i\sqrt{E}$ and solve simultaneously with equation (16) we get

$$F = \frac{(\sqrt{V_0-E} - i\sqrt{E})B - (\sqrt{V_0-E} + i\sqrt{E})C}{-2i\sqrt{E}} \tag{17}$$

and

$$D = \frac{(\sqrt{V_0-E} + i\sqrt{E})B + (i\sqrt{E}-\sqrt{V_0-E}\,)C}{2i\sqrt{E}} \tag{18}$$

As we found before, the value of B is negligible for certain parameters so let us set $B = 0$. Now we have the two equations

$$D = \frac{(1+i\sqrt{\frac{V_0-E}{E}})}{2}C \tag{19}$$

and

$$F = \frac{(1-i\sqrt{\frac{V_0-E}{E}})}{2}C. \tag{20}$$

Now we have D and F in terms of C, and C in terms of A. Our general aim is to find the transmission coefficient i.e the probability of tunneling. We know A is the coefficient of the wave equation of the outgoing wave because it has a positive power and is therefore moving from left to right. Further, D is the coefficient for the incoming wave, so we can see that these are the two coefficients that are important to us. So let us now write D in terms of A. Substituting the value from equation (14) into equation (19) and writing κ and k in their full forms gives

$$D = \frac{(1+i\sqrt{\frac{V_0-E}{E}})}{2}\frac{(\sqrt{2m}(\sqrt{V_0-E}-i\sqrt{E})}{2\sqrt{2m(V_0-E)}}Ae^{\frac{\sqrt{2m}}{\hbar}(\sqrt{V_0-E}+i\sqrt{E})L}. \tag{21}$$

Simplifying and rearranging to get D and A on the same side we have

$$\frac{D}{A} = \frac{1}{2}\left(1 + i\sqrt{\frac{V_0-E}{E}}\right)\frac{\sqrt{V_0-E}-i\sqrt{E}}{2\sqrt{V_0-E}}e^{\frac{\sqrt{2m}}{\hbar}(\sqrt{V_0-E}+i\sqrt{E})L}. \qquad (22)$$

In section three we discussed that the probability density could be found by finding the square of the magnitude of the wavefunction in the last region. For this problem we want to find the transmission coefficient, the ratio of the rate of outgoing particles to incoming particles. As a ratio, this is equal to $\frac{|A|^2}{|D|^2}$. Starting from equation (22) we get

$$\frac{|D|^2}{|A|^2} = \frac{1}{4}\left(1 + \frac{V_0-E}{E}\right)\left(\frac{V_0-E+E}{4(V_0-E)}\right)e^{2\sqrt{\frac{2m(V_0-E)}{\hbar^2}}L}$$
(23)

which simplifies to

$$\frac{|D|^2}{|A|^2} = \frac{V_0}{4E}\cdot\frac{V_0}{4(V_0-E)}e^{\sqrt{\frac{2m(V_0-E)}{\hbar^2}}2L}.$$
(24)

So

$$T = \frac{|A|^2}{|D|^2} = \frac{16E(V_0-E)}{V_0^2}e^{-2\sqrt{\frac{2m(V_0-E)}{\hbar^2}}L}. \qquad (25)$$

This is the probability of tunneling! We should be able to confirm two important pieces of information from equation (25). Firstly, we should see that when the length of the boundary L is large and when $V_0 - E$ is large, tunneling becomes less likely. Due to the negative power in the exponential factor, we can confirm that this is the case. Secondly, tunneling is certain when there is no potential barrier, so equation (25) should satisfy the condition that when $V_0 = 0,\ T = 1.$ Since we have V_0^2 in the denominator, this relationship will not be satisfied in our equation. This seems unusual but looking back at our initial wave functions, we assumed $V_0 > E$ from the beginning. The solution depends on $\sqrt{\frac{2m(V_0-E)}{\hbar^2}}$ being real, but when we have $V_0 = 0$ instead of $V_0 > E$, this condition breaks and the calculation is therefore invalid for that value.

Generally, this method can be used for most 1-D tunneling problems, provided we know the same pieces of information. Once one finds and substitutes the specific values for the particle being calculated for (mass and energy) as well as the potential of the system, a probability value can be found. However, this paper shows the base-level understanding of quantum tunneling and solves the barrier problem in the simplest, one-dimensional case. For applications of quantum tunneling to two-dimensional systems or to an atomic nucleus, a three-dimensional system, more advanced methods are required to predict the outcomes. Furthermore, we had assumed a ‘square’ potential for simplicity, but in actuality a potential barrier is not typically flat, and would need more intricate calculations. Qualitatively though, these systems behave in a similar way.

Given more time, some further explorable topics relating to the applications of quantum tunneling could include its role in flash memory where particles tunnel preferentially when voltage is applied. Specifically, the MOSFET devices used in this incorporate tunneling particles in their design which allows for “considerable power savings” [8]. Quantum tunneling also explains radioactive decay, where particles tunnel despite the attractive nuclear force they are seemingly bound within.

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