On the construction of a convex ideal polyhedron in hyperbolic 3-space

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Overview

1. Models of $\mathbb{H}^3$
2. Our question
3. Constructing a convex ideal cube

Models of $\mathbb{H}^3$

Models of $\mathbb{H}^3$

Upper Half-space Model: $\{(z,t) : z \in \mathbb{C}, t > 0\}$

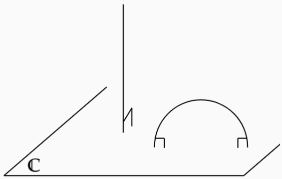


Figure 1: hyperbolic lines

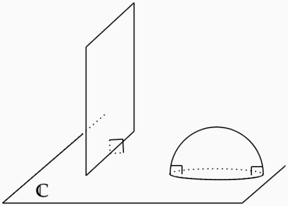


Figure 2: hyperbolic planes

- hyperbolic angles: same as euclidean angles
- hyperbolic lines: euclidean semicircles with bases on the boundary & vertical half-lines
- hyperbolic planes: euclidean hemispheres with bases on the boundary & vertical half-planes

Models of $\mathbb{H}^3$ (contd.)

Ball Model: $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}$

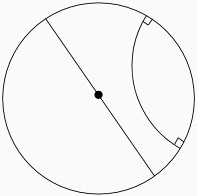


Figure 3: hyperbolic lines

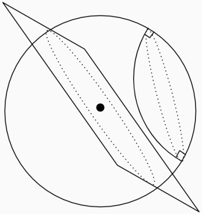


Figure 4: hyperbolic planes

- hyperbolic angles: same as euclidean angles
- hyperbolic lines: euclidean circular arcs orthogonal to the boundary & spherical diameters
- hyperbolic planes: euclidean spherical caps orthogonal to the boundary & planes containing the center of the ball

Example: A Cube in the Ball Model

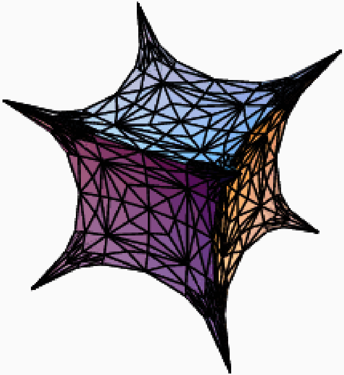


*image from Wolffram website

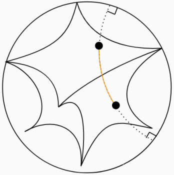
Our question

Given a set of appropriate internal dihedral angles,

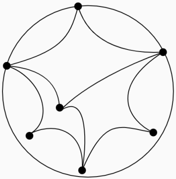
how do we construct a <u>convex</u> <u>ideal</u> polyhedron in $\mathbb{H}^3$?

Important things to note

- convex polyhedron



- ideal polyhedron



- appropriate set of dihedral angles $\rightarrow$
  a set of dihedral angles that belongs to a unique (up to isometry) convex ideal polyhedron[3]
- isometry $\rightarrow$
  distance preserving map
  Ex. isometries of euclidean plane: reflection, rotation, translation...

Our Question (revisited)

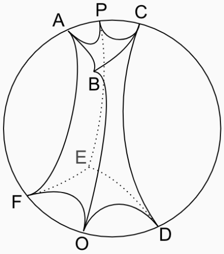
Given a set of appropriate internal dihedral angles,
how do we construct a convex ideal polyhedron in $\mathbb{H}^3$?

Constructing a convex ideal cube

Cuboid

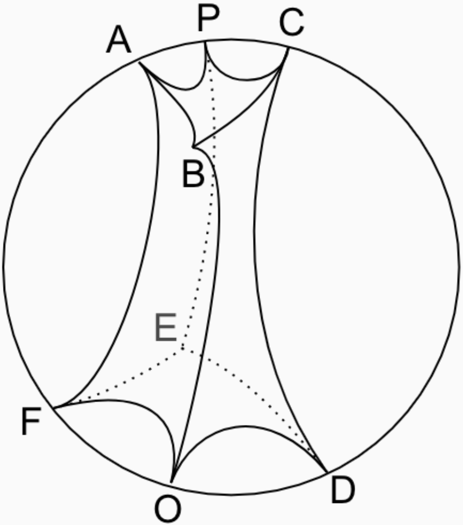
Cuboid:
a polyhedron with

- the same combinatorial structure as a cube
- six faces each consisting of four edges
- each vertex incident to three faces



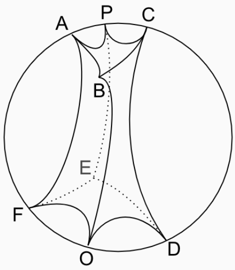
$\rightarrow$ I will use the word cube to mean cuboid.

Ball Model $\rightarrow$ Upper Half-space Model

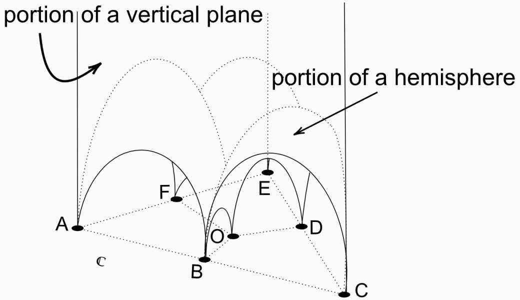


Ball Model → Upper Half-space Model (contd.)

Ball Model



Upper half-space model



Result of euclidean spherical inversion (center P & radius 2) and euclidean planar reflection (complex plane):

- Faces containing P → portions of vertical half-planes
- Faces not containing P → portions of hemispheres

Lemmas for Internal Dihedral Angles → Planar Angles

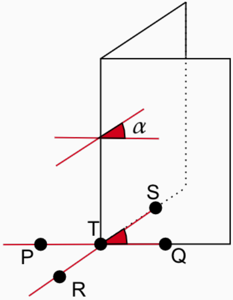


Figure 5: P-P

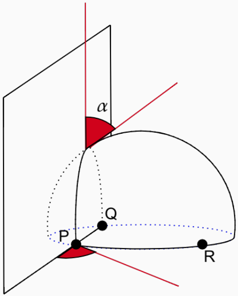


Figure 6: P-S

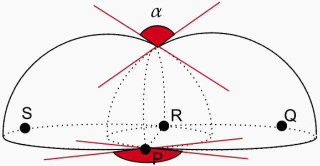
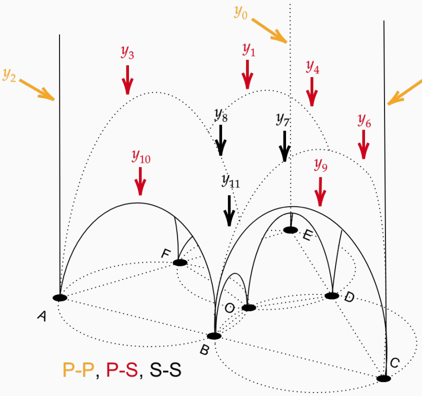


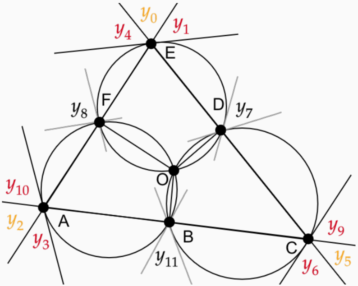
Figure 7: S-S

*P: plane, S: sphere

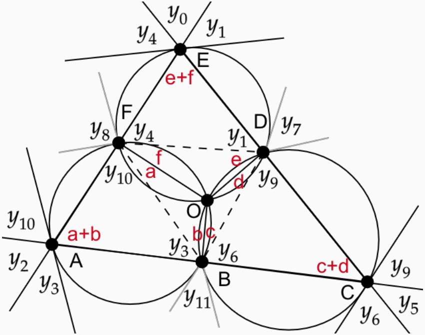
Internal Dihedral Angles → Planar Angles

internal dihedral angles: $\{y_0, y_1, \ldots, y_{11}\}$





System of Equations



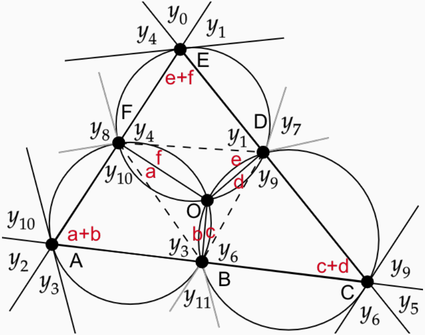
using results from euclidean plane geometry...

$$\text{P-P:}\quad \begin{cases} a+b=y_2 \\ c+d=y_5 \\ e+f=y_0 \end{cases}$$

$$\text{S-S:}\quad \begin{cases} a+d=y_{11} \\ c+f=y_7 \\ e+b=y_8 \end{cases}$$

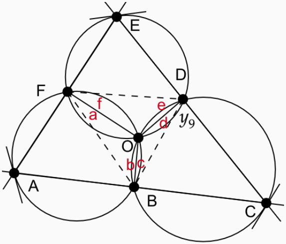
$$\text{P-S:}\quad \begin{cases} a+f+y_4+y_{10}=\pi \\ d+e+y_9+y_1=\pi \\ b+c+y_3+y_6=\pi. \end{cases}$$

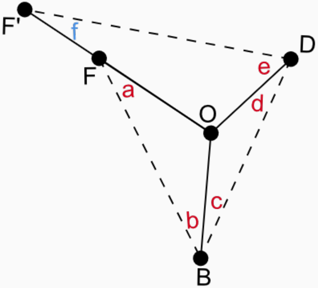
System of Equations (contd.)



$$\begin{cases} a = y_{11} - y_5 + y_7 - f \\ b = y_8 - y_0 + f \\ c = y_7 - f \\ d = y_5 - y_7 + f \\ e = y_0 - f \end{cases}$$

One more equation





We need $OF = OF'$.

Since

$$OF' = OF \cdot \frac{\sin a \cdot \sin c \cdot \sin e}{\sin b \cdot \sin d \cdot \sin f}$$

is true, we need

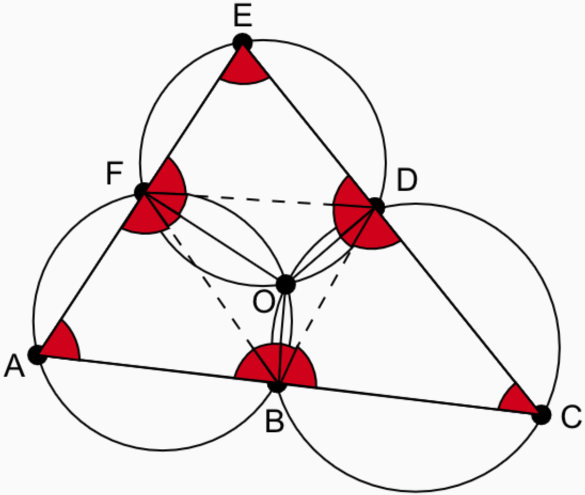
$$\underline{\sin f \cdot \sin d \cdot \sin b - \sin e \cdot \sin c \cdot \sin a = 0.}$$

$\rightarrow$ solve in terms of f

$\rightarrow$ Done.

Basically done!

Finally, we have obtained the locations of the vertices up to isometry.



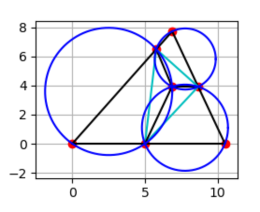
Completing the Construction

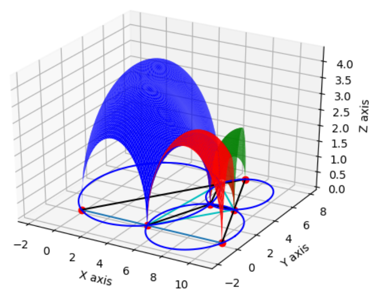
A set of internal dihedral angles:

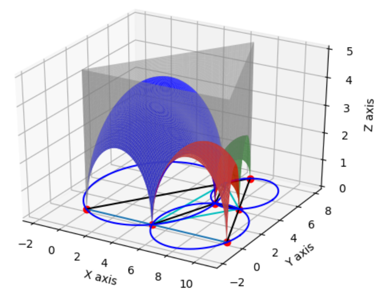
$$\begin{aligned}
y_0 &= \pi - 1.9748981268459183, & y_1 &= \pi - 2.7384076996659408, \\
y_2 &= \pi - 2.2979863709366652, & y_3 &= \pi - 1.4516735513314263, \\
y_4 &= \pi - 1.5698794806677308, & y_5 &= \pi - 2.0103008093970063, \\
y_6 &= \pi - 2.322152710378157, & y_7 &= \pi - 2.391153116133702, \\
y_8 &= \pi - 2.0931040561822227, & y_9 &= \pi - 1.9507317874044263, \\
y_{10} &= \pi - 2.5335253849114983, & y_{11} &= \pi - 1.7989281348636652.
\end{aligned}$$

Fix the length of an edge
$\rightarrow$ we chose $A(0, 0, 0)$ and $B(5, 0, 0)$

Convex Ideal Cube (upper half-space model)

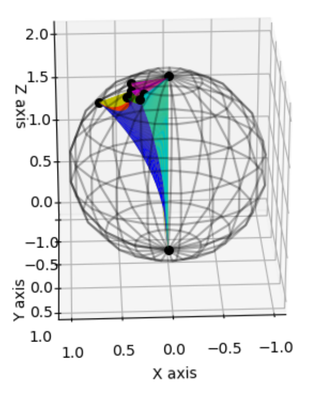


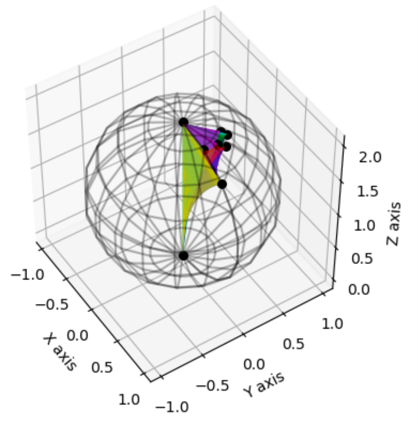




*Images produced through our Python code

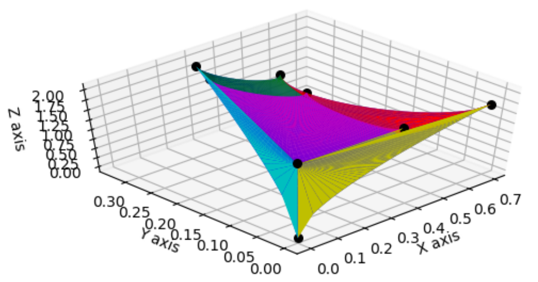
Convex Ideal Cube (ball model)





*Images produced through our Python code

Cube zoomed-in (different aspect ratio)



*Images produced through our Python code

Conclusion

- We may be able to extend the method to other polyhedra but may have more nonlinear equations.

References

[1] Cannon, J. W., Floyd, W. J., Kenyon, R., & Parry, W. R. (n.d.). *Hyperbolic Geometry (Vol. 31, Flavors of Geometry)*. MSRI Publications.
[2] Marden, A. (2007). *Outer circles: An introduction to hyperbolic 3-manifolds.* Cambridge: Cambridge University Press.
[3] Rivin, I. (1996). *A Characterization of Ideal Polyhedra in Hyperbolic 3-Space.* The Annals of Mathematics, 143(1), 51. doi:10.2307/2118652
[4] Hodgson, C. D., Rivin, I., & Smith, W. D. (1992). A characterization of convex hyperbolic polyhedra and of convex polyhedra inscribed in the sphere. Bulletin of the American Mathematical Society, 27(2), 246-252.
[5] Thurston, W. P., & Levy, S. (1997). *Three-dimensional geometry and topology.* Princeton, NJ: Princeton University Press.
[6] Online Mathematics Editor a fast way to write and share mathematics. (n.d.). Retrieved from https://www.mathcha.io/

Acknowledgments

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NCUWM

Thank you for listening!

Any questions?

Additional Slides

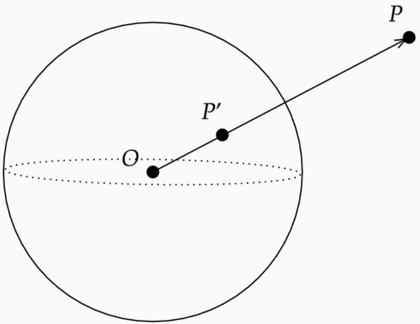
Additional Slides 1

Theorem (Rivin [4])
Let P be a polyhedral graph with weights $w(e)$ assigned to the edges. Let P^* be the planar dual of P, where the edge e^* dual to e is assigned the dual weight $w^*(e^*)$. Then P can be realized as a convex ideal polyhedron in $\mathbb{H}^3$ with dihedral angle $w(e) = \pi - w*(e*)$ at every edge e if and only if the following conditions hold:

> *Condition 1.* $0 < w(e^*) < \pi$ *for all edges* e^* *of* P^*.
>
> *Condition 2. If the edges* $e_1^*, e_2^*, ..., e_k^*$ *form the boundary of a face of* P^*, *then* $w(e_1^*) + w(e_2^*) + \cdots + w(e_k^*) = 2\pi$.
>
> *Condition 3. If* $e_1^*, e_2^*, ..., e_k^*$ *form a simple circuit which does not bound a face of* P^*, *then* $w(e_1^*) + w(e_2^*) + \cdots + w(e_k^*) > 2\pi$.

Additional Slides 2

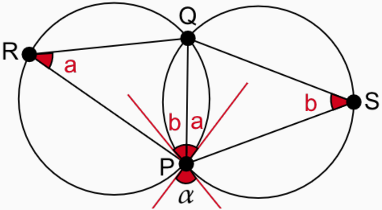
Spherical Inversion:

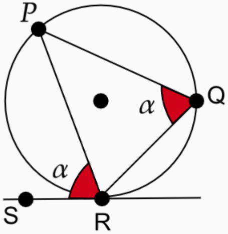


Let S be a sphere with center O and radius r. If a point P is not O, the image of P under inversion with respect to S is the point P' lying on the ray OP such that $OP \cdot OP' = r^2$.

Additional Slides 3

Euclidean plane geometry results:

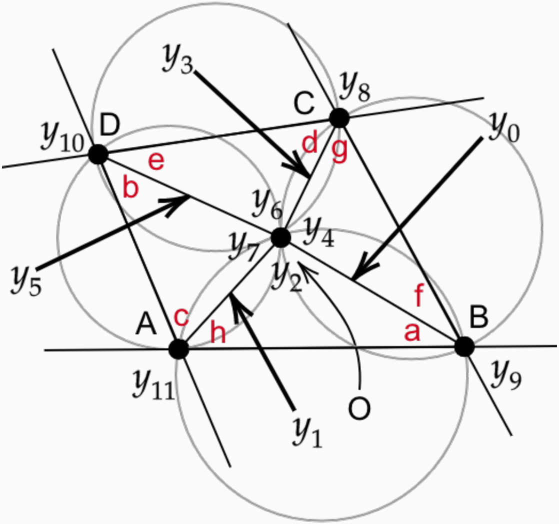




$$a + b = \alpha$$

Additional Slides 4

Vertices of hyperbolic “octahedron” in upper half-space model:



Additional Slides 5

Vertices of hyperbolic “dodecahedron” in upper half-space model: