Does non-stationary spatial data always require non-stationary random fields?

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Summary

Real world processes have spatially varying second-order structure, but is modeling this non-stationarity worth it?

The authors develop a novel model for non-stationary covariance structure and illustrate methods for parameterizing the model. They then apply their model to US precipitation data and compare predictions from their stationary and non-stationary models. They conclude by recommending careful consideration of the sources of non-stationarity and encourage balance between fitting complicated/flexible models and fitting simple/smarter models.

Classical Approaches to Non-stationarity and Anisotropy

- In class, we have seen non-stationarity in the mean, and how to account for it; the topic of this paper is to address non-stationarity in the covariance structure. Anisotropy is a common violation to non-stationarity, and refers to the setting where the association between two locations does not only depend upon distance, but also upon direction
- Addressing non-stationarity in the covariance > - In a seminal paper Sampson and Guttorp (1992) introduced an approach for non-stationarity through the **deformation method**: transform the geographic region D to a new region G If C denotes the isotropic covariance function on G, we have:

  $$\text{cov}(Y(s), Y(s')) = C(||g(s) - g(s')||)$$

  with C a standard class of covariance function, and

- In the paper we are presenting today, the authors introduce a novel approach building on the idea of a **local deformation** via **SPDE**

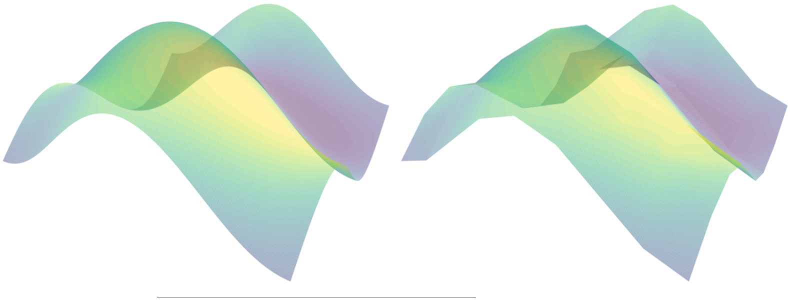
Stationary SPDE

The following equation defines a stochastic partial differential equation (SPDE), $u(\vec{s})$, whose solution is the Matérn covariance function

$$(\kappa^2 - \nabla \cdot \nabla)u(\vec{s}) = \sigma \mathcal{W}(\vec{s}), \qquad \vec{s} \in \mathbb{R}^2$$

Where κ and $\sigma > 0$ are constants, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)^T$ and $\mathcal{W}$ is a standard Gaussian white noise process. This correlation structure is isotropic because the Laplacian, $\Delta = \nabla \cdot \nabla$ is equal to the sum of the diagonal elements of the Hessian, is invariant to a change of coordinates that involves rotation and translation. The solution to this SPDE is a class of equations that have covariance described by the Matérn covariance function.

GMRF Approximation



The graph above displays a true continuously-indexed Gaussian Field and its discrete approximation

Model for Non-stationarity

The authors introduce a 2×2 matrix $\mathbf{H}$ into the SPDE which acts as a transformation of the grid on top of which we are measuring distance

$$(\kappa^2 - \nabla \cdot \mathbf{H}\nabla)u(\vec{s}) = \sigma\mathcal{W}(\vec{s}), \qquad \vec{s} \in \mathbb{R}^2$$

This results in an updated covariance function

$$r(\vec{s}_1, \vec{s}_2) = \frac{\sigma^2}{4\pi\kappa^2\sqrt{\det(\mathbf{H})}} \left(\kappa||\mathbf{H}^{-1/2}(\vec{s}_2 - \vec{s}_1)||\right) K_1 \left(\kappa||\mathbf{H}^{-1/2}(\vec{s}_2 - \vec{s}_1)||\right)$$

Parameters κ and $\mathbf{H}$ control the marginal variance and directionality of correlation, allowing σ to fall out of the SPDE formmula. The $\sqrt{\det(\mathbf{H})}$ that appears in the denominator of the covariance function is a consequence of the change of variable.

2D-Random Walk Penalty

To enforce smoothness of parameters across space, the authors introduce a second-order penalty into their model for the spatially-specific covariance parameters:

$$-\Delta\beta(\vec{s}) = \mathcal{W}_\beta(\vec{\int})/\sqrt{\tau_\beta}$$

where $\beta(\vec{s})$ is the location-specific value for parameter β and

$$\log(\beta(\vec{s})) = \sum_{i=1}^{k}\sum_{j=1}^{l}\alpha_{ij}f_{ij}(\vec{s})$$

where $\{\alpha_{ij}\}$ are the parameters for real-valued basis functions $\{f_{ij}\}$.

$$\vec{\alpha} \sim \mathcal{N}_{\|\updownarrow}\left(\vec{\imath}, \mathbf{Q}_{\mathrm{RW2}}^{-\infty}/\tau_\beta\right)$$

Full Hierachical Model

Observations: - Outcome: $\vec{y} = (y_1, \ldots, y_n)$, at locations $\vec{s_1}, \ldots, \vec{s_n}$
- Predictor: $X = (x(\vec{s_1}), \ldots, x(\vec{s_n}))$ - Spatial field $\vec{u}$ a GMRF -
$E = (e(\vec{s_1}), \ldots, e(\vec{s_n}))$

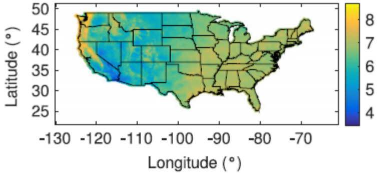
Stage 1: $\vec{y}|\vec{\beta}, \vec{u}, \log(\tau_{\text{noise}}) \sim \mathcal{N}_N(\mathbf{X}\vec{\beta} + \mathbf{E}\vec{u}, \mathbf{I}_N/\tau_{\text{noise}})$

Stage 2: $\vec{u}|\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3, \vec{\alpha}_4 \sim \mathcal{N}_{nm}(\vec{0}, \mathbf{Q}^{-1})$, $\quad \vec{\beta} \sim \mathcal{N}_p(\vec{0}, \mathbf{I}_p/\tau_\beta)$

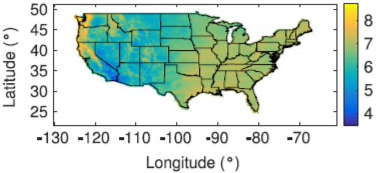
Stage 3: $\vec{\alpha}_i|\tau_i \sim \mathcal{N}_{kl}(\vec{0}, \mathbf{Q}_{\text{RW2}}^{-1}/\tau_i)$ for $i = 1, 2, 3, 4$

where $\tau_1, \tau_2, \tau_3, \tau_4$ and τ_{beta} are penalty parameters that must b

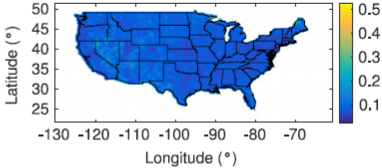
Comparing stationary and non-stationary models



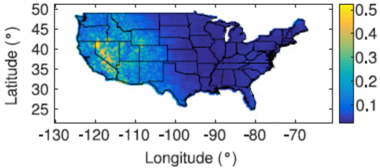
(a) Prediction for the stationary model.



(b) Prediction for the non-stationary model.



(c) Prediction standard deviations for the stationary model.



(d) Prediction standard deviations for the non-stationary model.

Implementation of the paper

We have fitted a stationary model to the data, getting the following results: