Vectores en Variedades

$X = X^i \frac{\partial}{\partial x^i}$ $\quad X[F] = X^i \frac{\partial}{\partial x^i} F|_p$

X^i componentes de X, $\{\frac{\partial}{\partial x^i}\} = \{e_i\}$ → Bases de $T_p(M)$

X → operador en $T_p(M)$

$$V = v^i \frac{\partial}{\partial x^i} = \tilde{v}^i \frac{\partial}{\partial y^i} = \tilde{v}^i \frac{\partial x^j}{\partial y^i}\frac{\partial}{\partial x^j}$$

$V^k = \tilde{V}^i \frac{\partial x^k}{\partial y^i}$ → vector transformado

Covectores, vectores duales o 1-formas

- 1-Forma. $\omega: T_pM \to \mathbb{R}$

$\langle \omega, v\rangle : T_pM \to \mathbb{R}$. esta es una 1-forma que nos permite constuir la 1-Forma.

- df: diferencial de f

$d(f+g) = df + dg$ $\qquad \langle df, v\rangle \equiv v[f]$ (1.1)

$d(\alpha f) = \alpha\, df$

$\langle df, v\rangle \equiv v[f] \equiv v^i \frac{\partial}{\partial x^i} f \in \mathbb{R}$

$df = \frac{\partial f}{\partial x^i} dx^i$ → diferencial en coordenadas x.

$df = f_i\, dx^i$ $\quad f_i$ → componentes de df
$\{dx^i\}$ → bases T_p^*M

$\langle dx^i, \frac{\partial}{\partial x^i}\rangle =$

De (1.1) $\langle df, v\rangle = \langle \frac{\partial f}{\partial x^i} dx^i, v^j \frac{\partial}{\partial x^j}\rangle = \left(v^k \frac{\partial}{\partial x^k}\right) f$

$v^j \frac{\partial f}{\partial x^i}\langle dx^i, \frac{\partial}{\partial x^j}\rangle = v^k \frac{\partial}{\partial x^k} f$ $\qquad \left(\frac{\partial f}{\partial x^i}\right) = f_i$

$\boxed{\langle dx^i, \frac{\partial}{\partial x^j}\rangle = \delta^i_j}$ (1.2)

1-Forma arbitraria ω:

$\omega = \omega_i dx^i \to \langle \omega, v\rangle = \langle \omega_i dx^i, v^k \frac{\partial}{\partial x^k}\rangle$

$= \omega_i v^k \langle dx^i, \frac{\partial}{\partial x^k}\rangle = \omega_i v^i$

$\langle \omega, v\rangle : T_p^*M \otimes T_pM \to \mathbb{R}$

- $\langle \omega_1 + \omega_2, v\rangle = \langle \omega_1, v\rangle + \langle \omega_2, v\rangle$
- $\langle \omega, v_1 + v_2\rangle = \langle \omega, v_1\rangle + \langle \omega, v_2\rangle$

$\omega = \underbrace{\omega_i} dx^i = \tilde{\omega}_i dy^i = \tilde{\omega}_i \underbrace{\frac{\partial y^i}{\partial x^k}} dx^k$

$\boxed{\omega_i = \tilde{\omega}_j \frac{\partial y^j}{\partial x^k}}$ (1.3)

Tensores en los productos $T_p^*M \otimes T_pM$

Tensor de tipo (q,r)

$$T = T^{\mu_1\mu_2\cdots\mu_q}{}_{\nu_1\nu_2\cdots\nu_r} \frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_q}} \otimes dx^{\nu_1} \otimes dx^{\nu_r}$$

$\otimes^q T_pM = T_pM \otimes \overbrace{\cdots}^{q\text{-veces}} \otimes T_pM$

Función multilineal: $\otimes^q T_pM \otimes^r T^*_pM \to \mathbb{R}$

$$T(\omega_1, \dots, \omega_r; v_1, \dots, v_q) = T^{\mu_1\mu_2\cdots\mu_r}{}_{\nu_1\nu_2\cdots\nu_q}\, \omega_{1\mu_1} \cdots \omega_{r\mu_r}\, v_1^{\nu_1} \cdots v_q^{\nu_q}$$

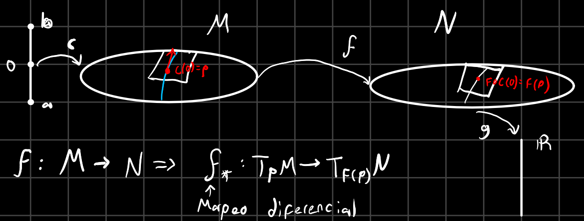
Caso particular: $\boxed{\omega(v) = \langle \omega, v\rangle = \omega_\mu v^\mu}$

$T \in \mathcal{T}^q_{r,p}(M)$: Conjunto de tensores de tipo (q,r) suaves en el punto p.

Ley de transformación ante cambios de coord. $x(z)$:

$$T^{\alpha_1\cdots\mu_q}{}_{\nu_1\cdots\nu_r} = \tilde{T}^{\alpha_1\cdots\alpha_q}{}_{\beta_1\cdots\beta_r} \frac{\partial x^{\mu_1}}{\partial z^{\alpha_1}} \cdots \frac{\partial x^{\mu_q}}{\partial z^{\alpha_q}} \frac{\partial z^{\beta_1}}{\partial x^{\nu_1}} \cdots \frac{\partial z^{\beta_r}}{\partial x^{\nu_r}}$$

Comportamiento de tensores ante mapeos



$f: M \to N \Rightarrow f_*: T_pM \to T_{f(p)}N$
↑ Mapeo diferencial

M y N no deben ser biyectivas, simplemente inyectivas.

$v \in T_pM \to f_* v \in T_{f(p)}N$

$(f_* v)[g] \equiv v[g \circ f]$ $\qquad g \circ f \in \mathcal{F}(M)$

$(f_* v)[g(f(p))] = v[g(f(p))]$

Diffeomorfismos de un parámetro

Dado un vector $X^\mu \in T_pM$ $\quad X = X^\mu \frac{\partial}{\partial x^\mu}$

si X^μ es $C^\infty \Rightarrow \mathfrak{X}(M)$ campos vectoriales suaves en M.

$$\frac{d}{dt}\sigma^\mu(t,x_0) = X^\mu(\sigma(t,x_0))$$

Ejemplo: $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$

$$\frac{d\sigma^\mu}{dt} = (-y, x) \rightsquigarrow \frac{dx}{dt} = -y,\ \frac{dy}{dt} = x$$

$$\underbrace{}_{X(\sigma(t,x_0))}$$

$$\frac{d^2x}{dt^2} = -x \Rightarrow x = C_1\cos(t) + C_2\,\mathrm{sen}(t)$$

$$x(0) = \underline{x_0 = C_1} \qquad C_2 = -y_0 \qquad \begin{array}{l} x = x_0\cos t - y_0\,\mathrm{sen}\,t \\ y = x_0\,\mathrm{sen}\,t + y_0\cos t \end{array}$$

$$\frac{dx}{dt} = -y,\ \frac{dy}{dt} = x$$

$$\frac{dx}{-y} = \frac{dy}{x} \rightarrow x\,dx + y\,dy = 0 \Rightarrow \frac{1}{2}(x^2+y^2) = \text{cte.}$$

$\sigma(t,x) : \sigma_t : \mathbb{R}\times M \longrightarrow M$ Flujo

$$\sigma(t,\sigma(s,x_0)) = \sigma(t+s,x_0)$$

$$\frac{d\sigma^\mu}{dt}[t,\sigma(s,x_0)] = X^\mu(\sigma(t,\sigma(s,x_0))) \qquad \sigma^\mu(0,\sigma(s,x_0)) = \sigma^\mu(s,x_0)$$

$$\frac{d}{dt}\sigma(t+s,x_0) = X(t+s,x_0)$$

$$\frac{d}{du}\sigma(u,x_0)\frac{du}{dt} = \frac{d}{d(t+s)}\sigma(t+s,x_0) = \frac{d}{dt}\sigma(t+s,x_0) = X(\sigma(t+s,x_0))$$

Estructura de grupo

$g_1\cdot g_2 \in G \qquad \exists\, \mathbb{1} \Rightarrow 1\cdot g = g\cdot 1 = g$

$g^{-1}\cdot g = \mathbb{1}$

- $\sigma_t(\sigma_s(x)) = \sigma_t \circ \sigma_s(x) = \sigma_{t+s}(x)$
- $\sigma_0 = 1.$ $\qquad$ Grupo de difeomorfismos de un parámetro
- $\sigma_{-t} = (\sigma_t)^{-1}$

$$\sigma^\mu(t,x) = e^{tX}x^\mu$$

Derivada de Lie

$$\mathcal{L}_X Y = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[(\sigma_{-\varepsilon})_* Y\big|_{\sigma_\varepsilon(x)} - Y|_x\right]$$

$$Y\big|_{\sigma_\varepsilon(x)} = Y^\mu(\sigma_\varepsilon(x))\,e_\mu\Big|_{x^\mu + \varepsilon X^\mu}$$

$$Y^\mu(x^\alpha + \varepsilon X^\alpha) = Y^\mu(x) + \varepsilon X^\alpha\frac{\partial}{\partial x^\alpha}Y^\mu$$

$$Y\big|_{\sigma_\varepsilon(x)} = \left(Y^\mu(x) + \varepsilon X^\alpha\frac{\partial}{\partial x^\alpha}Y^\mu\right)e_\mu\big|_{x+\varepsilon X}$$

$$(\sigma_{-\varepsilon})_* Y\big|_{\sigma_\varepsilon(x)} = \left(Y^\mu(x) + \varepsilon X^\beta\frac{\partial}{\partial x^\beta}Y^\mu\right)\left(\frac{\partial}{\partial x^\mu}(x^\alpha - \varepsilon X^\alpha)\right)e_\alpha\big|_x$$

$$= \left[Y^\mu(x) + \varepsilon X^\beta\partial_\beta Y^\mu\right]\left(\delta^\alpha_\mu - \varepsilon\frac{\partial X^\alpha}{\partial x^\mu}\right)e_\alpha\big|_x$$

$$= \left[Y^\alpha + \varepsilon\left(X^\beta\partial_\beta Y^\alpha - \varepsilon\, Y^\mu\partial_\mu X^\alpha\right)\right]e_\alpha\big|_x$$

$$\mathcal{L}_X Y = X^\beta\partial_\beta Y^\alpha - Y^\mu\partial_\mu X^\alpha.$$