AlmonacidTaller1

1. (Schwabl) Considere los operadores

$$p_i = \frac{\hbar}{i}\partial_i, \quad L_i = \epsilon_{ijk}x_j p_k, \quad \text{donde} \quad i,j,k = 1,2,3,$$

y donde:

$$\epsilon_{ijk} = \begin{cases} +1 & \text{si} \quad \text{sgn}\,\sigma(ijk) = +1 \\ -1 & \text{si} \quad \text{sgn}\,\sigma(ijk) = -1 \\ 0 & \text{si} \quad i = j \quad \text{para algún par } (i,\, j) \end{cases}$$

es el tensor de Levi-Civita o tensor totalmente antisimétrico. Calcule los conmutadores:

$$[p^2,\, f(x)], \quad [L_i,\, L_j],$$

donde $p^2 = p_x^2 + p_y^2 + p_z^2$.
Nota: Para el cálculo del segundo conmutador es conveniente usar las propiedades del tensor de Levi-Civita:

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.$$

si $p_i = \frac{\hbar}{i}\partial_i$ y $p^2 = p_x^2 + p_y^2 + p_z^2 = -\hbar^2 \partial_x^2$

$$[p^2, F(x)] \Rightarrow [p^2, F(x)]\Psi = \left(p_i[p_i, F(x)] + [p_i, F(x)]p_i\right)\Psi = \left(p_i(p_i F(x) - F(x)p_i) + (p_i F(x) - F(x)p_i)p_i\right)\Psi = \left(-\hbar^2 \partial_i \partial_i F(x) + \cancel{\hbar^2 \partial_i F(x)\partial_i} - \cancel{\hbar^2 \partial_i F(x) \partial_i} + \hbar^2 F(x)\partial_i \partial_i\right)\Psi = \boxed{\hbar^2\left(F(x)p^2 - p^2 F(x)\right)\Psi} \; //$$

$$[L_i, L_j] \Rightarrow [L_i, L_l] = \epsilon_{ijk}\epsilon_{lmn}[x_j p_k, x_m p_n] = i\hbar\, \epsilon_{ijk}\epsilon_{lmn}\left(-x_j \delta_{km} p_n + x_m \delta_{jn} p_k\right) = i\hbar\left(-\epsilon_{ijk}\epsilon_{nlk} x_j p_n + \epsilon_{kij}\epsilon_{lnj} x_m p_k\right) = i\hbar\left(-(\delta_{in}\delta_{jl} - \delta_{il}\delta_{jn})x_j p_n + (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})x_m p_k\right) = i\hbar\left(-x_j p_n \delta_{in}\delta_{jl} + x_j p_n \delta_{il}\delta_{jn} + x_m p_k \delta_{kl}\delta_{im} - x_m p_k \delta_{km}\delta_{il}\right)$$

$L_i = \epsilon_{ijk}\, \underbrace{x_j p_k}_{\text{mudos}}$

$$[AB, CD] = [AB, C]D + C[AB, D] = A[B, C]D + [A, C]BD + CA[B, D] + C[A, D]B$$

$$= x_j(-\delta_{km}p_n) + [x_j, x_m]p_k p_n + x_m x_j [p_k, p_n] + x_m \delta_{jn} p_k$$

$$= i\hbar\left(-x_l p_i + \cancel{x_n p_n \delta_{il}} + x_i p_l - \cancel{x_k p_k \delta_{il}}\right)$$

mudos

$$= i\hbar\left(x_i p_l - x_l p_i\right) = \boxed{i\hbar L_n} \; //$$

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$$A\,|\alpha_k, \beta_k\rangle = \alpha_k\,|\alpha_k, \beta_k\rangle$$

$$B\,|\alpha_k, \beta_k\rangle = \beta_k\,|\alpha_k, \beta_k\rangle$$

$$[A, B] = 0 \rightarrow AB = BA = A\sum \beta_j\,|\alpha_j, \beta_j\rangle = \sum \beta_j A|\alpha_j, \beta_j\rangle = \sum \beta_j\, \alpha_j\, |\alpha_j, \beta_j\rangle = \sum \beta_j\, |\beta_j\rangle A = BA \; //$$

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a

$H_{11} = \alpha\,|1\rangle\langle 1| = \alpha\,\langle 1|1\rangle\langle 1|1\rangle = \alpha$

$H_{12} = \alpha\,|1\rangle\langle 2| = \alpha\,\langle 1|1\rangle\langle 2|2\rangle = \alpha$

$H_{21} = -\alpha\,|2\rangle\langle 1| = -\alpha\,\langle 2|2\rangle\langle 1|1\rangle = -\alpha$

$H_{22} = \alpha\,|2\rangle\langle 2| = \alpha\,\langle 2|2\rangle\langle 2|2\rangle = \alpha$

$$\boxed{H = \begin{pmatrix} \alpha & \alpha \\ \alpha & -\alpha \end{pmatrix}}$$

b

$$\begin{vmatrix} \alpha - \lambda & \alpha \\ \alpha & -\alpha - \lambda \end{vmatrix} = -(\alpha - \lambda)(\alpha + \lambda) - \alpha^2 = \lambda^2 - \alpha^2 - \alpha^2 = \lambda^2 - 2\alpha^2 = 0$$

$\lambda = \pm\sqrt{2}\,\alpha$ • $\lambda_1 = \sqrt{2}\,\alpha$ • $\lambda_2 = -\sqrt{2}\,\alpha$

$$\boxed{|\lambda_1\rangle = \frac{1}{(4 - 2\sqrt{2})}\begin{bmatrix} 1 \\ \sqrt{2} - 1 \end{bmatrix} \qquad |\lambda_2\rangle = \frac{1}{(4 + 2\sqrt{2})}\begin{bmatrix} -1 \\ \sqrt{2} - 1 \end{bmatrix}} \; //$$