Teoría de perturbaciones cosmológicas

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$$S=\int\sqrt{g}\,\frac{M_p^2}{2}R\,d^4x \longrightarrow g_{\mu\nu} \Rightarrow ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

$$\underbrace{\Gamma^\alpha_{\mu\nu}}_{\text{Conexión}} = \frac{1}{2}g^{\alpha\sigma}(\partial_\mu g_{\nu\sigma}+\partial_\nu g_{\mu\sigma}-\partial_\sigma g_{\mu\nu})$$

Tensor de Curvatura

$$R^\lambda_{\mu\lambda\nu} = \partial_\mu\Gamma^\lambda_{\nu\alpha}-\partial_\nu\Gamma^\lambda_{\mu\alpha}+\Gamma^\sigma_{\lambda\alpha}\Gamma^\lambda_{\nu\sigma}-\Gamma^\sigma_{\nu\alpha}\Gamma^\lambda_{\mu\sigma}$$

contracción

$$R_{\mu\nu}=R^\lambda_{\mu\lambda\nu} \rightarrow \boxed{R=g^{\mu\nu}R_{\mu\nu}}$$

$$S\rightarrow S[g,\partial g,\partial^2 g]$$

$$G^\mu{}_\nu = 8\pi G\,\overbrace{T^\mu{}_\nu}^{\text{Tensor de energía-momento}}$$

$$=R^\mu{}_\nu-\frac{1}{2}\delta^\mu{}_\nu R$$

Tensor de Einstein

Teoría de perturbaciones $\rightarrow$ Necesidad de solución $\rightarrow$ Métrica FLDR

$$T_{\mu\nu}=\rho u_\mu u_\nu + P h_{\mu\nu}+2q_{(\mu}u_{\nu)}+\Sigma_{\mu\nu} \qquad T_{\mu\nu}=(\rho+P)u_\mu u_\nu+Pg_{\mu\nu}$$

$$h_{\mu\nu}=g_{\mu\nu}+u_\mu u_\nu \qquad q_\mu=-h_\mu T_{\sigma\rho}u^\rho$$

$$\Sigma_{\mu\nu}=h_{(\mu}{}^\sigma h_{\nu)}{}^\rho T_{\sigma\rho} \qquad P=\frac{1}{3}T_{\mu\nu}h^{\mu\nu}$$

Solución base: (FLDR)

$$ds^2=a^2(\eta)[d\eta^2-\gamma_{ij}dx^i dx^j],\quad \gamma_{ij}=\delta_{ij}\left(\frac{1}{1+\frac{1}{4}Kr^2}\right)^2 \qquad K=0,1,-1.\ \text{(Planas, cerradas, abiertas)}$$

$$\left(ds^2=dt^2-\underbrace{a^2(t)}_{\text{Factor de escala}}\gamma_{ij}dx^i dx^j\right)\rightarrow$$

$$a\,d\eta=dt \Rightarrow \underbrace{\eta}_{\text{tiempo conforme}}=\int_{t_0}^{t}\frac{dt}{a}$$

$$G^0{}_0=\frac{3}{a^2}(\mathcal{H}^2+K) \qquad \mathcal{H}=\frac{a'}{a} \qquad {}'\equiv\frac{d}{d\eta} \qquad R=-\frac{6}{a^2}(\mathcal{H}'+\mathcal{H}^2+K) \qquad G^i{}_j=\frac{1}{a^2}(2\mathcal{H}'+\mathcal{H}^2+K)\delta^i{}_j$$

Componentes de $g_{\mu\nu}$: 10 componentes $\rightarrow \delta g_{ij}$ $\qquad g_{\mu\nu}={}^{(0)}g_{\mu\nu}+\delta g^{\mu\nu}$

{escalares, vectoriales, tensoriales}

6 componentes

Ejercicios.

- Usando $\boxed{T_{\mu\nu}=(\rho+P)u_\mu u_\nu+Pg_{\mu\nu}}$ calcular la ec. de continuidad

$$\frac{dT_{00}}{d\eta}=-(4T^0_0-T)\frac{d\ln a}{d\eta}$$

$$T^\mu{}_\mu$$

$$\frac{d\rho}{dt}+3H(\rho+P)=0$$

$$H=\frac{\dot a}{a}$$

Perturbaciones escalares

Factor multiplicativo

$$\delta g^{(s)}_{\mu\nu}\rightarrow\delta[ds^2]=a^2\Big[\underbrace{\delta g_{00}}_{2\phi}d\eta^2-\delta g_{0i}dx^i d\eta+\overbrace{\delta g_{ij}}^{2\Psi}dx^i dx^j\Big]$$

$$+\ B_{,i} \qquad +\ E_{,ij}$$

Derivada covariante con respecto a i

$$\partial_i B \qquad B_{;i},\ \nabla_i B,\ \gamma^{ij}\nabla_j B$$

$$\delta g^{(s)}_{\mu\nu}=a^2(\eta)\begin{pmatrix}2\phi & -B_{,i}\\ -B_{,i} & +2(\Psi\delta_{ij}-E_{,ij})\end{pmatrix}$$

4 componentes

Perturbaciones vectoriales

$$u_i=\partial_i v+v_i \qquad \nabla\cdot v=0\rightarrow 2\ \text{componentes independientes}$$

$$\delta g^{(v)}_{\mu\nu}=\begin{pmatrix}0 & -S_i\\ -S_i & F_{i,j}+F_{j,i}\end{pmatrix}$$

$$V\rightarrow\Big\{\underset{2c}{S_i},\underset{2c}{F_i}\Big\}\rightarrow 4\ \text{componentes.}\ V_i$$

Perturbaciones tensoriales

$$T_{ij}=T_{ij}^{(s)}+T_{ij}^{(v)}+T_{ij}^{(t)} \qquad \delta g^{(t)}_{\mu\nu}=\begin{pmatrix}0 & 0\\ 0 & h_{ij}\end{pmatrix}$$

$$T_{ij}^{(s)}=\Big(\underbrace{-\partial_i\partial_j+\frac{1}{3}\delta^{ij}\nabla^2}_{-\frac{k_ik_j}{k^2}+\frac{1}{3}\delta_{ij}}\Big)$$

$(0 \;\; h_{ij})$

$$T_{ij}^{(S)} = \underbrace{\left(-\partial_i \partial_j + \frac{1}{3} \delta^{ij} \nabla^2 \right)}_{-\frac{K_i K_j}{K^2} + \frac{1}{3}\delta^{ij}}$$

$$T_{ij} \rightarrow \gamma^{ij} T_{ij} = 0, \quad \nabla_i T^{ij} = 0$$

Perturbaciones de la métrica

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$$g_{\mu\nu} = {}^{(0)}g_{\mu\nu} + \delta g_{\mu\nu} \qquad \delta g_{\mu\nu} \ll 1$$

$$\delta g_{\mu\nu} \longrightarrow \delta g_{\mu\nu}^{(s)} = a^2(\eta)\begin{pmatrix} 2\phi & -\partial_i B \\ -\partial_i B & (2\psi\delta_{ij} - D_{ij}E) \end{pmatrix} \quad 4_S$$

$$D_{ij} = \left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\nabla^2\right)$$

$$\delta g_{\mu\nu}^{(v)} = a^2(\eta)\begin{pmatrix} 0 & -S_i \\ -S_i & \nabla_i F_j + \nabla_j F_i \end{pmatrix} \quad 4_V$$

$$\delta g_{\mu\nu}^{(t)} = a^2(\eta)\begin{pmatrix} 0 & 0 \\ 0 & -h_{ij} \end{pmatrix} \quad 2_t$$

Variables y transformaciones de coordenadas

T. Rel. Gen. es invariante sobre difeomorfismos.

$$x^\mu \to z^\mu(x^\nu)$$

$$\tilde{x}^\mu = x^\mu + \xi^\mu(x)$$

- **Transformaciones activas y pasivas**
  - Pasivas $x^N \to \tilde{x}^N$ en $\mathcal{M}$ (cambios de coordenadas)

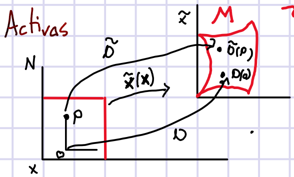
$(\mathcal{M}, x^\mu)$ — variedad espacio-tiempo

$$\delta Q \equiv Q(x^\alpha)\big|_p - \boxed{{}^{(0)}Q(x^\alpha)\big|_p}$$

$$\delta\tilde{Q} \equiv \tilde{Q}(\tilde{x}^\alpha)\big|_p - \boxed{{}^{(0)}\tilde{Q}(\tilde{x}^\alpha)\big|_p}$$

En t. pasivas son independientes del sist. de coordenadas

  - Activas



$$\left.\begin{aligned} \delta Q &\equiv Q(p) - {}^{(0)}Q(D^{-1}(p)) \\ \delta\tilde{Q} &\equiv \tilde{Q}(p) - {}^{(0)}Q(\tilde{D}^{-1}(p)) \end{aligned}\right\}$$

$$\Delta Q = \delta Q - \delta\tilde{Q} = \mathcal{L}_\xi \bar{Q}$$

Derivada de Lie

Transformación de coordenadas

$$\tilde{x}^\mu = x^\mu + \xi^\mu \qquad \xi^\mu = (\xi^0, \xi^i) = (\xi^0, \xi, \xi_{\perp i})$$

potencial; vorticidad

$$\xi^i = \xi^i_{tr} + \delta^{ij}\xi_{,j}, \qquad \xi^i_{tr} = \partial^i\xi$$

$$\eta \to \tilde{\eta} = \eta + \xi^0 \qquad x^i \to \tilde{x}^i = x^i + \delta^{ij}\nabla_j\xi$$

Variables y transformaciones de coordenadas

$$\tilde{g}_{\alpha\beta}(\tilde{x}) = \frac{\partial x^\mu}{\partial\tilde{x}^\alpha}\frac{\partial x^\nu}{\partial\tilde{x}^\beta}g_{\mu\nu}(x) = \left(\delta^\mu_\alpha - \partial_\alpha\xi^\mu\right)\left(\delta^\nu_\beta - \partial_\beta\xi^\nu\right)g_{\mu\nu}(x) = g_{\alpha\beta}(x) - \partial_\alpha\xi^\mu g_{\mu\beta} - \partial_\beta\xi^\nu g_{\alpha\nu} + O(\delta\xi^2) = g_{\alpha\beta}(\tilde{x} - \xi) - g_{\mu\beta}\partial_\alpha\xi^\mu - g_{\alpha\nu}\partial_\beta\xi^\nu$$

$$= g_{\alpha\beta}(\tilde{x}) - \left(\xi^\mu\partial_\mu g_{\alpha\beta} + g_{\mu\beta}\partial_\alpha\xi^\mu + g_{\alpha\nu}\partial_\beta\xi^\nu\right)$$

$$\tilde{x}^\mu - \xi^\mu = x^\mu \Rightarrow \frac{\partial x^\mu}{\partial\tilde{x}^\alpha} = \delta^\mu_\alpha - \frac{\partial\xi^\mu}{\partial\tilde{x}^\alpha} = \delta^\mu_\alpha - \frac{\partial\xi^\mu}{\partial x^\nu}\underbrace{\frac{\partial x^\nu}{\partial\tilde{x}^\alpha}}_{1} = \delta^\mu_\alpha - \frac{\partial\xi^\mu}{\partial x^\alpha}$$

$${}^{(0)}\cancel{\tilde{g}_{\alpha\beta}(\tilde{x})} + \delta\tilde{g}_{\alpha\beta}(\tilde{x}) = {}^{(0)}\cancel{g_{\alpha\beta}(\tilde{x})} + \delta g_{\alpha\beta}(\tilde{x}) - \mathcal{L}_\xi g_{\alpha\beta}$$

$$\delta\tilde{g}_{\alpha\beta}(x) = \delta g_{\alpha\beta}(x) - \mathcal{L}_\xi g_{\alpha\beta}$$

$$\delta\tilde{g}_{00} = \delta g_{00} - \mathcal{L}_\xi g_{00}$$

$$\delta\tilde{g}_{i0} = \delta g_{0i} - \mathcal{L}_\xi g_{0i}$$

$$\delta\tilde{g}$$