we have

$$EG - F^2 = \det\left(\begin{bmatrix} \mathbf{x}_u \cdot \mathbf{x}_u & \mathbf{x}_u \cdot \mathbf{x}_v \\ \mathbf{x}_v \cdot \mathbf{x}_u & \mathbf{x}_v \cdot \mathbf{x}_v \end{bmatrix}\right) = \det\left(\begin{bmatrix} \mathbf{x}_u \cdot \mathbf{x}_u & \mathbf{x}_u \cdot \mathbf{x}_v & 0 \\ \mathbf{x}_v \cdot \mathbf{x}_u & \mathbf{x}_v \cdot \mathbf{x}_v & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$
$$= \det\left(\begin{bmatrix} | & | & | \\ \mathbf{x}_u & \mathbf{x}_v & \mathbf{n} \\ | & | & | \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} | & | & | \\ \mathbf{x}_u & \mathbf{x}_v & \mathbf{n} \\ | & | & | \end{bmatrix}\right) = \left(\det\begin{bmatrix} | & | & | \\ \mathbf{x}_u & \mathbf{x}_v & \mathbf{n} \\ | & | & | \end{bmatrix}\right)^2,$$

which is the square of the volume of the parallelepiped spanned by $\mathbf{x}_u$, $\mathbf{x}_v$, and $\mathbf{n}$. Since $\mathbf{n}$ is a unit vector orthogonal to the plane spanned by $\mathbf{x}_u$ and $\mathbf{x}_v$, this is, in turn, the square of the area of the parallelogram spanned by $\mathbf{x}_u$ and $\mathbf{x}_v$. That is,

$$EG - F^2 = \|\mathbf{x}_u \times \mathbf{x}_v\|^2 > 0.$$

We remind the reader that we obtain the *surface area* of the parametrized surface $\mathbf{x}\colon U \to M$ by calculating the double integral

$$\int_U \|\mathbf{x}_u \times \mathbf{x}_v\| du dv = \int_U \sqrt{EG - F^2}\, du dv.$$

EXERCISES 2.1

1. Derive the formula given in Example 1(e) for the parametrization of the unit sphere.

♯2. Suppose $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$, $a \le t \le b$, is a parametrized curve on a surface M. Show that

$$\begin{aligned} \text{length}(\boldsymbol{\alpha}) &= \int_a^b \sqrt{\mathrm{I}_{\boldsymbol{\alpha}(t)}\big(\boldsymbol{\alpha}'(t), \boldsymbol{\alpha}'(t)\big)}\, dt \\ &= \int_a^b \sqrt{E(u(t), v(t))(u'(t))^2 + 2F(u(t), v(t))u'(t)v'(t) + G(u(t), v(t))(v'(t))^2}\, dt. \end{aligned}$$

Conclude that if $\boldsymbol{\alpha} \subset M$ and $\boldsymbol{\alpha}^* \subset M^*$ are corresponding paths in locally isometric surfaces, then $\text{length}(\boldsymbol{\alpha}) = \text{length}(\boldsymbol{\alpha}^*)$.

3. Compute I (i.e., E, F, and G) for the following parametrized surfaces.
   - *a. the sphere of radius a: $\mathbf{x}(u, v) = a(\sin u \cos v, \sin u \sin v, \cos u)$
   - b. the torus: $\mathbf{x}(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u)$ $(0 < b < a)$
   - c. the helicoid: $\mathbf{x}(u, v) = (u\cos v, u\sin v, bv)$
   - *d. the catenoid: $\mathbf{x}(u, v) = a(\cosh u \cos v, \cosh u \sin v, u)$

4. Find the surface area of the following parametrized surfaces.
   - *a. the torus: $\mathbf{x}(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u)$ $(0 < b < a), 0 \le u, v \le 2\pi$
   - b. a portion of the helicoid: $\mathbf{x}(u, v) = (u\cos v, u\sin v, bv)$, $1 < u < 3, 0 \le v \le 2\pi$
   - c. a zone of a sphere[3]: $\mathbf{x}(u, v) = a(\sin u \cos v, \sin u \sin v, \cos u)$, $0 \le u_0 \le u \le u_1 \le \pi$, $0 \le v \le 2\pi$

[3] You should obtain the remarkable result that the surface area of the portion of a sphere between two parallel planes depends only on the distance between the planes, not on where you locate them.

*5. Show that if all the normal lines to a surface pass through a fixed point, then the surface is (a portion of) a sphere. (By the normal line to M at P we mean the line passing through P with direction vector the unit normal at P.)

6. Check that the parametrization $\mathbf{x}(u, v)$ is conformal if and only if $E = G$ and $F = 0$. (Hint: For $\Longrightarrow$, choose *two* convenient pairs of orthogonal directions.)

*7. Check that a parametrization preserves area and is conformal if and only if it is a local isometry.

*8. Check that the parametrization of the unit sphere by stereographic projection (see Example 1(e)) is conformal.

9. *(Lambert's cylindrical projection)* Project the unit sphere (except for the north and south poles) radially outward to the cylinder of radius 1 by sending (x, y, z) to $(x/\sqrt{x^2+y^2}, y/\sqrt{x^2+y^2}, z)$. Check that this map preserves area locally, but is neither a local isometry nor conformal. (Hint: Let $\mathbf{x}(u, v)$ be the spherical coordinates parametrization of the sphere, and consider $\mathbf{x}^*(u, v) = (\cos v, \sin v, \cos u)$. Compare the parallelogram formed by $\mathbf{x}_u$ and $\mathbf{x}_v$ with the parallelogram formed by $\mathbf{x}^*_u$ and $\mathbf{x}^*_v$.)

♯10. Consider the "pacman" region M given by $\mathbf{x}(u, v) = (u \cos v, u \sin v, 0), 0 \le u \le R, 0 \le v \le V$, with $V < 2\pi$. Let $c = V/2\pi$. Let M^* be given by the parametrization

$$\mathbf{x}^*(u, v) = \big(cu \cos(v/c), cu \sin(v/c), \sqrt{1 - c^2}u\big), \quad 0 \le u \le R,\ 0 \le v \le V.$$

Compute that $E = E^*$, $F = F^*$, and $G = G^*$, and conclude that the mapping $\mathbf{f} = \mathbf{x}^* \circ \mathbf{x}^{-1} \colon M \to M^*$ is a local isometry. Describe this mapping in concrete geometric terms.

11. Consider the hyperboloid of one sheet, M, given by the equation $x^2 + y^2 - z^2 = 1$.
   a. Show that $\mathbf{x}(u, v) = (\cosh u \cos v, \cosh u \sin v, \sinh u)$, $u \in \mathbb{R}$, $0 \le v < 2\pi$, gives a parametrization of M as a surface of revolution.
   *b. Find two parametrizations of M as a ruled surface $\boldsymbol{\alpha}(u) + v\boldsymbol{\beta}(u)$.
   c. Show that $\mathbf{x}(u, v) = \left(\dfrac{uv+1}{uv-1}, \dfrac{u-v}{uv-1}, \dfrac{u+v}{uv-1}\right)$ gives a parametrization of M where *both* sets of parameter curves are rulings.

♯12. Given a ruled surface M parametrized by $\mathbf{x}(u, v) = \boldsymbol{\alpha}(u) + v\boldsymbol{\beta}(u)$ with $\boldsymbol{\alpha}' \ne 0$ and $\|\boldsymbol{\beta}\| = 1$.
   a. Check that we may assume that $\boldsymbol{\alpha}'(u) \cdot \boldsymbol{\beta}(u) = 0$ for all u. (Hint: Replace $\boldsymbol{\alpha}(u)$ with $\boldsymbol{\alpha}(u) + t(u)\boldsymbol{\beta}(u)$ for a suitable function t.)
   b. Suppose, moreover, that $\boldsymbol{\alpha}'(u)$, $\boldsymbol{\beta}(u)$, and $\boldsymbol{\beta}'(u)$ are linearly dependent for every u. Conclude that $\boldsymbol{\beta}'(u) = \lambda(u)\boldsymbol{\alpha}'(u)$ for some function λ. Prove that:
      (i) If $\lambda(u) = 0$ for all u, then M is a cylinder.
      (ii) If λ is a nonzero constant, then M is a cone.
      (iii) If λ and λ' are both nowhere zero, then M is a tangent developable. (Hint: Find the directrix.)

13. *(The Mercator projection)* Mercator developed his system for mapping the earth, as pictured in Figure 1.8, in 1569, about a century before the advent of calculus. We want a parametrization $\mathbf{x}(u, v)$ of the sphere, $u \in \mathbb{R}$, $v \in (-\pi, \pi)$, so that the u-curves are the longitudes and so that the parametrization is conformal. Letting (ϕ, θ) be the usual spherical coordinates, write $\phi = f(u)$ and $\theta = v$. Show that

FIGURE 1.8

conformality and symmetry about the equator will dictate $f(u) = 2\arctan(e^{-u})$. Deduce that

$$\mathbf{x}(u, v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u).$$

(Cf. Example 2 in Section 1 of Chapter 1.)

14. A parametrization $\mathbf{x}(u, v)$ is called a *Tschebyschev net* if the opposite sides of any quadrilateral formed by the coordinate curves have equal length.
    a. Prove that this occurs if and only if $\dfrac{\partial E}{\partial v} = \dfrac{\partial G}{\partial u} = 0$. (Hint: Express the length of the u-curves, $u_0 \le u \le u_1$, as an integral and use the fact that this length is independent of v.)
    b. Prove that we can locally reparametrize by $\tilde{\mathbf{x}}(\tilde{u}, \tilde{v})$ so as to obtain $\tilde{E} = \tilde{G} = 1$, $\tilde{F} = \cos\theta(\tilde{u}, \tilde{v})$ (so that the $\tilde{u}$- and $\tilde{v}$-curves are parametrized by arclength and meet at angle θ). (Hint: Choose $\tilde{u}$ as a function of u so that $\tilde{\mathbf{x}}_{\tilde{u}} = \mathbf{x}_u \big/ (d\tilde{u}/du)$ has unit length.)

15. Suppose $\mathbf{x}$ and $\mathbf{y}$ are two parametrizations of a surface M near P. Say $\mathbf{x}(u_0, v_0) = P = \mathbf{y}(s_0, t_0)$. Prove that $\operatorname{Span}(\mathbf{x}_u, \mathbf{x}_v) = \operatorname{Span}(\mathbf{y}_s, \mathbf{y}_t)$ (where the partial derivatives are all evaluated at the obvious points). (Hint: $\mathbf{f} = \mathbf{x}^{-1} \circ \mathbf{y}$ gives a $\mathcal{C}^1$ map from an open set around (s_0, t_0) to an open set around (u_0, v_0). Apply the chain rule to show $\mathbf{y}_s, \mathbf{y}_t \in \operatorname{Span}(\mathbf{x}_u, \mathbf{x}_v)$.)

16. (A programmable calculator, Maple, or Mathematica will be needed for parts of this problem.) A catenoid, as pictured in Figure 1.9, is parametrized by

$$\mathbf{x}(u, v) = (a \cosh u \cos v, a \cosh u \sin v, au), \quad u \in \mathbb{R},\ 0 \le v < 2\pi \quad (a > 0 \text{ fixed}).$$

    *a. Compute the surface area of that portion of the catenoid given by $|u| \le 1/a$. (Hint: $\cosh^2 u = \frac{1}{2}(1 + \cosh 2u)$.)
    b. Find the number $R_0 > 0$ so that for every $R \ge R_0$, there is at least one catenoid whose boundary is the pair of parallel circles $x^2 + y^2 = R^2$, $|z| = 1$. (Hint: Graph $f(t) = t\cosh(1/t)$.)
    c. For $R \ge R_0$, compare the area of the catenoid(s) with $2\pi R^2$ (the area of the pair of disks filling in the circles). For what values of R does the pair of disks have the least area? (You should display the results of your investigation in either a graph or a table.)

and so we have

$$E = 1, \quad F = 0, \quad G = f(u)^2, \quad \text{and} \quad \ell = f'(u)g''(u) - f''(u)g'(u), \quad m = 0, \quad n = f(u)g'(u).$$

By Exercise 2.2.1, then $k_1 = f'(u)g''(u) - f''(u)g'(u)$ and $k_2 = g'(u)/f(u)$. Thus,

$$K = k_1 k_2 = \big(f'(u)g''(u) - f''(u)g'(u)\big)\frac{g'(u)}{f(u)} = -\frac{f''(u)}{f(u)},$$

since from $f'(u)^2 + g'(u)^2 = 1$ we deduce that $f'(u)f''(u) + g'(u)g''(u) = 0$, and so

$$f'(u)g'(u)g''(u) - f''(u)g'(u)^2 = -(f'(u)^2 + g'(u)^2)f''(u) = -f''(u).$$

Note, as we observed in the special case of Example 8, that on every surface of revolution, the meridians and the parallels are lines of curvature. ∇

EXERCISES 2.2

*1. Check that if there are no umbilic points and the parameter curves are lines of curvature, then $F = m = 0$ and we have the principal curvatures $k_1 = \ell/E$ and $k_2 = n/G$. Conversely, prove that if $F = m = 0$, then the parameter curves are lines of curvature.

♯2. a. Show that the matrix representing the linear map $S_P : T_P M \to T_P M$ with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ is

$$\mathrm{I}_P^{-1}\mathrm{II}_P = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} \ell & m \\ m & n \end{bmatrix}.$$

(Hint: Write $S_P(\mathbf{x}_u) = a\mathbf{x}_u + b\mathbf{x}_v$ and $S_P(\mathbf{x}_v) = c\mathbf{x}_u + d\mathbf{x}_v$, and use the definition of ℓ, m, and n to get a system of linear equations for a, b, c, and d.)

b. Deduce that $K = \dfrac{\ell n - m^2}{EG - F^2}$.

3. Compute the second fundamental form II_P of the following parametrized surfaces. Then calculate the matrix of the shape operator, and determine H and K.
 a. the cylinder: $\mathbf{x}(u, v) = (a\cos u, a\sin u, v)$
 *b. the torus: $\mathbf{x}(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u)$ $(0 < b < a)$
 c. the helicoid: $\mathbf{x}(u, v) = (u\cos v, u\sin v, bv)$
 *d. the catenoid: $\mathbf{x}(u, v) = a(\cosh u\cos v, \cosh u\sin v, u)$
 e. the Mercator parametrization of the sphere: $\mathbf{x}(u, v) = (\operatorname{sech} u\cos v, \operatorname{sech} u\sin v, \tanh u)$
 f. Enneper's surface: $\mathbf{x}(u, v) = (u - u^3/3 + uv^2, v - v^3/3 + u^2v, u^2 - v^2)$

4. Find the principal curvatures, the principal directions, and asymptotic directions (when they exist) for each of the surfaces in Exercise 3. Identify the lines of curvature and asymptotic curves when possible.

*5. Prove by calculation that any one of the helices $\boldsymbol{\alpha}(t) = (a\cos t, a\sin t, bt)$ is an asymptotic curve on the helicoid given in Example 1(b) of Section 1. Also, calculate how the surface normal $\mathbf{n}$ changes as one moves along a ruling, and use this to explain why the rulings are asymptotic curves as well.

*6. Calculate the first and second fundamental forms of the pseudosphere (see Example 8) and check our computations of the principal curvatures and Gaussian curvature.

7. Show that a ruled surface has Gaussian curvature $K \le 0$.

8. a. Prove that the principal directions bisect the asymptotic directions at a hyperbolic point. (Hint: Euler's Formula.)
   b. Prove that if the asymptotic directions of M are orthogonal, then M is minimal. Prove the converse assuming M has no planar points.

9. Let $\kappa_n(\theta)$ denote the normal curvature in the direction making angle θ with the first principal direction.
   a. Show that $H = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \kappa_n(\theta)d\theta$.
   b. Show that $H = \dfrac{1}{2}\Big(\kappa_n(\theta) + \kappa_n\big(\theta + \dfrac{\pi}{2}\big)\Big)$ for any θ.
   c. (More challenging) Show that, more generally, for any θ and $m \ge 3$, we have $H = \dfrac{1}{m}\Big(\kappa_n(\theta) + \kappa_n\big(\theta + \dfrac{2\pi}{m}\big) + \cdots + \kappa_n\big(\theta + \dfrac{2\pi(m-1)}{m}\big)\Big)$.

10. Consider the ruled surface M given by $\mathbf{x}(u, v) = (v\cos u, v\sin u, uv)$, $v > 0$.
   a. Describe this surface geometrically.
   b. Find the first and second fundamental forms and the Gaussian curvature of M.
   c. Check that the v-curves are lines of curvature.
   d. Proceeding somewhat as in Example 6, show that the other lines of curvature are given by the equation $v\sqrt{1+u^2} = c$ for various constants c. Show that these curves are the intersection of M with the spheres $x^2 + y^2 + z^2 = c^2$. (It might be fun to use Mathematica to see this explicitly.)

11. The curve $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$ may arise by writing $\dfrac{dv}{du} = \dfrac{v'(t)}{u'(t)}$ and solving a differential equation to relate u and v either explicitly or implicitly.
   a. Show that $\boldsymbol{\alpha}$ is an asymptotic curve if and only if $\ell(u')^2 + 2mu'v' + n(v')^2 = 0$. Thus, if $\ell + 2m\frac{dv}{du} + n\big(\frac{dv}{du}\big)^2 = 0$, then $\boldsymbol{\alpha}$ is an asymptotic curve.
   b. Show that $\boldsymbol{\alpha}$ is a line of curvature if and only if $\begin{vmatrix} Eu' + Fv' & Fu' + Gv' \\ \ell u' + mv' & mu' + nv' \end{vmatrix} = 0$. Give the appropriate condition in terms of dv/du.
   c. Deduce that an alternative condition for $\boldsymbol{\alpha}$ to be a line of curvature is that
$$\begin{vmatrix} (v')^2 & -u'v' & (u')^2 \\ E & F & G \\ \ell & m & n \end{vmatrix} = 0.$$

12. a. Apply Meusnier's Formula to a latitude circle on a sphere of radius a to calculate the normal curvature.
   b. Apply Meusnier's Formula to prove that the curvature of any curve lying on a sphere of radius a satisfies $\kappa \ge 1/a$.

13. Prove or give a counterexample: If M is a surface with Gaussian curvature $K > 0$, then the curvature of any curve $C \subset M$ is everywhere positive. (Remember that, by definition, $\kappa \ge 0$.)

(iii) By (19), the induced metric g_{ij} on a surface $r = r(u, v)$ is given by

$$g_{12} = \langle r_u, r_v \rangle, \qquad g_{11} = \langle r_u, r_u \rangle, \qquad g_{22} = \langle r_v, r_v \rangle.$$

It is then easy to verify that $g = g_{11}g_{22} - g_{12}^2 = |[r_u, r_v]|^2$, whence the result. □

Using Definition 7.4.1 to give us a definition of area in an arbitrary 2-dimensional space with a Riemannian metric, we see that the concept of area, like arc length and angle, is defined in terms of the scalar product of vectors attached to each point, or, what amounts to the same thing, the metric g_{ij}.

7.5. Exercises

1. The torus T^2 in Euclidean 3-space can be realized as the surface of revolution obtained by revolving a circle about a straight line which it does not intersect, and which lies in the same plane. Find parametric equations for this torus, and calculate the induced metric on it.

2. Find the first fundamental form of the ellipsoid of revolution

$$\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1.$$

3. Find the metric induced on the surface of revolution

$$r(u, \varphi) = (\rho(u) \cos \varphi, \rho(u) \sin \varphi, z(u)).$$

Verify that its meridians ($\varphi =$ const.) and circles of latitude ($u =$ const.) form an orthogonal net. Find those curves which bisect the angles between the meridians and circles of latitude.

4. Find those curves on a sphere which intersect the meridians at a fixed angle α ("loxodromes"). Find the length of a loxodrome.

5. Let $F(x, y, z)$ be a smooth homogeneous function of degree n (i.e. $F(cx, cy, cz) = c^nF(x, y, z)$). Prove that away from the origin the induced metric on the conical surface $F(x, y, z) = 0$ is Euclidean.

§8. The Second Fundamental Form

8.1. Curvature of Curves on a Surface in Euclidean Space

Suppose we are given a surface in Euclidean 3-space and a non-singular point (x_0, y_0, z_0) on it. We shall assume initially that the z-axis is perpendicular to the tangent plane to the surface at the point (x_0, y_0, z_0), in which case the x-axis and y-axis will be parallel to it. The surface may then be given

8.4. Exercises

1. Find the surface all of whose normals intersect at a single point.

2. Calculate the second fundamental form for the surface of revolution

$$r(u, \varphi) = (x(u), \rho(u) \cos \varphi, \rho(u) \sin \varphi), \qquad \rho(u) > 0.$$

3. Calculate the Gaussian and mean curvatures of a surface given by an equation of the form

$$z = f(x) + g(y).$$

4. Prove that if the Gaussian and mean curvatures of a surface (as usual in Euclidean 3-space), are identically zero, then the surface is a plane.

5. Show that for the surface $z = f(x, y)$, the mean curvature is given by

$$H = \operatorname{div}\left(\frac{\operatorname{grad} f}{\sqrt{1 + |\operatorname{grad} f|^2}}\right).$$

6. Let S denote the surface swept out (i.e. "generated") by the tangent vector to a given curve with curvature $k(l)$. Prove that if the curve is twisted, but in such a way as to preserve $k(l)$, then the metric on the surface S is also preserved.

7. If the metric on a surface has the form

$$dl^2 = A^2\,du^2 + B^2\,dv^2, \qquad A = A(u, v), \qquad B = B(u, v),$$

then its Gaussian curvature is given by

$$K = -\frac{1}{AB}\left[\left(\frac{A_v}{B}\right)_v + \left(\frac{B_u}{A}\right)_u\right].$$

8. Show that the only surfaces of revolution with zero mean curvature are the plane and the catenoid (which is the surface obtained by revolving the curve $x = [\cosh(ay + b)]/a$ about the y-axis).

§9. The Metric on the Sphere

The equation of the sphere $S^2 \subset \mathbb{R}^3$ of radius R with centre at the origin is

$$x^2 + y^2 + z^2 = R^2. \tag{1}$$

In spherical co-ordinates r, θ, φ the sphere has the simple equation $r = R$ (with θ, φ arbitrary). It follows that each point on the sphere is determined by the corresponding values of θ and $\varphi (0 \le \theta \le \pi,\ 0 \le \varphi < 2\pi)$, so that θ, φ will serve as local co-ordinates of the sphere; however we shall need to exclude two points of the sphere from consideration, namely the north and south poles (where $\theta = 0, \pi$) since these are singular points of the spherical

From the first of the formulae (29) (with $\tilde{\varphi}$ replacing φ), and (34), it follows that the Gaussian curvature is $K = 4(ac - b\bar{b})$. By means of linear-fractional transformations (which the reader may like to construct), the formula (35) can be brought into the forms:

(i) $\dfrac{4R^4\,dz\,d\bar{z}}{(1 + |z|^2)^2}$ if $K = 4(ac - b\bar{b}) > 0$;

(ii) $dz\,d\bar{z}$ if $K = 4(ac - b\bar{b}) = 0$;

(iii) $\dfrac{4R^4\,dz\,d\bar{z}}{(1 - |z|^2)^2}$ if $K = 4(ac - b\bar{b}) < 0$,

which are the familiar forms of the metrics of the sphere, Euclidean plane, and hyperbolic plane respectively. □

13.4. Exercises

1. Suppose the metric on a surface has the form
$$dl^2 = dx^2 + f(x)\,dy^2,$$
where $f(x)$ is a positive (real-valued) function. Prove that this metric can be brought into conformal form $dl^2 = g(u, v)(du^2 + dv^2)$.

2. Prove that a 2-dimensional pseudo-Riemannian metric (of type (1, 1)) with real analytic coefficients, takes the form
$$dl^2 = \lambda(t, x)(dt^2 - dx^2)$$
after a suitable co-ordinate change.

§14. Transformation Groups as Surfaces in N-Dimensional Space

14.1. Co-ordinates in a Neighbourhood of the Identity

Consider the group $GL(n, \mathbb{R})$ of invertible matrices A, i.e.
$$A = (a^i_j), \qquad \det(a^i_j) \neq 0. \tag{1}$$

Condition (1) defines a region in the space $M(n, \mathbb{R})$ of all $n \times n$ real matrices. If we regard $M(n, \mathbb{R})$ as a vector space (under matrix addition, and multiplication of matrices by scalars), then from this point of view the general linear group is a region of the vector space $\mathbb{R}^{n^2} = M(n, \mathbb{R})$. There is a natural system of co-ordinates for this space, namely the entries a^i_j of the general