arc length s, since most concepts are defined only in terms of the derivatives of $\alpha(s)$.

It is convenient to set still another convention. Given the curve α parametrized by arc length $s \in (a, b)$, we may consider the curve β defined in $(-b, -a)$ by $\beta(-s) = \alpha(s)$, which has the same trace as the first one but is described in the opposite direction. We say, then, that these two curves differ by a *change of orientation*.

EXERCISES

1. Show that the tangent lines to the regular parametrized curve $\alpha(t) = (3t, 3t^2, 2t^3)$ make a constant angle with the line $y = 0$, $z = x$.

2. A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is called a *cycloid* (Fig. 1-7).

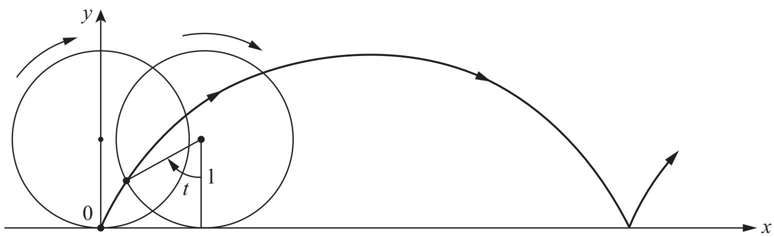

Figure 1-7. The cycloid.

***a.** Obtain a parametrized curve α: $R \to R^2$ the trace of which is the cycloid, and determine its singular points.

b. Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

3. Let $0A = 2a$ be the diameter of a circle S^1 and $0y$ and AV be the tangents to S^1 at 0 and A, respectively. A half-line r is drawn from 0 which meets the circle S^1 at C and the line AV at B. On $0B$ mark off the segment $0p = CB$. If we rotate r about 0, the point p will describe a curve called the *cissoid of Diocles*. By taking $0A$ as the x axis and $0Y$ as the y axis, prove that

a. The trace of

$$\alpha(t) = \left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2}\right), \qquad t \in R,$$

is the cissoid of Diocles ($t = \tan\theta$; see Fig. 1-8).

b. The origin (0, 0) is a singular point of the cissoid.

c. As $t \to \infty$, $\alpha(t)$ approaches the line $x = 2a$, and $\alpha'(t) \to 0, 2a$. Thus, as $t \to \infty$, the curve and its tangent approach the line $x = 2a$; we say that $x = 2a$ is an *asymptote* to the cissoid.

4. Let α: $(0, \pi) \to R^2$ be given by

$$\alpha(t) = \left(\sin t, \cos t + \log \tan \frac{t}{2}\right),$$

where t is the angle that the y axis makes with the vector $\alpha'(t)$. The trace of α is called the *tractrix* (Fig. 1-9). Show that

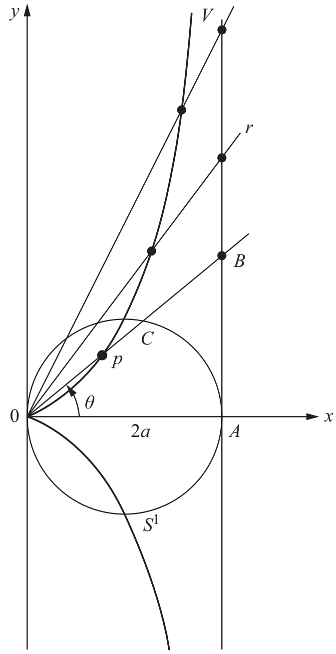

Figure 1-8. The cissoid of Diocles.

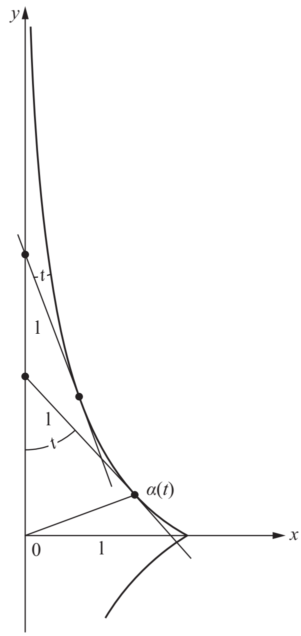

Figure 1-9. The tractrix.

a. α is a differentiable parametrized curve, regular except at $t = \pi/2$.

b. The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

5. Let α: $(-1, +\infty) \to R^2$ be given by

$$\alpha(t) = \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3}\right).$$

Prove that:

a. For $t = 0$, α is tangent to the x axis.

b. As $t \to +\infty$, $\alpha(t) \to (0, 0)$ and $\alpha'(t) \to (0, 0)$.

c. Take the curve with the opposite orientation. Now, as $t \to -1$, the curve and its tangent approach the line $x + y + a = 0$.

The figure obtained by completing the trace of α in such a way that it becomes symmetric relative to the line $y = x$ is called the *folium of Descartes* (see Fig. 1-10).

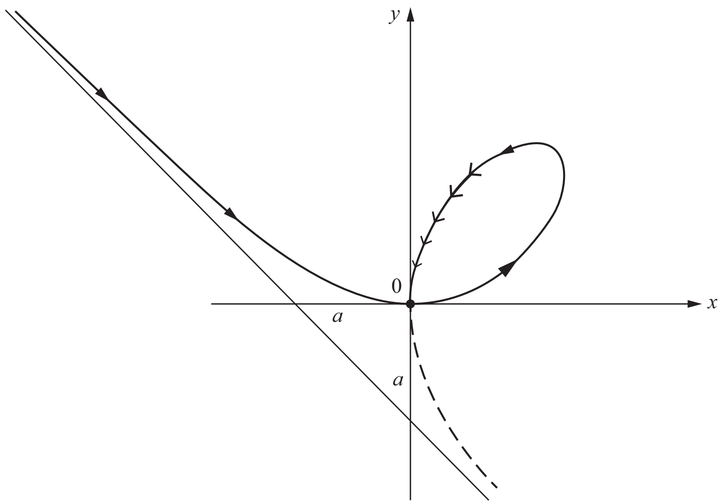

Figure 1-10. Folium of Descartes.

6. Let $\alpha(t) = (ae^{bt}\cos t, ae^{bt}\sin t)$, $t \in R$, a and b constants, $a > 0$, $b < 0$, be a parametrized curve.

a. Show that as $t \to +\infty$, $\alpha(t)$ approaches the origin 0, spiraling around it (because of this, the trace of α is called the *logarithmic spiral*; see Fig. 1-11).

b. Show that $\alpha'(t) \to (0, 0)$ as $t \to +\infty$ and that

$$\lim_{t\to+\infty} \int_{t_0}^{t} |\alpha'(t)|\, dt$$

is finite; that is, α has finite arc length in $[t_0, \infty)$.

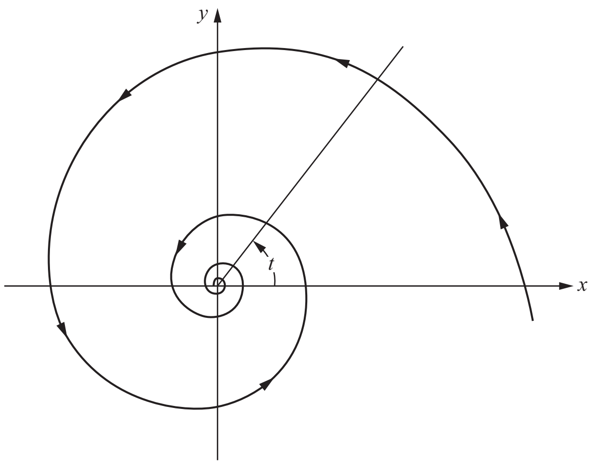

Figure 1-11. Logarithmic spiral.

7. A map $\alpha\colon I \to R^3$ is *called a curve of class* C^k if each of the coordinate functions in the expression $\alpha(t) = (x(t), y(t), z(t))$ has continuous derivatives up to order k. If α is merely continuous, we say that α is of class C^0. A curve α is called *simple* if the map α is one-to-one. Thus, the curve in Example 3 of Sec. 1-2 is not simple.

Let $\alpha\colon I \to R^3$ be a simple curve of class C^0. We say that α has a *weak tangent* at $t = t_0 \in I$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0)$ has a limit position when $h \to 0$. We say that α has a *strong tangent* at $t = t_0$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0 + k)$ has a limit position when $h, k \to 0$. Show that

a. $\alpha(t) = (t^3, t^2)$, $t \in R$, has a weak tangent but not a strong tangent at $t = 0$.

***b.** If $\alpha\colon I \to R^3$ is of class C^1 and regular at $t = t_0$, then it has a strong tangent at $t = t_0$.

c. The curve given by

$$\alpha(t) = \begin{cases} (t^2, t^2), & t \geq 0, \\ (t^2, -t^2), & t \leq 0, \end{cases}$$

is of class C^1 but not of class C^2. Draw a sketch of the curve and its tangent vectors.

***8.** Let $\alpha\colon I \to R^3$ be a differentiable curve and let $[a, b] \subset I$ be a closed interval. For every *partition*

$$a = t_0 < t_1 < \cdots < t_n = b$$

of $[a, b]$, consider the sum $\sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})| = l(\alpha, P)$, where P stands for the given partition. The norm $|P|$ of a partition P is defined as

$$|P| = \max(t_i - t_{i-1}), i = 1, \ldots, n.$$

Geometrically, $l(\alpha, P)$ is the length of a polygon inscribed in $\alpha([a, b])$ with vertices in $\alpha(t_i)$ (see Fig. 1-12). The point of the exercise is to show that the arc length of $\alpha([a, b])$ is, in some sense, a limit of lengths of inscribed polygons.

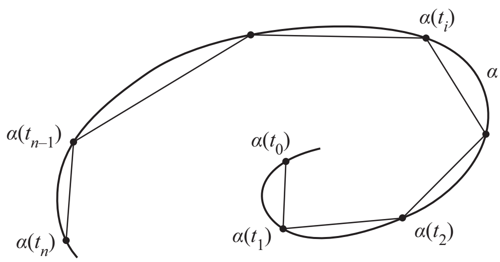

Figure 1-12

Prove that given $\epsilon > 0$ there exists $\delta > 0$ such that if $|P| < \delta$ then

$$\left| \int_a^b |\alpha'(t)|\, dt - l(\alpha, P) \right| < \epsilon.$$

9. a. Let $\alpha\colon I \to R^3$ be a curve of class C^0 (cf. Exercise 7). Use the approximation by polygons described in Exercise 8 to give a reasonable definition of arc length of α.

b. (*A Nonrectifiable Curve.*) The following example shows that, with any reasonable definition, the arc length of a C^0 curve in a closed interval may be unbounded. Let $\alpha\colon [0, 1] \to R^2$ be given as $\alpha(t) = (t, t\sin(\pi/t))$ if $t \neq 0$, and $\alpha(0) = (0, 0)$. Show, geometrically, that the arc length of the portion of the curve corresponding to $1/(n+1) \leq t \leq 1/n$ is at least $2/(n + \frac{1}{2})$. Use this to show that the length of the curve in the interval $1/N \leq t \leq 1$ is greater than $2\sum_{n=1}^{N} 1/(n+1)$, and thus it tends to infinity as $N \to \infty$.

10. (*Straight Lines as Shortest.*) Let $\alpha\colon I \to R^3$ be a parametrized curve. Let $[a, b] \subset I$ and set $\alpha(a) = p$, $\alpha(b) = q$.

a. Show that, for any constant vector v, $|v| = 1$,

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v\, dt \leq \int_a^b |\alpha'(t)|\, dt.$$

b. Set

$$v = \frac{q-p}{|q-p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \le \int_a^b |\alpha'(t)|\, dt;$$

that is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.

1-4. The Vector Product in R^3

In this section, we shall present some properties of the vector product in R^3. They will be found useful in our later study of curves and surfaces.

It is convenient to begin by reviewing the notion of orientation of a vector space. Two ordered bases $e = \{e_i\}$ and $f = \{f_i\}$, $i = 1, \dots, n$, of an n-dimensional vector space V have the *same orientation* if the matrix of change of basis has positive determinant. We denote this relation by $e \sim f$. From elementary properties of determinants, it follows that $e \sim f$ is an equivalence relation; i.e., it satisfies

1. $e \sim e$.
2. If $e \sim f$, then $f \sim e$.
3. If $e \sim f$, $f \sim g$, then $e \sim g$.

The set of all ordered bases of V is thus decomposed into equivalence classes (the elements of a given class are related by $\sim$) which by property 3 are disjoint. Since the determinant of a change of basis is either positive or negative, there are only two such classes.

Each of the equivalence classes determined by the above relation is called an *orientation* of V. Therefore, V has two orientations, and if we fix one of them arbitrarily, the other one is called the opposite orientation.

In the case $V = R^3$, there exists a natural ordered basis $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, and we shall call the orientation corresponding to this basis the *positive orientation* of R^3, the other one being the *negative orientation* (of course, this applies equally well to any R^n). We also say that a given ordered basis of R^3 is *positive* (or *negative*) if it belongs to the positive (or negative) orientation of R^3. Thus, the ordered basis e_1, e_3, e_2 is a negative basis, since the matrix which changes this basis into e_1, e_2, e_3 has determinant equal to -1.

where by Lemma 5.3.1 (with Δl playing the role of t), the matrix $B = (b_{ij})$ is skewsymmetric, i.e. B has the form

$$B = \begin{pmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{pmatrix}.$$

By the definition of the curvature k, and since $e_1 = v(l)$, $e_2 = n(l)$, we have that

$$\dot{e}_1(l) = \frac{de_1}{dl} = \frac{dv}{dl} = kn = ke_2.$$

Hence $b_{12} = k$ and $b_{13} = 0$, so that

$$B = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & -b_{32} \\ 0 & b_{32} & 0 \end{pmatrix}.$$

Writing $\varkappa$ for b_{32}, we obtain the Serret–Frenet formulae (22) (in the matrix form (24)).

The Serret–Frenet formulae for the planar case can be proven similarly.

5.4. Exercises

1. Find the "Serret–Frenet basis" v, n, b, and the curvature and torsion of the circular helix $r = (a \cos t, a \sin t, ct)$, $a > 0$, $c \neq 0$.

2. Find the curvature and torsion of the curves:
   (i) $r = e^t(\sin t, \cos t, 1)$;
   (ii) $r = a(\cosh t, \sinh t, t)$.

3. Find the curvature of the ellipse $x^2/a^2 + y^2/b^2 = 1$, at the points $(a, 0)$, $(0, b)$.

4. Find the curvature and torsion of the curves:
   (i) $r = (t^2\sqrt{\tfrac{3}{2}}, 2\text{-}t, t^3)$;
   (ii) $r = (3t\text{-}t^3, 3t^2, 3t + t^3)$.

5. Prove that a curve with identically zero curvature is a straight line.

6. Prove that a curve with identically zero torsion lies in some plane. Find the equation of the plane.

7. Describe the class of curves with constant curvature and torsion: $k(l) \equiv \text{const}$, $\varkappa(l) \equiv \text{const}$.

8. Describe the class of curves with constant torsion: $\varkappa(l) \equiv \text{const}$.

9. Prove that a curve $r = r(t)$ is planar if and only if $(\dot{r}, \ddot{r}, \dddot{r}) = 0$. (Here (ξ, η, ζ) denotes the "scalar triple product" $\langle \xi, [\eta, \zeta] \rangle$, which can be shown to be zero if and only if the vectors ξ, η, ζ are coplanar.)

10. Let S be the area (if finite) of the region bounded by a plane curve and a straight line parallel to the tangent line to the curve at a point on the curve, and at a distance h from it. Express $\lim_{h\to 0}(S^2/h^3)$ in terms of the curvature of the curve.

11. Show that for a smooth closed curve $C: r = r(l)$

$$\int_C (r\dot{k} - \varkappa b)\, dl = 0.$$

12. Show that the Serret–Frenet formulae can be put into the form

$$\dot{v} = [\zeta, v], \qquad \dot{n} = [\zeta, n], \qquad \dot{b} = [\zeta, b],$$

where ζ is a certain vector (called "Darboux' vector"). Find this vector.

13. Solve the equation $dr/dt = [\omega, r]$ where $r = r(t)$ and ω is a constant vector.

14. Prove that the curvature and torsion of a curve $r = r(l)$ are proportional (i.e. $k = c\varkappa$ for some constant c) if and only if there is a constant vector u such that $\langle u, v\rangle = \text{const}$.

15. Suppose that $r = r(l)$ is a curve with the property that every plane orthogonal to the curve (i.e. spanned by $n(l)$ and $b(l)$) passes through a fixed point x_0. Show that the curve lies on (the surface of) a sphere with centre at x_0.

16. Prove that a curve $r = r(l)$ lies on a sphere of radius R if and only if $\varkappa$ and k satisfy

$$R^2 = \frac{1}{k^2}\left(1 + \frac{(dk/dl)^2}{(\varkappa k)^2}\right).$$

17. Show that

$$\varkappa = -\frac{(\dot{r}, \ddot{r}, \dddot{r})}{|[\dot{r}, \ddot{r}]|}. \quad \text{(Cf. Exercise 9.)}$$

18. With any smooth curve $r = r(l)$ we can associate the curve $r = n(l)$ (where, as usual, $n(l)$ is the principal normal to the curve at the point on it corresponding to the value l of the parameter.) If l^* denotes the natural parameter for the latter curve, show that

$$\frac{dl^*}{dl} = \sqrt{k^2 + \varkappa^2}.$$

19. Let $r = r(l)$ be a space curve. Write

$$A = A(l) = \begin{pmatrix} 0 & k(l) & 0 \\ -k(l) & 0 & \varkappa(l) \\ 0 & -\varkappa(l) & 0 \end{pmatrix} = (a^i_j(l)).$$

Let the vectors $r_j = r_j(l)$ be the unique solutions of the system of equations

$$\frac{dr_j}{dl} = a^i_j r_i, \qquad j = 1, 2, 3,$$

satisfying the initial condition that $r_1(0)$, $r_2(0)$, $r_3(0)$ coincide with a fixed orthonormal basis.

(i) Show that the basis $r_1(l)$, $r_2(l)$, $r_3(l)$ is orthonormal for all l.

(ii) Define $r^*(l) = r_0 + \int_0^l r_1(l)\,dl$. Show that then $r_1(l) = v^*(l)$, $r_2(l) = n^*(l)$, $r_3(l) = b^*(l)$, where v^*, n^*, b^* are the tangent, normal and binormal to the curve $r = r^*(l)$, and show that the curvature and torsion of this curve are the same as those of the original, namely $k(l)$, $\varkappa(l)$.

20. Show that if a curve lies on a sphere and has constant curvature then it is a circle.

§6. Pseudo-Euclidean Spaces

6.1. The Simplest Concepts of the Special Theory of Relativity

Recall that the pseudo-Euclidean space $\mathbb{R}^n_{p,q}$, $p + q = n$, is by definition (3.2.1) the space equipped with co-ordinates $x^1, \dots, x^n$ in terms of which the square of the norm of a vector $\xi = (\xi^1, \dots, \xi^n)$ is given by the formula

$$|\xi|^2 = \langle \xi, \xi \rangle = \sum_{i=1}^{p} (\xi^i)^2 - \sum_{i=1}^{q} (\xi^{p+i})^2. \tag{1}$$

As already noted (in §3.2) for $n = 4$, $p = 1$, this space is termed *space-time of the special theory of relativity*, or *Minkowski space*. We denote this space by $\mathbb{R}^4_1$ rather than $\mathbb{R}^4_{1,3}$. We shall extend this term to cover also the spaces $\mathbb{R}^n_{1,n-1} = \mathbb{R}^n_1$, i.e. we shall call these spaces also Minkowski spaces (one for each dimension n).

By (1) the square of the length of a vector $\xi = (\xi^0, \xi^1, \xi^2, \xi^3)$ in $\mathbb{R}^4_1$ is given by

$$|\xi|^2 = \langle \xi, \xi \rangle = (\xi^0)^2 - (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2, \tag{2}$$

which quantity, as already noted in §3.2, may be positive, negative or zero. Those vectors ξ for which $|\xi| = 0$, form in $\mathbb{R}^4_1$ a cone called the *isotropic* or *light cone* (in Figure 7 the analogous cone in $\mathbb{R}^3_1$ is shown). Vectors inside this cone are just those whose squared length is positive, $|\xi|^2 > 0$; they are called *time-like vectors*. Those outside the cone are the ones with negative squared length, $|\xi|^2 < 0$, and are said to be *space-like*. In Figure 7 the time-like vectors are denoted by ξ_+, and the space-like vectors by ξ_-; vectors which, like ξ_0, lie on the light cone, have zero length and are called *isotropic* or *light vectors*.

We now consider the world-line of an arbitrary material point-particle (see §1.1). Such a particle will have a world-line in $\mathbb{R}^4_1$ of the form

$$x_0 = ct, \qquad x^1 = x^1(t), \qquad x^2 = x^2(t), \qquad x^3 = x^3(t). \tag{3}$$

Here the curve $x^1 = x^1(t)$, $x^2 = x^2(t)$, $x^3 = x^3(t)$ is just the usual trajectory of the point-particle in Euclidean 3-space $\mathbb{R}^3$. The tangent vector to the world-line (3) is given by

$$\xi = (c, \dot{x}^1, \dot{x}^2, \dot{x}^3). \tag{4}$$

6.3. Exercises

1. Define a "vector product" in the space $\mathbb{R}_1^3$ by

$$\xi \times \eta = (\xi^1\eta^2 - \xi^2\eta^1, \xi^0\eta^2 - \xi^2\eta^0, \xi^1\eta^0 - \xi^0\eta^1),$$

where $\xi = (\xi^0, \xi^1, \xi^2)$, $\eta = (\eta^0, \eta^1, \eta^2)$.

(i) Verify that for the basis vectors e_0, e_1, e_2 (where e_0 is time-like) we have

$$e_0 \times e_1 = -e_2, \qquad e_0 \times e_2 = e_1, \qquad e_1 \times e_2 = e_0.$$

(ii) Show that $\times$ is a bilinear antisymmetric operation and that Jacobi's identity holds:

$$\xi_1 \times (\xi_2 \times \xi_3) + \xi_3 \times (\xi_1 \times \xi_2) + \xi_2 \times (\xi_3 \times \xi_1) = 0.$$

(iii) Show that this vector product is preserved by the proper transformations in $O(1, 2)$.

2. Let $r = r(l)$ be a time-like curve in $\mathbb{R}_1^3$ such that $(\dot{r}(l))^2 = (\dot{r}^0)^2 - (\dot{r}^1)^2 - (\dot{r}^2)^2 \equiv 1$, and $\dot{r}^0 > 0$. Define vectors v, n, b by $v = \dot{r}$, $\dot{v} = kn$, $b = n \times v$. Prove the pseudo-Euclidean analogues of the Serret–Frenet formulae:

$$\dot{v} = kn,$$

$$\dot{v} = kv - \varkappa b,$$

$$\dot{b} = \varkappa n.$$

3. Establish a result analogous to that of Lemma 5.3.1, for the derivative of a transformation in $O(1, 2)$ depending on a parameter.

4. Solve in $\mathbb{R}_1^3$ the equation $\dot{r} = \omega \times r$, where ω is a constant vector.

5. Show that an orthogonal complement in $\mathbb{R}_1^{n+1}$ of a time-like vector is a space-like hyperplane (a hyperplane is an n-dimensional subspace). What are the possible orthogonal complements of a space-like vector? Of a light vector?