Taller#2

(a) The graph of a function $f: U \to \mathbb{R}$, $z = f(x, y)$, is parametrized by $\mathbf{x}(u, v) = (u, v, f(u, v))$. Note that $\mathbf{x}_u \times \mathbf{x}_v = (-f_u, -f_v, 1) \neq \mathbf{0}$, so this is always a regular parametrization.

$$X(u,v) = (u, v, F(u,v)) \qquad X_u = (1, 0, f_u) \qquad X_v = (0, 1, F_v)$$

$$X_u \times X_v = \begin{vmatrix} i & j & k \\ 1 & 0 & F_u \\ 0 & 1 & F_v \end{vmatrix} = (-F_u, -F_v, 1) \blacksquare$$

of the helix $\alpha(t) = (\cos t, \sin t, bt)$ to points on the helix:

$$\mathbf{x}(u, v) = (u\cos v, u\sin v, bv), \qquad u > 0,\ v \in \mathbb{R}.$$

Note that $\mathbf{x}_u \times \mathbf{x}_v = (b\sin v, -b\cos v, u) \neq \mathbf{0}$. The u-curves are rays and the v-curves are helices.

$$\vec{X}(u,v) = (u\cos v, u\sin v, bv) \qquad X_u = (\cos v, \sin v, 0) \qquad X_v = (-u\operatorname{sen} v, u\cos v, b)$$

$$X_u \times X_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u\operatorname{sen} v & u\cos v & b \end{vmatrix} = (b\sin v, b\cos v, u) \neq 0$$

v-curva: $\vec{X}(u,v)\big|_{u=cte=\alpha} = \vec{X}(\alpha, v) = (\alpha\cos v, \alpha\sin v, bv) \to$ trayectoria de hélice radio α

v-curva: $\vec{X}(u,v)\big|_{v=cte=\alpha} = \vec{X}(u,\alpha) = (u\cos(\alpha), u\sin(\alpha), b\alpha) \to$ vector $\rho = u$, $z = b\alpha$ rayo

by

$$\mathbf{x}(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u), \qquad 0 \le u, v < 2\pi.$$

Then $\mathbf{x}_u \times \mathbf{x}_v = -b(a + b\cos u)(\cos u\cos v, \cos u\sin v, \sin u)$, which is never $\mathbf{0}$.

$$X_u = (-b\operatorname{sen} u\cos v, -b\operatorname{sen} u\sin v, b\cos u)$$
$$X_v = (-(a+b\cos u)\sin v, (a+b\cos u)\cos v, 0)$$

$$X_u \times X_v = \begin{vmatrix} i & j & k \\ X_u^1 & X_u^2 & X_u^3 \\ X_v^1 & X_v^2 & 0 \end{vmatrix} = (-b(a+b\cos u)\cos u\cos v, -b(a+b\cos u)\cos u\sin v, -b(a+b\cos u)(\operatorname{sen} u\,\cancel{\cos^2 v} + \operatorname{sen} u\,\cancel{\operatorname{sen}^2 v}))$$

$$= -b(a+b\cos u)\overbrace{(\cos u\cos v, \cos u\sin v, \operatorname{sen} u)}^{\neq 0\ \forall (u,v)} \neq 0$$

Then $\mathbf{x}_u \times \mathbf{x}_v = -b(a + b\cos u)(\cos u\cos v, \cos u\sin v, \sin u)$, which is never $\mathbf{0}$.

(d) The standard parametrization of the unit sphere Σ is given by spherical coordinates $(\phi, \theta) \leftrightarrow (u, v)$:

$$\mathbf{x}(u, v) = (\sin u\cos v, \sin u\sin v, \cos u), \qquad 0 < u < \pi,\ 0 \le v < 2\pi.$$

Since $\mathbf{x}_u \times \mathbf{x}_v = \sin u(\sin u\cos v, \sin u\sin v, \cos u) = (\sin u)\mathbf{x}(u, v)$, the parametrization is regular away from $u = 0, \pi$, which we've excluded anyhow because $\mathbf{x}$ fails to be one-to-one at such points. The u-curves are the so-called lines of longitude and the v-curves are the lines of latitude on the sphere.

$X_u = (\cos u\cos v, \cos u\operatorname{sen} v, -\operatorname{sen} u)$, $X_v = (-\operatorname{sen} u\operatorname{sen} v, \operatorname{sen} u\cos v, 0)$

(e) Another interesting parametrization of the sphere is given by *stereographic projection*. (Cf. Exercise 1.1.1.) We parametrize the unit sphere less the north pole (0, 0, 1) by the xy-plane, assigning to each

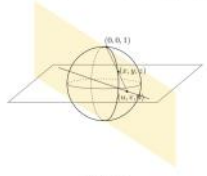


FIGURE 1.4

(u, v) the point ($\neq (0, 0, 1)$) where the line through $(0, 0, 1)$ and $(u, v, 0)$ intersects the unit sphere, as pictured in Figure 1.4. We leave it to the reader to derive the following formula in Exercise 1:

$$\mathbf{x}(u, v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right). \qquad \nabla$$

$$r(u,v) = K\frac{(u, v, -1)}{\sqrt{u^2+v^2+1}} + (0,0,1) \to \text{Recta}$$

$$\left(\frac{K(u,v,-1)}{\sqrt{u^2+v^2+1}} + (0,0,1)\right) = r$$

$$\left(\frac{Ku}{|r|}, \frac{Kv}{|r|}, \frac{|r|-K}{|r|}\right) = r \ (1)$$

π: $x^2 + y^2 + z^2 = 1 \to$ Esfera (restricción de r) (2)

Aplicando la restricción (2) para (1)

$$\frac{|Ku|^2}{|r|^2} + \frac{(Kv)^2}{|r|^2} + \frac{|r|^2 - 2|r|K + K^2}{|r|^2} = 1$$

$$K^2(u^2+v^2+1) - 2|r|K + \cancel{|r|^2} = \cancel{|r|^2}$$

Hallamos K que cumple

$$K^2(u^2+v^2+1) - 2|r|K = 0$$

$$K((u^2+v^2+1)K - 2|r|) = 0$$

$|r| = 1$

$$K = 0$$

$$K = \frac{2|r|}{u^2+v^2+1} = \frac{2}{\sqrt{u^2+v^2+1}}$$

$$|r| - K = \frac{\sqrt{u^2+v^2+1}}{1} - \frac{2}{\sqrt{u^2+v^2+1}} = \frac{u^2+v^2+1-2}{\sqrt{u^2+v^2+1}}$$

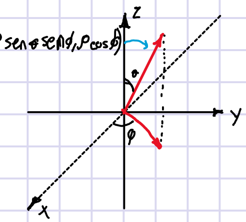
Reemplazamos K en (1) y llegamos al resultado: $r(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right) \blacksquare$

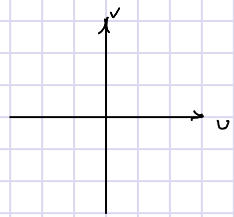
13. *(The Mercator projection)* Mercator developed his system for mapping the earth, as pictured in Figure 1.8, in 1569, about a century before the advent of calculus. We want a parametrization $\mathbf{x}(u, v)$ of the sphere, $u \in \mathbb{R}$, $v \in (-\pi, \pi)$, so that the u-curves are the longitudes and so that the parametrization is conformal. Letting (ϕ, θ) be the usual spherical coordinates, write $\phi = f(u)$ and $\theta = v$. Show that conformality and symmetry about the equator will dictate $f(u) = 2\arctan(e^{-u})$. Deduce that

$$\mathbf{x}(u, v) = (\operatorname{sech} u\cos v, \operatorname{sech} u\sin v, \tanh u).$$

$$r(\phi, \theta) = (\rho\cos\theta\operatorname{sen}\phi, \rho\operatorname{sen}\theta\operatorname{sen}\phi, \rho\cos\phi)$$

$$r(F(u), v) = (\rho\cos v\operatorname{sen} f(u), \rho\operatorname{sen} v\operatorname{sen} f(u), \rho\cos(F(u)))$$





$\phi = F(u)$. $\quad f(u) \in [0, \pi]$

$$\operatorname{atan}(g(u)) \in [-\pi/2, \pi/2] \to \operatorname{Im}(g(u)) \in [0, \infty]$$

$$\underbrace{2\operatorname{atan}(e^{-u}): [0, \infty] \to [0, \pi]}\quad g(u) = e^{-u}$$

$$X_u = \left(\rho\cos\theta\cos F(u)\frac{\partial f(u)}{\partial u}, \rho\operatorname{sen}\theta\cos(f(u))\frac{\partial f(u)}{\partial u}, -\rho\operatorname{sen}(F(u))\right)$$

$$\cos(F(u))\frac{\partial F(u)}{\partial u} = \cos(\phi)\frac{\partial\phi}{\partial u}$$

Since $\mathbf{x}_u \times \mathbf{x}_v = \sin u(\sin u \cos v, \sin u \sin v, \cos u) = (\sin u)\mathbf{x}(u, v)$, the parametrization is regular away from $u = 0, \pi$, which we've excluded anyhow because $\mathbf{x}$ fails to be one-to-one at such points. The u-curves are the so-called lines of longitude and the v-curves are the lines of latitude on the sphere.

$X_u = (\cos u \cos v, \cos u \operatorname{sen} v, -\operatorname{sen} u)$, $X_v = (-\operatorname{sen} u \operatorname{sen} v, \operatorname{sen} u \cos v, 0)$

$$X_u \times X_v = \left(\operatorname{sen}^2 u \cos v,\ \operatorname{sen}^2 u \operatorname{sen} v,\ \cos u \operatorname{sen} u \cancel{\cos^2 v} - \cos u \operatorname{sen} u \cancel{\operatorname{sen}^2 v} \right)$$

$$= \operatorname{sen} u \left(\operatorname{sen} u \cos v, \operatorname{sen} u \operatorname{sen} v, \cos u \right) = \operatorname{sen} u\, \vec{X}(u,v)$$

u-curvas: $(\alpha \operatorname{sen} u, \sqrt{1-\alpha^2} \operatorname{sen} u, \cos u)$ Curva de polo a polo

v-curvas: $(\alpha \cos v, \alpha \operatorname{sen} v, -\sqrt{1-\alpha^2})$ Curva cerrada en un plano trans.

$$\cos(F(u))\frac{\partial F(u)}{\partial u} = \cos(\phi)\frac{\partial \phi}{\partial u}$$