CS2214B - Final Exam

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Question 1

Let $\phi(n)$ be the number of even-sized subsets of X. We know that the number of subsets of X of size $k, k \leq n$ is just $\binom{n}{k}$. Therefore, we seek after a closed form of the following quantity:

$$\phi(n) = \sum_{k \in \mathbb{Z}_{\text{even}}}^{n-1} \binom{n}{k} = \binom{n}{1} + \binom{n}{3} + ... + \binom{n}{n-1} \tag{1}$$

Recall that from binomial theorem, we have:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \Rightarrow (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{2}$$

Additionally, substituting $a = 1, b = -1$ in the above expression results in:

$$0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - ... + \binom{n}{n-1} - \binom{n}{n} \tag{3}$$

Then, adding equations 2 and 3, we get:

$$2^n + 0 = \sum_{k=0}^{n} \binom{n}{k} - \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 2\binom{n}{0} + 2\binom{n}{2} + ... + 2\binom{n}{n-1} \tag{4}$$

Therefore, we conclude that:

$$2^n = 2\phi(n) \Longleftrightarrow \phi(n) = 2^{n-1} \tag{5}$$

We have found a closed form for our target function $\phi(n)$, so we are done.

Question 2

We begin by a sketch of the scenario:

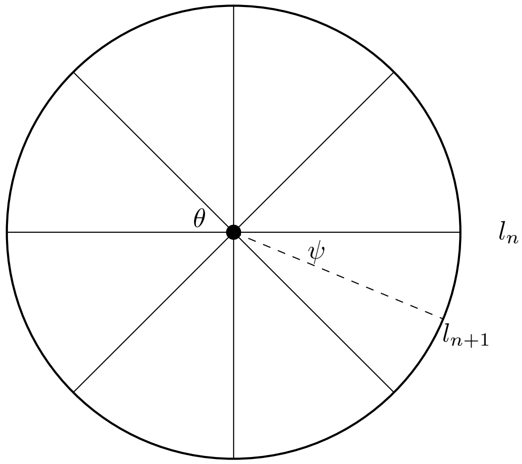

Figure 1: A sketch of the circle model with $n + 1$ lines

Note we have sketched the easiest scenario because in this case, all angles are equal. In fact, it would be counter-intuitive to look at a scenario that maximizes the angle of one pair at the expense of another. This is because it would lead to minimizing the other pair's angle. We use the pigeonhole principle here. Assume we construct a circle with equally-sized sectors. Each sector is a hole, and the lines are pigeons using the pigeonhole analogy. Then as we construct the holes, assume we put all lines $l_1, ..., l_n$ such that every adjacent pair has angle $\theta = \frac{2\pi}{n}$ since there are n equally-sized sectors. Adding that extra line l_{n+1} forces one of the sectors to have its area shrunk with its new angle $\psi < \theta = \frac{2\pi}{n}$ But, the angle of the new sector is just the angle between the lines l_n and l_{n+1}. This concludes the proof.

Question 3

Note that nothing is mentioned about the truth value of the propositions in S. Hence, we consider two cases:

1. $\forall p \in S, p \equiv \boldsymbol{T}$:

   In this case, R is reflexive because $p \wedge p \equiv p \equiv \boldsymbol{T}$ due to the Idempotent Law of Logical And. Additionally, R is symmetric because for given $p, q \in S, p \wedge q \equiv \boldsymbol{T} \equiv q \wedge p$ due to the Symmetry Law of Logical And. Finally, R is transitive because for given $p, q, r \in S, (p \wedge q) \wedge (q \wedge r) \Rightarrow (p \wedge r)$ due to the fact that $\boldsymbol{T} \Rightarrow \boldsymbol{T}$. Therefore, R is an equivalence relation if and only if all propositions in S are true. This case is trivial.

2. $\exists p \in S, p \equiv \boldsymbol{F}$:

   In this case, R is not reflexive because by existential instantiation, $\exists p : p \equiv \boldsymbol{F}$, hence $p \wedge p \equiv p \equiv \boldsymbol{F} \Rightarrow (p, p) \notin R$. As such, R cannot be an equivalence class when there is at least one false proposition in S.

Question 4

To proceed, we first re-write the formula for the binomial coefficient:

$$\binom{n}{k} = \frac{(1)(2)...(n)}{(1)(2)...(k)(1)(2)...(n-k)} = \frac{(n-k+1)(n-k+2)...(n)}{k!} = n\frac{(n-k+1)(n-k+2)...(n-1)}{k!}$$

Note that n can always be factored out of the numerator so long as $k \neq 0, n$ *(Condition 1)* and no integer $0 < r \leq k$ divides n *(Condition 2)*. We need *Condition 1* because it guarantees the non-cancellation of n by the denominator. We have this condition satisfied as a premise. We need *Condition 2* because assuming there is such an r that divdes n, then we get a smaller integer instead of n and we cannot guarantee divisibility. For example, take $\binom{4}{2} = \frac{4!}{2!2!} = 6 \equiv 2 \pmod 4$. Thankfully, *Condition 2* is satisfied by the given premises (primality of n and $k < n$) since to be a prime, it means no number $1 < k < n$ divides n. As such, we conclude that based on the premises given, $n|\binom{n}{k}$.

Question 5

We begin by the base step:

$$p_2 \equiv p_0 \to p_1 \equiv \boldsymbol{T} \to \boldsymbol{T} \equiv \boldsymbol{T}$$

Similarly, for p_3. Now, we proceed to the inductive step:

Assume for a given $n \in \mathbb{N}$, we have $p_{n-1} \equiv \boldsymbol{T}$ and $p_{n-2} \equiv \boldsymbol{T}$. Now, we use this fact (call it f) to prove that $p_n \equiv \boldsymbol{T}$. Note that by the second property of the given set, $p_n \equiv p_{n-1} \to p_{n-2}$ and this is false if and only if p_{n-2} is false. However, this is not the case due to f. Therefore p_n has to be true.

By structural inducton, we conclude that $p_n \equiv \boldsymbol{T}, \forall n \geq 2$.