CS2214B - Assignment 4

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Question 1

Let $\mathcal{P}(X)$ denote the power set of X. Then, we know that $|\mathcal{P}(X)| = 2^n$. (If this is not clear to the reader, this is shown by by noting that there is one empty set 2^0, and that if the number of $(k-1)$-element subsets of a set is 2^{k-1}, then the number of k-element subsets of that set is 2^k because for all the 2^{k-1} subsets, we can append a new element to them as a union, and we get $2 \cdot 2^{k-1} = 2^k$. Then, generalize the results by induction) Morever, we also know that:

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n$$

This can be shown using the famous Binomial Theorem:

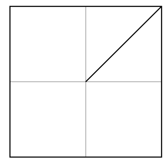
$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$$

Divide $\mathcal{P}(X)$ into disjoint subsets $A_r, r \in \{0, 1, ..., n\}$ such that $\bigcup\limits_{r=0}^{n} A_r = \mathcal{P}(X)$. Then, by the sum rule, we have $2^n = \sum\limits_{r=0}^{n} \binom{n}{r} = |\mathcal{P}(X)| = \left|\bigcup\limits_{r=0}^{n} A_r\right| = \sum\limits_{r=0}^{n} |A_r|$, thus for a given A_r, it must hold that $|A_r| = \binom{n}{k}$. But note the set A given must be one such A_r since all sets A contain all subsets of X with exactly r elements.

In addition, all strings of length n with exactly r 1's can be formed in $\binom{n}{r}$ way by the definition of combinations. Hence, $|B| = \binom{n}{r}$. Since $|A| = |B|$, by definition of cardinality, there is a bijection between A and B. For $j \in \mathbb{N}, 0 \leq j < \binom{n}{r}$ let the j^{th} element of A be a_j, and similarly, let the j^{th} string in B be b_j, then clearly, the function $f : A \to B$ such that $f(a_j) = b_j$ is a bijection, since $f^{-1} : B \to A$ exists and is defined as $f^{-1}(b_j) = a_j$. This concludes our proof.

Question 2

We begin by a sketch of the scenario:



Note our square is 2×2, and we divided it into a grid of four 1×1 squares. Using the Pigeonhole Principle analogy, these squares are going to be the holes. The pigeons in the analogy are going to be the 5 points. Therefore, if we place the points in the square, then we have at least 2 points in the same 1×1 square by the Pigeonhole Principle. To maximize the Euclidean distance of two points on a square, they must lie on two corners that form a diagonal. In our little sketch, we see that the line connecting two such points (distance maximizer) is just the hypotenuse of a right-angled triangle whose other sides are of length 1. By the Pythagorean Theorem, this hypotenuse has length $l = \sqrt{1^2 + 1^2} = \sqrt{2}$. Note that in fact, no interior point p_i is allowed to sit on the outer corners, hence the hypotenuse must be strictly smaller than $l = \sqrt{2}$. This concludes our proof.

Question 3

Denote the number of binary strings of length 7 and z 0's by $\mathcal{N}_7(z)$. Then:

$$\mathcal{N}_7(0) = \binom{7}{0} = 1$$

$$\mathcal{N}_7(1) = \binom{7}{1} = 7$$

$$\mathcal{N}_7(2) = \binom{7}{2} = 21$$

$$\mathcal{N}_7(3) = \binom{7}{3} = 35$$

$$T = \sum_{i=0}^{3} \mathcal{N}_7(i) = 64$$

Thus, using the definition of combinations and the sum rule, we obtained a total of 64 such strings.

Question 4

Recall the Binomial Theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

In this case, $a = x, b = x^{-2}, n = 100$, and we need to find k. Note that since $b^k = a^{-2k}$, then:

$$(x + x^{-2})^{100} = \sum_{k=0}^{100} \binom{100}{k} x^{100-k} x^{-2k} = \sum_{k=0}^{100} \binom{100}{k} x^{100-3k}$$

We need the value of k such that $100 - 3k = 94$. By inspection, $k = 2$. Then, the coefficient of x^{94} is $\binom{100}{2} = 4950$.