CS2214B - Assignment 3

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Question 1

Denote $\begin{bmatrix} x \\ y \end{bmatrix}$ by $\vec{v}$. We claim that the matrix $\mathbf{C} = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}$ satisfies the following substitution:

$$\mathbf{AB}\vec{v} = \mathbf{C}\vec{v}$$

Proof.

$$\mathbf{C}\vec{v} = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (ap+br)x + (aq+bs)y \\ (cp+dr)y + (cq+ds)y \end{bmatrix}$$

$$\mathbf{C}\vec{v} = \begin{bmatrix} a(px+qy) + b(rx+sy) \\ c(px+qy) + d(rx+sy) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} px+qy \\ rx+sy \end{bmatrix}$$

$$\mathbf{C}\vec{v} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p(x)+q(y) \\ r(x)+s(y) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{AB}\vec{v}$$

Hence, we have shown that matrix multiplication is a form of linear system substitution, with $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} px+qy \\ rx+sy \end{bmatrix}$ and $\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} (ap+br)x + (aq+bs)y \\ (cp+dr)y + (cq+ds)y \end{bmatrix}$. $\square$

Question 2

$$\sum_{k=0}^{n}(3k+1)^2 = 1 + \sum_{k=1}^{n}(3k+1)^2 = 1 + \sum_{k=0}^{n}(9k^2+6k+1)$$

$$= 1 + 9\sum_{k=1}^{n}k^2 + 6\sum_{k=1}^{n}k + \sum_{k=1}^{n}1 = 1 + \frac{9n(n+1)(2n+1)}{6} + \frac{6n(n+1)}{2} + n$$

$$= 1 + \frac{3n(n+1)(2n+1) + 6n(n+1) + 2n}{2}$$

$$= 1 + \frac{6n^3 + 15n^2 + 11n}{2}$$

$$= \frac{6n^3 + 15n^2 + 11n + 2}{2}$$

$$= f(n)$$

Question 3

Proof. Let $P(n)$ denote that 13 divides integers of the form $f(n) = 3^{n+1} + 4^{2n-1}$, for any positive n.

Base Case: $P(1)$, i.e. 13 divides 13, which is true.

Inductive Hypothesis: Assume $P(k)$ holds for $1 < k < n$. We prove it also holds for $k+1$.

Inductive Step:

$$f(k+1) = 3^{k+2} + 4^{2k+1}$$
$$f(k+1) = 3(3^{k+1}) + 4^2(4^{2k-1})$$
$$f(k+1) = 3(3^{k+1}) + 3(4^{2k-1}) + 13(4^{2k-1})$$
$$f(k+1) = 3f(k) + 13(4^{2k-1})$$

Since 13 divides $f(k)$ by the inductive hypothesis, it also divides any multiple of it. Similarly, since 13 divides 13, it must divide any multiple of it by the definition of divisibility. If 13 divides two numbers, it must divide their sum since 13 can be factored from both terms. That is, 13 divides $f(k+1)$. We have shown:

$$P(1) \land (\forall k \in \mathbb{Z}^+, k < n, P(k) \to P(k+1))$$

Thus, by the principle of induction, 13 divdes all positive integers of the form $f(n), n \in \mathbb{Z}^+$. □

Question 4

Proof. Let $P(n)$ denote that $\mathbf{A}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$ for any positive n.

Base Case: $P(1)$, i.e. $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, which is given.

Inductive Hypothesis: Assume $P(k)$ holds for $1 < k < n$. We prove it also holds for $k+1$.

Inductive Step:

$$A^{k+1} = AA^k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix}$$
$$A^{k+1} = \begin{bmatrix} F_{k+1} + F_k & F_k + F_{k-1} \\ F_{k+1} & F_k \end{bmatrix}$$
$$A^{k+1} = \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix}$$

Powers of matrices are well-defined since repeated multiplication of the same matrix from any direction results in the same power matrix. Additionally, we have used the fact that $F_{n+2} = F_n + F_{n+1}$, which is the recursive definition of a Fibbonaci sequence. We have shown:

$$P(1) \land (\forall k \in \mathbb{Z}^+, k < n, P(k) \to P(k+1))$$

Thus, by the principle of induction, $\mathbf{A}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}, \forall n \in \mathbb{Z}^+$. □