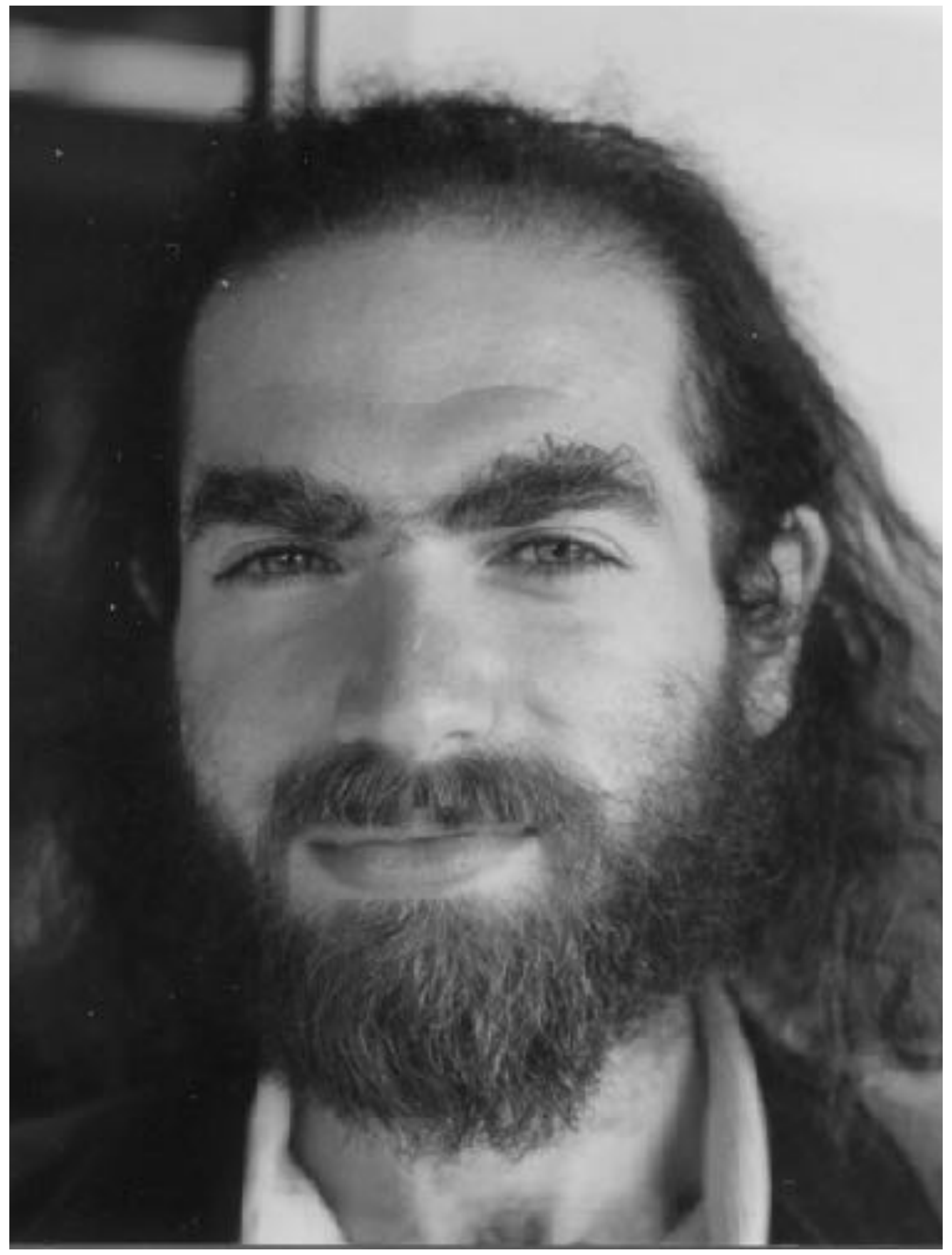
Niels Henrik Abel
(1802–1829)

Srinivasa Ramanujan
(1887–1920)

Alexander Grothendieck
(1928–2014)



Grigori Perelman
(b. 1966)

Pierre de Fermat
(1601–1665)

Blaise Pascal
(1623–1662)

Problem of Points

Pierre de Fermat
(1601–1665)

Consider a game of chance with two players who toss a fair coin. The players contribute equally to a prize pot, and agree in advance that the first player to have won a certain number of tosses will collect the entire prize. Now suppose that the game is interrupted by external circumstances before either player has achieved victory. How does one then divide the pot fairly?

Blaise Pascal
(1623–1662)

Problem of Points

Luca Pacioli
(1447– 1517)

Luca Pacioli considered such a problem in his 1494 textbook *Summa de arithmetica, geometrica, proportioni et proportionalità*. His method was to divide the stakes in proportion to the number of rounds won by each player, and the number of rounds needed to win did not enter his calculations at all.

Problem of Points

Luca Pacioli
(1447– 1517)

What if only one game is played?

Nicolo Tartaglia
(1499–1557)

Problem of Points

Luca Pacioli
(1447– 1517)

What if only one game is played?

Instead consider the size of the lead and the length of the game.

$$\frac{L}{T} + \frac{1}{2} : \frac{1}{2} - \frac{L}{T}$$

Nicolo Tartaglia
(1499–1557)

How did Fermat and Pascal solve it?

Pierre de Fermat
(1601–1665)

Consider a game of chance with two players who toss a fair coin. The players contribute equally to a prize pot, and agree in advance that the first player to have won a certain number of tosses will collect the entire prize. Now suppose that the game is interrupted by external circumstances before either player has achieved victory. How does one then divide the pot fairly?

Blaise Pascal
(1623–1662)

How to Win at a Casino

Lecture 1

Lodha Genius Programme 2025

How to Win at a Casino

Instructor: Aalok Thakkar

Teaching Assistants:

1. Abheri Banerjee
2. Saptak Bhattacharya
3. Upanshu Lakhani
4. Suraj Kumar Maharana
5. Vedika Navani
6. Vedant Rana
7. Rupsha
8. Sufiyan Ashraf Tejani

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7. Rupsha
8. Sufiyan Ashraf Tejani

Assessments:

Lecture 1: Exploration Task (Group)
Lecture 2: Problem Set (Group)
Lecture 3: In-class Auction
Lecture 4: Problem Set (Individual)
Lecture 5: In-class Casino Games

Optional Activity:

Screening: 21 (2008 film)
Thursday, June 12 at AC04-005

Exploration I

Pierre de Fermat
(1601–1665)

Consider a game of chance with two players who toss a fair coin. The players contribute equally to a prize pot, and agree in advance that the first player to have won a certain number of tosses will collect the entire prize. Now suppose that the game is interrupted by external circumstances before either player has achieved victory. How does one then divide the pot fairly?

Blaise Pascal
(1623–1662)

Consider this game:

I toss a coin. If it is heads, I give you ₹10. If it is tails, you give me ₹10.

Consider this game:

I toss a coin. If it is heads, I give you ₹10. If it is tails, you give me ₹10.

We continue playing this game till either of us runs out of money.

Consider this game:

I toss a coin. If it is heads, I give you ₹10. If it is tails, you give me ₹10.

We continue playing this game till either of us runs out of money.

Is this game *fair*?
Should you play this game?

Player A starts with ₹10 vs
Player B starts with ₹100?

Player A starts with M_A vs
Player B starts with M_B?

Christiaan Huygens

If player A has 0, then they lose.

If player A has $M_A + M_B$, then they win.

Christiaan Huygens

If player A has 0, then they lose.

If player A has $M_A + M_B$, then they win.

If player A has m, let us say the probability of winning is $P(m)$.

If player A has 0, then they lose. $P(0) = 0$

If player A has $M_A + M_B$, then they win. $P(M_A + M_B) = 1$

If player A has m, let us say the probability of winning is $P(m)$.

Christiaan Huygens

$$P(0) = 0$$

$$P(M_A + M_B) = 1$$

$$P(m) = \frac{1}{2}\left(P(m - 10) + P(m + 10)\right)$$

Christiaan Huygens

$$P(0) = 0$$

$$P(M_A + M_B) = 1$$

$$P(m) - P(m - 10) = P(m + 10) - P(m)$$

Christiaan Huygens

$$P(0) = 0$$

$$P(M_A + M_B) = 1$$

$$P(m) - P(m - 10) = P(m + 10) - P(m)$$

$$c = P(i) - P(i - 10)$$

Christiaan Huygens

Gambler's Ruin

$$P(M_A) = \frac{M_A}{M_A + M_B}$$

$$P(M_B) = \frac{M_B}{M_A + M_B}$$

Christiaan Huygens

Gambler's Ruin

Seeming fair processes can lead to unfair outcomes.

The House Always Wins

Exploration II

How would you make Gambler’s Ruin more fair?

Let's Start with Poker!

I have a well-shuffled deck of cards.

I deal you one card and I deal myself one card.

We decide the following order on the cards:

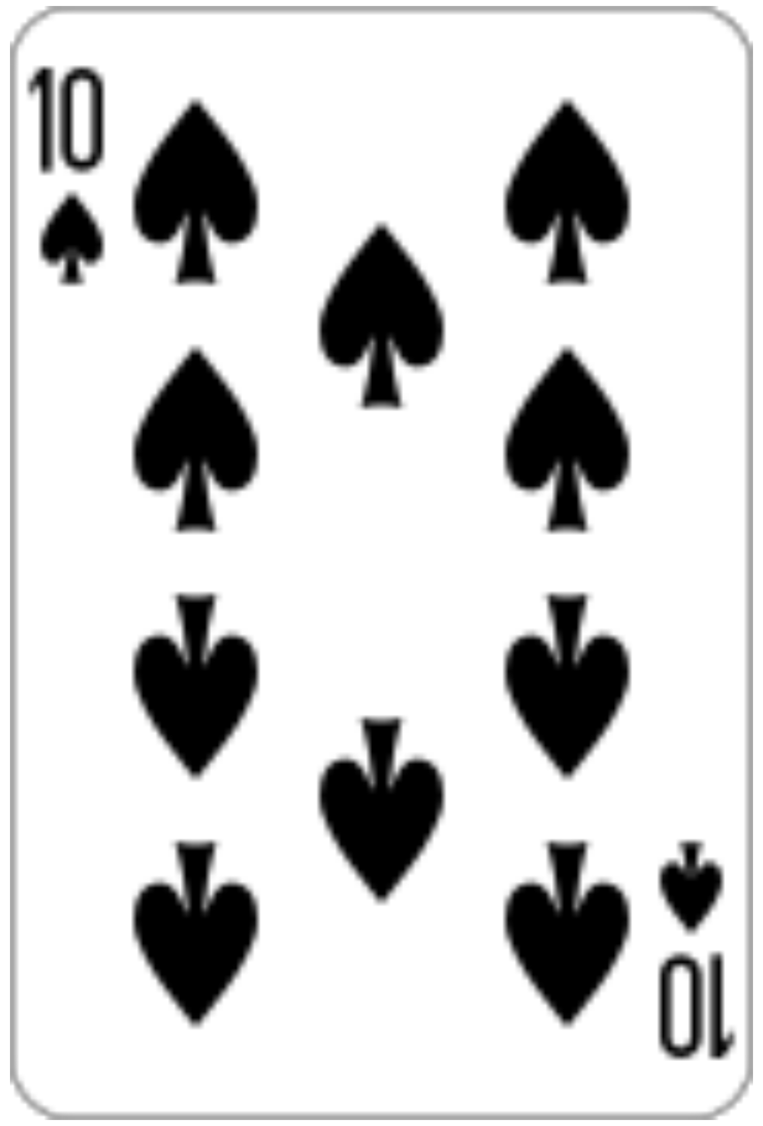
$A > K > Q > J > 10 > 9 > 8 > 7 > 6 > 5 > 4 > 3 > 2$

You and I each get one card face-down.

My Card

Your Card

My Card



Your Card

My Card

Your Card

My Card

Which one is higher?

Your Card

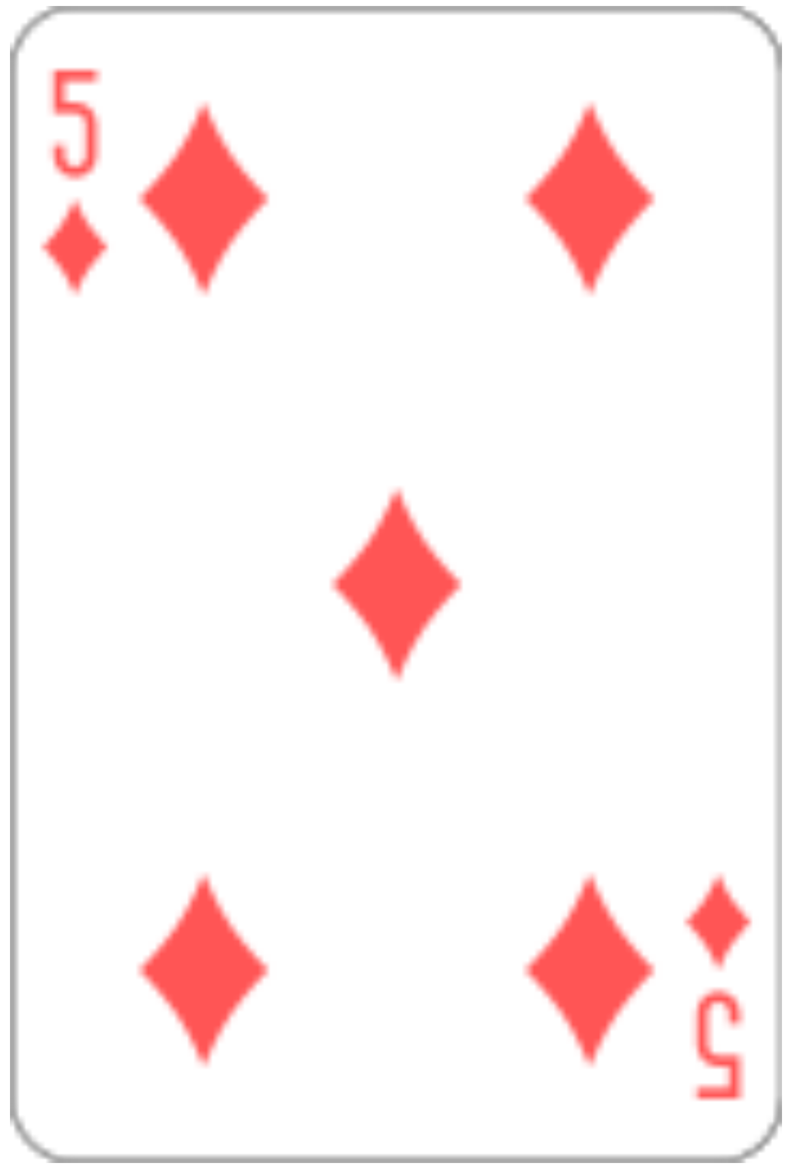
Which one is higher?

Case I: Your Card!

My Card

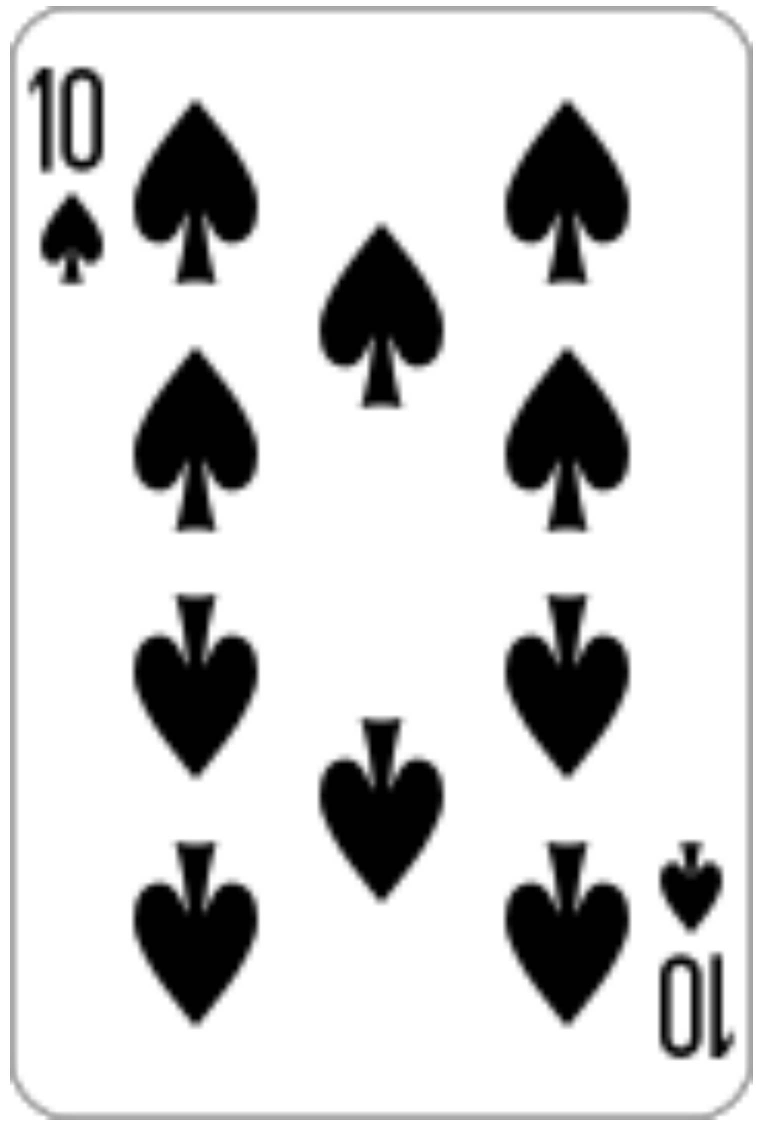
Your Card

Which one is higher?



My Card

Case I: Your Card!



Your Card

Which one is higher?

Case I: Your Card!

Case II: My Card!

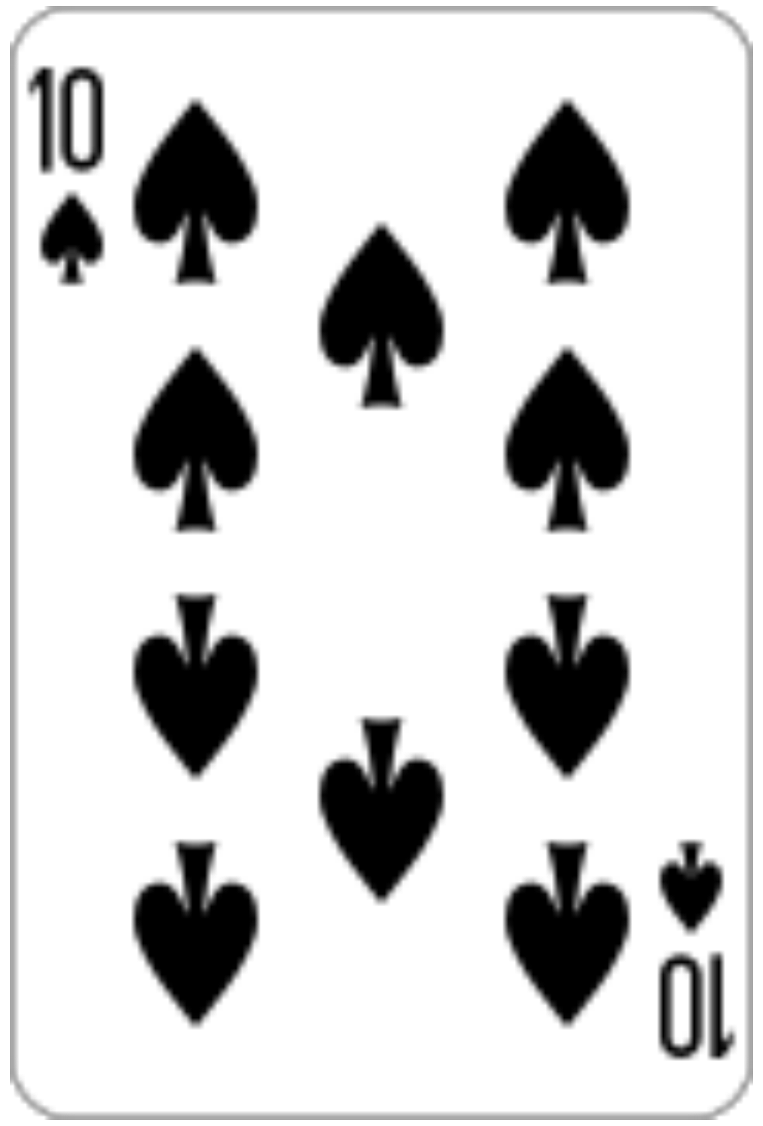
My Card

Your Card

Which one is higher?

Case I: Your Card!

Case II: My Card!



My Card

Your Card

Which one is higher?

Case I: Your Card!

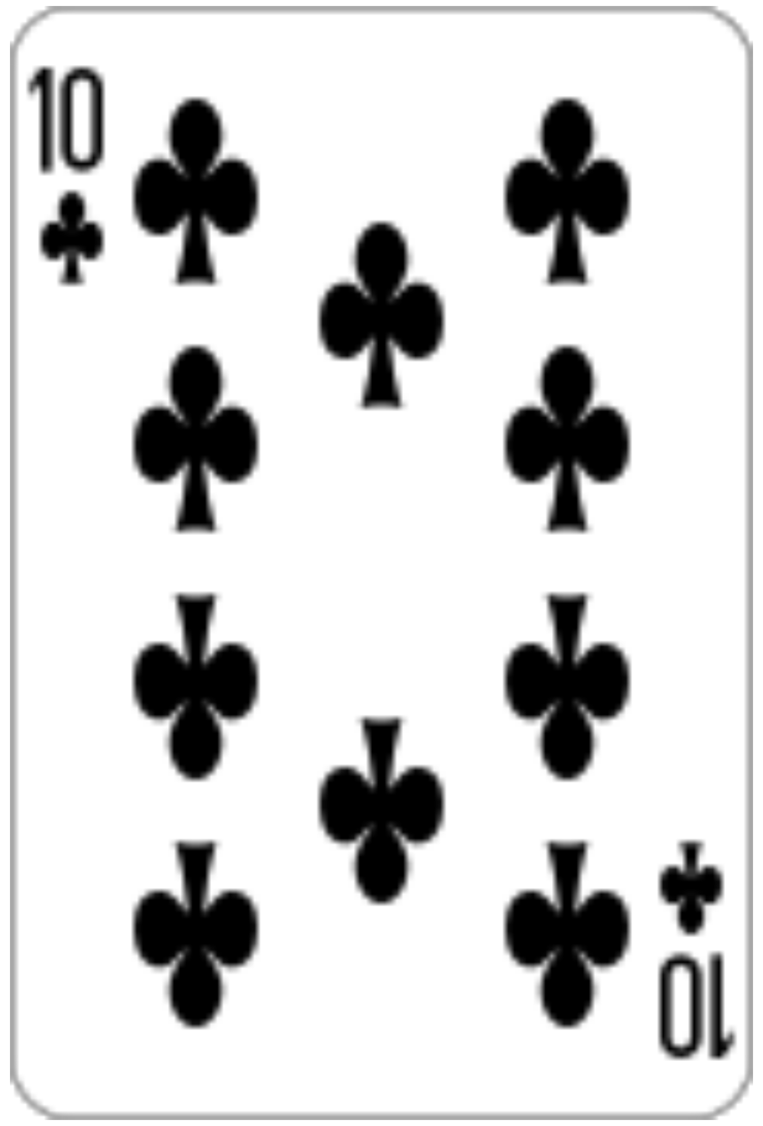
Case II: My Card!

Case III: None!

My Card

Your Card

Which one is higher?

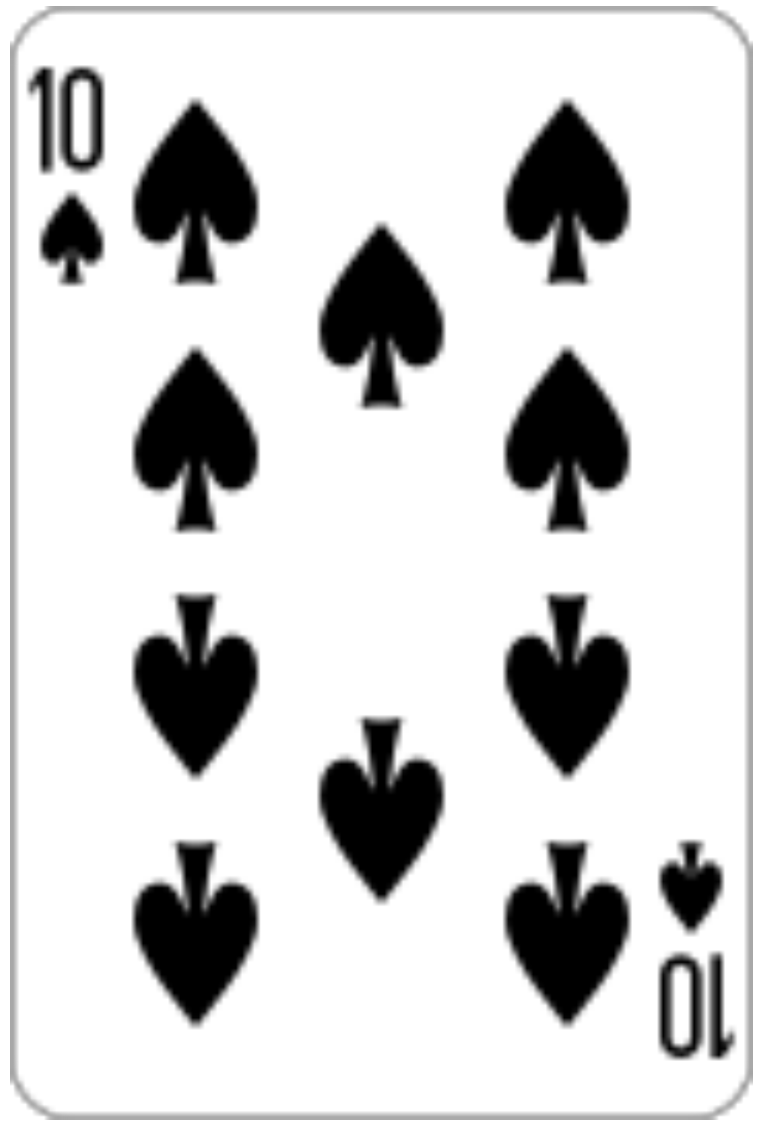


My Card

Case I: Your Card!

Case II: My Card!

Case III: None!



Your Card

Suppose you had to bet on these three cases. How would you bet money?

I have a well-shuffled deck of cards.

I deal you one card and I deal myself one card.

We decide the following order on the cards:

$$A > K > Q > J > 10 > 9 > 8 > 7 > 6 > 5 > 4 > 3 > 2$$

You and I each get one card face-down.

I have a **well-shuffled** deck of cards.

I deal you one card and I deal myself one card.

We decide the following order on the cards:

$A > K > Q > J > 10 > 9 > 8 > 7 > 6 > 5 > 4 > 3 > 2$

You and I each get one card face-down.

Which one is higher?

My Card

Case I: Your Card!

Your Card

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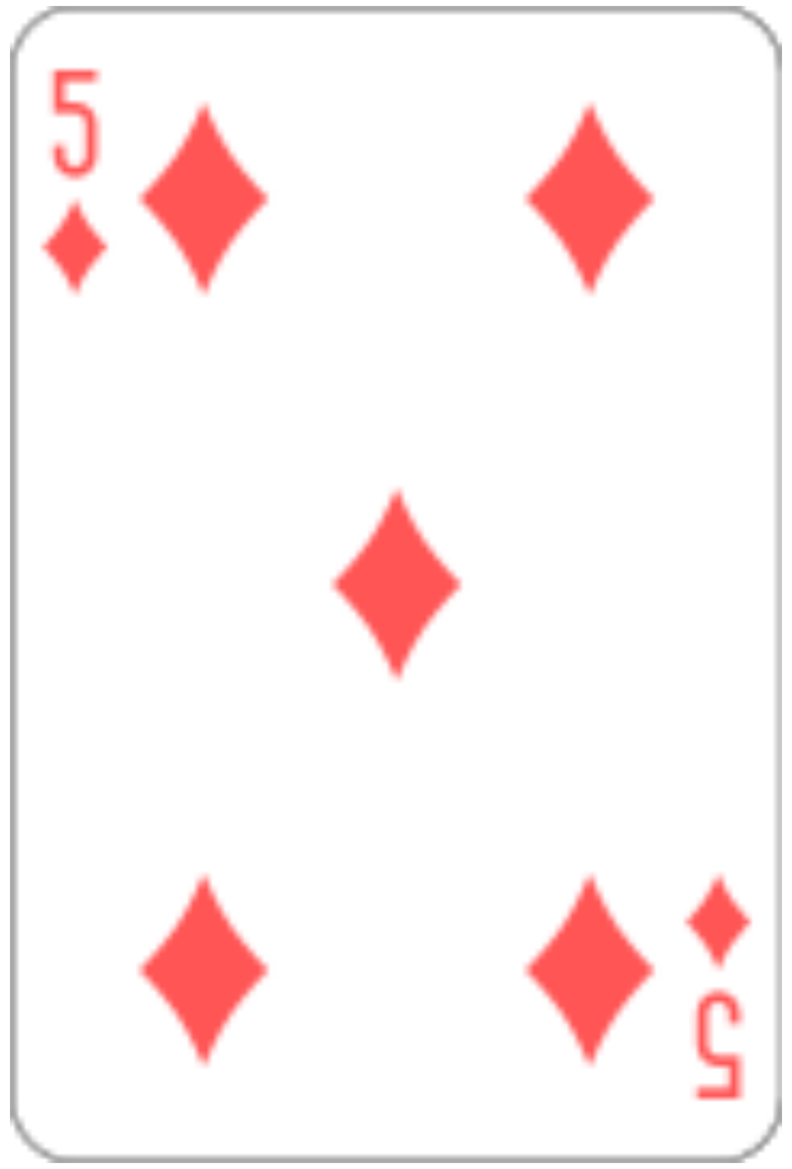
We decide the following order on the cards:

A > K > Q > J > 10 > **9** > **8** > **7** > **6** > **5** > **4** > **3** > **2**

You and I each get one card face-down.

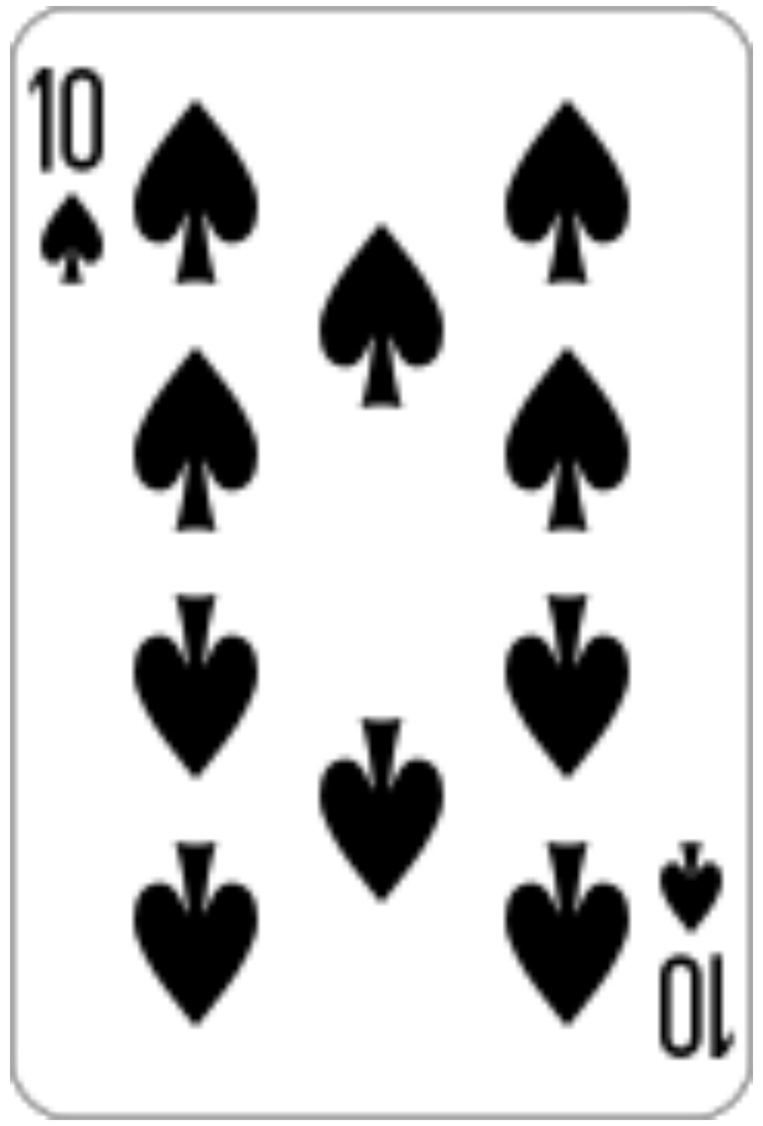
Which one is higher?



My Card

Case I: Your Card!

$$\frac{8}{13} \approx 61.54\,\%$$



Your Card

Which one is higher?

Case I: Your Card!

Case II: My Card!

My Card

Your Card

I have a **well-shuffled** deck of cards.

I deal you one card and I deal myself one card.

We decide the following order on the cards:

A $>$ **K** $>$ **Q** $>$ **J** $>$ 10 $>$ 9 $>$ 8 $>$ 7 $>$ 6 $>$ 5 $>$ 4 $>$ 3 $>$ 2

You and I each get one card face-down.

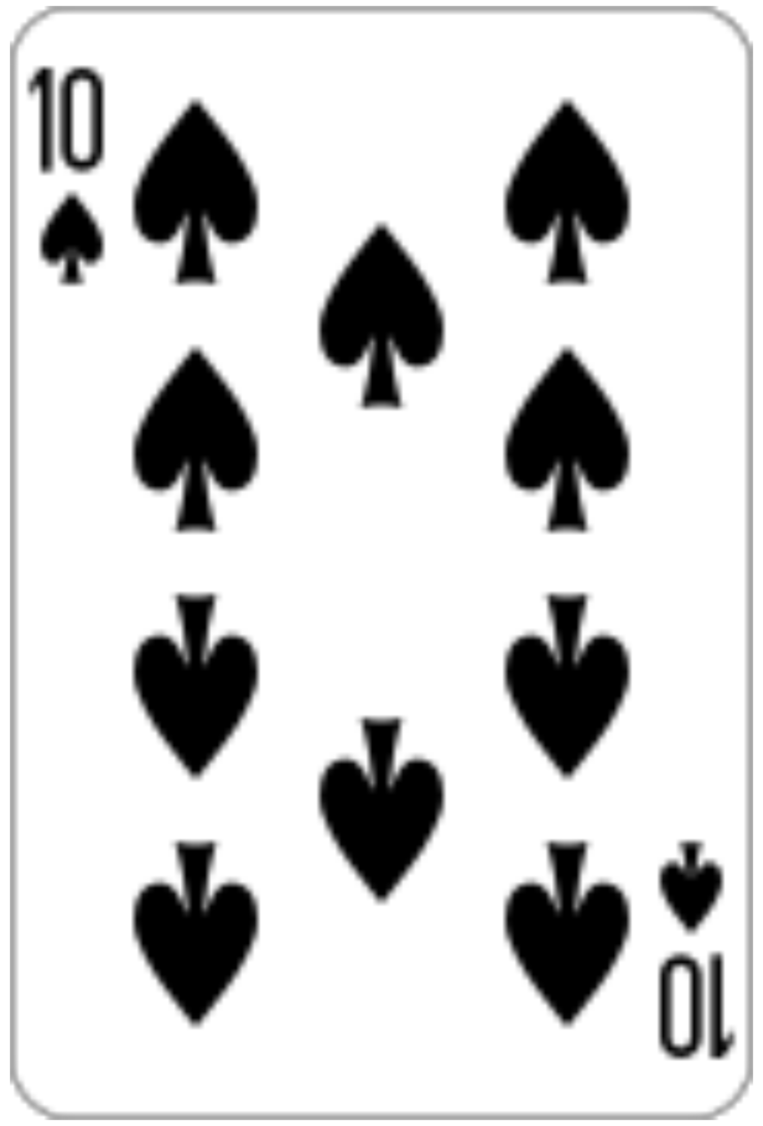
Which one is higher?

My Card

Case I: Your Card!

Case II: My Card!

$$\frac{4}{13} \approx 30.77\,\%$$



Your Card

Case I: Your Card!

$$\frac{8}{13} \approx 61.54\,\%$$

Case II: My Card!

$$\frac{4}{13} \approx 30.77\,\%$$

Why is this wrong?

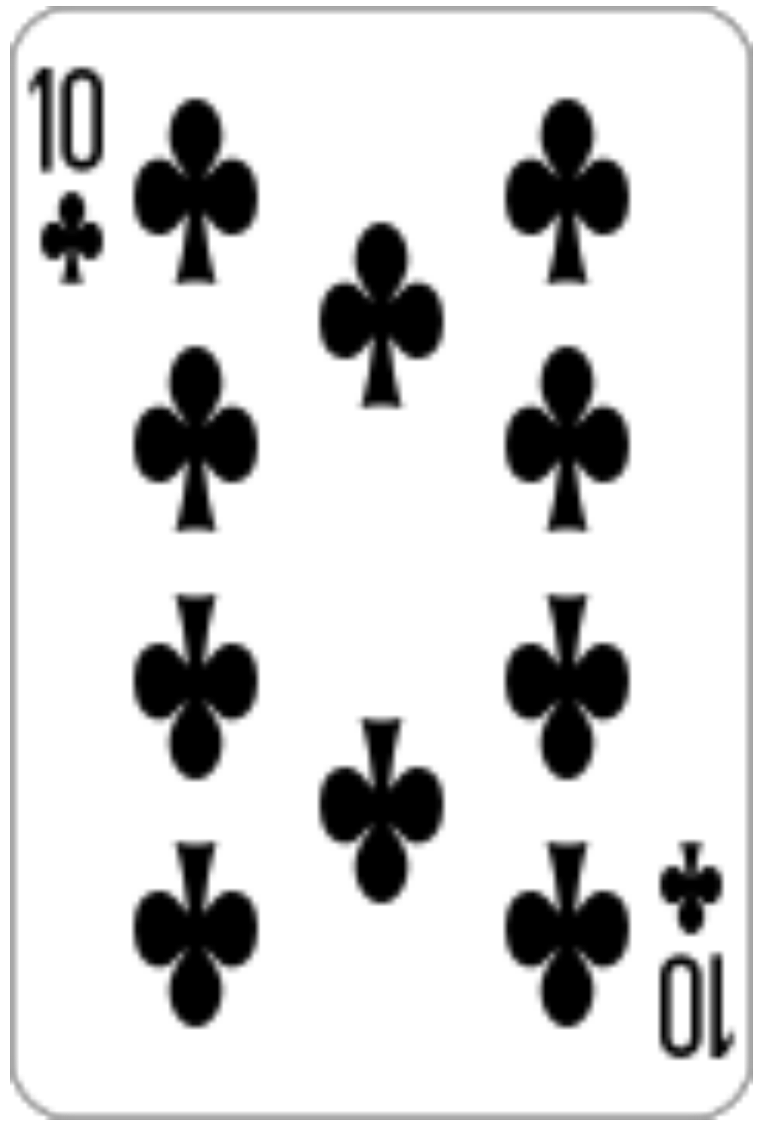
Case I: Your Card!

$$\frac{32}{51} \approx 62.74\,\%$$

Case II: My Card!

$$\frac{16}{51} \approx 31.37\,\%$$

Which one is higher?



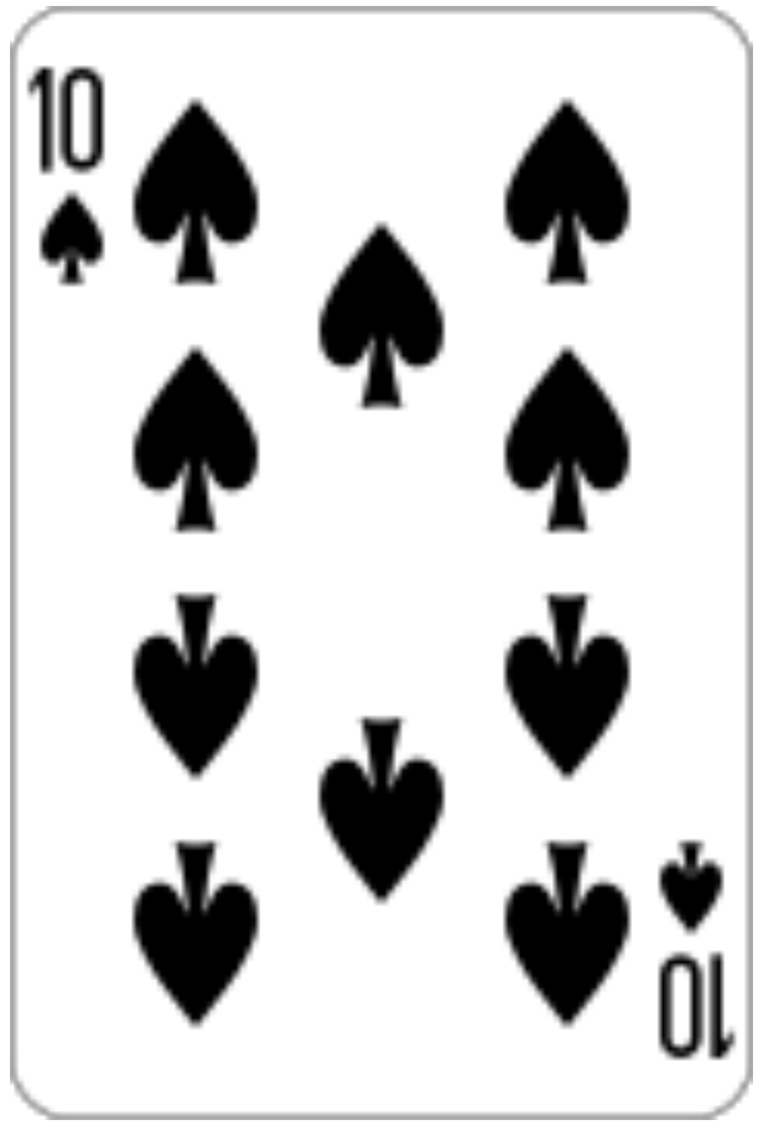
My Card

Case I: Your Card!

Case II: My Card!

Case III: None!

$$\frac{3}{51} \approx 5.88\,\%$$



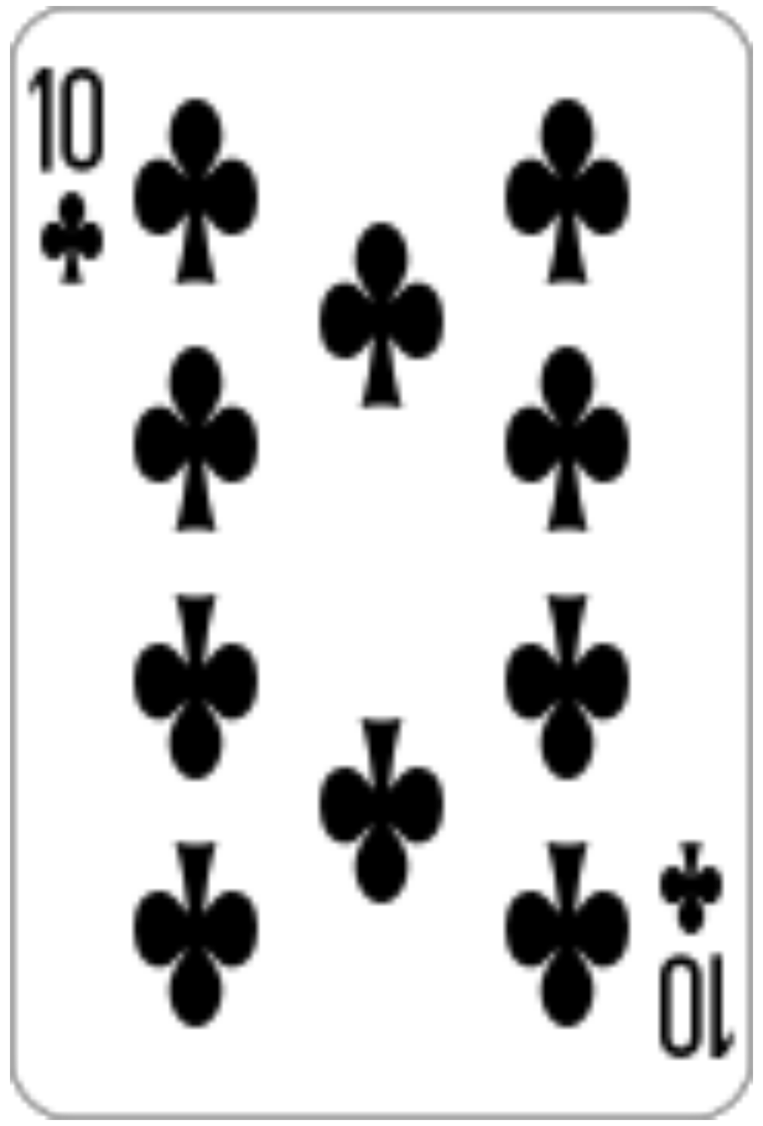
Your Card

Player X: Can either bet one token or fold

Player Y: If player X has folded, player Y also folds. If player X has bet, player Y can either bet or fold.

What is a good *strategy* for player X? What about player Y?

Which one is higher?

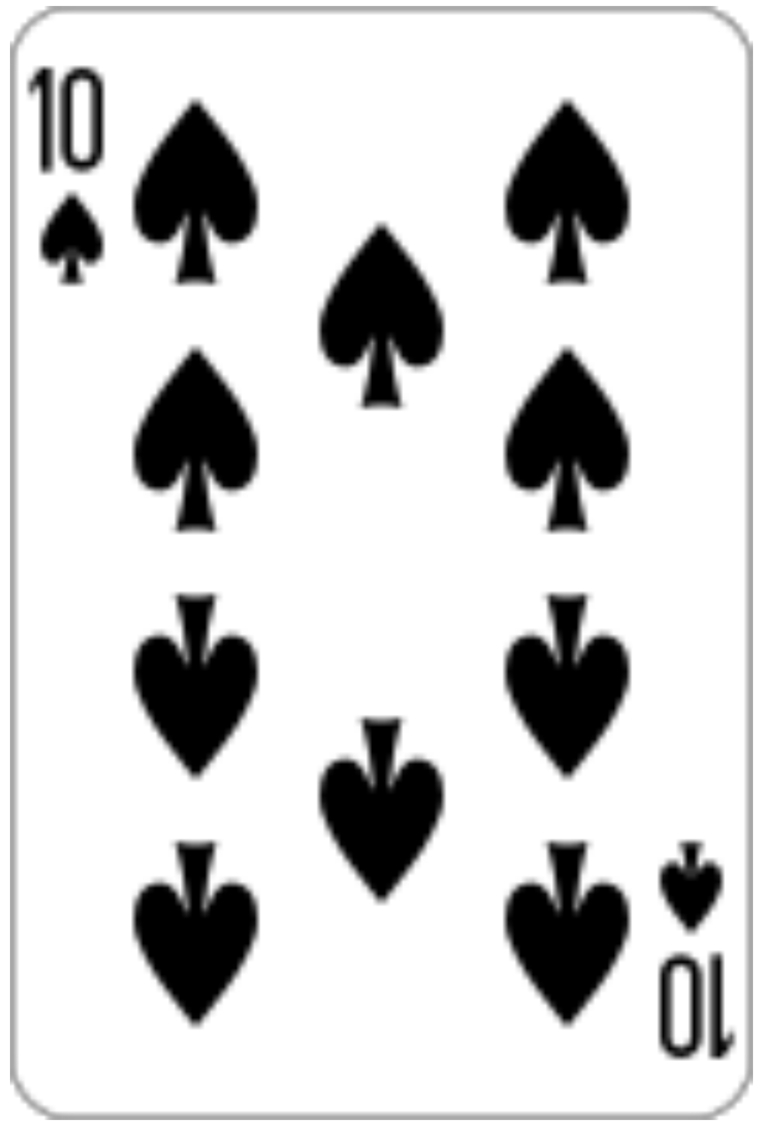


Case I: Your Card!

Case II: My Card!

Case III: None!

$$\frac{3}{51} \approx 5.88\,\%$$



My Card

Your Card

Under the assumption that the deck is well shuffled!

Exploration III

When is a deck of card well-shuffled?

How do you get a deck of well-shuffled cards?

Suppose you had to bet on these three cases. How would you bet money?

Case I: Your Card!

$$\frac{32}{51} \approx 62.74\,\%$$

Case II: My Card!

$$\frac{16}{51} \approx 31.37\,\%$$

Case III: None!

$$\frac{3}{51} \approx 5.88\,\%$$

What does any of this mean?

मदिरालय में कब से बैठा, पी न सका अब तक हाला
यत्न सहित भरता हूँ, कोई किंतु उलट देता प्याला |

मानव-बल के आगे निर्बल भाग्य सुना विद्यालय में
"भाग्य-प्रबल, मानव निर्बल" का पाठ पढाती मधुशाला ||

How do you know your math is right?

If I throw 4 dice at random, what is the probability that there will be at least 2 twos?

If I throw 4 dice at random, what is the probability that there will be at least 2 twos?

1 1 1 1

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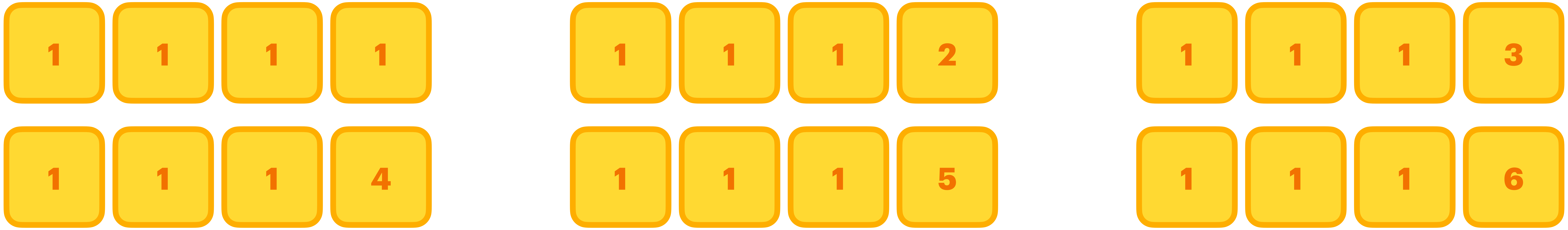

If I throw 4 dice at random, what is the probability that there will be at least 2 twos?

1	1	1	1	1	1	1	2	1	1	1	3
1	1	1	4	1	1	1	5	1	1	1	6
1	1	2	1	1	1	2	2	1	1	2	3
1	1	2	4	1	1	2	5	1	1	2	6

If I throw 4 dice at random, what is the probability that there will be at least 2 twos?

1	1	1	1	1	1	1	2	1	1	1	3
1	1	1	4	1	1	1	5	1	1	1	6
1	1	2	1	1	1	2	2	1	1	2	3
1	1	2	4	1	1	2	5	1	1	2	6
1	1	3	1	1	1	3	2	1	1	3	3
1	1	3	4	1	1	3	5	1	1	3	6

If I throw 4 dice at random, what is the probability that there will be at least 2 twos?

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$$P(k \geq 2) = P(k = 2) + P(k = 3) + P(k = 4)$$

If I throw 4 dice at random, what is the probability that there will be at least 2 twos?

$$P(k \geq 2) = P(k = 2) + P(k = 3) + P(k = 4)$$

$$P(k) = \binom{4}{k} \left(\frac{1}{6}\right)^{k} \left(\frac{5}{6}\right)^{4-k}$$

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$$P(k) = \binom{4}{k} \left(\frac{1}{6}\right)^{k} \left(\frac{5}{6}\right)^{4-k}$$

$$P(k \geq 2) = \binom{4}{2} \left(\frac{1}{6}\right)^{2} \left(\frac{5}{6}\right)^{2} + \binom{4}{3} \left(\frac{1}{6}\right)^{3} \left(\frac{5}{6}\right)^{1} + \binom{4}{4} \left(\frac{1}{6}\right)^{4} \left(\frac{5}{6}\right)^{0}$$

If I throw 4 dice at random, what is the probability that there will be at least 2 twos?

$$P(k \geq 2) = P(k = 2) + P(k = 3) + P(k = 4)$$

$$P(k) = \binom{4}{k} \left(\frac{1}{6}\right)^{k} \left(\frac{5}{6}\right)^{4-k}$$

$$P(k \geq 2) = \left(6 \times \frac{1}{36} \times \frac{25}{36}\right) + \left(4 \times \frac{1}{216} \times \frac{5}{6}\right) + \left(1 \times \frac{1}{1296} \times 1\right)$$

If I throw 4 dice at random, what is the probability that there will be at least 2 twos?

$$P(k \geq 2) = P(k = 2) + P(k = 3) + P(k = 4)$$

$$P(k) = \binom{4}{k} \left(\frac{1}{6}\right)^{k} \left(\frac{5}{6}\right)^{4-k}$$

$$P(k \geq 2) = \left(\frac{150}{1296}\right) + \left(\frac{20}{1296}\right) + \left(\frac{1}{1296}\right) = \frac{171}{1296} \approx 0.132$$

If I throw 4 dice at random, what is the probability that there will be at least 2 twos?

$$P(k \geq 2) = P(k = 2) + P(k = 3) + P(k = 4)$$

$$P(k) = \binom{4}{k}\left(\frac{1}{6}\right)^{k}\left(\frac{5}{6}\right)^{4-k}$$

$$P(k \geq 2) = \left(\frac{150}{1296}\right) + \left(\frac{20}{1296}\right) + \left(\frac{1}{1296}\right) = \frac{171}{1296} \approx 0.132$$

How do you *know*?

$$P(k) = \binom{4}{k}\left(\frac{1}{6}\right)^{k}\left(\frac{5}{6}\right)^{4-k}$$

How do you *know*?

What does *knowledge of uncertainty* even mean?

$$P(k) = \binom{4}{k}\left(\frac{1}{6}\right)^{k}\left(\frac{5}{6}\right)^{4-k}$$

$$D = \{e_1, e_2, e_3, \ldots, e_n\}$$

Let us say we observe two k out of n times.

$$D = \{e_1, e_2, e_3, \ldots, e_n\}$$

Let us say we observe two k out of n times.

Let $L(p)$ be the probability of observing data D given that the probability of getting a two is p.

$$L(p) = P(D \mid p)$$

$$D = \{e_1, e_2, e_3, \ldots, e_n\}$$

Let us say we observe two k out of n times.

Let $L(p)$ be the probability of observing data D given that the probability of getting a two is p.

$$L(p) = P(D \,|\, p)$$

Independent and Identically Distributed

$$D = \{e_1, e_2, e_3, \ldots, e_n\}$$

Let us say we observe two k out of n times.

Let $L(p)$ be the probability of observing data D given that the probability of getting a two is p.

$$L(p) = \prod_{i=1}^{n} P(e_i \mid p)$$

$$D = \{e_1, e_2, e_3, \ldots, e_n\}$$

Let us say we observe two k out of n times.

Let $L(p)$ be the probability of observing data D given that the probability of getting a two is p.

$$L(p) = \prod_{i=1}^{n} P(e_i \mid p)$$

Is this a fact?
Is this an assumption?

$$D = \{e_1, e_2, e_3, \ldots, e_n\}$$

Let us say we observe two k out of n times.

Let $L(p)$ be the probability of observing data D given that the probability of getting a two is p.

$$L(p) = \binom{n}{k} p^k (1-p)^{n-k}$$

Let us say we observe twos k out of n times.

$$D = \{e_1, e_2, e_3, \ldots, e_n\}$$

Let us say we observe two k out of n times.

Let $L(p)$ be the probability of observing data D given that the probability of getting a two is p.

$$L(p) = \binom{n}{k} p^k (1-p)^{n-k}$$

What if $p = 0$? What if $p = 0.1$? What if $p = 0.5$? What if $p = 1$?

$$D = \{e_1, e_2, e_3, \ldots, e_n\}$$

Let us say we observe two k out of n times.

Let $L(p)$ be the probability of observing data D given that the probability of getting a two is p.

$$L(p) = \binom{n}{k} p^k (1-p)^{n-k}$$

For what p, are we most likely to observe D?

$$D = \{e_1, e_2, e_3, \ldots, e_n\}$$

Let us say we observe two k out of n times.

Let $L(p)$ be the probability of observing data D given that the probability of getting a two is p.

$$L(p) = \binom{n}{k} p^k (1-p)^{n-k}$$

For what p, is $L(p)$ maximum?

Exploration IV

For what p, is $L(p)$ maximum? Prove it.

Weak Law of Large Numbers

If you take the average of a large number of independent and identically distributed random variables, that average will get closer and closer to the *expected value* (the true average) as you take more and more samples.

Weak Law of Large Numbers

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$$S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Weak Law of Large Numbers

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$$S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

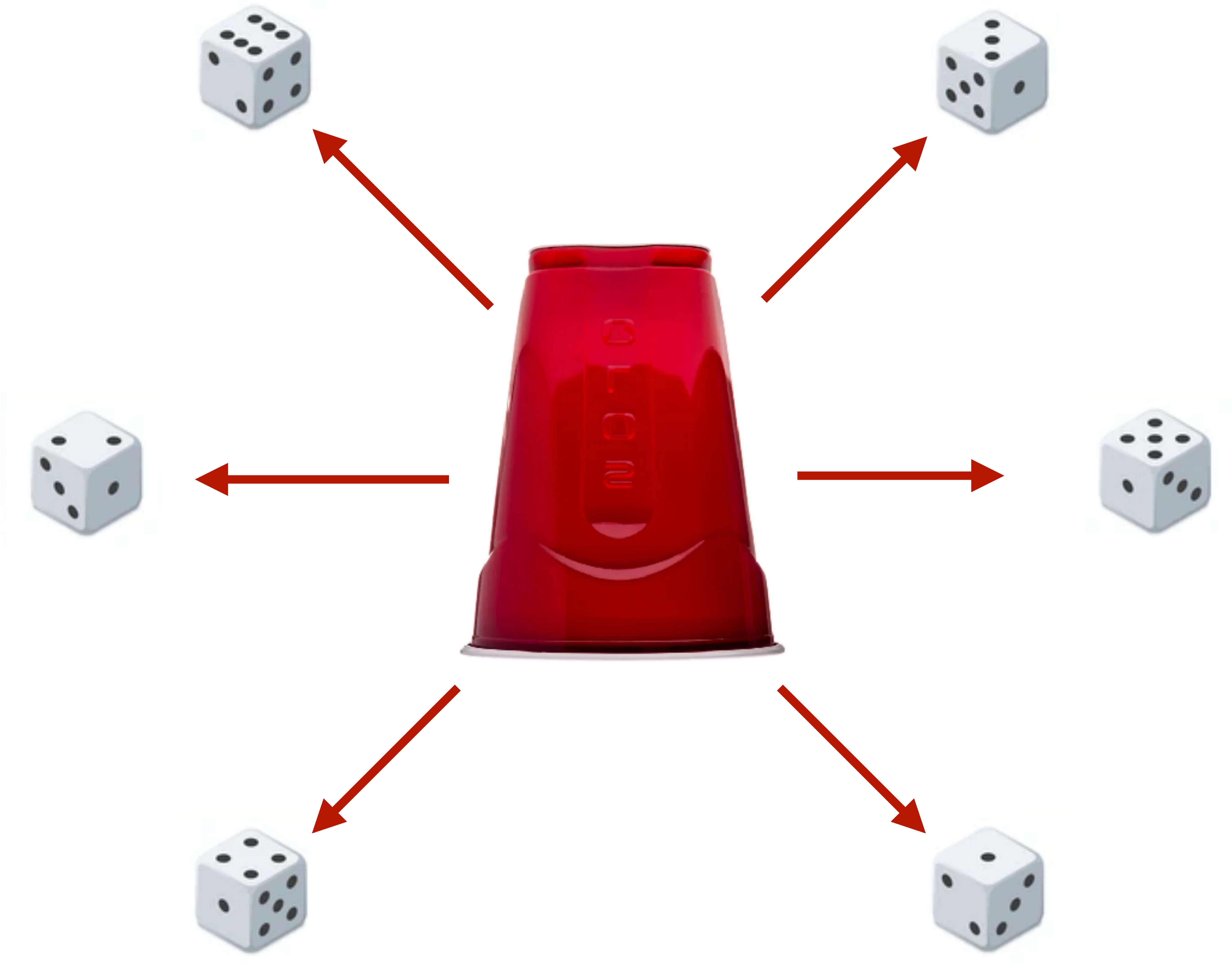
$$\lim_{n\to\infty} P\left(|S_n - \mathbb{E}(X)| \geq \varepsilon\right) = 0$$

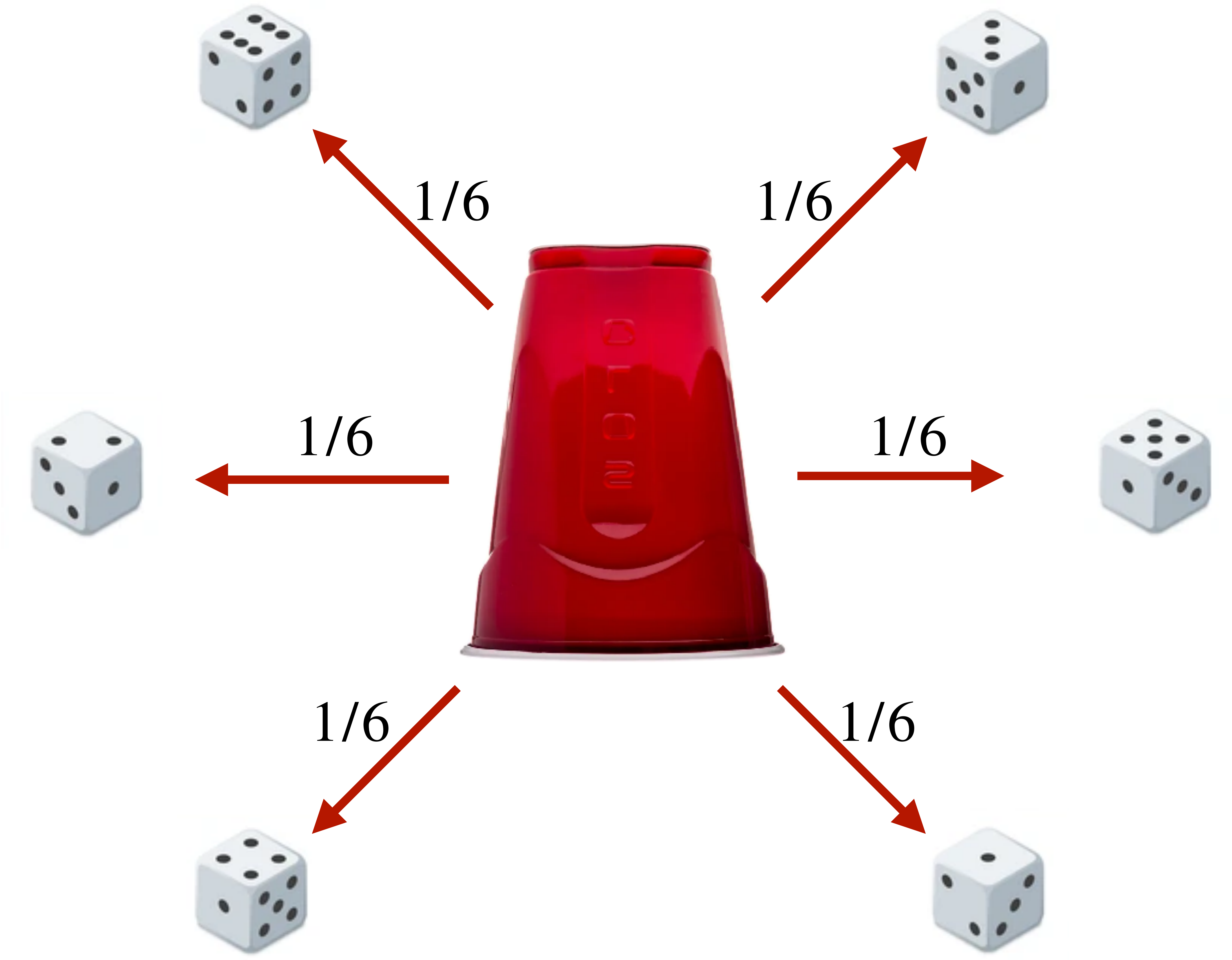
Weak Law of Large Numbers

If you take the average of a large number of independent and identically distributed random variables, that average will get closer and closer to the *expected value* (the true average) as you take more and more samples.

For all $\varepsilon > 0$:

$$\lim_{n\to\infty} P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}(X)\right| \geq \varepsilon\right) = 0$$

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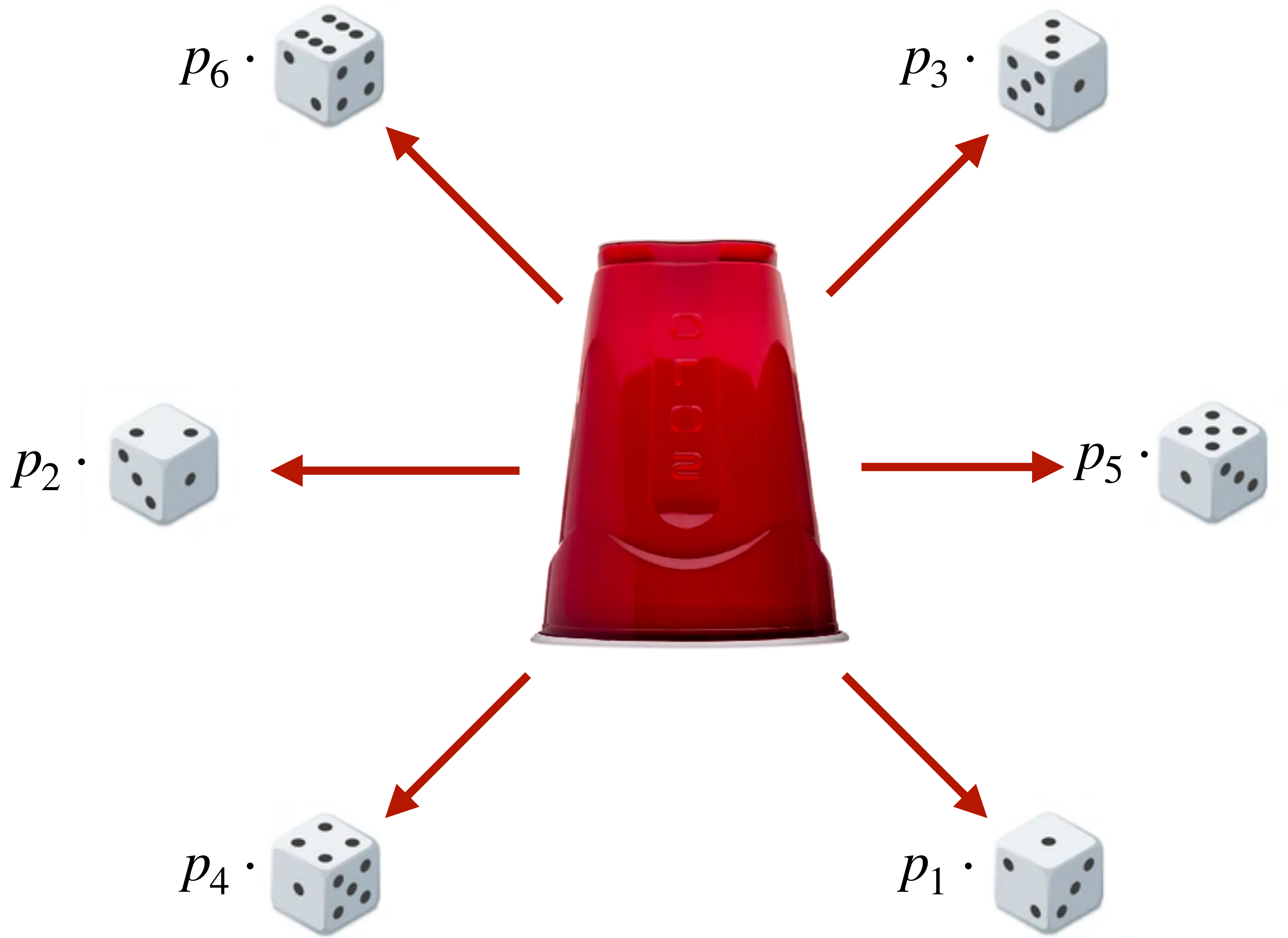
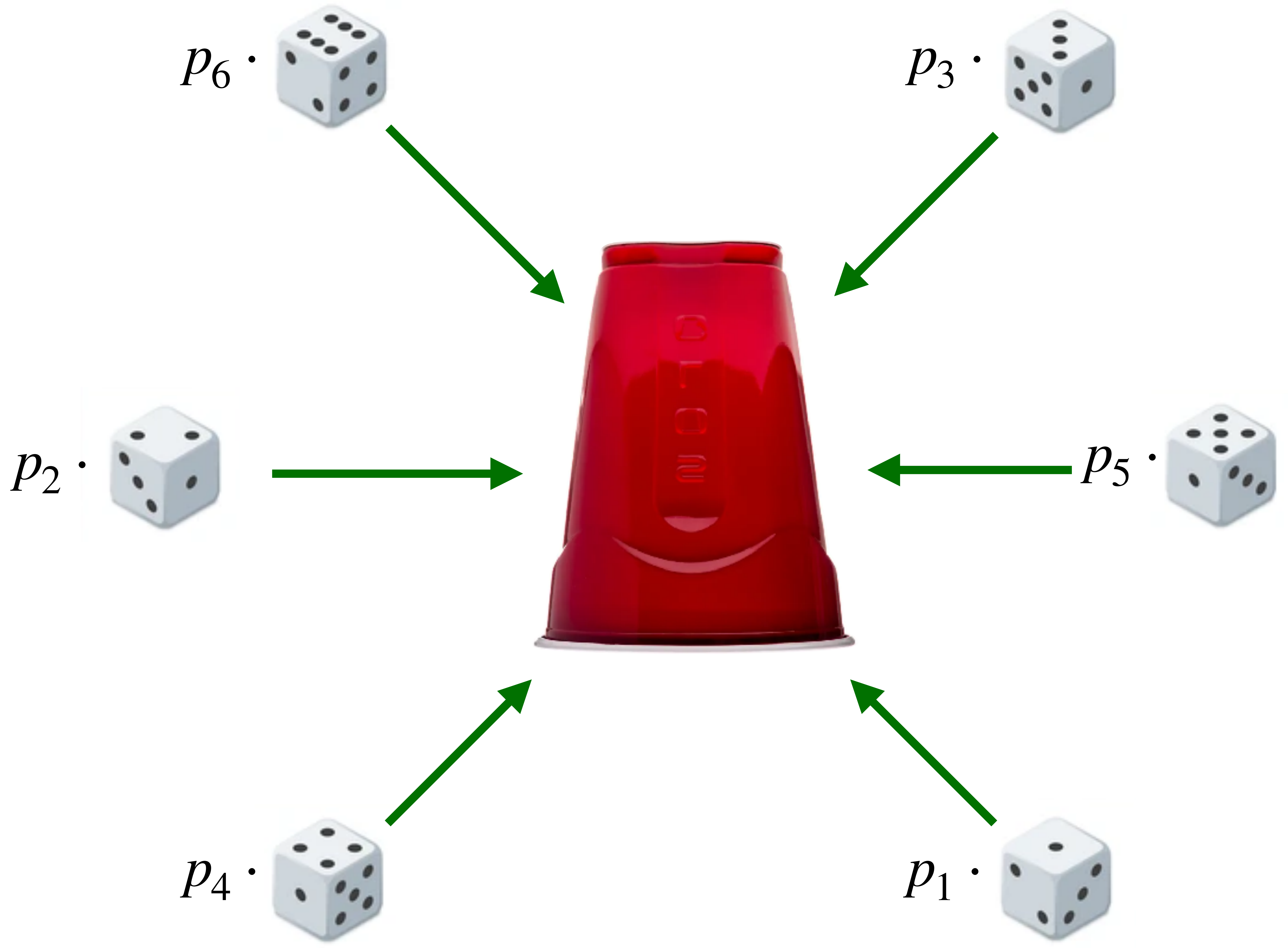


$p_6 \cdot$
$p_3 \cdot$
$p_2 \cdot$
$p_5 \cdot$
$p_4 \cdot$
$p_1 \cdot$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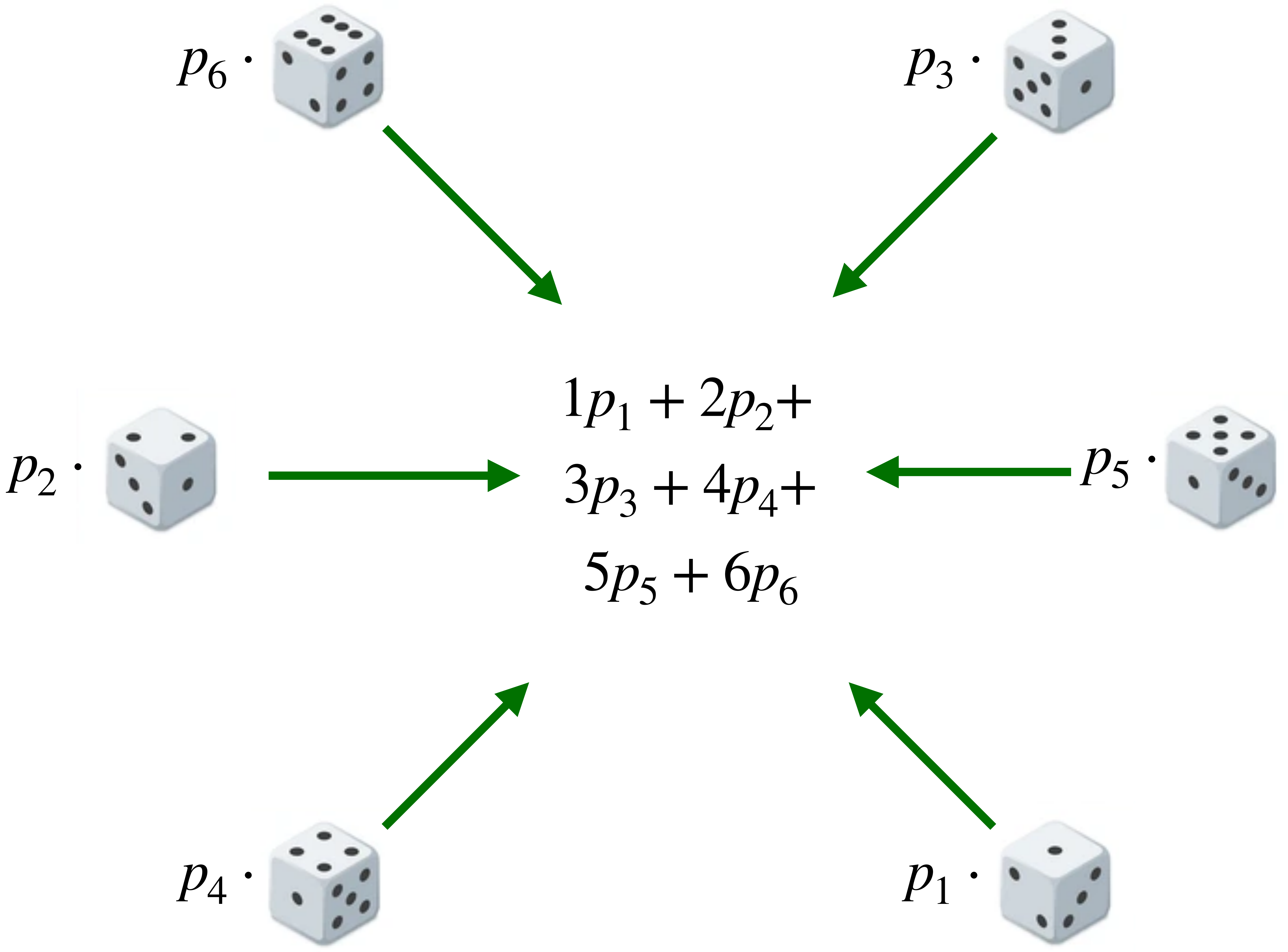
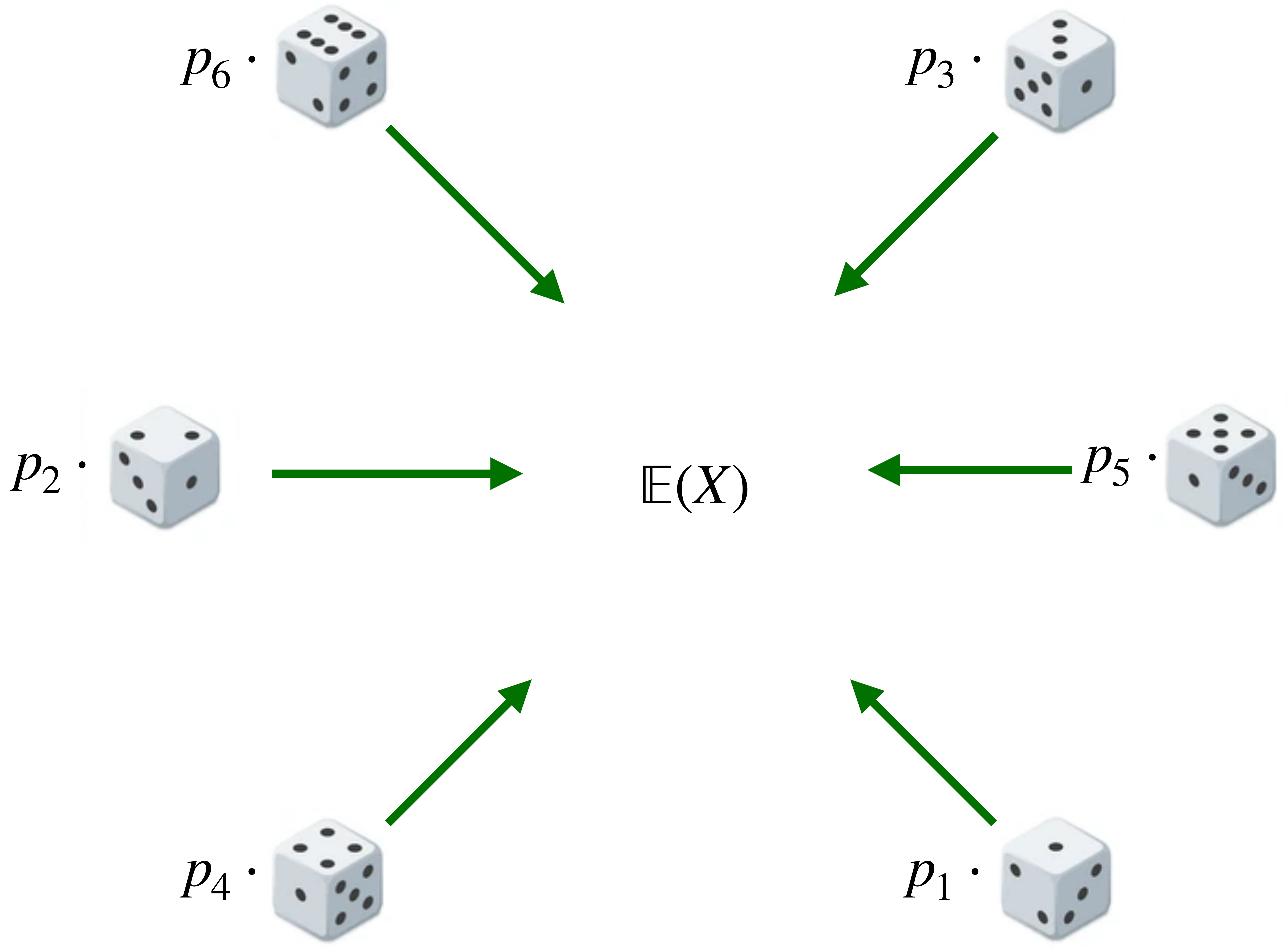
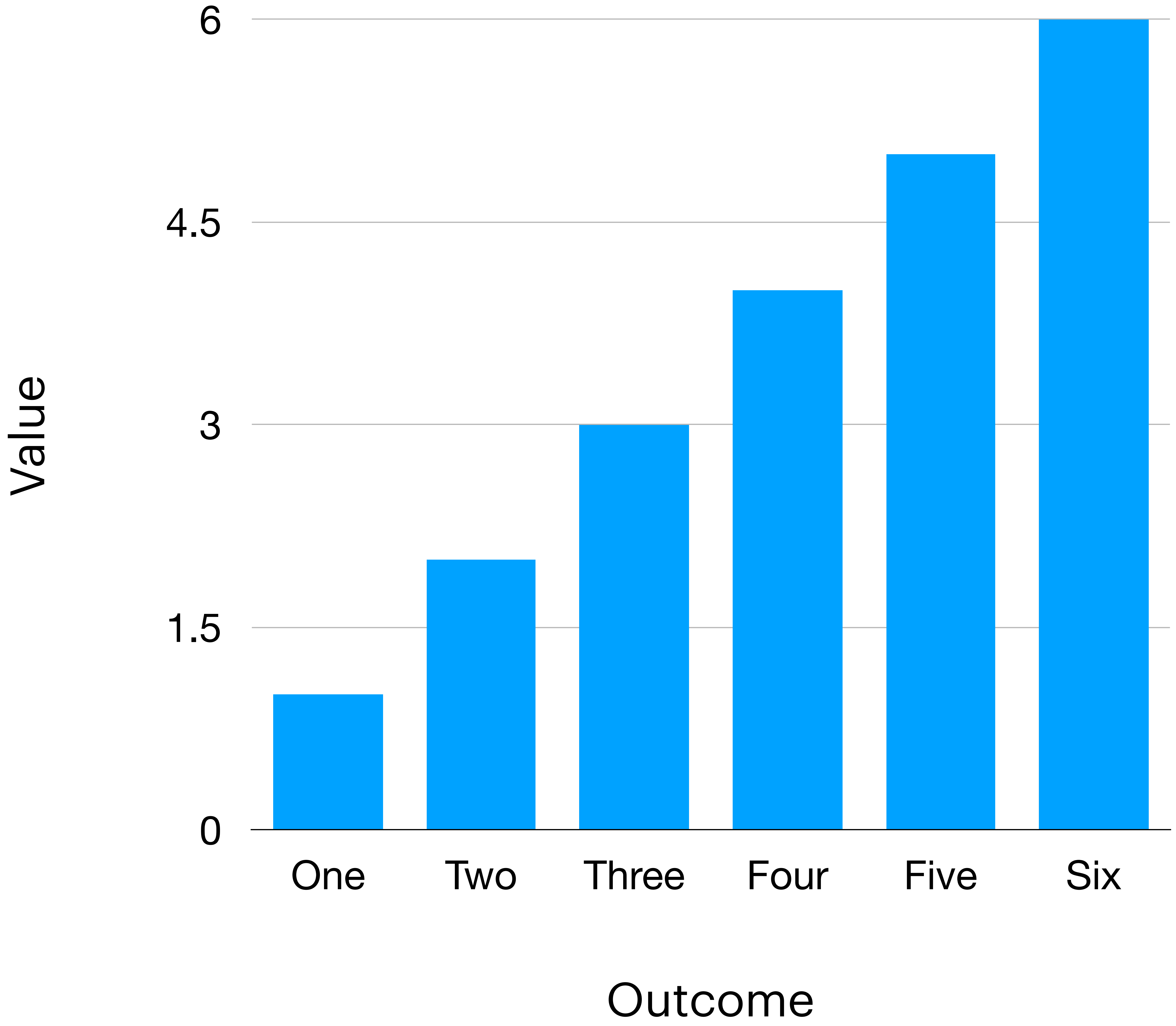
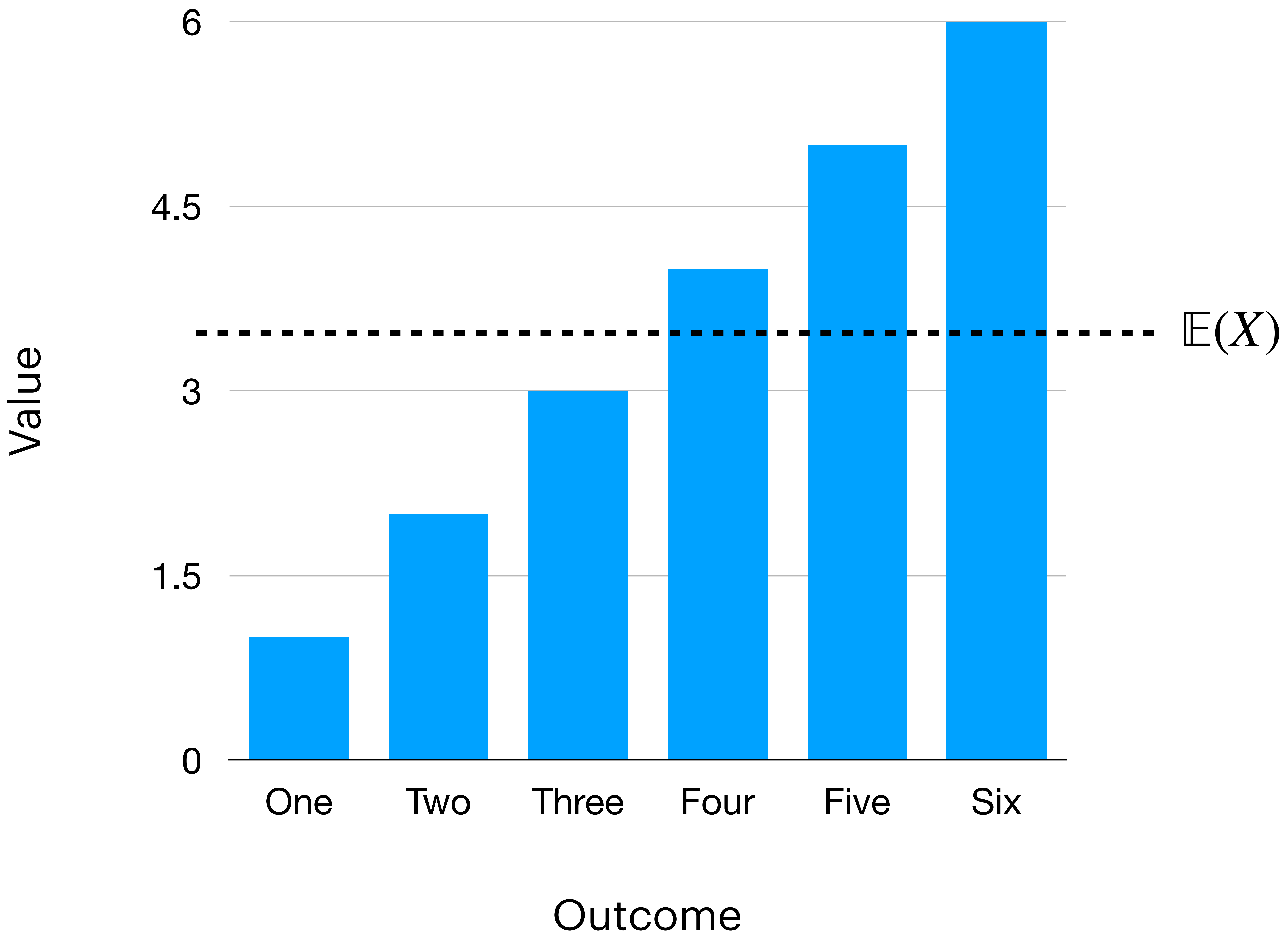
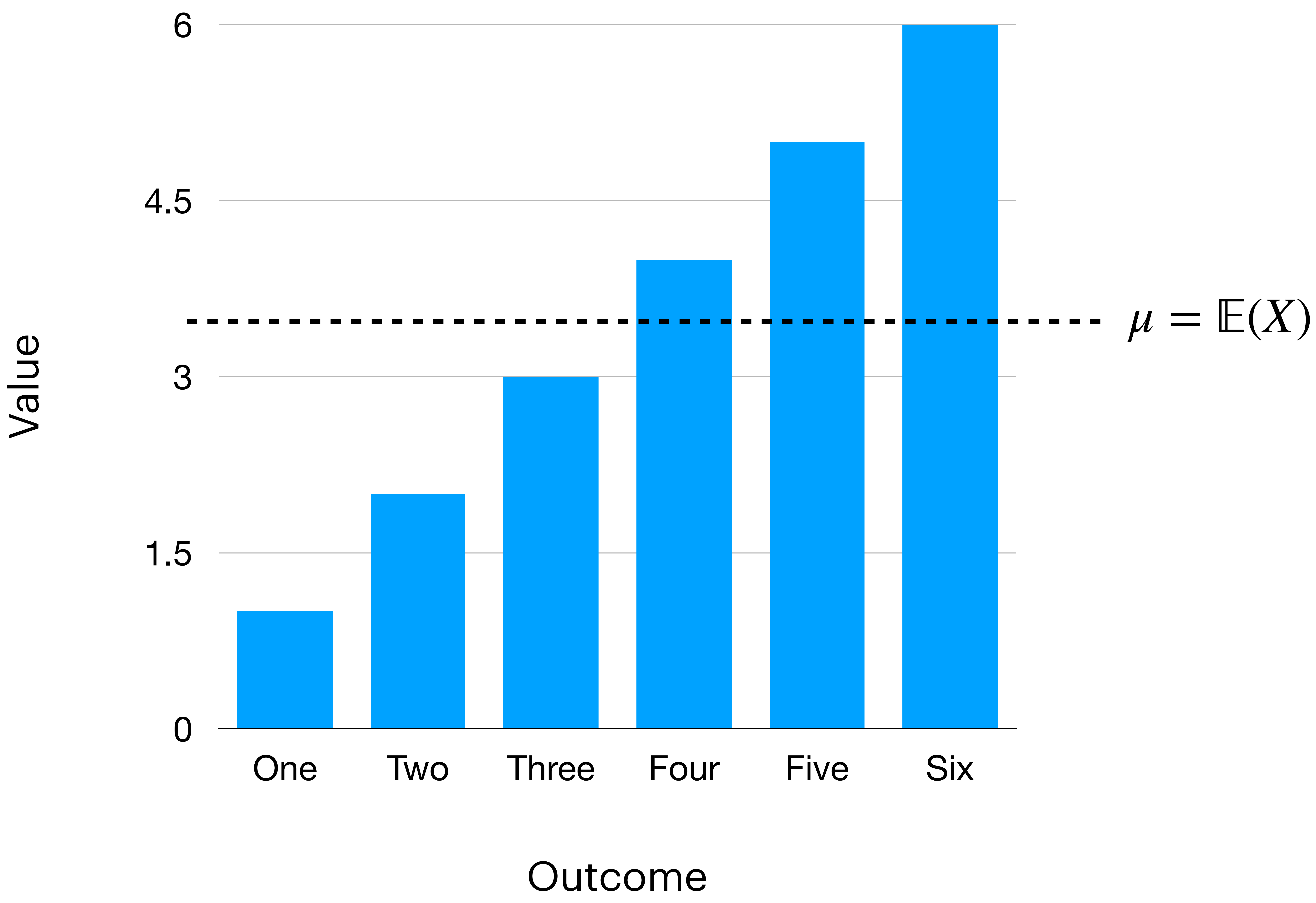
$p_1 \cdot$


$p_6 \cdot$
$p_3 \cdot$
$p_2 \cdot$
$1p_1 + 2p_2 +$
$3p_3 + 4p_4 +$
$5p_5 + 6p_6$
$p_5 \cdot$
$p_4 \cdot$
$p_1 \cdot$

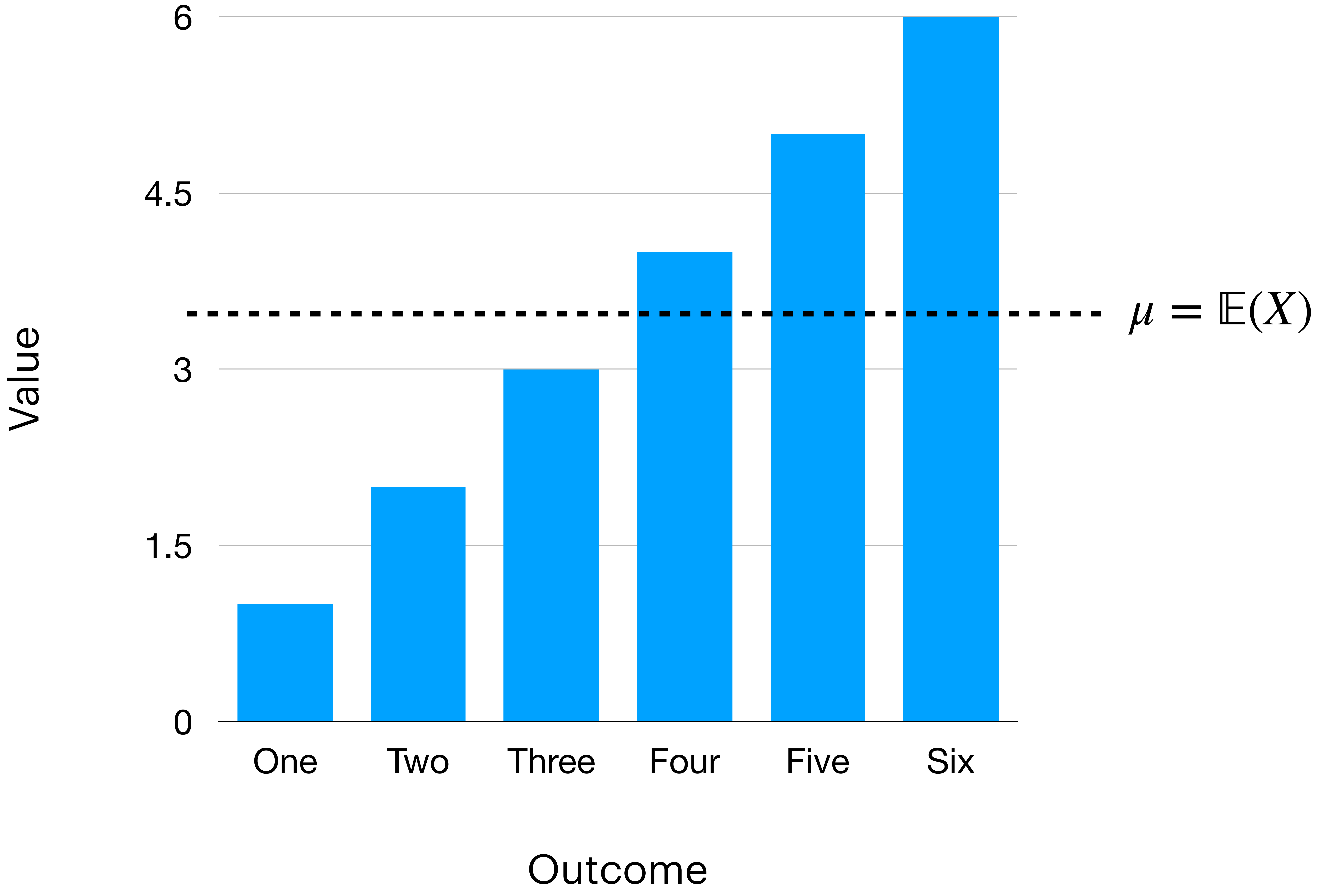

$p_6 \cdot$
$p_3 \cdot$
$p_2 \cdot$
$\mathbb{E}(X)$
$p_5 \cdot$
$p_4 \cdot$
$p_1 \cdot$


6
4.5
3
1.5
0
Value
One
Two
Three
Four
Five
Six
Outcome

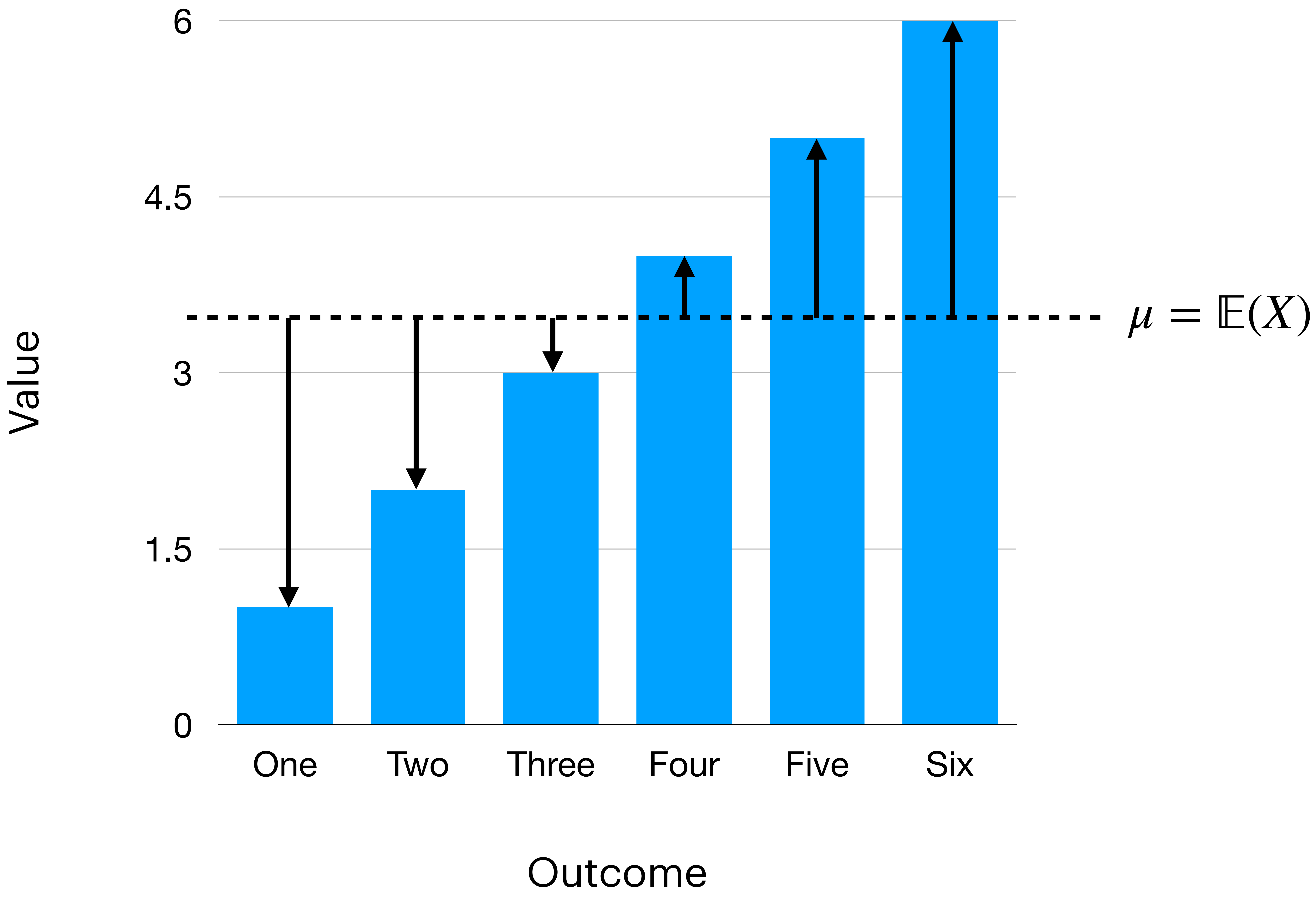

6
4.5
3
1.5
0
$\mathbb{E}(X)$
Value
One
Two
Three
Four
Five
Six
Outcome


6
4.5
3
1.5
0
Value
$\mu = \mathbb{E}(X)$
One
Two
Three
Four
Five
Six
Outcome

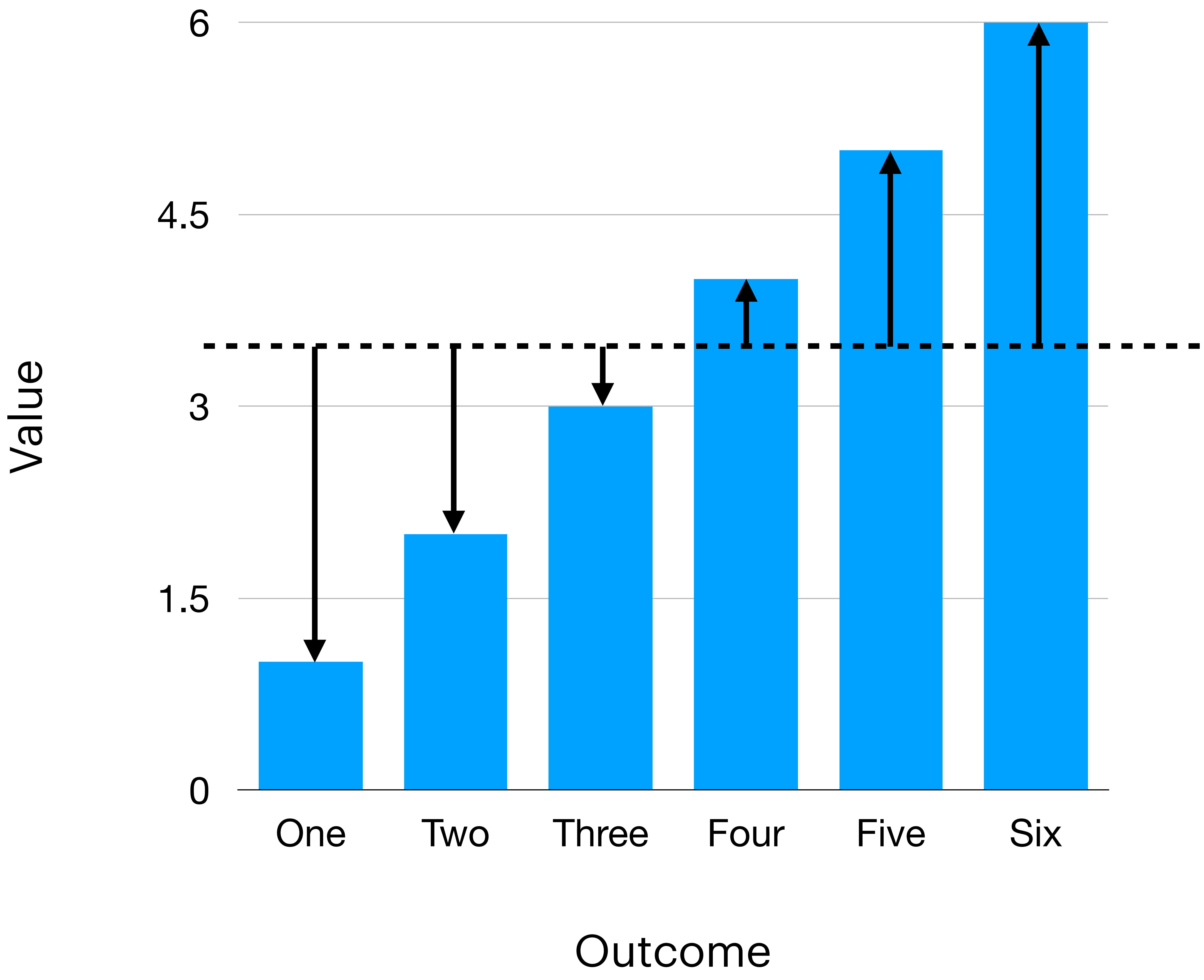
Underlying Distribution! Not the Sample!



Underlying Distribution! Not the Sample!



Underlying Distribution! Not the Sample!



$$\mu = \mathbb{E}(X)$$

$$\sigma^2 = \mathbb{E}\left[(X - \mu)^2\right]$$

Weak Law of Large Numbers

If you take the average of a large number of independent and identically distributed random variables, that average will get closer and closer to the *expected value* (the true average) as you take more and more samples.

$$S_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Sample Mean

$$\lim_{n \to \infty} P\left(| S_n - \mathbb{E}(X) | \geq \varepsilon \right) = 0$$

Expected Value

Weak Law of Large Numbers

If you take the average of a large number of independent and identically distributed random variables, that average will get closer and closer to the *expected value* (the true average) as you take more and more samples.

$$S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Sample Mean

$$\lim_{n\to\infty} P\left(|S_n - \mu| \geq \varepsilon\right) = 0$$

Distribution Mean

Weak Law of Large Numbers

If you take the average of a large number of independent and identically distributed random variables, that average will get closer and closer to the *expected value* (the true average) as you take more and more samples.

$$\lim_{n\to\infty} P\left(|S_n - \mu| \geq \varepsilon\right) = 0$$

How do we prove it?

Weak Law of Large Numbers

If you take the average of a large number of independent and identically distributed random variables, that average will get closer and closer to the *expected value* (the true average) as you take more and more samples.

$$\lim_{n \to \infty} P\left(|S_n - \mu| \geq \varepsilon\right) = 0$$

How do we prove it?

Empiricism? **Inference?** **Analogy?** **Testimony?**

Exploration V

Markov's Inequality $$P(X \geq a) \leq \frac{\mu}{a}$$

Chebyshev's Inequality For all $k > 0$, $$P\left(|X - \mu| \geq k\sigma\right) \leq \frac{1}{k^2}$$

Weak Law of Large Numbers For all $\varepsilon > 0$, $$\lim_{n\to\infty} P\left(|S_n - \mu| \geq \varepsilon\right) = 0$$

मदिरालय में कब से बैठा, पी न सका अब तक हाला
यत्न सहित भरता हूँ, कोई किंतु उलट देता प्याला |

मानव-बल के आगे निर्बल भाग्य सुना विद्यालय में
"भाग्य-प्रबल, मानव निर्बल" का पाठ पढाती मधुशाला ||

How do you know your math is right?

Exploration VI

Consider another one card game:

Player X: Can either bet one token or check

Player Y: If player X has not bet, player Y can also just check or bet one token. If player X has bet, then player Y must call.

Player X: If player Y bets, player X must call.

What is an optimal *strategy* for player X and Y?

We decide the following order on pairs of cards:

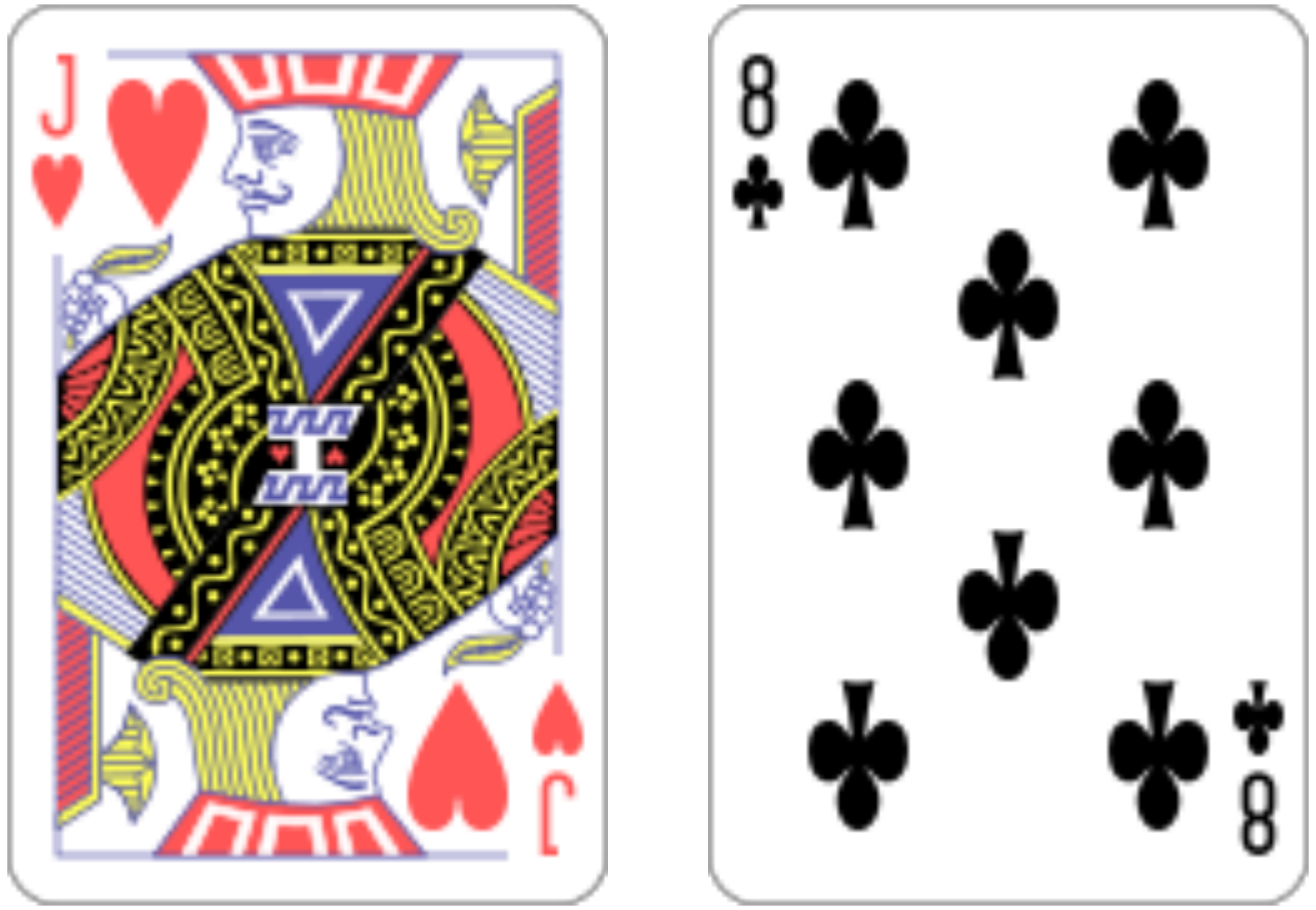
Pairs > No Pairs

The order for pairs remains:

$$A > K > Q > J > 10 > 9 > 8 > 7 > 6 > 5 > 4 > 3 > 2$$

The order for no pairs is determined by the high card.

Given a pair of cards, decide whether to bet or check.



We decide the following order on pairs of cards:

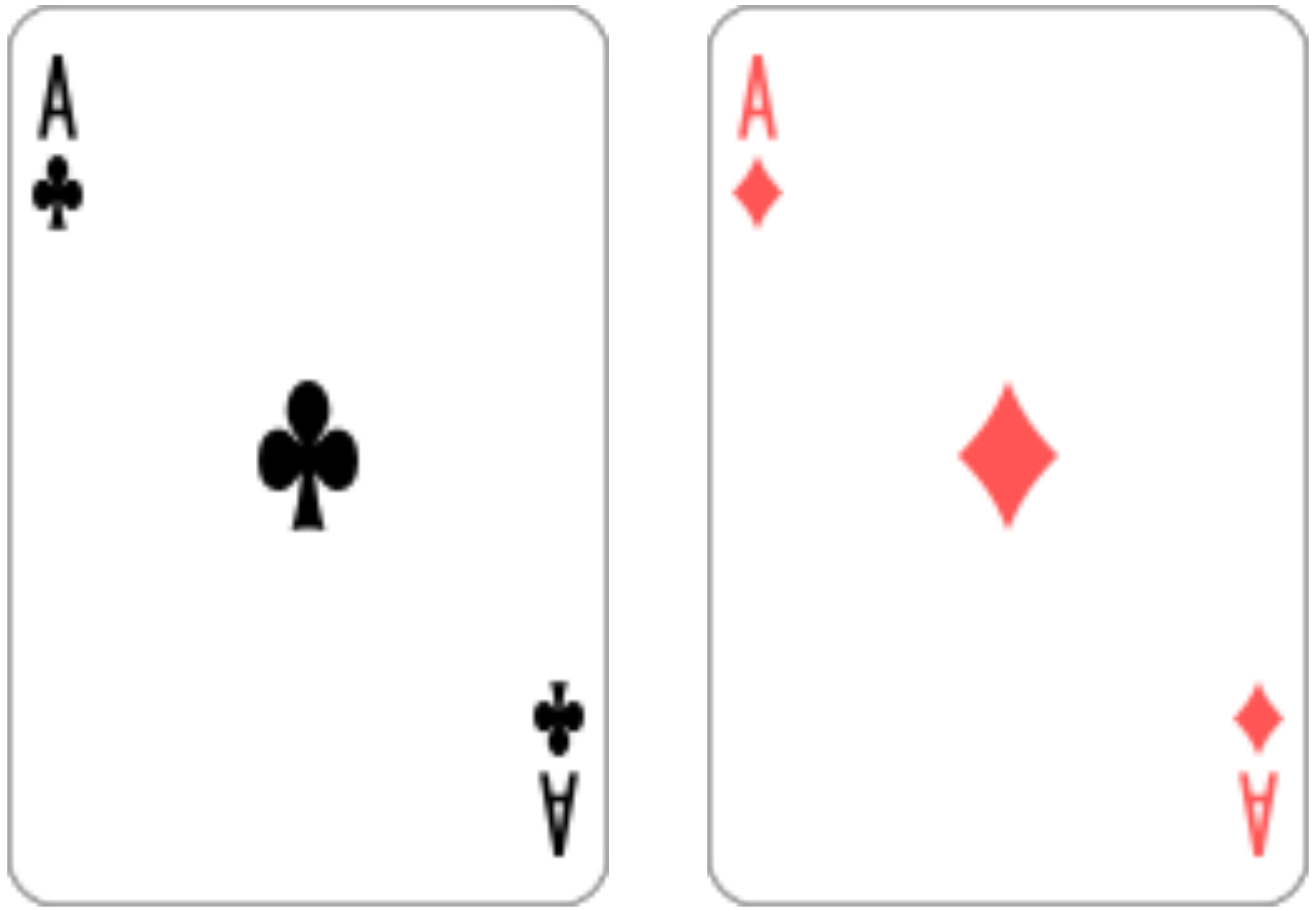
Pairs > No Pairs

The order for pairs remains:

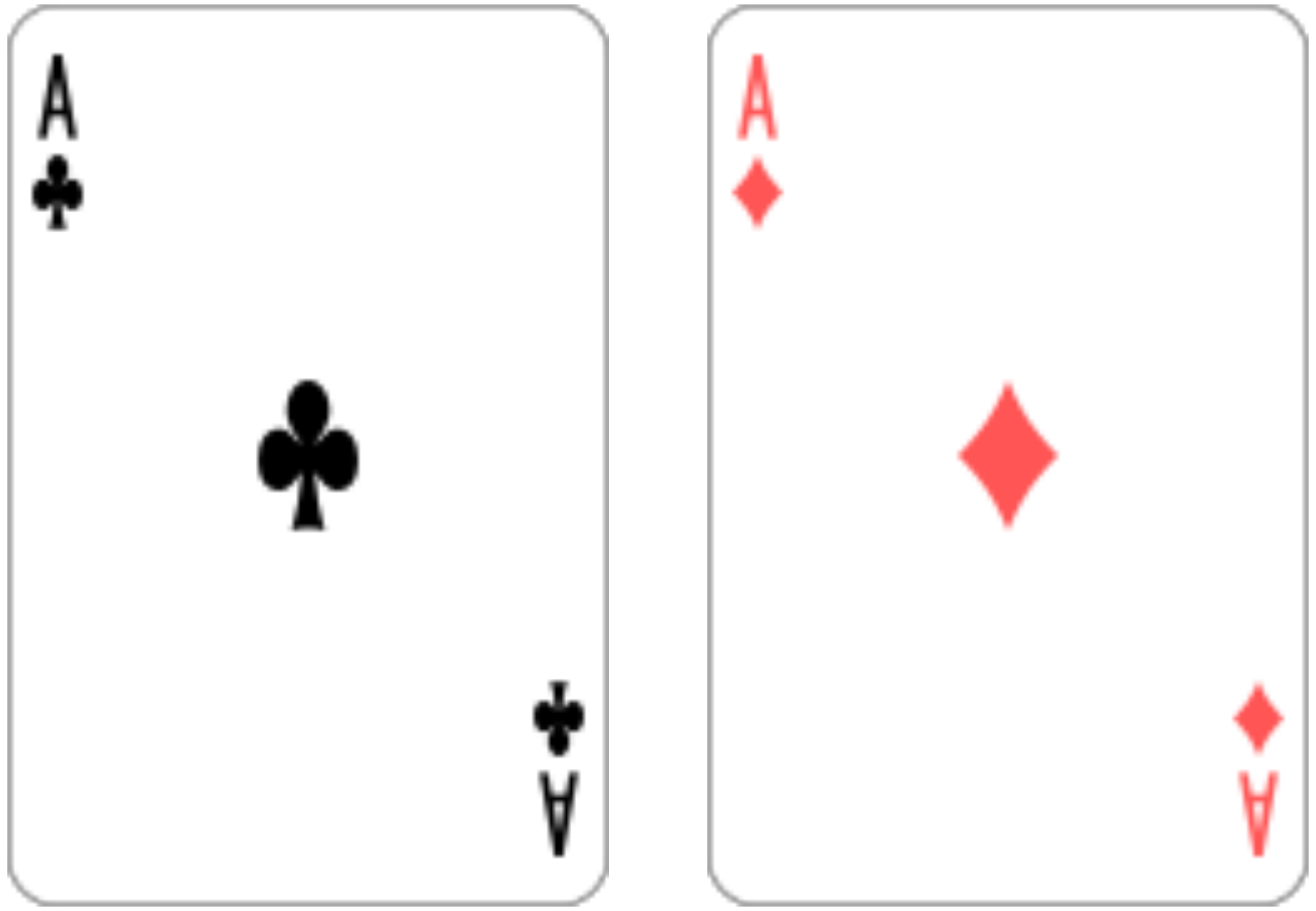
A > K > Q > J > 10 > 9 > 8 > 7 > 6 > 5 > 4 > 3 > 2

The order for no pairs is determined by the high card.

You should always bet on two aces?



What is the probability of getting two aces?



What is the probability of getting two aces?

$$\frac{12}{52 \times 51}$$

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Probability of picking an ace in one draw

Followed by another ace in the second draw

What is the probability of getting two aces?

$$\frac{12}{52 \times 51}$$

Probability of picking an ace in one draw

Followed by another ace in the second draw

$$\frac{4}{52} \times \frac{3}{51}$$

What is the probability of getting two aces?

In general:

$$P(A \cap B) = P(A) \times P(B)$$

What is the probability of getting two aces?

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Why is this wrong?

What is the probability of getting two aces?

In general:

$$P(A \cap B) = P(A) \times P(B)$$

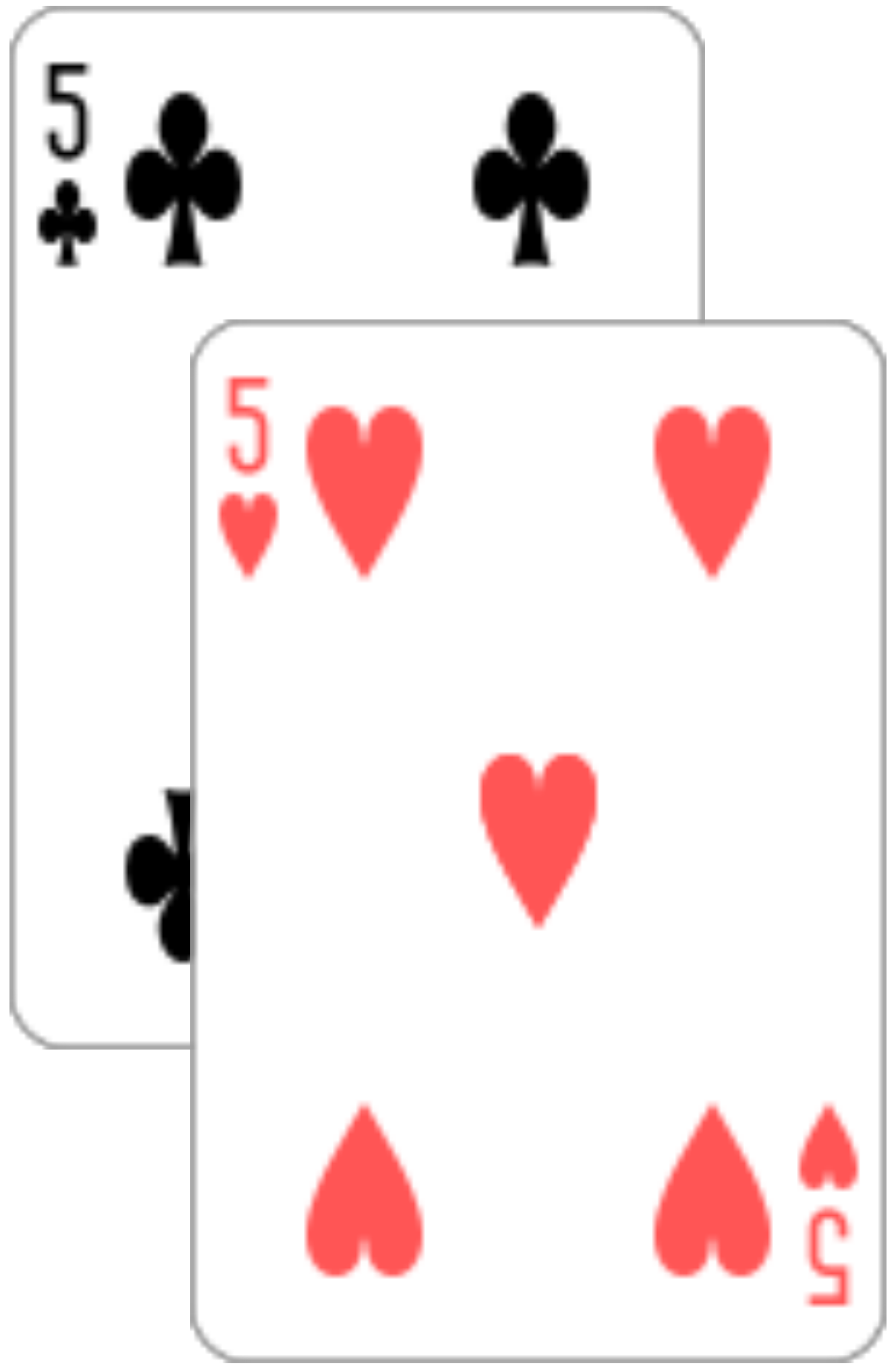
Why is this wrong?

Independence!

What is the probability of getting two aces?

When A and B are independent then:

$$P(A \cap B) = P(A) \times P(B)$$



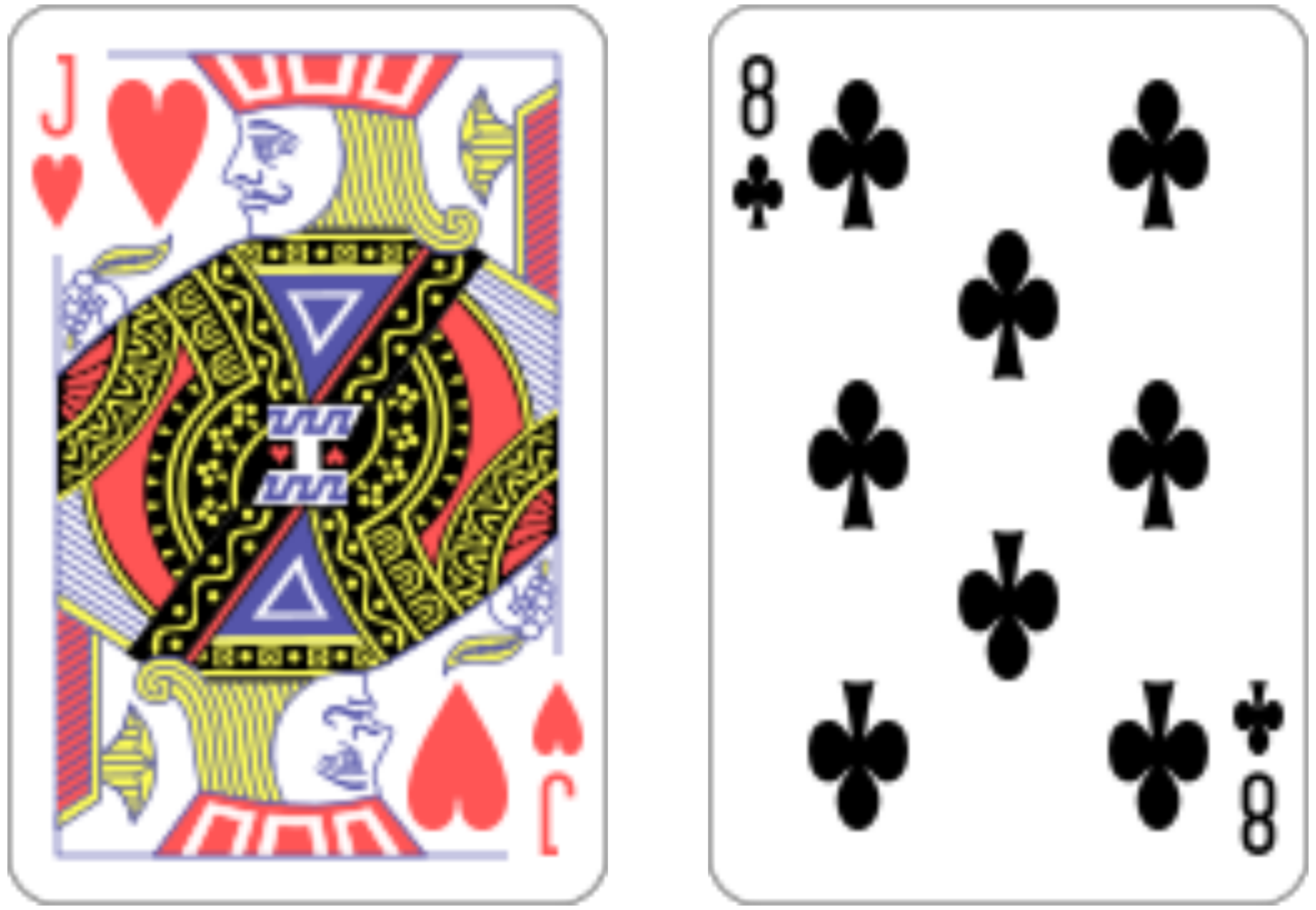
Back to our game on two cards!

Let us come up with a strategy?

Given a pair of cards, decide whether to bet or check.

Let us come up with a strategy?

Given a pair of cards, decide whether to bet or check.



What if the suit matters?

Pairs > Same Deck > No Pairs Different Suits

What if the suit matters?

Pairs > Same Deck > No Pairs Different Suits

What if there are three cards?

The Order of Hands for Teen Patti:

Three of a Kind

Straight Flush

Straight

Flush

Two of a Kind

High Card

Exploration VII

Find an optimal strategy for teen patti.