Simulation Methods - TD

2022

1 Box Muller Algorithm

1. Given X, Y two random variables, show that X and Y are two independent standard normal variables iif:

   (a) The random variable $R = X^2 + Y^2$ is exponentially distributed with rate $\mu = \frac{1}{2}$.

   (b) The point $\left(\frac{X}{\sqrt{R}}, \frac{Y}{\sqrt{R}}\right)$ is independent of R, and is uniformly distributed on the circle of radius 1 centered at the origin.

   Hint: use the following change of variables:

$$\begin{aligned} X &= \sqrt{R}\cos\theta \\ Y &= \sqrt{R}\sin\theta \end{aligned}$$

2. Explain how the previous properties are used in the Box-Muller algorithm, implement it.

3. Given U_1 and U_2 two independent r.v. from $\mathcal{U}([-1, 1])$, what is the conditional distribution of (U_1, U_2) given $U_1^2 + U_2^2 \leq 1$.

4. We define $Z = U_1^2 + U_2^2$. What is the conditional distribution of Z given $Z \leq 1$?

5. Explain how the previous properties are used in the modified Box-Muller algorithm, implement it.

6. Compare their speed.

2 Acceptance-Rejection Method

1. Let X be a standard normal variable, what is the distribution of $|X|$?

2. Let Y be an exponential r.v. with rate 1. We denote by $f_{|X|}$ (resp. f_Y) the PDF of $|X|$ (resp. Y), compute $\left\|\frac{f_{|X|}}{f_Y}\right\|_\infty$.

3. Implement the acceptance-rejection method to generate $|X|$ by generating Y. What is the acceptance probability? What is the average number of samples of Y needed to obtain a sample of $|X|$?

4. What is the conditional law of X given $|X|$?

5. Implement a way to generate a standard normal variable using the results of the previous questions. Compare its performance with the Box-Muller methods.

3 Black-Scholes Call

1. Recall the Black-Scholes dynamics for a risky asset (S_t).
2. Given $t \leq T$, what is the conditional distribution of S_T given S_t ?
3. Use the Monte Carlo method to compute the price of a 6M ATM Call option, using $S_0 = 100\$$ and yearly volatility $\sigma = 20\%$, and risk-free rate $r = 1\%$. What are the units of σ and r?
4. Give the 99% confidence interval.
5. Compare result to the Black-Scholes formula.
6. Same question with strikes ranging from 50% to 130%, with 10% steps.

4 Variance Reduction

1. How are the results affected using the antithetic variates method?
2. Use S_T as control variate in the previous exercise, what happens for the different options? What is the theoretical variance reduction if the control variate has correlation ρ with the original variable?
3. Use importance sampling by shifting the drift of S_T, what drift do we use to improve the estimation of the Monte Carlo method?

5 Asian Option ***

European Call and Put options' payoffs depend on the underlying price at maturity time T, An Asian option's payoff depends on the average price of the underlying asset over a certain period of time. Example: an Asian call's payoff is

$$(A - K)^+$$

where A is the arithmetic average defined by

$$A = \frac{1}{n} \sum_{i=1}^{n} S_{\frac{i}{n}T}$$

1. Assume that the underlying price has a Black-Scholes dynamics. Use the Monte Carlo method to price an Asian call option.
2. What is the distribution of $\left(W_{\frac{1}{n}T}, W_{\frac{2}{n}T}, \cdots, W_{\frac{n}{n}T}\right)$ where (W_t) is a Wiener process?
3. What is the distribution of $\frac{1}{n} \sum_{i=1}^{n} W_{\frac{i}{n}T}$?
4. Deduce the distribution of the geometric average G defined by
$$G = \left(\prod_{i}^{n} S_{\frac{i}{n}T} \right)^{1/n}.$$
5. Find the theoretical value of $\mathbb{E}\left[(G - K)^+\right]$.
6. Use $(G - K)^+$ as a control variate to compute $\mathbb{E}\left[(A - K)^+\right]$.

6 Van der Corput Sequence

1. Implement a method that computes the kth term Van der Corput sequence of base b.
2. Given a b-ary expansion $(a_j(k))$ of k, write an function that computes the b-ary expansion of $k+1$.
3. Use the previous function to implement an algorithm that returns the k first terms of a Van der Corput sequence of base b.
4. Use the previous method to compute the ATM option as in Exercice 3.
5. For different values of k, compute the associated standard error given by randomized QMC.
6. Plot the standard error vs computation time by varying k, for several values of b, compare the result to a plain Monte Carlo method.

7 Halton Sequence

1. Make a scatter plot of the 1000 first values of the 2D Halton sequence with bases $(2,3)$, $(2,4)$, $(2,113)$ and $(109,113)$.
2. An exchange option is a contract that gives its holder the option to exchange a stock S_1 for a stock S_2. What is the payoff of this option?
3. Assuming that S_1 and S_2 have a 2D Black-Scholes risk-neutral dynamics, with correlation ρ, compute the option price with Monte Carlo method.
4. Compute the option price with QMC method using a Halton sequence.
5. Same last question as in Exercice 6.

8 Copulas

1. Plot (surface+contour) bivariate independence, Gumbel, Clayton, comonotonicity and countermonotonicity copulas.
2. Implement algorithms to simulate from Gauss, Gumbel and Clayton copulas
3. Generate samples from a bivariate meta-Gauss distribution with $\mathcal{E}(1)$ marginals
4. Generate samples from a bivariate meta-Gumbel distribution with $\mathcal{N}(0,1)$ marginals, estimate its linear correlation, and compare its rank correlation coefficients with a bivariate Gaussian distribution with the same linear correlation coefficient.
5. Estimate and plot Kendall's tau, Spearman's rho for the Gauss, Gumbel and Clayton copulas for different parameters and compare with their theoretical values when available.

9 Bootstrap

1. Reproduce the study of the course (the data are available in DVO).
2. Using the same approach to compare the standard deviation between the two populations.