CENG 384 - Signals and Systems for Computer Engineers
Spring 2021
Homework 2

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1. (a) $x(t) - 5y(t) - 6\int y(t)\,dt = y'(t)$
$x'(t) - 5y'(t) - 6y(t) = y''(t)$
$x'(t) = y''(t) + 5y'(t) + 6y(t)$

(b) We can find y(t) by computing homogenous and particular solutions.

$y(t) = y_h(t) + y_p(t)$

To find $y_h(t)$ we assume $y_h(t) = C.e^{\alpha t}$, which is the general form.

$y_h'(t) = C\alpha e^{\alpha t}$ and $y_h''(t) = C\alpha^2 e^{\alpha t}$

$C\alpha^2 e^{\alpha t} + 5C\alpha e^{\alpha t} + 6Ce^{\alpha t} = 0$

$\alpha^2 + 5\alpha + 6 = 0$

Thus we find, $\alpha_1 = -2, \alpha_2 = -3$

$y_H(t) = C_1 e^{-2t} + C_2 e^{-3t}$

Then we find particular solution.

$y_p(t) = Ae^{-t}u(t) + Be^{-4t}u(t)$

$y_p'(t) = -Ae^{-t}u(t) - 4Be^{-4t}u(t)$

$y_p''(t) = Ae^{-t}u(t) + 16Be^{-4t}u(t)$

If we insert the variables to the equation, we get the following.

$Ae^{-t}u(t) + 16Be^{-4t}u(t) - 5Ae^{-t}u(t) - 20Be^{-4t}u(t) + 6Ae^{-t}u(t) + 6Be^{-4t}u(t) = e^{-t}u(t) + e^{-4t}u(t)$

$A = \frac{1}{2}$ and $B = \frac{1}{2}$

Thus we get, $y_p(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-4t}u(t)$

$y(t) = y_h(t) + y_p(t) = C_1 e^{-2t} + C_2 e^{-3t} + \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-4t}u(t)$

Due to initial rest, $y'(0) = y(0) = 0$

$y'(t) = -2C_1 e^{-2t} - 3C_2 e^{-3t} - \frac{1}{2}e^{-t}u(t) + -2e^{-4t}u(t)$

$y'(0) = -2C_1 - 3C_2 - \frac{1}{2} - 2 = 0$

$y(0) = C_1 + C_2 + 1 = 0$

$C_2 = -\frac{1}{2}$ and $C_1 = -\frac{1}{2}$

$y(t) = [-\frac{1}{2}e^{-2t} - \frac{1}{2}e^{-3t} - \frac{1}{2}e^{-t}u(t) + -\frac{1}{2}e^{-4t}]u(t)$

2. (a) $x[n] = \delta[n] + \delta[n-1]$, $y[n] = \delta[n-1]$

$x_1[n]$ can be written in terms of x[n]

$x_1[n] = x[n] - x[n-2]$

Similarly from linearity,

$y_1[n] = y[n] - y[n-2] = \delta[n-1] - \delta[n-3]$

(b) Since a certain solution method has not been suggested, we are going to apply Fourier transformation in this question to find the impulse response of the system.

Applying convolution property of the Fourier transform, we have the following.

$y[n] = x[n] * h[n] \rightarrow^{FT} Y(w) = X(w).H(w)$

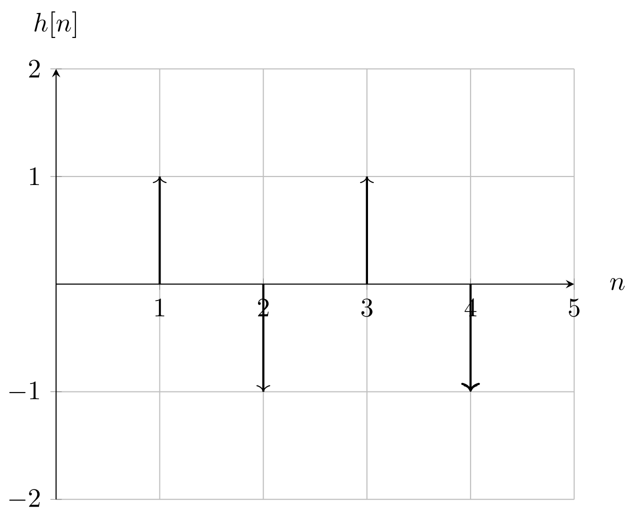
$x[n] = \delta[n] + \delta[n-1] \rightarrow^{FT} X(w) = 1 + e^{-jw}$

$y[n] = \delta[n-1] \rightarrow^{FT} Y(w) = e^{-jw}$

$H(w) = \frac{Y(w)}{X(w)} = \frac{e^{-jw}}{1+e^{-jw}} = \frac{1}{1+e^{jw}}$

$h[n] = (-1)^{n-1}u[n-1]$

Plot of $h[n]$

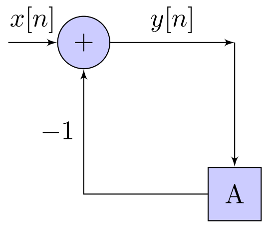


(c) $y[n] = \delta[n-1]$, $x[n] = \delta[n] + \delta[n-1]$

If we add y[n] and y[n+1] we get, $\delta[n] + \delta[n-1]$ which is equal to x[n]

$y[n] + y[n+1] = x[n]$

(d) Block diagram:



3. (a) $x[n] = \delta[n-3] + 2\delta[n+1]$, $h[n] = \delta[n-1] + 3\delta[n+2]$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Apply distributive property of convolution

$$x[n] * h[n] = \delta[n-3] * h[n] + 2\delta[n+1] * h[n]$$

$$= \delta[n-3] * \delta[n-1] + 3\delta[n-3] * \delta[n+2] + 2\delta[n+1] * \delta[n-1] + 6\delta[n+1] * \delta[n+2]$$

$$= \sum_{k=-\infty}^{\infty} \delta[k-3]\delta[n-k-1] + \sum_{k=-\infty}^{\infty} 3\delta[k-3]\delta[n-k+2] + \sum_{k=-\infty}^{\infty} 2\delta[k+1]\delta[n-k-1] + \sum_{k=-\infty}^{\infty} 6\delta[k+1]\delta[n-k+2]$$

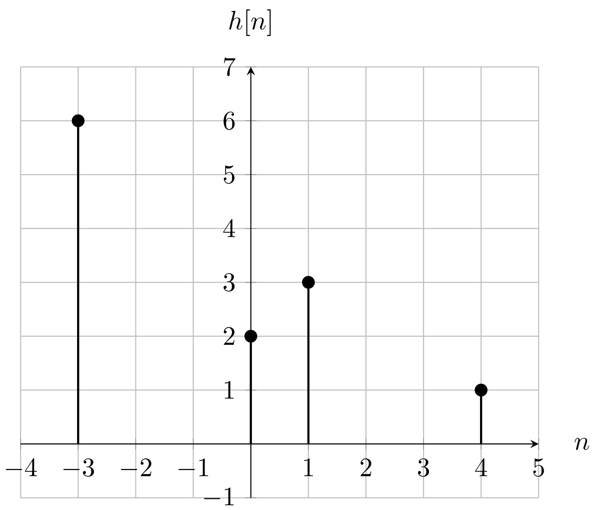
Now the "sifting" property of the δ function is that

$$\sum_{k=-\infty}^{\infty} f[k]\delta[a-k] = f[a]$$

Therefore the result is

$$= \delta[n-4] + 3\delta[n-1] + 2\delta[n] + 6\delta[n+3]$$

Plot of $x[n] * h[n]$



(b) $x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$ where

$x[n] = u[n+3] - u[n]$ and $h[n] = u[n-1] - u[n-3]$

As it can be seen from the equations x[n] and h[n] are square functions. When we convolute square functions we need to find some intervals, since multiplication of the function may show different behaviours in different intervals.

if $n-1 < -3$ then $x[k]h[n-k] = 0$ since no overlapping

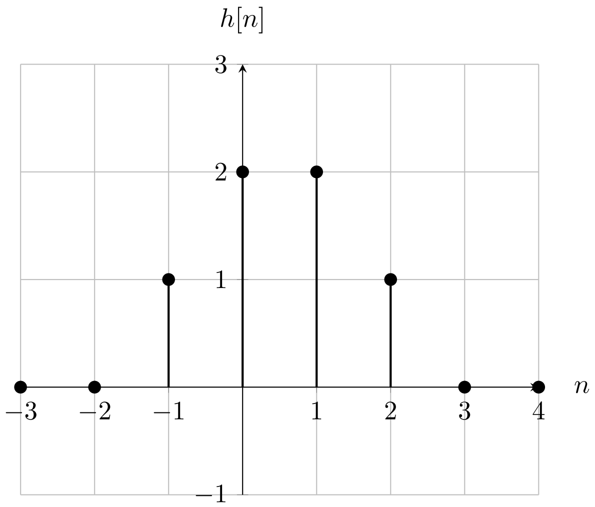
if $n-1 \geq -3$ and $n-3 < -3$, $-2 \leq n < 0$ then $x[k]h[n-k] = \sum_{k=-2}^{0}(n+2)$

if $n-3 \geq -3$ and $n-1 < 0$, $0 \leq n < 1$ then $x[k]h[n-k] = 2$

if $n-3 \leq 0$ and $n-1 > 0$, $1 \leq n < 3$ then $x[k]h[n-k] = \sum_{k=1}^{3}(3-n)$

if $n-3 \geq 0$ then 0

Plot of $x[n] * h[n]$



4. (a) $y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$

   $= \int_{-\infty}^{\infty} e^{-2(t-\tau)}u(t-\tau)e^{-3\tau}u(\tau)d\tau$

   $= \int_0^t e^{-2t+2\tau}e^{-3\tau}d\tau$

   $= e^{-2t}\int_0^t e^{-\tau}d\tau = e^{-2t}(1-e^{-t})u(t)$

   (b) $h(t) = e^{2t}u(t)$ and $x(t) = u(t) - u(t-2)$

   In order to calculate h(t) we need to divide the integration into three parts, since integration of $x(\tau)h(t-\tau)$ shows different behaviours in different intervals

   $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

   if $t < 0$ then y(t) = 0 since no overlap

   if $0 \leq t \leq 2$ then $y(t) = \int_0^t 1.e^{2(t-\tau)}d\tau$

   $= e^{2t}\int_0^t 1.e^{-2\tau}d\tau = e^{2t}.-\frac{1}{2}.(e^{-2t}-1)$

   $= -\frac{1}{2}(1-e^{2t}) = \frac{e^{2t}-1}{2}$

   if $t > 2$ then $y(t) = \int_0^2 1.e^{2(t-\tau)}d\tau$

   $= e^{2t}\int_0^2 e^{-2\tau}d\tau = \frac{e^{2t}}{2}(1-e^{-4})$

   As a result

   y(t)=0, $t < 0$
   $y(t) = \frac{e^{2t}-1}{2}$, $0 \leq t \leq 2$
   $y(t) = \frac{e^{2t}}{2}(1-e^{-4})$, $t > 2$

5. (a) $s[n] = nu[n]$

   We can obtain impulse response from step response as following.

   $h[n] = s[n] - s[n-1]$

$h[n] = nu[n] - (n-1)u[n-1]$

$h[n] = nu[n] - nu[n-1] + u[n-1]$

$h[n] = u[n-1]$

(b) $x[n] * h[n] = \delta[n] - \delta[n-1]$

$x[n] * u[n-1] = \delta[n] - \delta[n-1]$

$x[n] * u[n-1] = u[n] - u[n-1] - (u[n-1] - u[n-2])$

$x[n] * u[n-1] = u[n] - 2u[n-1] + 2u[n-2]$

By using the "sifting" property of delta function, we get the following.

$x[n] = \delta[n+1] - 2\delta[n] + \delta[n-1]$

(c) $x[n] = \delta[n+1] - 2\delta[n] + \delta[n-1]$

$y[n] = \delta[n] - \delta[n-1]$

$y[n+1] = \delta[n+1] - \delta[n]$

Then,

$y[n+1] - y[n] = \delta[n+1] - 2\delta[n] + \delta[n-1]$

As a result, $y[n+1] - y[n] = x[n]$

6. $s(t) = \frac{1}{2}t^2u(t)$

If we take the derivative, we get impulse response.

$h(t) = \frac{d}{dt}(s(t)) = tu(t)$

$y(t) = x(t) * h(t) = \int_{\infty}^{-\infty} x(\tau)h(t-\tau)d\tau$

for $t \leq 0$, $x(t) * h(t) = 0$

for $t > 0$, $\int_0^t e^{-\tau}(t-\tau)d\tau$

$= \int_0^t (te^{-\tau} - \tau e^{-\tau})d\tau = t\int_0^t e^{-\tau}d\tau - \int_0^t \tau e^{-\tau}d\tau$

If we integrate we find, $t\int_0^t e^{-\tau}d\tau = t(-e^{-t}+1)$ (1)

If we apply integral by parts for the rest of the equation,

Let $u = \tau$ and $dv = e^{-\tau}$

$du = d\tau$ and $v = -e^{-\tau}$

$\tau(-e^{-\tau}) - \int(-e^{-t})d\tau$

$= (-\tau e^{-\tau} - e^{-\tau}|_0^t) = -(-te^{-t} - e^{-t} - (-1)) = te^{-t} + e^{-t} - 1$ (2)

Combining the results (1) and (2) we get, $e^{-t}u(t) + tu(t) - u(t)$

7. (a) Impulse response of the parallel configuration $\delta(t-3)$ and $\delta(t-5)$ is $\delta(t-3) + \delta(t-5)$.

Impulse response of the parallel configuration and u(t) is $u(t) * (\delta(t-3) + \delta(t-5))$.

Therefore, $h(t) = u(t) * (\delta(t-3) + \delta(t-5))$

Convolving a signal with delta function leaves the signal unchanged.

However, a shift in the delta function results in the same shift in the signal.

$h(t) = u(t-3) + u(t-5)$

(b) $y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$

$y(t) = \int_{-\infty}^{\infty}(u(\tau-3) + u(\tau-5))e^{-3(t-\tau)}u(t-\tau)d\tau$

If $t \leq 3$, $x(t) * h(t) = 0$

If $3 < t < 5$, $x(t) * h(t) = \int_3^t e^{-3(t-\tau)d\tau} = e^{-3t}\int_3^t e^{3\tau}d\tau$

$= e^{-3t}(\frac{e^{3t}-e^9}{3})$

If $t \geq 5$, $x(t) * h(t) = \int_3^5 e^{-3(t-\tau)}d\tau + \int_5^t 2e^{-3(t-\tau)}d\tau$

$= e^{-3t}\int_3^5 e^{3\tau}d\tau + 2e^{-3t}\int_5^t e^{3\tau}d\tau$

$(e^{-3t}(\frac{e^{15}-e^9}{3}) + 2e^{-3t}(\frac{e^{3t}-e^{15}}{3}))$

$$y(t) = \begin{cases} 0, & -\infty < t \leq 3 \\ e^{-3t}(\frac{e^{3t}-e^9}{3}), & 3 < t \leq 5 \\ (e^{-3t}(\frac{e^{15}-e^9}{3}) + 2e^{-3t}(\frac{e^{3t}-e^{15}}{3})), & 5 < t < \infty \end{cases}$$

(c) $h(t) = u(t-3) + u(t-5)$

$\frac{dh(t)}{dt} = \frac{du(t-3)}{dt} + \frac{du(t-5)}{dt}$

$\frac{dh(t)}{dt}) = \delta(t-3) + \delta(t-5)$

$g(t) = (\delta(t-3) + \delta(t-5)) * x(t)$

By using distrubution property of convolution;

$g(t) = \delta(t-3) * x(t) + \delta(t-5) * x(t)$

By using the "sifting" property at the point of the occurrence, such that:

$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$. We get the following.

$g(t) = x(t-3) + x(t-5)$

if $x(t) = e^{-3t}u(t)$ then

$g(t) = e^{-3(t-3)}u(t-3) + e^{-3(t-5)}u(t-5)$